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Constructing All Magic Squares of Order Three

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Abstract

We find by applying MacMahon's partition analysis that all magic squares of order three, up to rotations and reflections, are of two types, each generated by three basis elements. A combinatorial proof of this fact is given.

Key words: magic squares, linear Diophantine equations

1 Introduction and Main Results

A *magic square* of *order* n is an n by n matrix with distinct entries in $\mathbb{N}$, the set of nonnegative integers, such that every row sum, column sum, and (two) diagonal sum is equal to the same number m, the *magic number*. Adding 1 to every entry will give us a *traditional magic square* which has only positive entries. A magic square is *pure* if the entries are the consecutive numbers from 0 to n^2-1, and hence it has magic number $3\binom{n+1}{3}$. *Weak magic squares*, magic squares without the restriction of distinct elements, have been studied in [1; 2; 4; 6] by using the rich theory of counting solutions of a system of linear Diophantine equations, or equivalently, counting lattice points of a convex polytope. For further references, see [8, Chapter 4.6]. These methods also apply to counting magic squares, but give no obvious reason why a simple solution as in Theorem 1 exists.

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[1] The author would like to thank the anonymous referees for helpful comments. This work was done during the author's stay at Brandeis University in USA.

Magic squares have been objects of study for centuries. As Pickover wrote in his book[7, p. 60]:

> *. . . the holy grail of magic squares creation would be to discover a method that would generate every possible arrangement for a square of a given size. Such a solution is probably not discoverable.*

This "holy grail" could be achieved by first finding the complete generating function (which is a rational function) for magic squares of a given size, and then writing the generating function as a sum of simple rational functions, the series expansion of which has only nonnegative coefficients.

We achieve this for magic squares of order 3, as given in Theorem 1.

Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & 4 \\ 2 & 6 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 3 \\ 2 & 2 & 2 \\ 1 & 4 & 1 \end{bmatrix}, \tag{1}$$

$$T_1 = \begin{bmatrix} 7 & 0 & 5 \\ 2 & 4 & 6 \\ 3 & 8 & 1 \end{bmatrix}, \qquad T_2 = \begin{bmatrix} 8 & 0 & 7 \\ 4 & 5 & 6 \\ 3 & 10 & 2 \end{bmatrix}. \tag{2}$$

Then they are related as follows:

$$B = C + D, \qquad T_1 = B + C, \qquad T_2 = B + D. \tag{3}$$

If we let C' be obtained from C by reflecting in the vertical axis, then we have one more relation: $D = C + C'$. It is straightforward to check that A, C, and D are linearly independent.

In fact, A, C, D are the three basis elements that generate all magic squares of order 3, and T_1 is the unique magic square with magic number 12 up to rotations and reflections.

Theorem 1 *Every magic square of order three, up to rotations and reflections, can be written uniquely as either $T_1+iA+jB+kC$ or $T_2+iA+jB+kD$, where i, j, k are nonnegative integers.*

Corollary 2 *Up to rotations and reflections, traditional magic squares are generated by $i'A + j'C + k'D$ for positive integers i', j', k' with $j' \neq k'$.*

Theorem 1 says that the set of magic squares, as lattice points, is a disjoint union of $16 = 8 \cdot 2$ *polyhedrons* that are isomorphic to $\mathbb{N}^3$, where the factor

8 is the order of the dihedral group of rotations and reflections. We will give a combinatorial proof of this result in the next section. It is well-known that the magic number of a magic square is always a multiple of 3. Let s mod 3 be the remainder of s when divided by 3.

Corollary 3 *The number of magic squares of order* 3 *with magic number* $3s$ *and its associated generating function are given by*

$$\begin{aligned}\frac{8t^4}{(1-t)^2(1-t^3)} &+ \frac{8t^5}{(1-t)(1-t^2)(1-t^5)} \\ &= \sum_{s\geq 0}\left(2s^2 - \frac{20}{3}s + 1 - (-1)^s + \frac{8}{3}(s \bmod 3)\right)t^s \\ &= 8\left(t^4 + 3t^5 + 4t^6 + 7t^7 + 10t^8 + 13t^9 + 17t^{10} + \cdots\right).\end{aligned}$$

2 A Combinatorial Proof

In what follows, magic squares are always of order 3 unless specified otherwise.

Let M be a magic square with magic number m. We write

$$M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}, \tag{4}$$

where

C1: Every row sum, column sum, and diagonal sum is equal to m.
C2: The entries of M are distinct nonnegative integers.

Rotating or reflecting M will give us different magic squares. Without loss of generality, we can assume that c_3 is smaller than a_1, a_3 and c_1, and that $c_1 < a_3$. Also by subtracting A times the minimal entry of M from M, we can assume that 0 is an entry of M. Then M satisfies the following two extra conditions:

C3: One of the entries of M is 0.
C4: $c_3 < a_1, a_3, c_1$, and $c_1 < a_3$.

In fact, **C4** can be replaced with

C4′: $c_3 < c_1 < a_3 < a_1$,

which follows from the equalities of the sum of the two diagonals.

If M satisfies the above four conditions, then we say that M is a *reduced* magic square. It is well-known that the magic number m is $m = 3b_2$, as can be easily seen by subtracting the equalities for the first and third row from the sum of the equalities for the two diagonals and the second column. Let $m = 3s$ or equivalently $s = b_2$.

Lemma 4 *If M is a reduced magic square, then $a_2 = 0$, and $c_2 = 2s$.*

PROOF. Since $b_2 = s = m/3$, where m is the magic number, the last statement $c_2 = 2s$ follows from $a_2 = 0$, which is what we are going to show now.

In a reduced magic square M, a_1 and a_3 are the largest two entries among the four corners a_1, a_3, c_1, c_3. It follows from the equalities of the first and third row sums and column sums that $a_2 < b_1, c_2, b_3$. We see that $b_2 = s \geq 4$, since all entries are distinct nonnegative integers. It remains to show that c_3 cannot be 0. Assume that c_3 equals 0. Then $a_1 = 2s$ by the equality of the diagonal sum $a_1 + b_2 + c_3 = 3s$. By investigating the equalities of the first row sum and the first column sum, we get $a_3 < s - 1$ and $c_1 < s - 1$, contradicting the equality for the diagonal (a_3, b_2, c_1).

Lemma 5 *A reduced magic square M can be uniquely written as $T_1 + \alpha C + \beta D$, where $\alpha \geq -1$ and $\beta \geq 0$ are integers.*

PROOF. To see the existence, we use Lemma 4. Assuming that $c_3 = r$ and $b_2 = s$, we obtain all the entries of M by the condition **C1** for row sums, column sums, and diagonal sums:

$$M = \begin{bmatrix} 2s - r & 0 & s + r \\ 2r & s & 2s - 2r \\ s - r & 2s & r \end{bmatrix}.$$

Comparing the above matrix with

$$M = T_1 + \alpha C + \beta D = \begin{bmatrix} 7 + 2\alpha + 3\beta & 0 & 5 + \alpha + 3\beta \\ 2 + 2\beta & 4 + \alpha + 2\beta & 6 + 2\alpha + 2\beta \\ 3 + \alpha + \beta & 8 + 2\alpha + 4\beta & 1 + \beta \end{bmatrix}, \qquad (5)$$

we solve uniquely for α and β:

$$\alpha = s - 2r - 2, \text{ and } \beta = r - 1.$$

Consequently,

$$s = \alpha + 2\beta + 4 \text{ and } r = \beta + 1.$$

Comparing (4) with (5), we get that $c_1 = 3 + \alpha + \beta$ and $c_3 = 1 + \beta$. Since $c_3 \neq a_2 = 0$, we have $\beta \geq 0$. Then by condition **C4′**, especially $c_1 > c_3 > 0$, we deduce that $\alpha \geq -1$. This completes the proof of the existence.

The uniqueness follows from the above proof, and also from the fact that C and D are linearly independent.

We are now ready to give the proof of our main theorem.

Proofs of Theorem 1 and Corollary 2. We first show the uniqueness. When written in terms of basis elements A, C and D by using (1, 2, 3), we have

$$T_1 + iA + jB + kC = iA + (j + k + 2)C + (j + 1)D, \tag{6}$$
$$T_2 + iA + jB + kC = iA + (j + 1)C + (j + k + 2)D. \tag{7}$$

Thus if written as $iA + j'C + k'D$, then $j' > k'$ corresponds to the "T_1" class magic squares, and $j' < k'$ corresponds to the "T_2" class magic squares. This not only shows the uniqueness, but also shows Corollary 2 if we let $i' = i + 1$.

Given a magic square M, let i be the minimum of the entries of M. Then up to rotations and reflections, we can assume $M' = M - iA$ to be a reduced magic square. By Lemma 5, M' can be uniquely written as $T_1 + \alpha C + \beta D$, with $\alpha \geq -1$ and $\beta \geq 0$. We need to show that M equals either (6) or (7).

If $\alpha \geq \beta \geq 0$, M' can be rewritten (recall that $B = C + D$) as $T_1 + \beta B + (\alpha - \beta)C$. Hence we let $j = \beta \geq 0$ and $k = \alpha - \beta \geq 0$.

If $\alpha < \beta$, M' can be rewritten (recall that $T_1 + D = T_2 + C$) as

$$T_1 + \alpha B + (\beta - \alpha)D = T_2 + C + \alpha B + (\beta - \alpha - 1)D = T_2 + (\alpha + 1)B + (\beta - \alpha - 2)D.$$

Thus we let $j = \alpha + 1 \geq 0$ and $k = \beta - \alpha - 2 \geq -1$. It only remains to exclude the case $k = -1$, which is equivalent to $\beta = \alpha + 1$. But in this case

$$M' = T_1 + (\beta - 1)C + \beta D = \begin{bmatrix} 5 + 5\beta & 0 & 4 + 4\beta \\ 2 + 2\beta & 3 + 3\beta & 4 + 4\beta \\ 2 + 2\beta & 6 + 6\beta & 1 + \beta \end{bmatrix},$$

which is not a magic square because of having equal entries.

3 Further Discussion

The combinatorial proof in the previous section seems unlikely to be applicable to magic squares of higher order. We describe how we discovered Theorem 1 by using MacMahon's partition analysis, which has been restudied by Andrews and his coauthors in a series of papers (see e.g., [3; 4; 5]).

MacMahon's idea is to use new variables to replace linear constraints. For example, if we want to count nonnegative integral solutions of the linear equation $a_1 + a_2 - a_3 = 0$, we can simply write the generating function as

$$\sum_{\substack{a_1,a_2,a_3 \geq 0 \\ a_1+a_2-a_3=0}} x_1^{a_1} x_2^{a_2} x_3^{a_3} = \sum_{a_1,a_2,a_3 \geq 0} \operatorname*{CT}_{\lambda} \lambda^{a_1+a_2-a_3} x_1^{a_1} x_2^{a_2} x_3^{a_3}$$

$$= \operatorname*{CT}_{\lambda} \frac{1}{(1-\lambda x_1)(1-\lambda x_2)(1-x_3/\lambda)},$$

where CT_λ means to take the constant term in λ. Then the counting problem is converted to evaluating the constant term of a special rational function, which can be done by computer as in [3; 5; 9]. For a rigorous description about how the above works in the general situation of a field of iterated Laurent series, the reader is referred to [9].

Using a computer we can easily obtain the generating function of weak magic squares of order 3:

$$G = \frac{(1 - tx_4x_7x_9x_6x_2x_3x_5x_8x_1)(1 + tx_4x_7x_9x_6x_2x_3x_5x_8x_1)^2}{(1 - tx_1x_5x_9{x_4}^2{x_8}^2{x_3}^2)(1 - tx_7x_5x_3{x_4}^2{x_2}^2{x_9}^2)}$$
$$\times \frac{1}{(1 - tx_7x_5x_3{x_1}^2{x_8}^2{x_6}^2)(1 - tx_1x_5x_9{x_7}^2{x_2}^2{x_6}^2)},$$

where the exponent of t represents $m/3$ since the m is always divisible by 3, and the exponents in $x_1, \dots x_9$ represents $a_1, a_2, a_3, b_1, \dots$.

To obtain the generating function for magic squares, we shall take only terms in G that have different exponents in the x's. This can be done by inclusion and exclusion, but an alternative iterative way is easy to perform. First, we eliminate those terms with the same exponents in x_1 and x_2; next, we eliminate those terms with same exponents in x_1 and x_3, and so on.

To eliminate those terms with same exponents in x_i and x_j from G' (a possible generating function in a middle step), we subtract from G' the diagonal $\operatorname{diag}_{x_i,x_j} G'$ with respect to x_i and x_j, where

$$\operatorname{diag}_{x,y} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} b_{r,s} x^r y^s = \sum_{r=0}^{\infty} b_{r,r} x^r y^r,$$

and we use the formula for a rational power series $F(x,y)$:

$$\operatorname{diag}_{x,y} F(x,y) = \operatorname*{CT}_{\lambda_1,\lambda_2} \frac{1}{1 - xy/(\lambda_1\lambda_2)} F(\lambda_1,\lambda_2).$$

The generating function of all magic squares of order 3 is still complicated. We can add the extra constraints that $c_3 < c_1 < a_3 < a_1$ to eliminate rotations and reflections. It suffices to find a way to add the constraint that the exponent of x_9 is smaller than that of x_7. The other constraints can be added iteratively. We omit the details here.

Finally we obtain the generating function of desired magic squares:

$$\begin{aligned} &\frac{t^4 {x_7}^3 {x_5}^4 {x_3}^5 {x_1}^7 {x_8}^8 {x_6}^6 x_9 {x_4}^2}{(1 - t x_7 x_5 x_3 {x_1}^2 {x_8}^2 {x_6}^2)\,(1 - t x_4 x_7 x_9 x_6 x_2 x_3 x_5 x_8 x_1)} \\ &\quad\times \frac{(1 + t x_1 x_5 x_9 {x_4}^2 {x_8}^2 {x_3}^2 - 2t^2 {x_5}^2 x_9 {x_4}^2 {x_8}^4 {x_1}^3 {x_3}^3 x_7 {x_6}^2)}{(1 - t^2 {x_5}^2 x_9 {x_4}^2 {x_8}^4 {x_1}^3 {x_3}^3 x_7 {x_6}^2)\,(1 - t^3 {x_7}^2 {x_5}^3 {x_1}^5 {x_8}^6 {x_3}^4 {x_6}^4 x_9 {x_4}^2)}. \end{aligned} \tag{8}$$

We observe that the factor inside the parenthesis of the numerator can be rewritten as

$$\begin{aligned} &1 + t x_1 x_5 x_9 {x_4}^2 {x_8}^2 {x_3}^2 - 2t^2 {x_5}^2 x_9 {x_4}^2 {x_8}^4 {x_1}^3 {x_3}^3 x_7 {x_6}^2 \\ &= (1 - t^2 {x_5}^2 x_9 {x_4}^2 {x_8}^4 {x_1}^3 {x_3}^3 x_7 {x_6}^2) + t x_1 x_5 x_9 {x_4}^2 {x_8}^2 {x_3}^2 (1 - t x_7 x_5 x_3 {x_1}^2 {x_8}^2 {x_6}^2). \end{aligned}$$

For M given by (4), we let x^M denote $x_1^{a_1} x_2^{a_2} \cdots x_9^{c_3}$. Recalling (1), (2), and (3), we can then rewrite (8) as

$$\begin{aligned} &\frac{t^4 x^{T_1}(1 - t^2 x^D) + t^5 x^{T_2}(1 - t x^C)}{(1 - t x^C)(1 - t x^A)(1 - t^2 x^D)(1 - t^3 x^B)} \\ &\qquad = \frac{t^4 x^{T_1}}{(1 - t x^C)(1 - t x^A)(1 - t^3 x^B)} + \frac{t^5 x^{T_2}}{(1 - t x^A)(1 - t^2 x^D)(1 - t^3 x^B)}. \end{aligned}$$

Theorem 1 is thus concluded.

The order 4 case would be truly hard. The difficulty lies in the fact that there are 880 pure magic squares of order 4 (up to rotations and reflections), which suggests that there will be at least 880 simple rational functions. The current package as provided in [9] is not powerful enough to find an explicit generating function for magic squares of order 4 analogous to (8).

References

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[3] G. E. Andrews, P. Paule, and A. Riese, *MacMahon's partition analysis III: the Omega package*, European J. Combin., **22** (2001), 887–904.

[4] G. E. Andrews, P. Paule, A. Riese, and V. Strehl, *MacMahon's partition analysis V: Bijections, recursions, and magic squares*, Algebraic combinatorics and applications (Gößweinstein, 1999), 1–39, Springer, Berlin, 2001.

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[7] C. Pickover, *The Zen of Magic Squares, Circles, and Stars: An Exhibition of Surprising Structures Across Dimensions*, Princeton University Press, Princeton, NJ, 2002.

[8] R. P. Stanley, *Enumerative Combinatorics*, 2nd ed., vol. 1, Cambridge University Press, 1997.

[9] G. Xin, *A fast algorithm for MacMahon's partition analysis*, Electron. J. Combin. **11** (2004), Research Paper 58, 20 pp. (electronic).

Response to Reviewers

I would like to thank the referee's helpful comments. I adopt most of the suggestions.

----Action for the first report-----

- Section 1, line 2: "s" at the end of "integer".

GX: Changed.

- Section 1, line 3: "is equal to" rather than "equals to".

GX: Changed.

- Two lines before Theorem 1: delete "s" at the end of "squares" and "numbers".

GX: Changed

- In Corollary 3, it couldn't hurt to insert the generating function
  as a sum of the two terms
  $8t^4/(1-t)^2/(1-t^3) + 8t^5/(1-t)/(1-t^2)/(1-t^3)$
  at the beginning for clarity.

GX: I think replacing the generating function by the above formula is more suitable.

- Page 4, next to last line: It seems to me that the condition
  $\beta >= 0$ comes not from C4' but from the fact that the entries are
  distinct and $a_2=0$ and $c_3 > 0$, so $1+\beta >0$.

GX: I have made the changes. What you said is right, but we still need C4' to claim the other equality.

- In the proof of Theorem 1, the remark "it is straightforward to check
  that ... give different magic squares for all nonnegative integers
  i,j,k" is glossing over things a bit. There is a lot to check,
  albeit all straightforward. It must be checked that the squares are
  magic, that
  $T_1 + iA + jB + kC = T_1 + i'A + j'B + k'C$ iff $(i,j,k)=(i',j',k')$,

that

$T_2 + iA + jB + kD = T_2 + i'A + j'B + k'D$ iff $(i,j,k)=(i',j',k')$,

and that it is never true for nonnegative integers i,j,k,i',j',k', that

$T_1 + iA + jB + kC = T_2 + i'A + j'B + k'D$.

In checking these I looked for nice arguments that gave some insight into the structure of the solutions. For example, no square in the "T_1" class can be in the "T_2" class, since squares in the first class have $b_1 < c_1$, whereas those in the second have $b_1 > c_1$.

GX: I have made changes accordingly. It is clearer to write them in terms of basis elements A, C, D.

- Page 6, next-to-last paragraph, line 2: "be" should be "by";
  line 4: insert "the" before "same".

GX: inserted.

- Page 6, last paragraph:, line 2: delete "by".

GX: deleted.

- Page 7, last paragraph, line 1: I think "truely" should be "truly".
  line2: add "s" to the end of "square".

GX: corrected.

----Action for the second report-------

Comments to the author:

- Corollary 2: I recommend to include its proof into the paper. (It goes nicely and could be kept sufficiently short.)

GX: I include the proof together with the proof of Theorem 1. For the statement of the corollary, I added ``up to rotations and reflections".

- Bottom of page 3: "It is well-known that the magic number m is $m=3 b_2$." This fact (already used at the top of the same page) is

the basis for all mathematical considerations in this note.
Therefore a reference to a proof of it should be given.

GX: I added a one sentence explanation of this fact.

- On page 6, concerning computer evaluation, a reference to [7] is given. The Omega package by Riese et al. provides an alternative which should be cited as well. I am referring to the following algorithmic versions of MacMahon's partition analysis:

  G.E. Andrews, P. Paule, and A. Riese,
  "MacMahon's Partition Analysis III: The Omega Package",
  European J. Combin. 22 (2001), 887-904,

and its improved version,

  G.E. Andrews, P. Paule, and A. Riese,
  "MacMahon's Partition Analysis VI: A New Reduction
  Algorithm", Ann. Comb. 5 (2001), 251-270.

Corresponding implementations have been released before [7]. In addition, a general efficiency comparison between the latter implementation and [7] is still open - at least to the reviewer's opinion.

GX: I have added the two references and cited them in the second paragraph of section 3.

- The paper should be checked once again with respect to typos. (E.g., on page 2, "... is the unique magic squares with magic numbers 12 up to rotations ...")

GX: I have checked twice for typos.