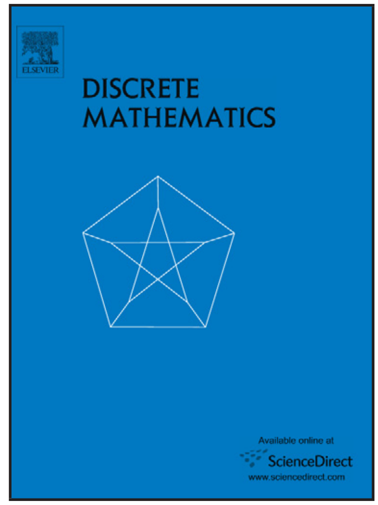

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2-BINARY TREES: BIJECTIONS AND RELATED ISSUES

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Abstract

A *2-binary tree* is a binary rooted tree whose root is colored black and the other vertices are either black or white. We present several bijections concerning different types of 2-binary trees as well as other combinatorial structures such as ternary trees, non-crossing trees, Schröder paths, Motzkin paths and Dyck paths. We also obtain a number of enumeration results with respect to certain statistics.

Keywords: Binary tree, Ternary tree, Non-crossing tree, Schröder path, Motzkin path, Dyck path.

AMS Mathematical Subject Classifications: 05A05, 05A15, 05C05.

1. Introduction

Trees play an important role in combinatorics [19] and appear in a large number of applications in other branches of mathematics, in physics, in biology, *etc.* For example, trees are natural structures for representing hierarchical data (see *e.g.* [7]). A *rooted tree* is a tree with a distinct vertex called the *root*. In what follows, we always draw a rooted tree with the root on the top level. A vertex w is said to be a *child* or *successor* of a vertex v if w is on the next lower level connected to v; the vertex v is then said to be the *parent* of w. The *degree* of v is the total number of its children. A *leaf* is a vertex with degree 0, that is a vertex with no child. A rooted tree in which the children of each vertex are ordered is also called a *plane tree*. A classical algorithm is used throughout this paper to run over all the vertices in a plane tree, which is the *preorder traversal* (or the worm crawling around the tree, see [2], [9, p.239], [12, pp. 21-27] and [19, pp. 33-34] for details and applications), i.e. visiting the root and then the subtrees from left to right recursively. A *binary tree* is a plane tree in which each vertex has at most two children and each child of a vertex is designated as its *left* or *right child*. Define the *leftmost (resp. rightmost) path* of a vertex v in a binary tree to be the longest path starting from v with each vertex being the left (resp. right) child of its parent. Now we give our main combinatorial structure as follows.

Definition 1.1. *A 2-binary tree is a binary tree with each of its vertices colored with one of two colors, for instance, black or white and the root is colored black.*

We would like to emphasize that 2-binary trees always refer to black rooted trees; if the root is allowed to be white also, we will refer to 2-binary trees with white root permitted.

It is well known that the number of binary trees with n vertices is the n-th *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$ (see [18, Sequence A000108] and [19, Exe. 6.19]). Hence, the number of 2-binary trees with n vertices equals $\frac{2^{n-1}}{n+1}\binom{2n}{n}$ for $n \geq 1$. According to the definition, an edge e in a 2-binary tree is of the following eight types: , and . We say a 2-binary tree T is *e-free* if and only if there is no edge of type e in T.

In 1994, Pallo [14] considered a special class of 2-binary trees, namely *hybrid binary trees*, which are 2-binary trees with no restriction on the color of the root and no edge of type . A straightforward computation gives that the number of hybrid binary trees with n vertices equals $\sum_{j=n+1}^{2n+1} \frac{1}{j}\binom{j}{n+1}\binom{j}{2n+1-j}$. In this paper, we consider other classes of 2-binary trees, and give their enumerations and statistical properties in a bijective way with other combinatorial structures. For this purpose we need the following definitions.

A *ternary tree* is a plane tree in which each vertex has degree 0 or 3, and each child of a vertex is designated as its *left*, *middle*, or *right* child (see [10, 15]). In the literature, this kind of tree is often called complete ternary tree.

A *non-crossing tree* is a plane tree drawn with n vertices on a circle such that the edges lie entirely within the circle and do not cross. Non-crossing trees have been studied by Chen et al. [1], Deutsch et al. [5, 6], Flajolet et al. [8], Hough [11], Noy et al. [13], and Panholzer et al. [15].

A *Schröder path of length* $2n$ is a lattice path going from $(0,0)$ to $(2n,0)$ consisting of up steps $U=(1,1)$, down steps $D=(1,-1)$ and horizontal steps $H=(2,0)$, which never goes below the x-axis. The number of Schröder paths of length $2n$ is called the n-th Schröder number S_n ([18, Sequence A006318] and [19, Exe. 6.39]).

A *little Schröder path of length* $2n$ is a Schröder path of length $2n$ with no peak at level 1, where a *peak* of a lattice path is an up step followed immediately by a down step, say UD, and its level is defined by the y-coordinate of the intersection point of its up and down steps. The number of little Schröder paths of length $2n$ is called the n-th little Schröder number L_n ([18, Sequence A001003] and [19, Exe. 6.39]). In our paper, we give a new combinatorial explanation for the well-known relation $S_n = 2L_n$ for $n \geq 1$ (see [3, 17, 20] for the previous proofs).

A *Motzkin path of length* n is a lattice path going from $(0,0)$ to $(n,0)$ consisting of up steps $U=(1,1)$, down steps $D=(1,-1)$ and horizontal steps $H=(1,0)$, which never goes below the x-axis. The number of Motzkin paths of length n is called the n-th Motzkin number M_n ([18, Sequence A001006] and [19, Exe. 6.38]).

A *Dyck path of length* $2n$ is a lattice path going from $(0,0)$ to $(2n,0)$ consisting of up steps $U=(1,1)$ and down steps $D=(1,-1)$, which never goes below the x-axis. It is known that the number of Dyck paths of length $2n$ is the n-th Catalan number C_n.

In Section 2, we give a bijection between the set of -free 2-binary trees with n vertices and the set of ternary trees with n internal vertices. In Section 3, we define a representative set for -free 2-binary trees with n vertices and then construct a bijection between this representative set and the set of non-crossing trees with $n+1$ vertices on a circle. In Section 4, we relate 2-binary trees with many kinds of lattice paths. More precisely, we present not only a bijection between the set of $\{\diagdown, \diagdown\}$-free 2-binary trees with $n+1$ vertices and the set of Schröder paths of length $2n$, but also a bijection between the set of $\{\diagdown, \diagdown\}$-free 2-binary trees with n vertices and the set of little Schröder paths of length $2n$. Moreover, we create a bijection between the set of $\{\diagup, \diagup, \diagup, \diagdown\}$-free 2-binary trees with n vertices and the set of Motzkin paths of length n. Finally we conclude this paper by describing a bijection between the set of $\{\diagup, \diagup, \diagup\}$-free 2-binary trees with n vertices and the set of Dyck paths of length $2n$.

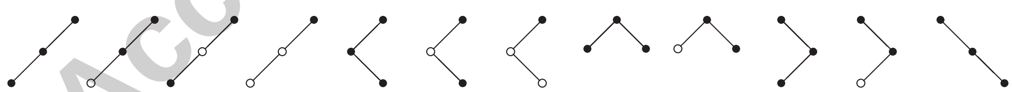
2. 2-binary trees and ternary trees

In this section, we present a bijection between the set of $\diagdown$-free 2-binary trees with n vertices and the set of ternary trees with n internal vertices.

We begin by pointing out that from an enumerative point of view there are only two interesting cases to consider for any e-free 2-binary tree, where $e \in \{\diagup, \diagup, \diagup, \diagup, \diagdown, \diagdown, \diagdown, \diagdown\}$. Given an e-free 2-binary tree T with $e \in \{\diagup, \diagup, \diagup, \diagup\}$, all the other e-free 2-binary trees with $e \in \{\diagdown, \diagdown, \diagdown, \diagdown\}$ can be obtained by the following map $f : T \to T'$, where $T' = f(T)$ is the tree obtained from T by defining the left (resp. right) child of each vertex in T' as the right (resp. left) child of that vertex in T. Clearly, $f^2 = id$ and a 2-binary tree T is e-free if and only if the 2-binary tree $T' = f(T)$ is $f(e)$-free.

Recall that Pallo [14] proved that the generating function for hybrid binary trees, which is denoted by $h(x)$, is given by $1 + \frac{xg^2(x)}{1-xg(x)}$, where $g(x)$ satisfies $g(x) = (1 + xg(x))(1 + xg^2(x))$. Using the Lagrange inversion formula, see [19, Sec. 5.4] and [21, Sec. 5.1], we know that the number of hybrid binary trees with n edges is $\sum_{j=n+1}^{2n+1} \frac{1}{j}\binom{j}{n+1}\binom{j}{2n+1-j}$ (see also Sequence A007863 in [18]). What's more, the generating function for $\diagdown$-free 2-binary trees with n vertices is $1 + xh^2(x)$ and the generating function for $\diagdown$-free 2-binary trees is $h(x)(1 - xh(x))$, and their enumerations are given by Sequence A011270 and Sequence A011272 in [18], respectively. Hence we only need to enumerate the remaining two cases, namely $\diagdown$-free 2-binary trees and $\diagdown$-free 2-binary trees.

We now consider the set of $\diagdown$-free 2-binary trees. As an example, the twelve $\diagdown$-free 2-binary trees with 3 vertices are illustrated below:



Theorem 2.1. *There is a bijection between the set of $\diagdown$-free 2-binary trees with n vertices and the set of ternary trees with n internal vertices.*

Proof. We recursively define a map α from the set of $\diagdown$-free 2-binary trees with n vertices to the set of ternary trees with n internal vertices. Let T be a $\diagdown$-free 2-binary tree with n vertices. We have three steps for the map α. In each step, we use α_i ($i = 1, 2, 3$) to denote the map.

Step 1:

We can decompose T into several $\diagdown$-free 2-binary subtrees by eliminating the edges $\diagup$ and $\diagup$ in the leftmost path of the black root v_1. Then for each subtree, the leftmost path of the root consists of a black root and consecutive white vertices. Hence, according to the decomposition, T can be displayed uniquely as the left structure in Figure 1, where B_ℓ and B_r are $\searrow$-free 2-binary trees and R_i with $2 \le i \le m$ are $\searrow$-free 2-binary trees with white root permitted (all of them can be empty also). In this case, we proceed to construct an internal vertex in the ternary tree from a vertex in T as illustrated in Figure 1, where the right picture in Figure 1 is the ternary tree corresponding to $\alpha_1(T)$ with B'_ℓ (resp. B'_r, R'_i) corresponding to the ternary tree of B_ℓ (resp. B_r, R_i).

Furthermore, the position of R'_i in Figure 1 will be determined by the color of the root of R_i. If the root of R_i is black, we assign the subtree $R'_i = \alpha_1(R_i)$ as the middle subtree of v_i. Otherwise, if the root of R_i is white, we assign the subtree $R'_i = \alpha_2(R_i)$ as the right subtree of v_i, where α_2 will be determined by Step 2.

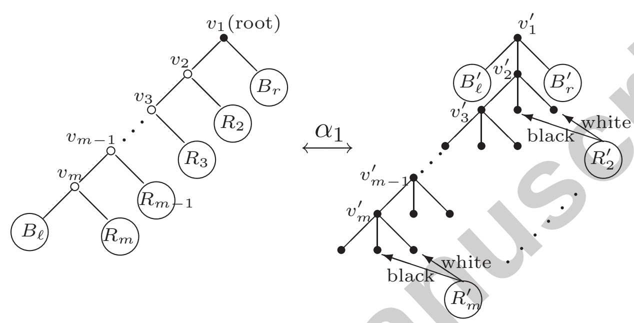


FIGURE 1. Step 1 of the bijection $\alpha = \alpha_1$

Step 2:

For a white rooted $\searrow$-free 2-binary tree R_i, the rightmost path of its root w_1 can be represented uniquely as consecutive white vertices $w_1, w_2, \cdots, w_m$. We illustrate the map $\alpha = \alpha_2$ in Figure 2. As before, B_r is a $\searrow$-free 2-binary tree, and we assign its corresponding ternary tree $B'_r = \alpha_1(B_r)$ as the middle subtree of w'_m, which is the internal vertex corresponding to w_m. The L_j $(1 \le j \le m)$ are $\searrow$-free 2-binary trees with white root permitted, and we denote the ternary tree corresponding to L_j by L'_j. Similar to the previous case, the position of L'_j

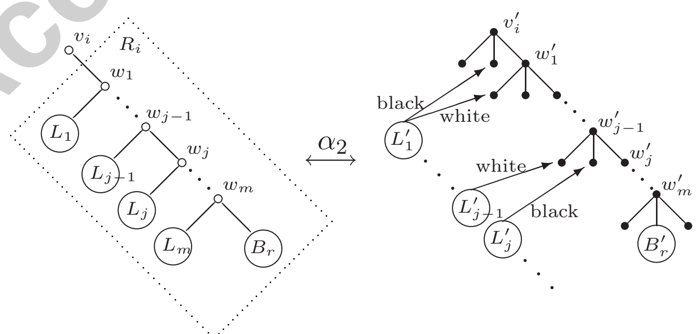


FIGURE 2. Step 2 of the bijection $\alpha = \alpha_2$

in Figure 2 will be determined by the color of the root of L_j. If it is black, we assign the subtree $L'_i = \alpha_1(L_i)$ as the middle subtree of w'_{i-1}. In particular, the L'_1 will be placed as the middle subtree of v'_i. Otherwise, if the root of L_i is white, we assign the corresponding subtree $L'_i = \alpha_3(R_i)$ as the left subtree of w_i, where α_3 will be determined by Step 3.

Step 3:

Now we consider the map on L_j when its root is white. Similar to Step 1, we decompose L_j into a $\diagdown$-free binary tree $\overline{L_j}$ with white root whose leftmost part of the root consists of consecutive white vertices, and B_l is a $\diagdown$-free 2-binary tree. Then B_l will be mapped to the middle subtree $B'_l = \alpha_1(B_l)$ of w'_{j-1}, and $\overline{L_j}$ will be mapped to the left subtree $\alpha_1(\overline{L_j})$ of w'_j, where α_1 is applied on the structure of Figure 1 neglecting substructures v_1, B_l, and B_r.

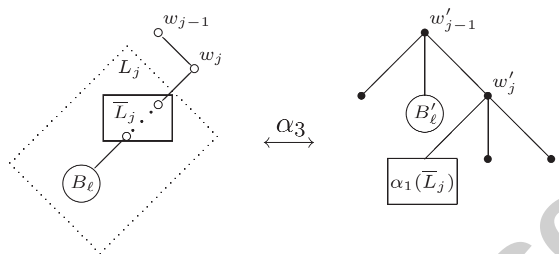


FIGURE 3. Step 3 of the bijection $\alpha = \alpha_3$

In conclusion, we get a ternary tree with n internal vertices following the above constructions recursively. It is not hard to check that in each step α_i is reversable.

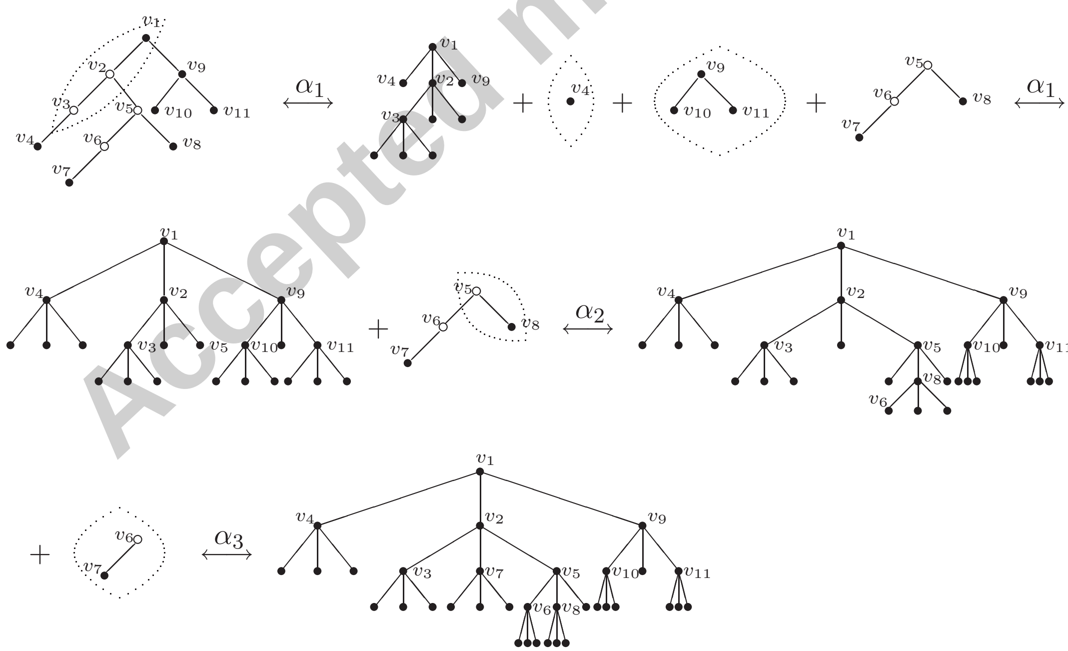


FIGURE 4. 2-binary trees and ternary trees

In order to show that α is a bijection, we construct the reverse map of α. Given a ternary tree Q with root v'_1, one can find the path $v'_1 v'_2 v'_3 \cdots v'_m$ such that v'_2 is the middle child of v'_1

and v'_i is the left child of v'_{i-1} with $3 \leq i \leq m$. By the reverse map of α_1, we may have a $\searrow$-free 2-binary subtree. Then go on to search the ternary subtree with root v'_i for $2 \leq i \leq m$ with no need of considering its left subtree. We may have a path $v'_i w'_1 w'_2 \cdots w'_m$ such that w'_1 is the right child of v'_i and w'_j is the right child of w'_{j-1} for $2 \leq j \leq m$. We may create a $\searrow$-free 2-binary subtree according to the reverse maps of α_2 and α_3. Thus, we have shown that α is a bijection. □

Figure 4 illustrates an example of the bijection α step by step, where the subtrees in dash ellipses are going to be transferred by the bijection α in the case of α_i with $1 \leq i \leq 3$.

It is well known (see [10, 15]) that the number of ternary trees with n internal vertices is the generalized Catalan number $\frac{1}{2n+1}\binom{3n}{n}$ (see [18, Sequence A001764]). Hence we have proved the following result.

Corollary 2.2. *The number of $\searrow$-free 2-binary trees with n vertices is* $\frac{1}{2n+1}\binom{3n}{n}$.

3. 2-binary trees and non-crossing trees

In this section, we enumerate $\searrow$-free 2-binary trees in terms of non-crossing trees on a circle.

We claim that the definition of the *leftmost path* of a non-crossing tree coordinates with that of an ordinary tree, that is, beginning with the root and then each vertex except for the root is the leftmost child of its parent. The *length* of the path is defined as the number of its edges. For example, the non-crossing tree of 12 vertices with root v'_0 in Figure 5 has the leftmost path $v'_0 v'_1 v'_2 v'_3$ of length 3.

We now give the following bijection that leads to an enumeration of $\searrow$-free 2-binary trees.

Theorem 3.1. *The number of $\searrow$-free 2-binary trees with n vertices equals the sum of the leftmost path's length of all non-crossing trees with $n+1$ vertices on a circle.*

Proof. We prove the theorem in steps.

We first, decompose the set of $\searrow$-free 2-binary trees with n vertices into several subsets, where all trees in a subset have the same number of vertices on the rightmost path of the root and exactly the same subtrees except for that path itself. Since the 2-binary trees are $\searrow$-free, the rightmost path of any vertex consists of consecutive black vertices (if any) followed immediately by consecutive white vertices (if any). In particular, the rightmost path of the root is of this type. Hence the cardinality of each subset equals the number of vertices on the rightmost path of the root.

Secondly, in each subset, we choose the unique representative whose vertices on the rightmost path of the root are all black. We then form a set with all these representatives and call it *representative set*.

Finally, we construct a bijection β between the representative set of $\searrow$-free 2-binary trees with n vertices and the set of non-crossing trees with $n+1$ vertices on a circle, where we map the vertices on the rightmost path of the root of a representative to the edges of the leftmost path in the corresponding non-crossing tree. Then the theorem holds immediately.

We now establish the bijection β. It is constructed by the following three steps, where in each step we concern with the rightmost path of some vertices in a $\searrow$-free 2-binary tree and we use the clockwise orientation for the corresponding non-crossing tree.

Step 1:

For a $\diagdown$-free 2-binary tree T in the representative set, we consider its rightmost path $v_1 v_2 \cdots v_m$ where v_1 is the root of T. In this case, we place $m+1$ vertices $v'_0, v'_1, \cdots, v'_m$ clockwise on the circle, where v'_0 is the root of the corresponding non-crossing tree. Then we connect the vertices from v'_{i-1} to v'_i $(1 \le i \le m)$ to form a unique path, which turns out to be the leftmost path of the corresponding non-crossing tree. What's more, the edge $v'_{i-1}v'_i$ and the circle form an area $A_{i-1,i}$, where there is no edge within.

Step 2:

We create the corresponding edges of the left subtree L_i of v_i in the area $A_{i-1,i}$ in a non-crossing way as follows. Let $w_1 w_2 \cdots w_k$ be the rightmost path of the root w_1 in L_i. We place their corresponding vertices $w'_1, w'_2, \cdots, w'_k$ clockwise on the arc of $A_{i-1,i}$. Then we connect the vertices v'_{i-1} and w'_j $(1 \le j \le k)$ if w_j is black. Otherwise, we connect the vertices v'_i and w'_j. For the other vertices not mapped yet, we recursively use the following Step 3 to finish the construction.

Step 3:

Let $u_1 \cdots u_\ell u_{\ell+1} \cdots u_h$ with $1 \le \ell \le h$ be the rightmost path of the root u_1 in the left subtree of w_j, where $u_1, \cdots, u_\ell$ are black vertices, while $u_{\ell+1}, \cdots, u_h$ are white vertices. We place their corresponding vertices as a sequence of consecutive vertices $u'_1, \cdots, u'_\ell, w'_j, u'_{\ell+1}, \cdots, u'_h$ clockwise on the arc of $A_{i-1,i}$. That is to say, w'_j is the only intermediate vertex which separates the sequence $u'_1, \cdots, u'_h$. Now connect each of the vertices $u'_1, \cdots, u'_h$ with w'_j. Obviously these edges are non-crossing and do not cross with any other edges.

Using these steps, we obtain a non-crossing tree Q with $n+1$ vertices on a circle. It is obvious that each step is reversable.

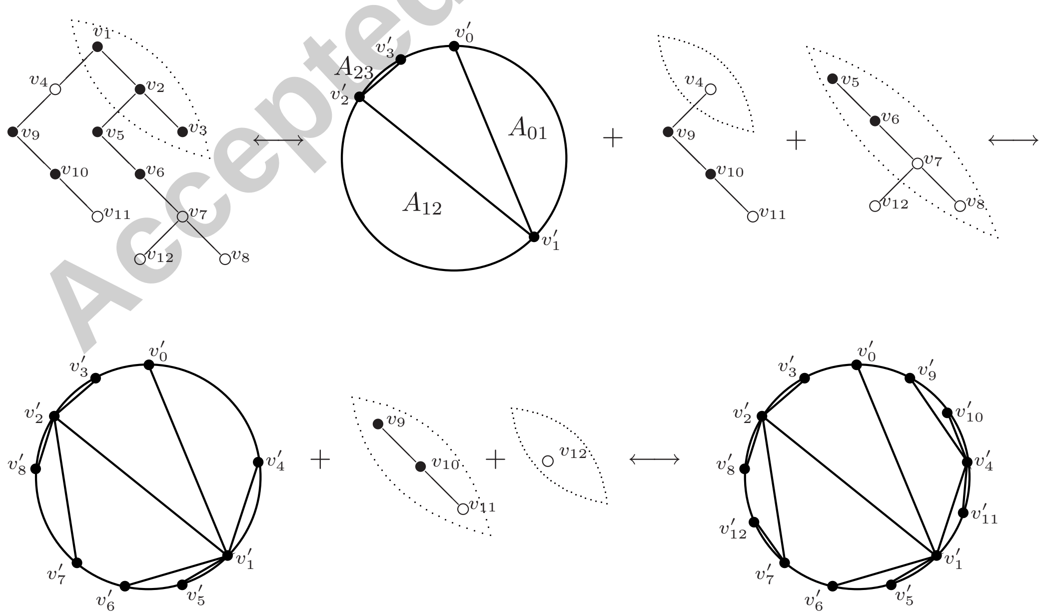


FIGURE 5. 2-binary trees and non-crossing trees

To show that β is a bijection, we proceed to construct the inverse map of β. Let Q be a non-crossing tree with $n+1$ vertices on a circle. We can find the leftmost path $v'_0v'_1\cdots v'_m$ with root v'_0 which maps to the rightmost path $v_1v_2\cdots v_m$ of the ●╲●-free 2-binary tree in which all the vertices are black. Then for all the children w'_j of the vertices on path $v'_0v'_1\cdots v'_m$ in the area $A_{i-1,i}$, we read them off clockwise and map them to the rightmost path of v_i, where the corresponding vertex w_j is colored black (resp. white) if $v'_{i-1}w'_j$ (resp. $v'_iw'_j$) is an edge in Q. Then for all the children u'_ℓ of w'_j, we can use the reverse map of Step 3 to construct a rightmost path of w_j. Recursively using the reverse map of Step 3, we can map the other vertices to the vertices in the ●╲●-free 2-binary tree. Thus we have that β is a bijection which completes the proof. □

An example of the bijection β is shown in Figure 5.

4. 2-binary trees and lattice paths

In this section, we investigate several enumerations of 2-binary trees characterized by the e-free property. It turns out that they are related to Schröder paths, Motzkin paths, and Dyck paths. Given a subset $E \subseteq$ {●╱●, ○╱●, ●╱○, ○╱○, ●╲●, ●╲○, ○╲●, ○╲○}, we say that a 2-binary tree is *E-free* if it does not contain any edge $e \in E$.

4.1 {●╲●, ●╲○}-free 2-binary trees and Schröder paths

In this subsection, we construct a bijection γ_1 between the set of {●╲●, ●╲○}-free 2-binary trees with $n+1$ vertices and the set of Schröder paths of length $2n$. Let T be a {●╲●, ●╲○}-free 2-binary tree with $n+1$ vertices. For each vertex in T except for the root, we proceed to create a step in the corresponding Schröder path with respect to the preorder traversal. If T contains no white vertex, it is a left path $v_rv_1v_2\cdots v_n$, where v_r is the root of T, and we put it in correspondence with n horizontal steps. Now suppose T contains at least one white vertex. Since T does not contain any edge of type {●╲●, ●╲○}, it can be described uniquely as

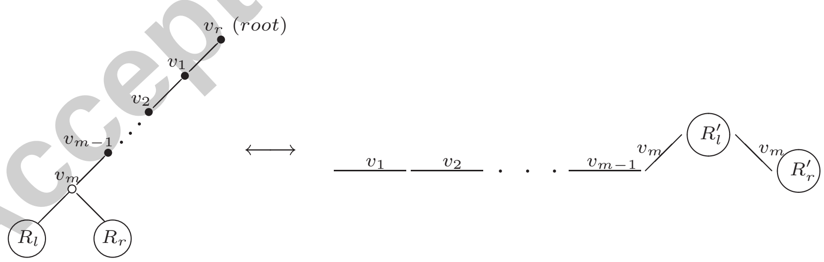


Figure 6. The bijection γ_1

the first structure in Figure 6, where R_l and R_r are {●╲●, ●╲○}-free 2-binary trees with white root permitted. We construct a Schröder path of length $2n$ from the preorder traversal as follows. First, delete the root of T. Each time we visit a black vertex, we get an horizontal step $H = (2, 0)$. Each time we visit a white vertex, we get $UR'_lDR'_r$, where U and D are up steps and down steps, respectively, and R'_l (resp. R'_r) is the Schröder path corresponding to R_l (resp. R_r). The process is also illustrated in Figure 6. Conversely, given a Schröder path, we may decompose it uniquely into segments as the second structure in Figure 6, namely the *first return decomposition* of a Dyck path [4]. More precisely, read the Schröder path from left to right and find the first up step and the first down step that returns to the x-axis. Clearly, we may reverse the above procedure to construct a $\{\diagdown, \diagdown\}$-free 2-binary tree, and we have the following result.

Theorem 4.1. *There is a bijection between the set of* $\{\diagdown, \diagdown\}$*-free* 2*-binary trees with* $n+1$ *vertices and the set of Schröder paths of length* $2n$.

An example of the above bijection is Shown in Figure 7.

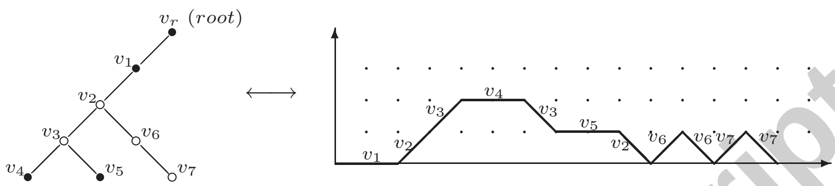


FIGURE 7. 2-binary trees and Schröder paths

For a Schröder path of length $2n$, we may first choose k horizontal steps $H = (2, 0)$ from all its $2n - k$ steps, which is the binomial coefficient $\binom{2n-k}{k}$, then the left $2n - 2k$ steps from a Dyck path of length $2n - 2k$, which is enumerated by C_{n-k}. According to the bijection γ_1, each black vertex except for the root in a $\{\diagdown, \diagdown\}$-free 2-binary tree corresponds to a horizontal step in the Schröder path. We can then formulate the following, more detailed, result.

Corollary 4.2. *The number of* $\{\diagdown, \diagdown\}$*-free* 2*-binary trees with* $n+1$ *vertices is the* n*-th Schröder number* S_n. *The number of* $\{\diagdown, \diagdown\}$*-free* 2*-binary trees with* $n+1$ *vertices and* $k+1$ *black vertices equals* $\binom{2n-k}{k} C_{n-k}$.

4.2 $\{\diagdown, \diagdown\}$-free 2-binary trees and little Schröder paths

In this subsection, we first construct a bijection γ_2 between the set of $\{\diagdown, \diagdown\}$-free 2-binary trees of n vertices with white root permitted and the set of Schröder paths of length $2n$. Based on this bijection, we then describe a bijection γ_3 between the set of $\{\diagdown, \diagdown\}$-free 2-binary trees with n vertices and the set of little Schröder paths of length $2n$.

Let T be a $\{\diagdown, \diagdown\}$-free 2-binary tree of n vertices with white root permitted. It is obvious that any vertex v in T may have a left subtree R_l and a right subtree W_r, where R_l is a $\{\diagdown, \diagdown\}$-free 2-binary tree with white root permitted, while W_r is a $\{\diagdown, \diagdown\}$-free 2-binary tree with white root. We proceed now to construct a Schröder path of length $2n$ by the preorder traversal. Each time we visit a black vertex, we get $R'_l H W'_r$, where H denotes a horizontal step and R'_l (resp. W'_r) is the corresponding Schröder path of R_l (resp. W_r). Each time we visit a white vertex, we get $U R'_l D W'_r$, where U and D are up steps and down steps respectively. The process is also illustrated in Figure 8. Conversely, given a Schröder path, we may reverse the above procedure to construct a $\{\diagdown, \diagdown\}$-free 2-binary tree with white root permitted. That is to say, read the Schröder path from right to left and find the first horizontal step H on the x-axis if it exist, then the path can be decomposed uniquely as $R'_l H W'_r$ where R'_l is a Schröder path and W'_r is a Schröder path with no horizontal step on the x-axis. Hence we can reverse the map γ_2 in Figure 8 to create a black vertex with subtrees and proceed with the construction recursively. If there is no horizontal step on the x-axis, the Schröder path can be decomposed uniquely as $UR_l'DW_r'$. Then we can also reverse γ_2 as illustrated in Figure 8 to construct a white vertex with subtrees and proceed recursively. Thus we have the following result.

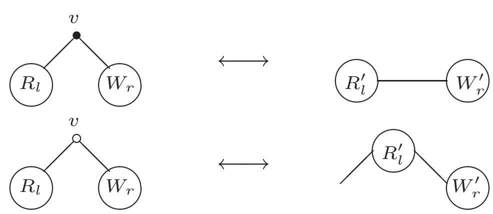


FIGURE 8. The bijection γ_2

Theorem 4.3. *There is a bijection between the set of {⟍, ⟍}-free 2-binary trees of n vertices with white root permitted and the set of Schröder paths of length $2n$.*

According to the bijection γ_2, each black vertex in a {⟍, ⟍}-free 2-binary tree with white root permitted corresponds to a horizontal step in a Schröder path. Similar to the arguments in the pervious subsection, we can derive the following result.

Corollary 4.4. *The number of {⟍, ⟍}-free 2-binary trees of n vertices with white root permitted is the n-th Schröder number S_n. Moreover, the number of {⟍, ⟍}-free 2-binary trees of n vertices with white root permitted and k black vertices equals $\binom{2n-k}{k}C_{n-k}$.*

An example of the above bijection is given in Figure 9.

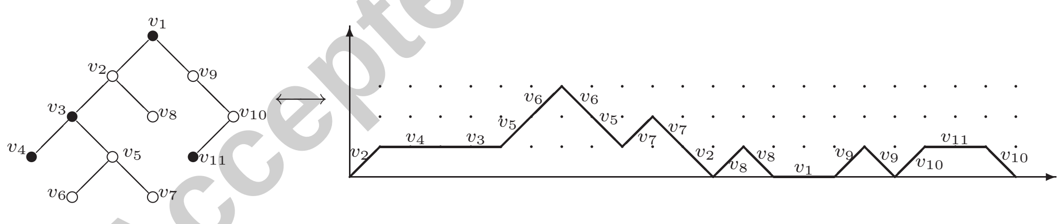


FIGURE 9. 2-binary trees and Schröder paths

Note that white vertices and black vertices have the same distribution in the set of { ⟍, ⟍}-free 2-binary trees with white root permitted. Hence over all of these trees, half of them are white rooted, while the other half are black rooted. According to the bijection γ_2, a {⟍ , ⟍}-free 2-binary tree with white root corresponds to a Schröder path with no horizontal steps on the x-axis. Hence the number of such Schröder paths is $S_n/2$. There is a simple bijection, which is constructed by replacing each horizontal step H by a peak UD. This transforms Schröder paths with no horizontal step on the x-axis into little Schröder paths. Hence we derive the equality $S_n = 2L_n$ for $n \geq 1$ (see, for example, [19, pp. 178, 219, 256]), which has been proved bijectively by Deutsch[3], Shapiro and Sulanke[17], and Sulanke[20] recently, and the following property holds.

Corollary 4.5. *Both the number of* {●⟍●, ○⟍●}*-free* 2*-binary trees of* n *vertices and the number of such trees with white root are counted by the little Schröder number* L_n*, which is also the number of Schröder paths with no horizontal steps on* x*-axis.*

In addition, according to the bijection γ_2, the number of consecutive black vertices beginning from the root on the leftmost path of a {●⟍●, ○⟍●}-free 2-binary tree equals the number of horizontal steps on x-axis in the corresponding Schröder path.

Now we consider {●⟍●, ○⟍●}-free 2-binary trees with the aim of giving a bijection γ_3 between them and little Schröder paths, which turns out to be a pure combinatorial proof of the above results.

Let T be a {●⟍●, ○⟍●}-free 2-binary tree with n vertices. If the root v_1 of T has a left {●⟍●, ○⟍●}-free 2-binary subtree B whose root is black (by definition) and an empty right subtree, then it corresponds to $B'H$, where H is a horizontal step and B' is the corresponding little Schröder path of B. The map here obviously does not create a peak at level one. Otherwise, if v_1 has a white rooted left subtree or a nonempty right subtree, then T can be decomposed uniquely as illustrated in Figure 10, since T dose not contain any edge of type {●⟍●, ○⟍●}. In Figure 10,

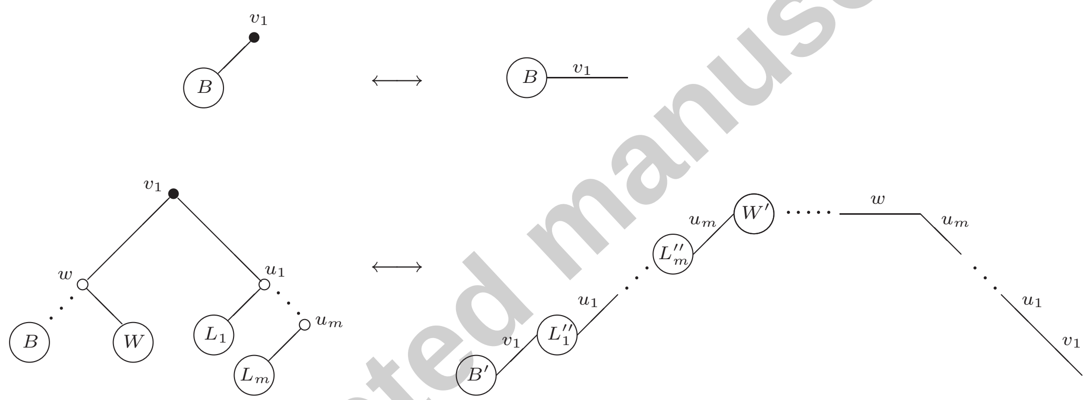


FIGURE 10. The bijection γ_3

B is a {●⟍●, ○⟍●}-free 2-binary tree and its black root is the first black vertex (except for the root v_1) that appears in the preorder traversal, W is a {●⟍●, ○⟍●}-free 2-binary tree with white root, L_i $(1 \leq i \leq m)$ are {●⟍●, ○⟍●}-free 2-binary trees with white root permitted. We remark that w (with W) and u_i (with L_i) can not be both empty except in the final case. Hence, the correspondence is constructed by mapping each vertex in T to a pair of U and D steps or an H step, which are illustrated in Figure 10, where B' (resp. W') is the little Schröder path corresponding to B (resp. W) and the bijection γ_2 is used to transfer L_i to $L''_i = \gamma_2(L_i)$. It is not hard to get the reverse map in this case. Recall that a peak is an up step followed immediately by a down step. Since w and u_i can not be both empty in this case, then the Schröder path we obtained does not possess any peak at level one. Hence recursively we get a little Schröder path P of length $2n$.

Conversely, given a little Schröder path, we may decompose it uniquely into segments as shown in Figure 10 in the following way. Read the little Schröder path from right to left. If it starts with a horizontal step, then create a black vertex with a left subtree and use the procedure recursively. If it starts with a down step, find the next up step on its left that starts from the x-axis and recursively create the tree as indicated in Figure 10. Thus we have the following result.

Theorem 4.6. *There is a bijection between the set of* {╲, ╲}*-free* 2*-binary trees with* n *vertices and the set of little Schröder paths of length* $2n$.

An example of the above bijection γ_3 is given in Figure 11.

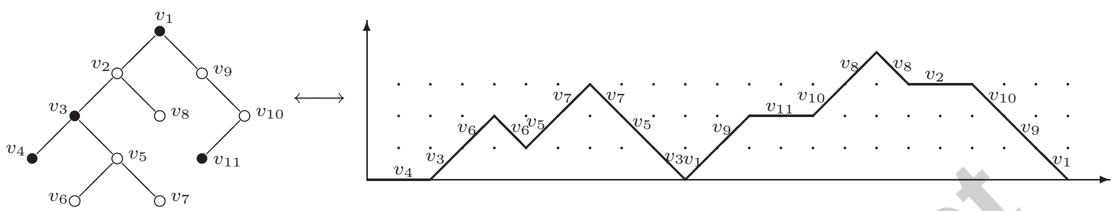


FIGURE 11. 2-binary trees and little Schröder paths

To study some statistics using the bijection γ_3, we define the *rightmost height* of a little Schröder path of length $2n$ as the number of consecutive down steps that end at $(2n, 0)$. For example, the rightmost height of the little Schröder path in Figure 11 is 3. Let T be a {╲, ╲}-free 2-binary tree and P its corresponding little Schröder path. Then we have:

- Assume that the root of T has an empty right subtree. If the left child of the root is black then the last step of P is a horizontal step, and the rightmost height of P is 0, as shown in the first map in Figure 10. If the left child of the root is white, then the rightmost height of P is 1, as shown in the second map in Figure 10.
- Assume that the root of T has an nonempty right subtree. Then the rightmost height of P equals the number of vertices on the rightmost path of T including the root, as shown in the second map in Figure 10.

Note that according to Theorems 4.3 and 4.6, we again have a brand new way to arrive at the equality $S_n = 2L_n$.

4.3 {⁄, ⁄, ⁄, ╲}-free 2-binary trees and Motzkin paths

In this subsection, we present a bijection θ between the set of {⁄, ⁄, ⁄, ╲}-free 2-binary trees with n vertices and the set of Motzkin paths of length n, which leads us to an exact enumeration for the number of white vertices in a {⁄, ⁄, ⁄, ╲}-free 2-binary tree.

Let T be a {⁄, ⁄, ⁄, ╲}-free 2-binary tree with n vertices. If T contains no white vertex, it is a right path $v_1 v_2 \cdots v_n$, that is to say, v_{i+1} is the right child of v_i for all $1 \leq i \leq n-1$. We then put this in correspondence to a Motzkin path with n horizontal steps. Now suppose T contains at least one white vertex. Since T does not contain any edge of type {⁄, ⁄, ⁄, ╲}, it can be decomposed uniquely as the first structure in Figure 12, where R_l and R_r are {⁄, ⁄, ⁄, ╲}-free 2-binary trees which can be empty also. Bearing in mind that T has a black root and does not have edges of type {⁄, ⁄, ╲}, each white vertex must be incident with an edge ╲. We proceed to construct a Motzkin path of length n by the preorder traversal as follows. Each time we visit a black vertex not incident with an edge ╲, we get horizontal step. And each time we visit a black vertex followed immediately by a white vertex which is incident with an edge ╲, we get $UR_l'DR_r'$, where R_l' (resp. R_r') is the Motzkin path corresponding to R_l (resp. R_r). The process is also illustrated in Figure 12.

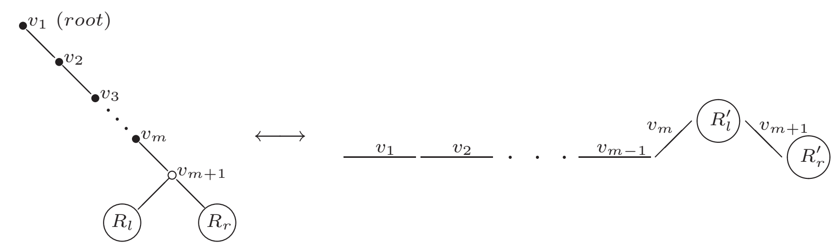


FIGURE 12. The bijection θ

Conversely, given a Motzkin path, we may decompose it uniquely into segments as shown in the second structure in Figure 12, according to the first return decomposition of a Dyck path [4], and we may reverse the above procedure to construct a {⟋, ⟋, ⟋, ⟍}-free 2-binary tree. Thus we have the following result.

Theorem 4.7. *There is a bijection between the set of* {⟋, ⟋, ⟋, ⟍}*-free 2-binary trees with n vertices and the set of Motzkin paths of length n.*

An example of the bijection θ is shown in Figure 13.

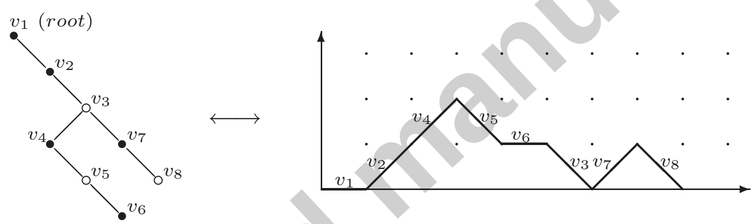


FIGURE 13. 2-binary trees and Motzkin paths

For a Motzkin path of length n, we may first choose $2k$ ($0 \leq k \leq \lfloor \frac{n}{2} \rfloor$) steps to form a Dyck path of length $2k$, and then fill the left $n - 2k$ positions with horizontal steps. This gives the binomial coefficient $\binom{n}{2k} C_k$. According to the bijection θ, each white vertex in a {⟋, ⟋, ⟋, ⟍}-free 2-binary tree corresponds to a down (up) step in the Motzkin path. Hence we can formulate the following result.

Corollary 4.8. *The number of* {⟋, ⟋, ⟋, ⟍}*-free 2-binary trees with n vertices is the n-th Motzkin number M_n. Moreover, the number of* {⟋, ⟋, ⟋, ⟍}*-free 2-binary trees with n vertices and k white vertices equals $\binom{n}{2k} C_k$.*

4.4 {⟋, ⟋, ⟋}-free 2-binary trees and Dyck paths

In this subsection, we present a bijection ϕ between the set of {⟋, ⟋, ⟋}-free 2-binary trees with n vertices and the set of Dyck paths of length $2n$. Furthermore, we study the statistic of black vertices in {⟋, ⟋, ⟋}-free 2-binary trees.

Let T be a {⟋, ⟋, ⟋}-free 2-binary tree with n vertices. Then any black vertex in T has no left subtree, while any white vertex in T may have a {⟋, ⟋, ⟋}-free 2-binary subtree. Now we proceed to construct a Dyck path of length $2n$ by the preorder traversal inductively as follows. Each time we visit a black vertex, we create a UD. And each time we visit a white vertex with left subtree T', we create $P'UPD$, where P is the nonempty Dyck path we get before visiting this white vertex and P' is the Dyck path corresponding to T'. Conversely, given a Dyck path, we read it from right to left. If we encounter a peak UD, we map it to a black vertex. If we encounter more than to consecutive down steps, we may decompose the Dyck path uniquely as $P'UPD$ with nonempty Dyck path P and Dyck path P'. Then we reverse the above procedure to construct a white vertex with left subtree in a {○/●, ●/●, ○/○}-free 2-binary tree. Thus we have the following result.

Theorem 4.9. *There is a bijection between the set of {○/●, ●/●, ○/○}-free 2-binary trees with n vertices and the set of Dyck paths of length $2n$.*

An example of the bijection ϕ is shown in Figure 14.

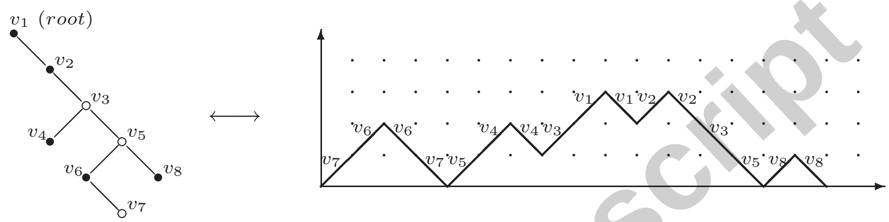


FIGURE 14. 2-binary trees and Dyck paths

According to the bijection ϕ, each black vertex in a {○/●, ●/●, ○/○}-free 2-binary tree corresponds to a peak in the Dyck path. It is known [4] that the number of Dyck paths of length $2n$ with k peaks is the Narayana number (see [16, 20] and [19, Exe. 6.36]), $N_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$. Hence we can formulate the following result.

Corollary 4.10. *The number of {○/●, ●/●, ○/○}-free 2-binary trees with n vertices is C_n. Moreover, the number of {○/●, ●/●, ○/○}-free 2-binary trees with n vertices and k black vertices is $N_{n,k}$.*

We would like to remark that there are still many types of 2-binary trees related to other combinatorial structures. These trees give new combinatorial interpretations of a number of sequences from Sloane's EIS [18]. Some of these are given in the table below.

E	Sequence for E-free 2-binary trees in [18]	E	Sequence for E-free 2-binary trees in [18]
{○/●, ●\○}	A000108	{○/●, ●/●}	A006318
{○/●, ○/○}	A001003	{○/●, ●/●, ●\●}	A073155
{○/●, ○/○, ●\●}	A109081	{○/●, ○/○, ○\○}	A001002
{●/●, ○/○, ●/○}	A000245	{●/●, ●\●, ●/○}	A052706
{○/●, ○/○, ●/○}	A002212	{○/●, ●/●, ●/○}	A007317
{○/●, ●\●, ○/○}	A049124	{○/●, ●\●, ●/○}	A106228
{○/●, ●/●, ●\●, ●/○}	A052709	{○/●, ●/●, ○/○, ○\○}	A001006
{○/●, ●/●, ○/○, ●\●}	A105633	{○/●, ●/●, ●/○, ○\●}	A014137
{●/●, ●\●, ○/○, ●/○}	A002026	{○/●, ●\●, ○/○, ●/○}	A025242
{○/●, ●/●, ●\●, ○/○, ●/○}	A000045	{○/●, ●/●, ●\●, ○/○, ○/○}	A007477

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