Path Graphs Versus Line Graphs - A Survey *

Xueliang Li and Yan Liu

Center for Combinatorics and LPMC, Nankai University

Tianjin 300071, P.R. China. Email: lxl@nankai.edu.cn

Abstract

The line graph transformation may be the most widely studied among all graph transformations, the concept of which was first introduced by Whitney. There are many results on this topic published in various journal papers. The path graph transformation is a natural and interesting generalization of line graph transformation. As one will see in the survey, properties of path graphs are sometimes very different from those of line graphs. Some problems become very complicated and substantially difficult for path graphs. In this paper, we will survey results on path graphs versus line graphs. We classify the results into the following categories: some basic facts, determination problem, characterization problem, traversability, connectivity. For results not in these categories, we skip them because otherwise the paper would become too long.

Keywords: line graph, path graph, (edge-, vertex-, induced) isomorphism, traversability, connectivity

AMS subject classification 2000: 05C60, 05C75, 05C38, 05C40, 05C45

1 Introduction

The *line graph* $L(G)$ of a graph G is defined as a graph whose vertices are the edges of G, with two vertices adjacent if and only if the corresponding edges are adjacent in G. The line graph transformation may be the most widely studied among all graph transformations. But at first people studied it with different names, such as interchange graph, derived graph, conjugate graph or representative graph. The name of line graphs was first introduced by Harary and Norman [20] in 1960. However the first to study the concept is in 1932 by Whitney [57], showed that for connected graphs edge-isomorphism

*Supported by NSFC, PCSIRT and the "973" program.

implies (vertex-)isomorphism except for K_3 and $K_{1,3}$. There are plenty of results on the line graphs. For a classical survey we refer to [22]. Later, there are many old and new generalizations of line graphs. For example, *total graphs, middle graphs, clique graphs, triangle graphs, path graphs* and *super line graphs*, etc. In this survey, we mainly focus our attention on path graphs versus line graphs. For more interesting generalizations of line graphs, we refer the reader(s) to [4, 5, 6, 7, 3, 50, 61].

Broersma and Hoede [12] generalized the concept of line graphs to that of path graphs. Let P_k and C_k denote a path and a cycle with k vertices, respectively. Denote by $\Pi_k(G)$ the set of all paths of G on k vertices ($k \geq 1$). The *path graph* $P_k(G)$ of a graph G has vertex set $\Pi_k(G)$ and edges joining pairs of vertices that represent two paths P_k, the union of which forms either a path P_{k+1} or a cycle C_k in G. If $k = 2$, then the P_2-graph is exactly the line graph. The way of describing a line graph stresses the adjacent concept, whereas the way of describing a path graph stresses the concept of the path generation by consecutive paths.

For a graph transformation, there are two general problems, which are formulated by Grünbaum [18]. We state them here for path graphs.

Determination Problem. Determine which graphs have a given graph as their P_k-graphs.

Characterization Problem. Characterize those graphs that are P_k-graphs of some graphs.

Solutions of the determination and characterization problems for path graphs will be given in Sections 3 and 4, respectively. For line graphs and P_3-graphs, these two problems have been well studied. For $k \geq 4$, the problems become more difficult. Although the determination and characterization problems for P_k-graphs for $k \geq 4$ have not been completely solved, there are also some results for graphs with higher degree. In the last two sections, Sections 5 and 6, we will present some results on the traversability and connectivity of path graphs. For other results and more literature related to the path graphs, we refer the readers to references [26, 28, 32, 35, 48, 49, 51, 52, 53, 54].

Throughout the paper, we follow [15] for terminology and notations. In each of the sections, we classify the results into four categories: line graphs, line digraphs, path graphs and directed path graphs.

2 Basic Facts on Path Graphs

In this section, we introduce some basic observations about line graphs and P_3-graphs. Several definitions and results of P_3-graphs can be viewed as a counterpart with respect to line graphs. First, some examples of line graphs are given in Figure 2.1.

Theorem 2.1 ([22]). *Let G be a graph with n vertices and m edges, and $d(v)$ denote the*

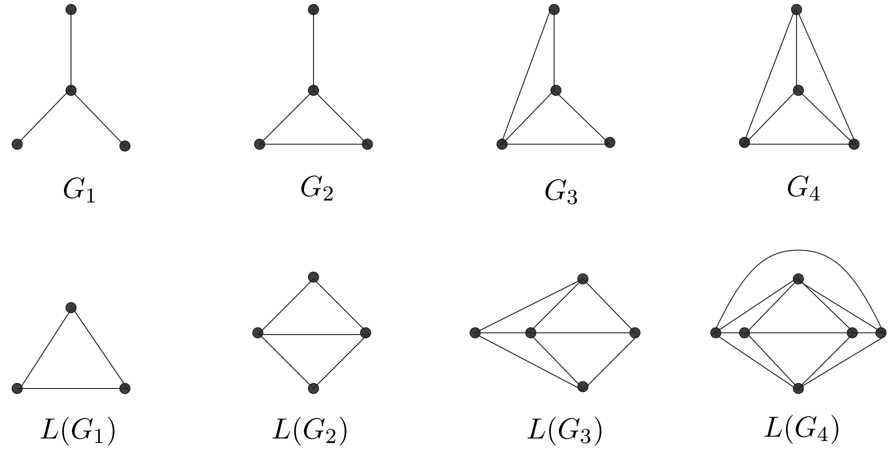


Figure 2.1

degree of a vertex v in G. Then

(i) $L(G)$ *has* m *vertices and* $\sum_v \frac{1}{2}d(v)^2 - m$ *edges;*

(ii) the degree in $L(G)$ *of an edge* uv *of* G *is* $d(u) + d(v) - 2$*;*

(iii) G *and* $L(G)$ *have the same number of connected components if* G *has no isolated vertices;*

(iv) If G *is connected and* $n \geq 4$*, then* $L(G) \cong K_n$ *if and only if* $G \cong K_{1,n}$*.*

Theorem 2.2 ([12])**.** *Let* $G = (V, E)$ *be a graph,* $P_3(G) = (V^{'}, E^{'})$ *its path graph, and let* $I(G)$ *be the set of vertices of* G *with degrees greater than 1. Then*

$$|V^{'}| = \sum_{v \in I(G)} \binom{d(v)}{2}$$

$$|E^{'}| = \frac{1}{2}\sum_{v \in V}[(d(v) - 1) \sum_{u \in N(v)} (d(u) - 1)].$$

In [20], Harary and Norman introduced the notion of line digraphs for digraphs. The *line digraph* $\vec{L}(D)$ of a digraph D, defined naturally and similarly, has as its vertex set the set of arcs of D; ab is an arc of $\vec{L}(D)$ if and only if there are vertices u, v, w in D with $a = uv$ and $b = vw$. By a natural way, Broersma and Li [13] generalized the concept of line digraphs to that of directed path graphs.

Let k be a positive integer, and denote by $\vec{P}_k$ or $\vec{C}_k$ a directed path or a directed cycle on k vertices, respectively. Let D be a digraph containing at least one directed path $\vec{P}_k$. Denote by $\vec{\Pi}_k(D)$ the set of all $\vec{P}_k$'s of D. Then the *directed* $\vec{P}_k$*-graph* of D, denoted by $\vec{P}_k(D)$, is the digraph with vertex set $\vec{\Pi}_k(D)$; pq is an arc of $\vec{P}_k(D)$ if and only if there is a $\vec{P}_{k+1}$ or $\vec{C}_k$ $v_1v_2 \cdots v_{k+1}$ in D (with $v_1 = v_{k+1}$ in the case of a $\vec{C}_k$) such that $p = v_1v_2 \cdots v_k$ and $q = v_2 \cdots v_kv_{k+1}$. Note that $\vec{P}_1(D) = D$ and $\vec{P}_2(D) = \vec{L}(D)$.

For any vertex v in a digraph D, denote by $N_D^-(v)$ and $N_D^+(v)$ the *in-neighborhood* and *out-neighborhood* of v in D, respectively; and denote by $d^-(v)$ and $d^+(v)$ the *in-degree* and *out-degree* of v, respectively. v is a *source* or *sink* if $d^-(v) = 0$ or $d^+(v) = 0$, respectively. The *underlying graph* $U(D)$ of a digraph D is the graph (or multigraph) obtained from D by replacing each arc by an (undirected) edge joining the same pair of vertices. A digraph D is called *strongly connected* if, for each pair of vertices v and w, there is a directed path in D from v to w, and *connected* if there is a path from v to w in $U(D)$.

Next we give some basic results on line digraphs and directed $\vec{P}_3$-graphs.

Theorem 2.3 ([22]). *Let D be a digraph with n vertices (none of which is isolated), and m arcs. Then*

(i) $\vec{L}(D)$ has m vertices and $\sum d^+(v)d^-(v)$ arcs;

(ii) the out-degree in $\vec{L}(D)$ of an arc vw in D is $d^+(w)$ and the in-degree is $d^-(v)$;

(iii) $\vec{L}(D) \cong \vec{P}_{n-1}$ if and only if $D \cong \vec{P}_n$;

(iv) $\vec{L}(D) \cong \vec{C}_n$ if and only if $D \cong \vec{C}_n$;

(v) $\vec{L}(D) \cong \vec{K}_{r,s}$ if and only if D consists of r in-coming arcs and s out-going arcs (no loops) at some vertex.

Theorem 2.4 ([13]). *Let D be a digraph containing at least one $\vec{P}_3$. Then*

(i) $\vec{P}_3(D)$ contains no $\vec{C}_2$;

(ii) Each C_3 in $U(\vec{P}_3(D))$ is a $\vec{C}_3$ in $\vec{P}_3(D)$;

(iii) Each C_4 in $U(\vec{P}_3(D))$ is induced (has no chords) and is a $\vec{C}_4$ or is oriented with alternating arc directions in $\vec{P}_3(D)$;

(iv) No C_k ($k \geq 5$) of $U(\vec{P}_3(D))$ is both induced and oriented with alternating arc directions in $\vec{P}_3(D)$.

3 Isomorphisms of Path Graphs

First, let us recall some notations and definitions.

A graph *isomorphism* from G to G' is a bijection $f : V(G) \to V(G')$ such that two vertices are adjacent in G if and only if their images are adjacent in G'. Denote the set of all isomorphisms of G onto G' by $\Gamma(G, G')$.

An *edge isomorphism* from G to G' is a bijection $f : E(G) \to E(G')$ such that two edges are adjacent in G if and only if their images are adjacent in G'. Let $\Gamma_e(G, G')$ denote the set of all edge isomorphisms of G onto G', and it is easy to see that $\Gamma(L(G), L(G')) = \Gamma_e(G, G')$.

For $f \in \Gamma(G, G')$, define $f^* : E(G) \to E(G')$ by $f^*(uv) = f(u)f(v)$, and call f^* induced by (vertex-) isomorphism f. Let $\Gamma^*(G, G') = \{f^* : f \in \Gamma(G, G')\}$.

For digraphs, the above definitions are naturally given as follows. Let D and D' be two digraphs. An *isomorphism* of D onto D' is a bijection $f : V(D) \to V(D')$ such that

$uv \in A(D)$ if and only if $f(u)f(v) \in A(D')$. To stress the head-to-tail adjacency, for two arcs $a, b \in A(D)$, define that a *hits* b if $a = vw$ and $b = wz$. An *arc-isomorphism* of D onto D' is a bijection $f : A(D) \to A(D')$ such that $a \in A(D)$ hits $b \in A(D)$ if and only if $f(a) \in A(D')$ hits $f(b) \in A(D')$. Hence an arc-isomorphism of D onto D' is an isomorphism of $\vec{L}(D)$ onto $\vec{L}(D')$. An arc-isomorphism f of D onto D' is *induced by an isomorphism* of D onto D' if there exists an isomorphism f^* of D onto D' such that $f(uv) = f^*(u)f^*(v)$ for each arc uv of D.

This section is the main body of the paper, and contains many results. We divide it into several subsections, each dealing with a subject.

3.1 Fixed Point of a P_k-Transformation

For a graph operator, one of the first problems considered is to determine the fixed point of the function. So here we discuss which graphs are isomorphic to their path graphs. It is well-known that a connected graph G is isomorphic to its line graph $L(G)$ if and only if G is a cycle. For $k = 3$, Broersma and Hoede [12] proved the following result.

Theorem 3.1.1 ([12]). *A connected graph G is isomorphic to its path graph $P_3(G)$ if and only if G is a cycle.*

Later, Li and Zhao gave a similar result for the case $k = 4$.

Theorem 3.1.2 ([43]). *A connected graph G is isomorphic to its path graph $P_4(G)$ if and only if G is a cycle of length at least four.*

On the other hand, Knor and Niepel, using the iterated path graphs, got even stronger results for P_3-graphs and P_4-graphs in [28, 30], respectively.

The *iterated path graph* is defined by $P_k^1(G) = P_k(G)$, and $P_k^n(G) = P_k(P_k^{n-1}(G))$ for $n > 1$, $k \geq 2$. In Figure 2.1, it is clear to see that $L(G_2) \cong G_3$, so that $L(G_3) \cong L(L(G_2))$.

Theorem 3.1.3 ([56]). *If G is connected and $L^n(G) \cong G$ for some n, then $L(G) \cong G$ and G is a cycle.*

Theorem 3.1.4 ([28]). *Let G be a graph and n a number for $n \geq 1$, such that G isomorphic to $P_3^n(G)$. Then each component of G is a cycle.*

Theorem 3.1.5 ([30]). *Let G be a graph and n a number for $n \geq 1$, such that G isomorphic to $P_4^n(G)$. Then each component of G is a cycle of length greater than or equal to 4.*

For line digraphs, the solution of this problem is as a result on periodic iterated line digraphs. Our discussion is based on the work of Balconi [10] and Hemminger [21]. But the first published result is due to Harary and Norman [20] in the following.

Theorem 3.1.6 ([20]). *Let D be a connected digraph. Then $\vec{L}(D) \cong D$ if and only if every vertex has out-degree 1 or every vertex has in-degree 1.*

For some positive integers n and k, define that a digraph D is *$\vec{L}$-periodic* if $\vec{L}^{n+k}(D) \cong \vec{L}^n(D)$. The following theorem characterizes the property of periodic digraphs.

Theorem 3.1.7 ([22]). *Let D be a digraph. Then*

(i) $\vec{L}^n(D)$ is a null graph for some n if and only if D has no directed cycles;

(ii) The order of $\vec{L}^n(D)$ gets arbitrarily large if and only if D has two directed cycles joined by a directed path (possibly of length 0);

(iii) D is $\vec{L}$-periodic if and only if D has directed cycles, no two of which are joined by a directed path.

(iv) If D is strongly connected, and if $\vec{L}^n(D) \cong D$ for some n, then $\vec{L}(D) \cong D$, and D is a directed cycle.

Broersma and Li [13] concluded that the only connected digraphs D with $\vec{P}_3(D) \cong D$ consist of a directed cycle with in-trees or out-trees attached to its vertices, with at most one nontrivial tree per vertex, and only one type of nontrivial trees, where a directed tree T of D is an *out-tree of D* if $V(T) = V(D)$ and precisely one vertex of T has in-degree zero (the root of T), while all other vertices of T have in-degree one, and an *in-tree of D* is defined analogously with respect to out-degrees. They also had the following result.

Theorem 3.1.8 ([13]). *Let D be a connected digraph without sources or sinks. If D has an in-tree or an out-tree, then $\vec{P}_3(D) \cong D$ if and only if $D \cong \vec{C}_n$ for some $n \geq 3$. Hence, if D is strongly connected, then $\vec{P}_3(D) \cong D$ if and only if $D \cong \vec{C}_n$ for some $n \geq 3$.*

Analogous to line digraphs above, Broersma and Li defined *$\vec{P}_3$-periodic* if $\vec{P}_3^{n+k}(D) \cong \vec{P}_3^n(D)$ for some positive integers n and k, and obtained the following results.

Theorem 3.1.9 ([13]). *Let D be a digraph. Then*

(i) $A(\vec{P}_3^n(D)) = \emptyset$ for some n if and only if D has no directed cycles except for $\vec{C}_2$'s.

(ii) $|V(\vec{P}_3^n(D))|$ gets arbitrarily large for sufficiently large n if and only if D has two directed cycles of length at least 3 joined by a directed path (possibly of length 0).

(iii) D is $\vec{P}_3$-periodic if and only if D has directed cycles of length at least 3, no two of which are joined by a directed path (possibly of length 0).

(iv) If D is strongly connected and $\vec{P}_3^n(D) \cong D$ for some $n \geq 1$, then $\vec{P}_3(D) \cong D$ and D is a directed cycle.

3.2 Whitney's Theorem

In 1932, Whitney first solved the determination problem for line graphs, referred as "Whitney's theorem" which is a lemma used to show that 3-connected planar graphs have unique duals.

Theorem 3.2.1 ([22]). *If G and H are graphs, then* *(i)* $\Gamma^*(G,H) \subseteq \Gamma_e(G,H)$;

(ii) *the mapping* $T : \Gamma(G,H) \to \Gamma^*(G,H)$ *given by* $T(f) = f^*$ *is one-to-one if and only if* G *has at most one isolated vertex and no isolated edges.*

Here we state the idea of the proof of "Whitney's theorem" given by Jung [24]. First we introduce some definitions and a lemma. Any subset of the set of edges incident to a vertex v of a graph G is called a *star* in G, and let $S(v)$ denote the set of all edges incident to v. A mapping $\sigma : E(G) \to E(H)$ is called *star-preserving* if the set $\sigma(S)$ is a star in H whenever S is a star in G.

Lemma 3.2.2 ([24]). *If* G *and* H *are connected graphs and* $\sigma : E(G) \to E(H)$ *is a bijection, then* σ *is induced by an isomorphism of* G *onto* H *if and only if* σ *and* σ^{-1} *preserve stars.*

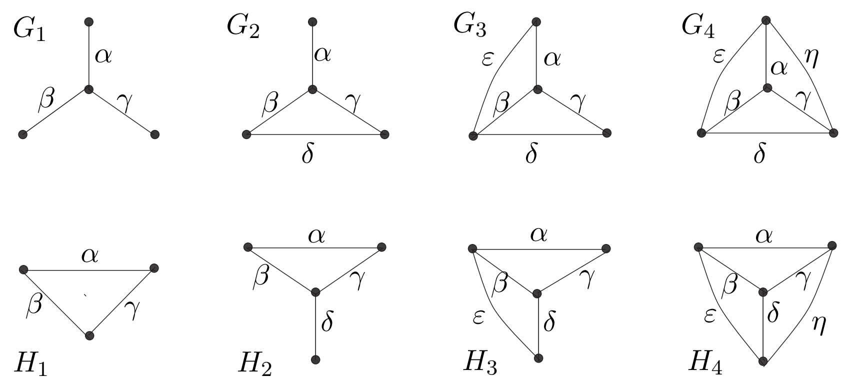


Figure 3.2

Then from Lemma 3.2.2 and Theorem 2.1 (*iv*), the following result holds.

Theorem 3.2.3 ([57]). *If* G *and* H *are connected graphs, then, except for the four cases shown in Figure 3.2, each edge isomorphism of* G *onto* H *is induced by an isomorphism of* G *onto* H.

Corollary 3.2.4 ([57]). *If* G *and* H *are connected graphs, then* $L(G) \cong L(H)$ *if and only if either* $G \cong H$ *or* $\{G, H\}$ *is, up to isomorphism, the unordered pair* $\{K_3, K_{1,3}\}$.

For line digraphs, Harary and Norman [20] got the following results.

Theorem 3.2.5 ([20]). *Let* D *and* D' *be two digraphs without sources or sinks. Then every arc-isomorphism of* D *onto* D' *is induced by an isomorphism of* D *onto* D'*, hence* $\vec{L}(D) \cong \vec{L}(D')$ *if and only if* $D \cong D'$.

Corollary 3.2.6 ([20]). *If D is a connected digraph with at most one source and at most one sink, then its automorphism group is isomorphic to that of its line digraph.*

With no restriction to line digraphs, Ouyang and Ouyang [47] gave a necessary and sufficient condition for a digraph to be determined uniquely by its line digraph.

We follow the definitions in [47]. Define an operation *"Splitting Vertices"* on a digraph D as follows: Let $v \in V(D)$ be a source with out-arcs $vu_1, \cdots, vu_k$. First replace v by two (or more) new vertices v_1, v_2, and then split the out-arcs $vu_1, \cdots, vu_k$ into two (or more) disjoint (nonempty) sets $v_1u_1, \cdots, v_1u_{k_1}$, $v_2u_{k_1+1}, \cdots, v_2u_k$. Similar operation can be defined to apply it on a sink of D. Of course the reverse operation of combining sources or sinks also preserves the $\vec{P}_k$-structure, as long as sources or sinks do not have common out-neighbors or in-neighbors, respectively.

Denote by $B(D)$ the digraph obtained from D by "Splitting Vertices" on all sources and sinks of D such that all sources and sinks of $B(D)$ have degree 1. Then Ouyang and Ouyang proved the following theorem for D and $\vec{L}(D)$ with no isolated vertices.

Theorem 3.2.7 ([47]). *Let D and D' be two digraphs. $\vec{L}(D) \cong \vec{L}(D')$ if and only if $B(D) \cong B(D')$.*

3.3 P_3-Isomorphisms of Undirected Graphs

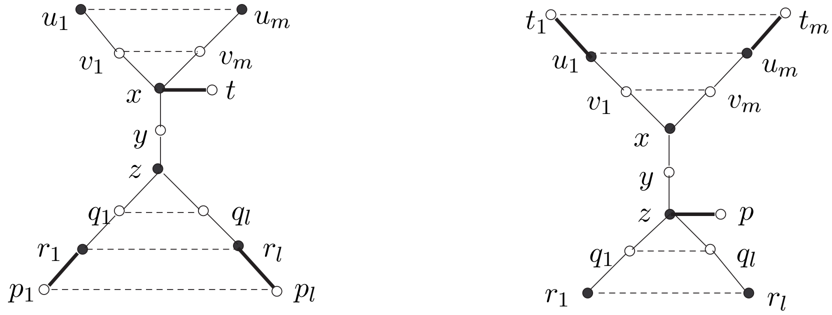
In [12], two infinite classes of graphs were given to show that pairs of nonisomorphic connected graphs have isomorphic connected P_3-graphs, which are shown in Figure 3.3 and 3.4, respectively. These classes of graphs show that Whitney's result on line graphs has no similar counterpart with respect to P_3-graphs. It is easy to find more pairs of connected nonisomorphic graphs that have isomorphic P_3-graphs. For example, $P_3(S(K_{1,3})) = C_6 = P_3(C_6)$, where $S(K_{1,3})$ is the graph resulting from $K_{1,3}$ by subdividing every edge of $K_{1,3}$ exactly once.

Hence, Broersma and Hoede [12] proposed the following two problems.

Problem 1. Characterize all pairs of nonisomorphic connected graphs with isomorphic connected P_3-graphs.

Problem 2. Whether there exist triples of mutually nonisomorphic connected graphs with isomorphic connected P_3-graphs?

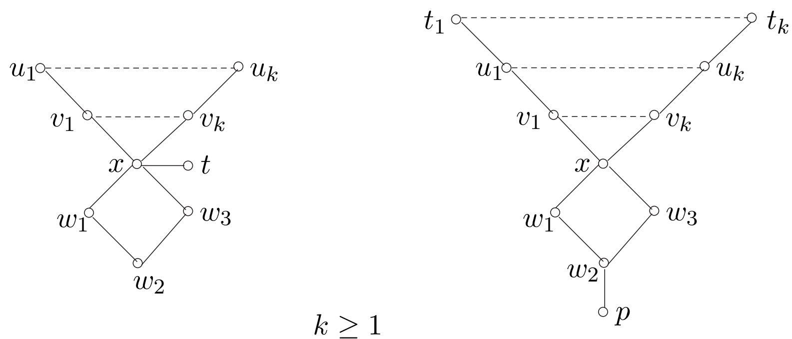
By now both of these problems have been solved. For the first problem, Li [38, 39] showed that the P_3-transformation is one-to-one for all connected graphs of minimum degree at least 3. Later, Aldred, Ellingham, Hemminger and Jipsen [2] characterized that two graphs with isomorphic P_3-graphs are either isomorphic or part of three exceptional families, and thus the determination problem for $k = 3$ was completely solved. Recently, according to Aldred, Ellingham, Hemminger and Jipsen's result, Li and Liu [40] gave a negative answer to the second problem, i.e., there is no such triples.



$m > l \geq 0$

$T_{m,l}$ $T'_{m,l}$

Figure 3.3 Pairs of nonisomorphic trees $T_{m,l}$ and $T'_{m,l}$ with isomorphic P_3-graphs.



$k \geq 1$

U_k U'_k

Figure 3.4 Pairs of nonisomorphic unicyclic graphs U_k and U'_k with isomorphic P_3-graphs.

3.3.1 Li's Results

Noticing that all examples above have minimum degree at most 2, so Li conjectured that for graphs with larger minimum degree, if their P_3-graphs are isomorphic, then they themselves must be isomorphic. In fact, Li proved that it holds for all graphs with minimum degree at least 3.

We will follow Beineke and Hemminger's treatment of Whitney's theorem in [22], which in turn reflects Jung's ideas in [24], to introduce the following notations and the corresponding results in [44].

Shorten $\Gamma(P_k(G), P_k(G'))$ to $\Gamma_k(G, G')$ and call the members *P_k-isomorphisms*. Obviously, $\Gamma_1(G, G') = \Gamma(G, G')$ and $\Gamma_2(G, G') = \Gamma_e(G, G')$. Let $f \in \Gamma(G, G')$ and $x_1 x_2 \cdots x_k$ be a P_k-path in G, then $f(x_1)f(x_2)\cdots f(x_k)$ is a P_k-path in G'. Define a mapping $f^* : \Pi_k(G) \to \Pi_k(G')$ by $f^*(x_1 x_2 \cdots x_k) = f(x_1)f(x_2)\cdots f(x_k)$ and call f^* the *mapping*

$(*)$-*induced by* f. Let $\Gamma_k^*(G,G^{'}) = \{f^* : f \in \Gamma(G,G^{'})\}$. Note that $\Gamma_2^*(G,G^{'}) = \Gamma^*(G,G^{'})$.

For a non-negative integer d, denote by $\mathcal{G}_d$ the class of all connected graphs with minimum degree at least d.

Theorem 3.3.1 ([44]). *Let* $G, G^{'} \in \mathcal{G}_k$ $(k \geq 3)$. *Then*

(i) $\Gamma_k^*(G,G^{'}) \subseteq \Gamma_k(G,G^{'})$;

(ii) the mapping $T : \Gamma(G,G^{'}) \to \Gamma_k^*(G,G^{'})$ *given by* $T(f) = f^*$ *is one-to-one.*

Now, we introduce two results in [38], which are crucial for the proofs of the following theorems. Before giving them, some definitions are needed. If $P_3 = uvw$, then v is called the *middle vertex* of the path. The set of all the P_3-paths with a common middle vertex v is denoted by $S(v)$ and any subset of $S(v)$ is called a *star* at v. A mapping $f : \Pi_3(G) \to \Pi_3(G^{'})$ is called *star-preserving* if the set $f(S(v))$ is a star in $G^{'}$ for every vertex v of G.

Lemma 3.3.2 ([38]). *Let* $G, G^{'} \in \mathcal{G}_3$ *and* $f : \Pi_3(G) \to \Pi_3(G^{'})$ *be a mapping. Then* f *is induced by an isomorphism from* G *onto* $G^{'}$ *if and only if* f *and* f^{-1} *are star-preserving* P_3*-isomorphisms.*

Lemma 3.3.3 ([38]). *Let* $G, G^{'} \in \mathcal{G}_3$ *and* f *be a* P_3*-isomorphism from* G *to* $G^{'}$. *Then* f *is star-preserving if and only if for every edge* $e = uv$ *of* G, $f(x_1uv), \cdots, f(x_ruv)$ *have a common middle vertex and* $f(uvy_1), \cdots, f(uvy_s)$ *have a common middle vertex, where* $x_1, \cdots, x_r$ *and* $y_1, \cdots, y_s$ *are neighbors of* u *and* v, *respectively.*

Using Lemmas 3.3.2 and 3.3.3, the following three theorems hold.

Theorem 3.3.4 ([38]). *Let* G *and* $G^{'}$ *be two graphs with minimum degree at least 3 and with* $max\{d(u), d(v)\} \geq 4$ *for every edge* $e = uv$ *of* G *and* $G^{'}$. *Then every* P_3*-isomorphism* f *from* G *to* $G^{'}$ *is induced by an isomorphism from* G *to* $G^{'}$. *Thus, if* $P_3(G)$ *is isomorphic to* $P_3(G^{'})$, *then* G *is isomorphic to* $G^{'}$.

Corollary 3.3.5 ([38]). *Let* G *and* $G^{'}$ *be two graphs with minimum degree at least 4. Then* $P_3(G)$ *is isomorphic to* $P_3(G^{'})$ *if and only if* G *is isomorphic to* $G^{'}$.

Theorem 3.3.6 ([38]). *Let* G *and* $G^{'}$ *be graphs with minimum degree at least 3 and without* C_4. *Then every* P_3*-isomorphism* f *from* G *to* $G^{'}$ *is induced by an isomorphism from* G *to* $G^{'}$.

A graph G is called *double-claw-free* if for every edge uv of G, $N_G(u)\backslash\{v\}$ or $N_G(v)\backslash\{u\}$ induces a subgraph containing at least one edge.

Theorem 3.3.7 ([38]). *Let* G *and* $G^{'}$ *be two double-claw-free graphs with minimum degree at least 3. Then every* P_3*-isomorphism* f *from* G *to* $G^{'}$ *is induced by an isomorphism from* G *to* $G^{'}$.

In [39], Li extended the above results, and then obtained a stronger result that every P_3-isomorphism from G to G' is induced by an isomorphism from G to G' with one exception. In order to prove this result, the following lemma is needed.

Lemma 3.3.8 ([39]). *Let $G, G' \in \mathcal{G}_3$ and assume that the maximum degree $\triangle(G) \geq 4$ or G have a triangle. If f is a P_3-isomorphism from G to G', then f is star-preserving.*

Then the main result follows from Lemmas 3.3.2 and 3.3.8.

Theorem 3.3.9 ([39]). *Let $G, G' \in \mathcal{G}_3 \setminus \{K_{3,3}\}$. Then f is a P_3-isomorphism from G to G' if and only if f is induced by an isomorphism of G onto G'. Furthermore, for any $G, G' \in \mathcal{G}_3$, $P_3(G)$ is isomorphic to $P_3(G')$ if and only if G is isomorphic to G'.*

From Theorems 3.3.1 and 3.3.9, the next conclusion follows immediately.

Corollary 3.3.10 ([39]). *Let $G \in \mathcal{G}_3 \setminus \{K_{3,3}\}$. Then the automorphism group of $P_3(G)$ is isomorphic to that of G.*

3.3.2 Aldred, Ellingham, Hemminger and Jipsen's Results

From the above Subsection 3.3.1, we know that the only cases that remain open for the first problem are cases where some degree is 1 or 2 in a graph. Aldred, Ellingham, Hemminger and Jipsen identified all P_3-isomorphisms for graphs with no degree restriction.

We follow [2] for the terminology and notations. If τ_i is a P_k-isomorphism from G_i to H_i for $i = 1$ and 2, then τ_1 and τ_2 are *equivalent* if there are isomorphisms σ and ρ from G_1 to G_2 and H_1 to H_2, respectively, such that $\tau_1 = (\rho^*)^{-1} \circ \tau_2 \circ \sigma^*$. For α in $\Pi_3(G)$, let $m(\alpha)$ denote the middle vertex of α, let $S(a)$ be the set of all P_3's with middle vertex a, and define $a \vdash b$ to be the set of all P_3's in $S(a)$ with an end at b, where $a \vdash b$ is empty if $a \not\sim b$. The set $a \vdash b$, if nonempty, is called a *bundle* with a as its *middle* and b as its *base.* If $R \subseteq \Pi_3(G)$ then the P_3-isomorphism τ is said to *disperse* R if $m(\tau(\alpha)) \neq m(\tau(\beta))$ for some $\alpha, \beta \in R$. A vertex of degree 1 is also called *terminal.* Define an *i-thorn* to be a P_3 with exactly i ($i = 1$ or 2) terminal ends in G, and a *thorn* to be a 1- or 2-thorn. A P_3 in G is called *terminal* if it has degree 1 in $P_3(G)$. Let $T_i(G)$ be the set of i-thorns in G.

Any swap of two 2-thorns is a P_3-isomorphism, called a *2-thorn swap*, or *T-swap* for short. Consider two 1-thorns abc and abd where $deg(a) \geq 2$ and $deg(c) = deg(d) = 1$, swapping abc and abd gives a P_3-isomorphism, called a *bundle 1-thorn swap*, or *B-swap* for short. Suppose $abcde$ is a P_5 in G with both abc and cde terminal 1-thorns, i.e., $deg(a) = deg(e) = 1$ and $deg(c) = 2$, swapping abc and cde gives a P_3-isomorphism, called a *split 1-thorn swap*, or *S-swap* for short.

For distinct $a, b \in V(G)$, let $D_{a,b}$ denote the subgraph of G consisting of the union of all P_3's with ends a and b and with middle vertex of degree 2 in G. If $D_{a,b}$ is nonempty, then call it a *diamond* with ends a and b. Usually write $V(D_{a,b}) - \{a, b\}$ as $\{c_1, c_2, \cdots, c_k\}$

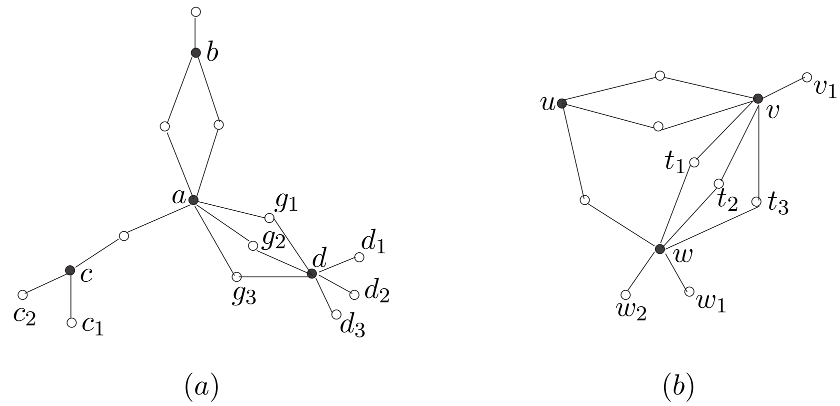


Figure 3.5 Illustrating swaps, diamond inflations and Whitney type P_3-isomorphisms.

and say that $D_{a,b}$ is a *trivial diamond* if $k = 1$, and *nontrivial* otherwise; denote by k the *width* of $D_{a,b}$, and refer to $D_{a,b}$ as a *k-diamond.* Note that if $a \sim b$, the edge ab is not included in $D_{a,b}$. To distinguish the two possibilities, the diamond $D_{a,b}$ is *braced* if $a \sim b$ and *unbraced* otherwise.

Suppose that $D_{a,b}$ is a nontrivial diamond with vertices labeled as above. For $1 \leq i < j \leq k$, the P_3's ac_ib are called *diamond paths* while the pair of P_3's c_iac_j and c_ibc_j is called a *diamond pair* associated with ac_ib and ac_jb. A P_3 of the form cad where $deg(c) = deg(d) = 2$ that is not one of a diamond pair (thus c and d are in different diamonds) is called a *diamond connector.* Suppose c_iac_j and c_ibc_j are a diamond pair, swapping c_iac_j and c_ibc_j gives a P_3-isomorphism, called a *diamond pair swap*, or *D-swap* for short.

Two P_3-isomorphisms τ_i from G_i to H_i for $i = 1$ and 2, are *T-related* if (i) G_1 and G_2 differ only in their star components, as do H_1 and H_2; (ii) $|T_2(G_1)| = |T_2(G_2)|$; and (iii) $\tau_1(\alpha) = \tau_2(\alpha)$ for every $\alpha \in \Pi_3(G_1) - T_2(G_1) = \Pi_3(G_2) - T_2(G_2)$. If τ_1 and τ_2 are P_3-isomorphisms from G to H, then τ_1 and τ_2 are *B-related* if $\tau_2^{-1} \circ \tau_1$ is the identity or a composition of B-swaps. The *S-related* and *D-related* are defined similarly. They use joins of these four equivalence relations: for example, two P_3-isomorphisms are *TBSD-related* if you can get from one to the other by a chain of zero or more T-, B-, S- and/or D-relations. Other joins will be denoted by analogous notation.

The following is the main result of [2]. The definition for each type in the Main Theorem will be given in the successive parts.

Main Theorem ([2]). *Let τ be a P_3-isomorphism from G to H such that at least one of G or H is connected. Then τ is one of the following:*

(i) T-related to a P_3-isomorphism of generalized $K_{3,3}$ type;

(ii) of special Whitney type;

(iii) D-related to a P_3-isomorphism of Whitney type 3, 4, 5 or 6;

(iv) *D-related to a P_3-isomorphism of bipartite type; or*

(v) *TBSD-related to an induced P_3-isomorphism.*

Part 1. Generalized $K_{3,3}$ Type

The following construction shows that even P_3-isomorphisms between graphs of minimum degree 3 may not be induced.

Construction on $K_{3,3}$. Let vertex sets $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ be a partition of $K_{3,3}$ and define $\tau_0 : \Pi_3(K_{3,3}) \to \Pi_3(K_{3,3})$ by $\tau_0(u_i v_i u_j) = u_k v_k u_j$, $\tau_0(v_i u_i v_j) = u_k u_k v_j$, $\tau_0(u_i v_j u_k) = u_i v_j u_k$ and $\tau_0(v_i u_j v_k) = v_i u_j v_k$ for each i, j and k with $\{i, j, k\} = \{1, 2, 3\}$.

Theorem ($K_{3,3}$). *Let G and H be connected graphs of minimum degree at least 2 and let τ be a P_3-isomorphism from G to H. If τ disperses $a \vdash b$ where $deg(b) \geq 3$, then $G \cong H \cong K_{3,3}$ and τ is equivalent to τ_0 as in the Construction on $K_{3,3}$.*

Construction of the Generalized $K_{3,3}$ Pairs. Relax the minimum degree restriction of the above theorem, and allow terminal vertices in G and H, then six more P_3-isomorphisms τ that disperse bundles will be obtained. All seven are listed below, using the following notation. Write $(c,d)ab(e,f) \mapsto uvwxu$ if G contains the edges ab, ac, ad, be, bf, H contains the C_4 $uvwxu$, and τ maps $cab \mapsto xuv$, $dab \mapsto vwx$, $abe \mapsto uvw$ and $abf \mapsto wxu$. Also write $abc(d,e) \mapsto uvwxy$ if G contains the edges ab, bc, cd, ce, H contains the P_5 $uvwxy$, and τ maps $abc \mapsto vwx$, $bcd \mapsto uvw$ and $bce \mapsto wxy$. This notation will be reversed (e.g., $abcda \mapsto (w,x)uv(y,z)$) as needed.

(i) $(c,d)ab(e,f) \mapsto u_1v_1u_2v_2u_1$, and cad and ebf map to P_3 components of H.

(ii) $(c,d)ab(e,f) \mapsto u_1v_1u_2v_2u_1$, $kebfh \mapsto yv_3u_1(v_1,v_2)$, and cad maps to a P_3 component.

(iii) $(c,d)ab(e,f) \mapsto u_1v_1u_2v_2u_1$, $(k,l)eb(a,f) \mapsto u_1v_1u_2v_3u_1$, $(h,i)fb(a,e) \mapsto u_1v_2u_2v_3u_1$, and cad, kel and hfi map to P_3 components.

(iv) $(c,d)ab(e,f) \mapsto u_1v_1u_2v_2u_1$, $ecadg \mapsto xu_3v_1(u_1,u_2)$, and $cebfh \mapsto yv_3u_1(v_1,v_2)$. Note that G and H are connected and isomorphic.

(v) $(c,d)ab(e,f) \mapsto u_1v_1u_2v_2u_1$, $ebfhe \mapsto (v_1,v_2)u_1v_3(y,z)$, and cad maps to yv_3z. Again G and H are connected and isomorphic.

(vi) $(c,d)ab(e,f) \mapsto u_1v_1u_2v_2u_1$, $(c,d)eb(a,f) \mapsto u_1v_1u_2v_3u_1$, $(h,i)fb(a,e) \mapsto u_1v_2u_2v_3u_1$, $aceda \mapsto (w,x)u_3v_1(u_1,u_2)$, and hfi maps to wu_3x. Again G and H are connected and isomorphic.

(vii) The Construction on $K_{3,3}$; $G \cong H \cong K_{3,3}$.

Either τ or τ^{-1} as in cases (i) through (vii) above, or any equivalent P_3-isomorphism, is said to be of *generalized $K_{3,3}$ type.*

Part 2. Special Whitney Type and Whitney Type 3, 4, 5 and 6

Begin with a rather general idea which will be used here and in the next part. Suppose F is a graph. A *diamond inflation* of F is a graph obtained by replacing each edge $ab \in E(F)$ by an unbraced s_{ab}-diamond $D_{a,b}$ ($s_{a,b} \geq 1$), and adding t_a terminal edges incident with each $a \in V(F)$ ($t_a \geq 0$).

Now suppose φ is an edge isomorphism between graphs F and $F^{'}$, and suppose I and $I^{'}$ are diamond inflations of F and $F^{'}$, respectively, with the following property: for every $ab \in E(F)$, if $\varphi(ab) = uv$ then (i) $s_{uv} = s_{ab}$ and (ii) $t_u + t_v = t_a + t_b$. Obtain G and H from I and $I^{'}$, respectively, by adding star components to one of them (if necessary) to make the number of 2-thorns equal. Then define a P_3-isomorphism τ from G to H, as follows. Suppose $ab \in E(F)$ and $\varphi(ab) = uv$. Let $\tau|\Pi_3(D_{a,b})$ be induced by any isomorphism from $D_{a,b}$ to $D_{u,v}$: the two diamonds are the same size by (i). For any diamond path α of $D_{a,b}$, the $t_a + t_b$ terminal 1-thorns adjacent to α can be mapped arbitrarily to the $t_u + t_v$ terminal 1-thorns adjacent to $\tau(\alpha)$, since the numbers are equal by (ii). The 2-thorns of G can be mapped arbitrarily to the 2-thorns of H. This only leaves the diamond connectors: the image of each diamond connector is uniquely determined by the images of the two diamond paths which are its neighbors. Then τ is said a *diamond inflation* of φ.

Theorem (Whitney [57]). *Suppose that φ is an edge isomorphism from G to H where G and H are both connected. If φ is not induced, then $i = |E(G)| = |E(H)| \in \{3, 4, 5, 6\}$, G and H are isomorphic to W_i and $W_i^{'}$ in some order, and φ is equivalent to φ_i or φ_i^{-1}, where*

(i) $W_6 \cong W_6^{'} \cong K_4$, *with* $V(W_6) = \{a, b, c, d\}$, $V(W_6^{'}) = \{u, v, w, x\}$, *and* φ_6 *maps* $ab \mapsto uv$, $ac \mapsto uw$, $ad \mapsto vw$, $bc \mapsto ux$, $bd \mapsto vx$ *and* $cd \mapsto wx$;

(ii) $W_5 = W_6 - cd$, $W_5^{'} = W_6^{'} - wx$ *and* $\varphi_5 = \varphi_6|E(W_5)$;

(iii) $W_4 = W_6 - \{bd, cd\}$, $W_4^{'} = W_6^{'} - \{vx, wx\}$ *and* $\varphi_4 = \varphi_6|E(W_4)$; *and*

(iv) $W_3 = W_6 - \{bc, bd, cd\} \cong K_{1,3}$, $W_3^{'} = W_6^{'} - x \cong K_3$, *and* $\varphi_3 = \varphi_6|E(W_3)$.

Construction on the Whitney Graphs. A P_3-isomorphism τ is said to be of *Whitney type i* if τ or τ^{-1} is equivalent to a diamond inflation of φ_i as above for $i = 3, 4, 5, 6$.

The Special Whitney Type. It is easy to see that $P_3(S(K_{1,3})) \cong C_6$. Rotation of this C_6 by one step is a noninduced P_3-isomorphism from $S(K_{1,3})$ to itself; then say this or any equivalent P_3-isomorphism is of *special Whitney type*.

Part 3. Bipartite Type

Construction on a Bipartite Graph. Start with a positive integer k and an arbitrary bipartite graph F with at least one edge and with bipartition (A, B). Let I and $I^{'}$ be different diamond inflations of F, where each edge e is inflated to a diamond of the same width s_e both times, but in producing I each vertex v has t_v terminal edges added, while in producing I' it has t_v' terminal edges added, where

$$t_v' = \begin{cases} t_v - k & \text{if } v \in A \\ t_v + k & \text{if } v \in B \end{cases} \tag{3.3.1}$$

Thus, it need $t_v \geq k$ for all $v \in A$. Let φ be the identity edge isomorphism from F to itself. Clearly φ, I and I' satisfy condition (i) of Diamond Inflation, and condition (ii) is satisfied because each edge of F has the form ab with $a \in A$ and $b \in B$, so that $t_a' + t_b' = (t_a - k) + (t_b + k) = t_a + t_b$. Therefore a P_3-isomorphism τ can be obtained by diamond inflation; τ is in general not induced. Then say τ and τ^{-1}, or any equivalent P_3-isomorphisms, are of *bipartite type*.

Some Examples. Note that the Broersma and Hoede examples in Figure 3.3 and 3.4 are fairly simple, albeit quite illustrative, cases of this construction. They may be obtained by starting with a tree F of diameter 3, whose vertices are the solid vertices of the Figure 3.3. To obtain $I = T_{m,l}$ from F, replace all edges with unbraced 1-diamonds (that is, subdivide the edges of F), add a single terminal edge at each vertex of $A = \{x, r_1, r_2, \cdots, r_l\}$, and add nothing at each vertex of $B = \{z, u_1, u_2, \cdots, u_m\}$. To obtain $I' = T_{m,l}'$, take $k = 1$ in the construction. The added terminal edges for I and I' are emboldened in the Figure 3.3. The graphs of Broersma and Hoede's Figure 3.4 are equally simple, differing only in that the starting graph is $K_{1,n+1}$ and one of the replacing diamonds is a 2-diamond, again an unbraced one.

At last, one special case is easily stated, which is actually Theorem 3.3.9.

Corollary ([2]). *If τ is a P_3-isomorphism from G to H, where G has minimum degree at least 3, then $G \cong H$. Moreover, τ is induced unless τ is equivalent to τ_0 as in the Construction on $K_{3,3}$.*

3.3.3 Negative Answer to Problem 2

For the second problem, Li and Liu [40] proved that there is no triple of mutually connected graphs with isomorphic connected P_3-graphs. In order to solve this problem, just to consider the original graphs G and H are nonisomorphic connected graphs with $T_2(G) = T_2(H) = \emptyset$. They analyzed the types in the Main Theorem above case by case in details, which are stated in the following.

By the definition of generalized $K_{3,3}$ type in Part 1, it is easy to check that there is no pair of nonisomorphic connected graphs with isomorphic connected P_3-graphs from (i) to (vii).

For the special Whitney type, it is a noninduced P_3-isomorphism from $S(K_{1,3})$ to itself. So go on considering the third type (i.e., Whitney type 3, 4, 5 and 6). Follow the label as Whitney Theorem in Part 2. Denote by t_z the number of terminal edges incident with z for z in $\{a, b, c, d\}$ or $\{u, v, w, x\}$. For Whitney type P_3-isomorphisms, according to condition

(ii) of Diamond Inflation, gives one equation from each pair of corresponding edges of the original Whitney graphs. Then there is a same solution for all four types:

$$\begin{cases} t_u = \frac{1}{2}(t_a + t_b + t_c - t_d) \\ t_v = \frac{1}{2}(t_a + t_b - t_c + t_d) \\ t_w = \frac{1}{2}(t_a - t_b + t_c + t_d) \\ t_x = \frac{1}{2}(-t_a + t_b + t_c + t_d) \quad \text{(except for type 3)} \end{cases} \tag{3.3.2}$$

Because it requires connected P_3-graphs, for the four equations in (3.3.2), t_z must equal to 0 or 1 for every $z \in \{a, b, c, d\} \cup \{u, v, w, x\}$. Write $(t_a, t_b, t_c, t_d) \mapsto (t_u, t_v, t_w, t_x)$. If t_a, t_b, t_c, $t_d = 0$ or 1, then the corresponding solutions for t_u, t_v, t_w, t_x by (3.3.2). For example: (1, 0, 0, 1) $\mapsto$ (0, 1, 1, 0) denotes that $t_a = 1$, $t_b = t_c = 0$ and $t_d = 1$ correspond to solutions $t_u = 0$, $t_v = t_w = 1$ and $t_x = 0$ by (3.3.2). So it is easy to check that there are only the following eight cases satisfying $t_z = 0$ or 1 for every $z \in \{a, b, c, d\} \cup \{u, v, w, x\}$:

(i) (0, 0, 0, 0) $\mapsto$ (0, 0, 0, 0).
(ii) (1, 1, 1, 1) $\mapsto$ (1, 1, 1, 1) (except for type 3).
(iii) (1, 1, 0, 0) $\mapsto$ (1, 1, 0, 0) ($ab \mapsto uv$).
(iv) (1, 0, 1, 0) $\mapsto$ (1, 0, 1, 0) ($ac \mapsto uw$).
(v) (1, 0, 0, 1) $\mapsto$ (0, 1, 1, 0) ($ad \mapsto vw$).
(vi) (0, 1, 1, 0) $\mapsto$ (1, 0, 0, 1) ($bc \mapsto ux$) (except for type 3).
(vii) (0, 1, 0, 1) $\mapsto$ (0, 1, 0, 1) ($bd \mapsto vx$) (except for type 3 or 4).
(viii) (0, 0, 1, 1) $\mapsto$ (0, 0, 1, 1) ($cd \mapsto wx$) (except for type $3, 4$ or 5).

If a P_3-isomorphism τ or τ^{-1} is equivalent to a diamond inflation of φ_i in Part 2, and falls into one of the cases (i) through (viii) above, then τ is said to be of *special Whitney type* i for $i = 3, 4, 5$ or 6.

The discussion of bipartite type is similar to that of Whitney type. Also follow the label in Part 3. If the P_3-graphs of I and I' are connected, then t_v, $t'_v = 0$ or 1 for every $v \in A \cup B$. In equations (3.3.1), $k \leq t_v (v \in A)$, so $k = 0$ or 1. If $k = 0$, then $I \cong I'$. If $k = 1$, then $t_u = 1$ for all $u \in A$ and $t_v = 0$ for all $v \in B$. Otherwise, if there is a vertex $u_0 \in A$ with $t_{u_0} = 0$ or a vertex $v_0 \in B$ with $t_{v_0} = 1$, then $t'_{u_0} = -1$ or $t'_{v_0} = 2$ by (3.3.1). Therefore there is a P_3-isomorphism τ_0 from I to I', where $t_u = 1$ and $t'_u = 0$ for all $u \in A$, $t_v = 0$ and $t'_v = 1$ for all $v \in B$, respectively. Then say τ_0 and τ_0^{-1}, or any equivalent P_3-isomorphisms, are of *special bipartite type*.

For the last type, if it requires connected P_3-graphs, then the original graph and its P_3-graph are one-to-one.

From the arguments above, Li and Liu got the following corollary which is essential to solve the second problem.

Corollary 3.3.11 ([40]). *Let τ be a P_3-isomorphism from G to H, where G and H are nonisomorphic connected graphs with an isomorphic connected P_3-graph. Then τ is one of the following:*

(i) D-related to a P_3-isomorphism of special Whitney type $3, 4, 5$ *or* 6*; or*

(ii) D-related to a P_3-isomorphism of special bipartite type.

Theorem 3.3.12 ([40]). *There is no triple of mutually nonisomorphic connected graphs with an isomorphic connected P_3-graph.*

3.3.4 Directed Case

For the directed path graphs, Broersma and Li in [13] proved a result on $\vec{P}_3$-isomorphisms with additional assumption concerning the digraphs D and D'. This additional assumption is the nature counterpart of the assumption in Theorem 3.2.5 that D and D' contain no sources or sinks.

Before giving the result, we introduce some terminology concerning isomorphisms. A $\vec{P}_3$*-isomorphism* of D onto D' is an isomorphism of $\vec{P}_3(D)$ onto $\vec{P}_3(D')$. A $\vec{P}_3$-isomorphism f of D onto D' is *induced by an arc-isomorphism* of D onto D' if there exists an arc-isomorphism f^* of D onto D' such that $f(uvw) = f^*(uv)f^*(vw)$ for each $\vec{P}_3 = uvw$ of D.

Theorem 3.3.13 ([13]). *Let D and D' be two connected digraphs. If for each arc $a = uv \in A(D) \cup A(D')$ these exist arcs $b = xu$ and $c = vy$ in the same digraph with $x \neq v$ and $y \neq u$, then every $\vec{P}_3$-isomorphism of D onto D' is induced by an arc-isomorphism of D onto D'.*

With no restriction to original digraphs, analogous to line digraphs, Li, Liu and Zhao [42] also found a necessary and sufficient condition for digraphs D and D' with isomorphic $\vec{P}_3$-graphs.

Now, we introduce some definitions and two operations in [42]. For an arc vw, we say that v is *adjacent to* w, and w is *adjacent from* v. The pair of arcs a, b is called a $\vec{C}_2$*-pair* if a and b form a $\vec{C}_2$ in D. Let $\{uv, vu\}$ be a $\vec{C}_2$-pair, then v is called a *pseudo-source with respect to* u if $N^-(v) = \{u\}$ and $d^+(v) \geq 2$, v is called a *pseudo-sink with respect to* u if $N^+(v) = \{u\}$ and $d^-(v) \geq 2$, and v is called an *end* if $N^+(v) = N^-(v) = \{u\}$. Denote by S_v and T_v the sets of all sources adjacent to, and sinks adjacent from a vertex v, respectively; and by X_v and Y_v the sets of all pseudo-sources adjacent to, and pseudo-sinks adjacent from a vertex v, respectively. If $\vec{P}_3 = uvw$, then v is called the *middle vertex* of $\vec{P}_3$. Any subset of the set of all the $\vec{P}_3$-paths with a common middle vertex v is called a *star* at v. Let $S(v)$ denote the set of all such $\vec{P}_3$-paths, and $S_2(v)$ the set of all the $\vec{P}_3$-paths, with a common middle vertex v, which are isolated vertices in $\vec{P}_3(D)$. Let $S_2(D) = \cup_{v \in V(D)} S_2(v)$, $S_1(v) = S(v) \backslash S_2(v)$.

Take the following three properties (a), (b) and (c) as one *property* $\mathcal{P}$: (a) D has no ends. (b) All sources and ends of D are degree 1. (c) For any arc $uv \in A(D)$, there exists a $\vec{P}_3$-path wxu or vyz in D with $x \neq v$ and $y \neq u$.

In the following two operations A and B on a digraph D are stated, which also preserve the $\vec{P}_3$-structure of D.

Operation A.

1. For $v \in V(D)$, $N^-(v) = S_v \cup X_v \cup Y_v$.

Let $S_v = \{s_1, \cdots, s_n\}$ and $d^+(s_i) = 1$ for $1 \leq i \leq n$. Let $X_v = \{x_1, \cdots, x_r\}$ and $Y_v = \{y_1, \cdots, y_s\}$.

(i) $N^-(v) = S_v \neq \emptyset$.

So $X_v = Y_v = \emptyset$ and $T_v = \emptyset$ by property (c) of $\mathcal{P}$. Let $N^+(v) = \{u_1, \cdots, u_k\}$. If the in-neighborhood of a vertex v consists of only sources, then do the following:

First, delete vertices $s_1, \cdots, s_n$, then v is a source with out-arcs $vu_1, \cdots, vu_k$. Do the operation "Splitting Vertices" at v. Replace v by k vertices $v_1, \cdots, v_k$ and split the out-arcs $vu, \cdots, vu_k$ into k arcs $v_1u_1, \cdots, v_ku_k$. Finally, join k sets of n independent vertices by arcs, each to one vertex of $\{v_1, \cdots, v_k\}$.

(ii) $N^-(v) = S_v \cup X_v$ and $X_v \neq \emptyset$.

Then $Y_v = \emptyset$ and $T_v = \emptyset$ by property (c) of $\mathcal{P}$. Let $N^+(v) = \{u_1, \cdots, u_k\} \cup X_v$. If the in-neighborhood of a vertex v consists of sources and pseudo-sources with respect to v, then do the following:

Delete vertices $s_1, \cdots, s_n$ and arcs $x_1v, \cdots, x_rv$. Then replace v by two vertices v_1, v_2 and split the arcs $vu_1, \cdots, vu_k, vx_1, \cdots, vx_r$ into $v_1u_1, \cdots, v_1u_k, v_2x_1, \cdots, v_2x_r$. Finally, add two sets of $n+r$ and $n+r-1$ independent vertices such that each is adjacent to v_1 and v_2, respectively. Then it is easy to see that the in-neighborhoods of v_1 and v_2 both consist of sources. Continue to do the operation at v_1 and v_2 similar to (i).

(iii) $N^-(v) = S_v \cup X_v \cup Y_v$ and $Y_v \neq \emptyset$.

If $N^+(v) \backslash (X_v \cup Y_v \cup T_v) \neq \emptyset$, then there exists a $\vec{P}_3$-path vuw in D with $u \notin X_v$. Thus do the following: delete the arcs $vy_1, \cdots, vy_s$ and add $s-1$ independent vertices such that all of them are adjacent from v.

Otherwise, let $N^+(v) = X_v \cup Y_v \cup T_v$. If $X_v \neq \emptyset$, then do the following: delete the arcs $x_1v, \cdots, x_rv$ and $vy_1, \cdots, vy_s$ and add two sets of $r-1$ and $s-1$ independent vertices such that each is adjacent to and from v, respectively. If $X_v = \emptyset$, then $N^+(v) = Y_v \cup T_v$. Then, do the operation at v similar to (ii).

2. Let $v \in V(D)$, $N^+(v) = T_v \cup Y_v \cup X_v$. Do the operation at v similar to 1.

Denote by $\mathcal{F}$ the set of digraphs obtained from a cycle C_n by replacing each edge uv of C_n by two arcs uv and vu, $n \geq 3$. Denote by $\mathcal{F}^*$ the set of connected digraphs consisting of $\vec{C}_2$-pairs $\{x_ix_{i+1}, x_{i+1}x_i\}$ $(x_1 = x_{k+1})$, and $N^-(x_i) = \{x_{i-1}, x_{i+1}\} \cup S_{x_i}$, $N^+(x_i) = \{x_{i-1}, x_{i+1}\} \cup T_{x_i}$, for $i = 1, \cdots, k$, $k \geq 3$.

Operation B.

1. If $D \in \mathcal{F}^*$, then replace D by two digraphs D_1 and D_2 as follows: D_1 is obtained from D by deleting the arcs $x_1x_2, \cdots, x_{k-1}x_k$, and D_2 by deleting the arcs $x_kx_{k-1}, \cdots, x_2x_1$ from D.

2. There are $\vec{C}_2$-pairs $\{x_ix_{i+1}, x_{i+1}x_i\}$ in D for $i = 1, \cdots, k-1$ ($k \geq 2$), and $N^-(x_i) = \{x_{i-1}, x_{i+1}\} \cup S_{x_i}$, $N^+(x_i) = \{x_{i-1}, x_{i+1}\} \cup T_{x_i}$, for $i = 2, \cdots, k-1$. Now consider the in-neighborhoods and out-neighborhoods of x_1 and x_k. If x_1 and x_k satisfy one of the following conditions, then some operations will be done at x_1 and x_k.

(i) $N^-(x_1) = \{x_2\} \cup S_{x_1} \cup X_{x_1}$ and $N^-(x_k) = \{x_{k-1}\} \cup S_{x_k} \cup X_{x_k}$.

(ii) $N^+(x_1) = \{x_2\} \cup T_{x_1} \cup Y_{x_1}$ and $N^+(x_k) = \{x_{k-1}\} \cup T_{x_k} \cup Y_{x_k}$.

(iii) $N^-(x_1) = \{x_2\} \cup S_{x_1} \cup X_{x_1}$ and $N^+(x_k) = \{x_{k-1}\} \cup T_{x_k} \cup Y_{x_k}$.

Without loss of generality, let x_1 and x_k satisfy condition (i), then do the following:

First, replace x_1 and x_k by x_{11}, x_{12} and x_{k1}, x_{k2}, respectively, such that $N^-(x_{11}) = \{x_2\} = N^+(x_{11})$, $N^-(x_{k1}) = \{x_{k-1}\} = N^+(x_{k1})$, $N^-(x_{12}) = S_{x_1} \cup X_{x_1}$, $N^+(x_{12}) = N_D^+(x_1) \backslash \{x_2\}$, $N^-(x_{k2}) = S_{x_k} \cup X_{x_k}$ and $N^+(x_{k2}) = N_D^+(x_k) \backslash \{x_{k-1}\}$. Then denote by H the component containing x_{11} and x_{k1}. Replace H by two digraphs H_1 and H_2, where H_1 is obtained from H by deleting the arcs $x_{11}x_2, x_2x_3, \cdots, x_{k-1}x_{k1}$, H_2 by deleting $x_{k1}x_{k-1}, \cdots, x_3x_2, x_2x_{11}$ from H. Let $|S_{x_i}| = n_i$ and $|X_{x_i}| = r_i$ for $i = 1, k$. Then add two sets of $n_1 + r_1$ and $n_k + r_k$ independent vertices such that each is adjacent to x_{11} in H_2 and x_{k1} in H_1, respectively. Finally, identify x_{12} to x_{11} in H_1, and identify x_{k2} to x_{k1} in H_2. At last, observe the out-neighborhoods of x_{12} and x_{k2}. If the out-neighborhoods of x_{12} and x_{k2} consist of sinks and pseudo-sources with respect to x_{12} and x_{k2}, respectively, then do the operation at x_{12} and x_{k2} similar to operation A.

For the cases (ii) and (iii), do the operation at v similar to (i) as above.

Now denote by $C(D)$ the digraph resulting from D by doing the operations A and B, and satisfying the property $\mathcal{P}$. If necessary, add one or more $\vec{P}_3$-paths to either $C(D)$ or $C(D')$ to obtain digraphs $\tilde{C}(D)$ and $\tilde{C}(D')$ such that $|S_2(\tilde{C}(D))| = |S_2(\tilde{C}(D'))|$, and then $\vec{P}_3(\tilde{C}(D)) \cong \vec{P}_3(\tilde{C}(D'))$.

Theorem 3.3.14 ([42]). *Let D and D' be two digraphs with $|S_2(D)| = |S_2(D')|$. Then $\vec{P}_3(D) \cong \vec{P}_3(D')$ if and only if $C(D) \cong C(D')$.*

3.4 P_k-Isomorphisms for $k \geq 4$

At first, we give some pairs of nonisomorphic connected graphs with isomorphic P_k-graphs in [44]. Denote by A, B and G_k the graphs shown in Figure 3.6. One can see that $P_4(A) = C_4 = P_4(C_4)$, $P_4(B) = 3C_4 = P_4(K_4)$, where $3C_4$ is the graph obtained by taking three disjoint copies of C_4 together. $P_k(G_k) = C_{4k-8} = P_k(C_{4k-8})$ for $k \geq 4$. $P_k(S^{k-2}(K_{1,3})) = C_{3k-3} = P_k(C_{3k-3})$ for $k \geq 2$, where $S^n(G)$ is the graph resulting from G by subdividing every edge of G n times.

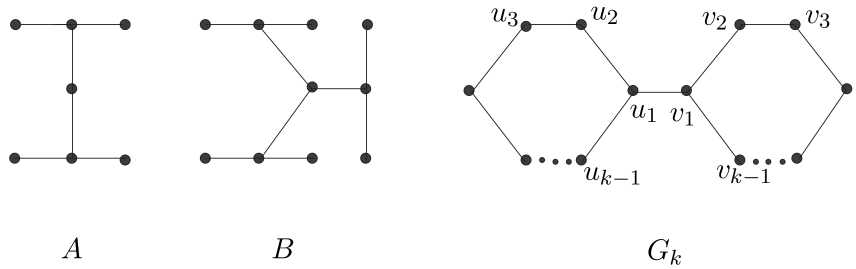


Figure 3.6

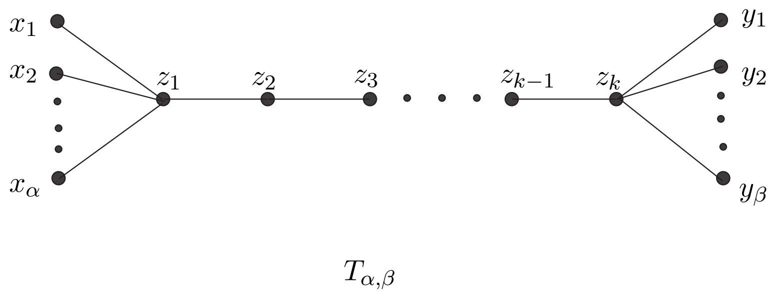


Figure 3.7

Actually, let γ be an integer and $\alpha + \beta = \gamma$ be a nonnegative 2-partition of γ, then it is easy to get $[\frac{\gamma}{2}] + 1$ nonisomorphic trees as shown in Figure 3.7. Obviously, for $k \geq 4$ the P_k-graphs of all these $[\frac{\gamma}{2}] + 1$ trees are isomorphic to the star $K_{1,\gamma}$. So, for $k \geq 4$ there exist arbitrarily many nonisomorphic connected graphs with isomorphic connected P_k-graphs, as γ goes arbitrarily large.

It is not difficult to find more pairs of nonisomorphic connected graphs with isomorphic connected P_k-graphs. However, to characterize all pairs of such graphs is also a very difficult problem.

In [44], Li and Zhao conjectured that for graphs with minimum degree at least k, the P_k-transformation ($k \geq 4$) is one to one, and then they proved it.

In order to prove this conjecture, some definitions and results are required. The following conclusion is as a consequence of Theorems 3.2.1 and 3.2.3.

Theorem 3.4.1 ([44])**.** *If G and G' are connected graphs with $|V(G)|$ and $|V(G')|$ at least 5, then the mapping T given by $T(f) = f^*$ is a bijection of $\Gamma(G, G')$ onto $\Gamma_e(G, G')$.*

For $f \in \Gamma_{k-2}(G, G')$ with $k \geq 4$, let $u_1u_2\cdots u_k$ be a P_k in G. In general, the graph-union $f(u_1\cdots u_{k-2}) \cup f(u_2\cdots u_{k-1}) \cup f(u_3\cdots u_k)$ needs not to be a P_k in G'. For example, let G be a P_k, and let G' be as in Figure 3.8. Then define a mapping $f : \Pi_{k-2}(G) \to \Pi_{k-2}(G')$ by $f(u_1u_2\cdots u_{k-2}) = v_3\cdots v_{k-1}v_1$, $f(u_2\cdots u_{k-2}u_{k-1}) = v_2v_3\cdots v_{k-1}$ and $f(u_3\cdots u_{k-1}u_k) =$

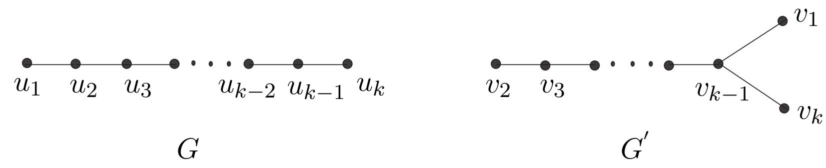


Figure 3.8

$v_3 \cdots v_{k-1}v_k$. Obviously, $f \in \Gamma_{k-2}(G, G')$, but $f(u_1 \cdots u_{k-2}) \cup f(u_2 \cdots u_{k-1}) \cup f(u_3 \cdots u_k)$ needs not be a P_k in G'.

Let $f \in \Gamma_{k-2}(G, G')$, $k \geq 4$, and $u_1u_2 \cdots u_k$ be any P_k in G, and let $v_1v_2 \cdots v_k$ be any P_k in G'. If $f(u_1 \cdots u_{k-2}) \cup f(u_2 \cdots u_{k-1}) \cup f(u_3 \cdots u_k)$ is a P_k in G', and $f^{-1}(v_1 \cdots v_{k-2}) \cup f^{-1}(v_2 \cdots v_{k-1}) \cup f^{-1}(v_3 \cdots v_k)$ is a P_k in G, then define a mapping $f^+ : \Pi_k(G) \to \Pi_k(G')$ by $f^+(u_1u_2 \cdots u_k) = f(u_1 \cdots u_{k-2}) \cup f(u_2 \cdots u_{k-1}) \cup f(u_3 \cdots u_k)$ and call f^+ the *mapping (+)-induced by* f. Let $\Gamma_k^+(G, G') = \{f^+ | f \in \Gamma_{k-2}(G, G')\}$. Note that f^+ is not defined for a connected graph unless it has at least one P_k.

Theorem 3.4.2 ([44])**.** *Let $G, G' \in \mathcal{G}_k$ $(k \geq 4)$. Then $\Gamma_k^+(G, G') \subseteq \Gamma_k(G, G')$.*

Let $P_k = u_1u_2 \cdots u_k \in \Pi_k(G)$, $k \geq 3$, then $u_2u_3 \cdots u_{k-1}$ is called the *middle* P_{k-2} of the P_k path $u_1u_2 \cdots u_k$. Denote by $S(u_2u_3 \cdots u_{k-1})$ the set of all the P_k with a common middle $P_{k-2} = u_2u_3 \cdots u_{k-1}$. Any subset of $S(u_2u_3 \cdots u_{k-1})$ is called a *generalized double star*, or a *GDS*, at $u_2u_3 \cdots u_{k-1}$. A mapping $f : \Pi_k(G) \to \Pi_k(G')$ is called *GDS-preserving* if the set $f(S(u_2u_3 \cdots u_{k-1}))$ is a GDS in G' for every $P_{k-2} = u_2u_3 \cdots u_{k-1}$ of G. For the case $k = 4$, speak of *middle edge*, *double star*, *double star-preserving* instead speaking of middle P_{k-2}-path, generalized double star (GDS), GDS-preserving.

Theorem 3.4.3 ([44])**.** *Let $G, G' \in \mathcal{G}_k$, $k \geq 4$, and let $f \in \Gamma_k(G, G')$. Then f is (+)-induced by a P_{k-2}-isomorphism from G to G' if and only if f and f^{-1} are GDS-preserving P_k-isomorphisms.*

The above result is proved by Theorem 3.4.2, which is used to prove the following conclusion.

Theorem 3.4.4 ([44])**.** *Let $G, G' \in \mathcal{G}_k$, $k \geq 4$. Then $f \in \Gamma_k(G, G')$ if and only if f is (+)-induced by a P_{k-2}-isomorphism from G onto G'.*

By induction on k and Theorems 3.4.1 and 3.4.4 as well as Theorem 3.3.9, Li and Zhao got the following result.

Theorem 3.4.5 ([44])**.** *Let $G, G' \in \mathcal{G}_k$ with $k \geq 4$. Then $f \in \Gamma_k(G, G')$ if and only if f is $(*)$-induced by an isomorphism of G onto G', i.e., $P_k(G)$ is isomorphic to $P_k(G')$ if and only if G is isomorphic to G'.*

From Theorems 3.3.1 and 3.4.5, the following corollary holds.

Corollary 3.4.6 ([44]). *Let $G, G^{'} \in \mathcal{G}_k$, $k \geq 4$. Then the P_k-transformation is one-to-one.*

Now, we focus our attention on the case $k = 4$ and graphs with minimum degree at least 3, then P_4-transformation is one-to-one by [41, 43].

We follow the definitions in [43]. For $f \in \Gamma_e(G, G^{'})$, define a mapping $\hat{f}$ by $\hat{f}(tuvw) = f(tu)f(uv)f(vw)$ for a P_4-path $tuvw$ in G, and call $\hat{f}$ the *mapping induced by* f. Let $\hat{\Gamma}(G, G^{'}) = \{\hat{f} | f \in \Gamma_e(G, G^{'})\}$. Obviously $\hat{\Gamma}(G, G^{'}) = \Gamma_4^{+}(G, G^{'})$.

Theorem 3.4.7 ([43]). *If $G, G^{'} \in \mathcal{G}_3$, then*

(1) $\hat{\Gamma}(G, G^{'}) \subseteq \Gamma_4(G, G^{'})$.

(2) the mapping $T : \Gamma_e(G, G^{'}) \rightarrow \hat{\Gamma}(G, G^{'})$ given by $T(f) = \hat{f}$ is one-to-one.

The following theorem relax the minimum degree condition of Theorem 3.4.3 to $\delta \geq 3$.

Theorem 3.4.8 ([43]). *Let $G, G^{'} \in \mathcal{G}_3$ and let $f : \Pi_4(G) \rightarrow \Pi_4(G^{'})$ be a bijective mapping. Then f is induced by an edge-isomorphism from G to $G^{'}$ if and only if f and f^{-1} are double star-preserving P_4-isomorphisms.*

Lemma 3.4.9 ([43]). *Let $G, G^{'} \in \mathcal{G}_3$ and let f be a P_4-isomorphism from G to $G^{'}$. Assume G and $G^{'}$ satisfy one of the following conditions:*

(1) if u is a vertex of some triangle in G, then $d(u) \geq 4$;

(2) G and $G^{'}$ do not contain any C_4 as a subgraph.

Then f is double star-preserving if and only if for every P_3-path tuv of G, $f(x_1tuv), \cdots, f(x_rtuv)$ have a common middle edge and $f(tuvy_1), \cdots, f(tuvy_s)$ have a common middle edge, where $x_i \in N(t) \backslash \{u, v\}$ for $1 \leq i \leq r$, $y_j \in N(v) \backslash \{t, u\}$ for $1 \leq j \leq s$.

Then from Theorem 3.4.8 and Lemma 3.4.9, Li and Zhao proved the following result.

Theorem 3.4.10 ([43]). *Let $G, G^{'} \in \mathcal{G}_3$. Assume G and $G^{'}$ satisfy one of the following two conditions:*

(1) if u is a vertex of some triangle in G, then $d(u) \geq 4$;

(2) G and $G^{'}$ do not contain any C_4 as a subgraph.

Then $f \in \Gamma_4(G, G^{'})$ if and only if f is induced by an edge isomorphism from G to $G^{'}$, i.e., $P_4(G)$ is isomorphic to $P_4(G^{'})$ if and only if the line graph $L(G)$ is isomorphic to $L(G^{'})$.

By Theorems 3.2.3, 3.4.7 and 3.4.10, the following results are immediate.

Theorem 3.4.11 ([43]). *Let $G, G^{'} \in \mathcal{G}_3$. Assume G and $G^{'}$ satisfy one of the following two conditions:*

(1) if u is a vertex of some triangle in G, then $d(u) \geq 4$;

(2) G and G' do not contain any C_4 as a subgraph.

Then $f \in \Gamma_4(G, G')$ if and only if f is induced by an isomorphism of G to G', i.e., $P_4(G)$ is isomorphic to $P_4(G')$ if and only if G is isomorphic to G'.

Corollary 3.4.12 ([43])**.** *Let $G, G' \in \mathcal{G}_3$. Assume G and G' satisfy one of the following two conditions:*

(1) if u is a vertex of some triangle in G, then $d(u) \geq 4$;

(2) G and G' do not contain any C_4 as a subgraph.

Then the P_4-transformation is one-to-one.

In [41], Li and Liu characterized all P_4-isomorphisms for graphs with $\delta \geq 3$.

Denote by $\mathcal{H}$ the set of all graphs obtained from n copies of K_4 by identifying one corresponding edge of each copy of K_4, where $n \geq 1$.

Lemma 3.4.13 ([41])**.** *Let $G, G' \in \mathcal{G}_3 \backslash \mathcal{H}$ and let f be a P_4-isomorphism from G to G'. Then f is double star-preserving.*

By Theorem 3.4.8 and Lemma 3.4.13, the next result follows immediately.

Theorem 3.4.14 ([41])**.** *Let $G, G' \in \mathcal{G}_3 \backslash \mathcal{H}$. Then $f \in \Gamma_4(G, G')$ if and only if f is induced by an edge isomorphism from G to G', i.e., $P_4(G)$ is isomorphic to $P_4(G')$ if and only if the line graph $L(G)$ is isomorphic to $L(G')$.*

Then the following theorem is immediate from Theorems 3.2.3 and 3.4.14.

Theorem 3.4.15 ([41])**.** *Let $G, G' \in \mathcal{G}_3 \backslash \mathcal{H}$. Then $f \in \Gamma_4(G, G')$ if and only if f is induced by an isomorphism from G to G', i.e., $P_4(G)$ is isomorphic to $P_4(G')$ if and only if G is isomorphic to G'.*

From the proof of Lemma 3.4.13, it is easy to get a result as follows.

Corollary 3.4.16 ([41])**.** *Let $G \in \mathcal{H}$ and $G' \in \mathcal{G}_3$. If f is a P_4-isomorphism from G to G', then G is isomorphic to G'.*

Then the main result follows by Theorems 3.4.7 and 3.4.15 and Corollary 3.4.16.

Corollary 3.4.17 ([41])**.** *Let $G, G' \in \mathcal{G}_3$, then the P_4-transformation is one-to-one.*

For the case $k = 4$, if the graphs have lower degree 1 or 2, the determination problem for P_4-graphs has not been solved. Also, if the graphs with $\delta < k$, $k \geq 5$, then the determination problem for P_k-graphs seems more difficult. But if one wants to solve this problem, the first thing is to give a necessary and sufficient condition such that a connected graph has a connected P_k-graph, $k \geq 5$, which is still open.

4 Characterizations of Path Graphs

For line graphs, it is obvious that not all graphs are line graphs. For example, $K_{1,3}$ is not a line graph. Now we state three characterizations of line graphs. One is due to Krausz [34], the second to van Rooij and Wilf [56], and the third to Beineke [11]. There are also several other characterizations mentioned in [22].

Define a collection $\mathcal{K}$ of subgraphs of a graph H to be a *Krausz partition* of H if it has the following three properties: (i) each member of $\mathcal{K}$ is a complete graph; (ii) every edge of H is in exactly one member of $\mathcal{K}$; (iii) every vertex of H is in exactly two members of $\mathcal{K}$. A triangle is called *odd* if there is some vertex adjacent to an odd number of its vertices; otherwise it is called *even*. The graph $K_4 - K_2$ is obtained from K_4 by deleting an edge.

Theorem 4.1 ([22]). *Let H be a graph. Then the following statements are equivalent:*

(i) H is a line graph;

(ii) H has a Krausz partition;

(iii) H does not have $K_{1,3}$ as an induced subgraph, and any induced subgraph isomorphic to $K_4 - K_2$ has one of its triangles even;

(iv) H does not contain an induced subgraph isomorphic to any of the graphs of Figure 4.9.

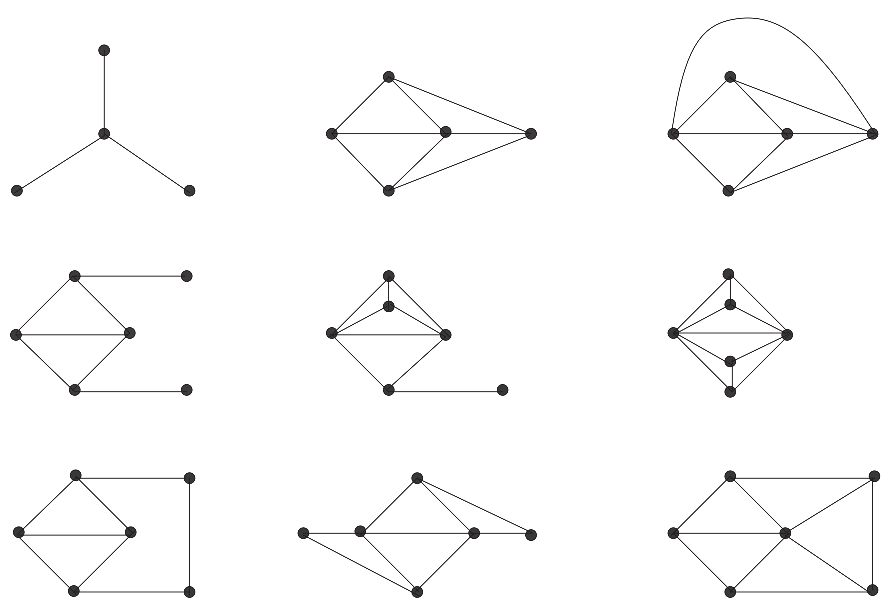


Figure 4.9

For line digraphs, of the characterizations in the following theorem, (*ii*) is due to Harary and Norman [20], (*iii*) to Heuchenne [23], and (*iv*) and (*v*) to Richards [55].

Some definitions are needed. If A and B are two sets of vertices (not necessarily disjoint, but not both empty), then the digraph $\vec{K}(A,B)$ has vertex set $A \cup B$ and arc set $A \times B$. A collection $\{S_i\}_{i\in I}$ of (possibly empty) subsets of a set S is called a *general partition* of S if $S = \bigcup_{i\in I} S_i$, and if $S_i \cap S_j = \emptyset$ whenever $i \neq j$.

Theorem 4.2 ([22]). *Let F be a digraph, let $A(F)$ be its arc set, and let M be its adjacency matrix. Then the following statements are equivalent:*

(i) F is a line digraph;

(ii) there exist two general partitions $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ of $V(F)$ such that $A(F) = \bigcup_{i\in I} \vec{K}(A_i, B_i)$;

(iii) if vw, uw and ux are arcs of F, then so is vx;

(iv) any two rows of M are either identical or orthogonal;

(v) any two columns of M are either identical or orthogonal.

For P_3-graphs, Broersma and Hoede [12] gave a solution to the characterization problem, which contained a flaw. Later, Li and Lin [36] presented a corrected form of the characterization stated as follows.

A path P_3 with middle vertex v in G is called an *induced vertex of v* in $P_3(G)$. The set of all such induced vertices, denoted by $I(v)$, is called the *set induced by v*. Clearly, if the degree of v in G is d, then $|I(v)| = \binom{d}{2}$, and such set is called a *binomial set.*

Theorem 4.3 ([36]). *A graph K is the P_3-graph of a graph G if and only if K is simple and*

(i) the vertices of K can be partitioned into binomial sets of independent vertices in such a way that

(ii) the edges of K can be partitioned into sets of edges that induce complete bipartite graphs B with classes each belonging to one binomial set with order $d-1$ if the order of the binomial set is $\binom{d}{2}$

(iii) each vertex of K belongs to at most two of the graphs B, and

(iv) the classes of vertices, of two graphs B, that belong to the same binomial set have exactly one vertex in common.

Broersma and Hoede gave the characterization of P_3-graphs with conditions (i), (ii), (iii) and (iv)*, where (iv)* is that no vertex of K belongs to more than one triangle. This conclusion is not true and there is a counterexample. By the following lemma, Li and Lin got the necessary condition (iv) for a P_3-graph, which improved the condition (iv)*.

Lemma 4.4. *A binomial set S of order $\binom{d}{2}$, $d \geq 3$, can be expressed as the union of d subsets of $d-1$ elements with the property that two subsets have exactly one element in common; and moreover, each element belongs to precisely two subsets.*

To end this section, we propose an open problem for further study, i.e., how to give a characterization for directed $\vec{P}_3$-graphs ?

5 Traversability of Path Graphs

Harary and Nash-Williams [19] gave the relationship between eulerian graphs and hamiltonian line graphs stated as follows.

Theorem 5.1 ([19]). *Let G be a graph. Then*

(i) $L(G)$ is hamiltonian if and only if G has a closed trail incident with each edge;

(ii) $L(S^1(G))$ is hamiltonian if and only if G has a spanning closed trial;

(iii) $L(S^2(G))$ is hamiltonian if and only if G is eulerian.

The following two results are also about hamiltonian line graphs. More results are given in [14, 16].

Theorem 5.2 (Nebeský [46]). *If G is a graph of order 5 or more, then G or $\bar{G}$ has a hamiltonian line graph.*

Theorem 5.3 (Kotzig [33] and Martin [45]). *A cubic graph G is hamiltonian if and only if $L(G)$ has two edge-disjoint hamiltonian cycles.*

The hamiltonian characterization for line digraphs was observed by Aigner [1] and by Kasteleyn [25].

Theorem 5.4 ([1] and [25]). *Let D be a strongly connected digraph. Then*

(i) $\vec{L}(D)$ is eulerian if and only if $d^-(v) = d^+(w)$ for each arc vw of D;

(ii) $\vec{L}(D)$ is hamiltonian if and only if D is eulerian.

For P_3-graphs, Broersma and Hoede [12] considered the trees and unicyclic graphs with hamiltonian P_3-graphs, and obtained the following result.

Theorem 5.5 ([12]). *If T is a tree with $\triangle(T) \leq 3$, then $P_3(T)$ is hamiltonian if and only if T is a 1-2-tree, where T is a 1-2 tree if $\triangle(T) = 3$ and if every vertex with degree 1 has a neighbor with degree 2 and vice versa.*

Thus Broersma and Hoede conjectured that the 1-2-trees are precisely the trees with hamiltonian P_3-graphs. They checked that the following conjecture is true for $\triangle(T) = 4$.

Conjecture 5.6 ([12]). *If T is a tree with $\triangle(T) \geq 4$, then $P_3(T)$ is not hamiltonian.*

For unicyclic graphs, Broersma and Hoede proved the following result. Before giving it, some definitions are required. Let G be a unicyclic graph containing the cycle C. Every component of $G - V(C)$ has a unique neighbor on the cycle called its *source*. A component of $G - V(C)$ together with its source and the edge joining the source to a vertex of the component is called a *beam*. A beam with one edge is a *1-beam*; a beam is a *2-beam* if it is isomorphic to P_3 or if it can be obtained from P_3 by repeatedly applying the following

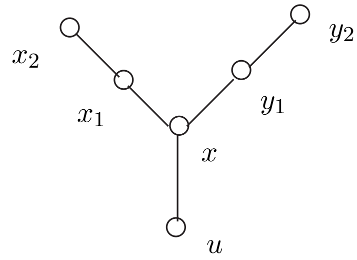


Figure 5.10

procedure: for some endvertex u other than the source with neighbor v, add new vertices u', v', and v'' and edges uu', vv', and $v'v''$. If $C = v_1v_2\cdots v_kv_1$, then a *2-interval* of C is a sequence $\{v_i, v_{i+1}, \cdots, v_j\}$ such that all v_r, $i \le r \le j$, are sources of a 2-beam, and v_{i-1} and v_{j+1} are no sources of a 2-beam (indices modulo k).

A unicyclic graph G with a cycle $C = v_1v_2\cdots v_kv_1$ and $\triangle(G) \le 3$ is a *1-2-corona* if all beams are 1-beams or 2-beams, sources of 1-beams are not adjacent to vertices of C that are no sources, and if every 2-interval $I = \{v_i, v_{i+1}, \cdots, v_j\}$ has the following property: if $|I|$ is odd, then precisely one of v_{i-1} and v_{j+1} is a source of a 1-beam ($v_{i-1} \ne v_{j+1}$); if $|I|$ is even, then either both v_{i-1} and v_{j+1} are sources of a 1-beam, or both v_{i-1} and v_{j+1} are no sources of a 1-beam (indices modulo k).

Theorem 5.7 ([12])**.** *Let G be a unicyclic graph with $\triangle(G) \le 3$. Then $P_3(G)$ is hamiltonian if and only if G is a 1-2-corona.*

Broersma and Hoede also found that there exist unicyclic graphs with maximum degree 4 and hamiltonian P_3-graphs. And thus they gave a conjecture as follows.

Conjecture 5.8 ([12])**.** *If G is a unicyclic graph with $\triangle(G) \ge 5$, then $P_3(G)$ is not hamiltonian.*

Later, Conjectures 5.6 and 5.8 were solved by Yu [58], and also by Yuan and Lin [59]. Here we state the solutions of these two conjectures by Yu.

For a graph G, a Y is a subgraph of G as shown in Figure 5.10 with $d_G(u) \ge 3$ and $d_Y(v) = d_G(v)$ for every $v \in Y\backslash\{u\}$. Let $H = G\backslash\{x_2, y_1, y_2\}$. Then $P_3(G)$ is hamiltonian if and only if $P_3(H)$ is hamiltonian. Because of this fact, a graph G is *reduced* if it does not contain any Y and $P_3(G)$ is hamiltonian. If $\Delta(G) = 2$, then G is reduced if and only if G is a cycle. So from now on, $\Delta(G) \ge 3$. Therefore, in order to characterize all trees and unicyclic graphs G with $\Delta(G) \ge 3$ and $P_3(G)$ hamiltonian, only need to characterize reduced graphs.

Let G be either a tree or a unicyclic graph. If G is a unicyclic graph, then let C be the unique cycle of G; if G is a tree, then let C be a vertex of G with the maximum degree

$\Delta(G)$. Note that if G is a tree, every component of $G\backslash C$ has the same source C. For a vertex x on C ($x = C$ if G is a tree), x is a *2-edge-source* if x is the source of exactly two components (of $G\backslash C$), each of which is a single edge, x is a *1-edge-source* if x is the source of exactly one component (of $G\backslash C$), which is a single edge, and x is a *0-source* if $G\backslash x$ is connected.

The following theorem is essential in characterizing the reduced graphs.

Theorem 5.9 ([58]). *Let G be a reduced graph. Then, (1) If G is a unicyclic graph, then $\triangle(G) \leq 4$; and if $d(x) = 4$, then x is a 2-edge-source and (2) If G is a tree, then $G = S(K_{1,3})$, the subdivision of $K_{1,3}$.*

Since adding or deleting any Y does not change $\triangle(G)$, Conjectures 5.6 and 5.8 are true. By Theorem 5.9, $S(K_{1,3})$ is the unique reduced tree. All reduced unicyclic graphs can also be characterized with the help of the operation defined below.

A *4-interval* on C is a maximal set $\{x_i, x_{i+1}, \cdots, x_j\}$ (of consecutive vertices on C) such that x_k is a 2-edge-source for $k = i, i+1, \cdots, j$, and neither x_{i-1} nor x_{j+1} is a 0-source. A 4-interval is *even* if it has even number of vertices. Let $I = \{x_i, x_{i+1}, \cdots, x_j\}$ be an even 4-interval of G on C, and let $I^+ = I \cup \{$ components of $G\backslash C$ with sources in $I\}$. Let H be the graph obtained from $G\backslash I^+$ by adding two vertices u, w, and edges wx_{i-1}, wx_{j+1}, and uw. Then G is obtained from H by an *addition* of an even 4-interval. Now the following result holds.

Theorem 5.10 ([58]). *Let G be a unicyclic graph with unique cycle C. Then G is reduced if and only if one of the following is true:*

(a) C is a 4-interval;

(b) C has exactly one 1-edge-source and all other vertices on C are 2-edge-sources;

(c) G is a reduced graph with $\triangle(G) \leq 3$;

(d) G can be obtained from a reduced graph H with $\triangle(H) \leq 3$ by a series of additions of even 4-intervals.

Broersma and Hoede characterized reduced unicyclic graphs G with $\triangle(G) \leq 3$ in Theorem 5.7, thus all unicyclic graphs with $P_3(G)$ hamiltonian have been characterized.

For directed $\vec{P}_3$-graphs, Broersma and Li [13] obtained some results on the (directed) eulerian tours and (directed) hamiltonian cycles stated in the following.

Here two definitions are required. Given a digraph D, denote by $Asym(D)$ the graph obtained from D by deleting all $\vec{C}_2$'s, i.e., by deleting all $\vec{C}_2$-pairs $\{uv, vu\} \subseteq A(D)$. A eulerian tour T of D is a *$\vec{C}_2$-tour* if the arcs of each $\vec{C}_2$ of D are successive arcs in T.

Theorem 5.11 ([13]). *Let D be a digraph such that $Asym(\vec{L}(D))$ is strongly connected. Then* *(i)* $\vec{P}_3(D)$ *is eulerian if and only if* $d^-(ab) = d^+(bc)$ *for each* $\vec{P}_3$ *abc in* D*;*

(ii) $\vec{P}_3(D)$ *is hamiltonian if and only if* $\vec{L}(D)$ *has a* $\vec{C}_2$*-tour;*

(iii) $\vec{P}_3(D)$ *contains a 2-factor if and only if* $\vec{L}(D)$ *is eulerian, or, equivalently if* $d^-(v) = d^+(w)$ *for each arc vw in* D*;*

(iv) $\vec{P}_3(D)$ *is hamiltonian if* $d^-(v) = d^+(w)$ *for each arc vw in* D*, and* D *contains no* $\vec{C}_2$*.*

6 Connectivity and Edge-Connectivity of Path Graphs

Denote by κ and λ the connectivity and edge-connectivity of a graph G, respectively.

Chartrand and Stewart [17] gave the bounds on the connectivity and edge-connectivity of line graphs stated as follows.

Theorem 6.1 ([17]). *For any graph* G*,*

(i) $\kappa(L(G)) \geq \lambda(G)$*;*

(ii) $\lambda(L(G)) \geq 2\lambda(G) - 2$*;*

(iii) if $\lambda(G) \neq 2$*, then* $\lambda(L(G)) = 2\lambda(G) - 2$ *if and only if there exist two adjacent vertices in* G *with degree* $\lambda(G)$*.*

Theorem 6.2 (Zamfirescu [60]).

(i) If $\delta(L(G)) \leq \lambda(G)[\frac{1}{2}(\lambda(G)+1)]$*, then* $\lambda(L(G)) = \delta(L(G))$*.*

(ii) If $\delta(L(G)) \geq \lambda(G)[\frac{1}{2}(\lambda(G)+1)]$*, then* $\lambda(G)[\frac{1}{2}(\lambda(G)+1)] \leq \lambda(L(G)) \leq \delta(L(G))$*.*

The above result is on the edge-connectivity of line graphs, and the following two theorems give the lower bounds for the connectivity of iterated line graphs.

Theorem 6.3 (Knor and Niepel [31]).

(i) Let G *be a connected graph with minimum degree* $\delta(G) \geq 3$*. Then* $\kappa(L^2(G)) \geq \delta - 1$*.*

(ii) Let G *be a graph with* $\kappa(G) \geq 4$*. Then* $\kappa(L^2(G)) \geq 4\delta(G) - 6$*.*

Theorem 6.4 ([31]). *Let* G *be a connected graph with* $\delta(G) \geq 3$*. Then*

(i) If $\lambda(G) \geq 2$ *or* $\delta(G) \geq 5$*, then* $\kappa(L^2(G)) \geq 4$*.*

(ii) If $\lambda(G) = 1$ *and* $3 \leq \delta(G) \leq 4$*, then* $\kappa(L^3(G)) \geq 4$*.*

The connectedness of line digraphs is due primarily to Aigner [1] stated as follows.

Theorem 6.5 ([1]). *Let* D *be a digraph with at least three vertices (none of which is isolated). Then*

(i) $\vec{L}(D)$ *is strongly connected if and only if* D *is strongly connected;*

(ii) $\vec{L}(D)$ *is unilaterally connected if and only if (a)* D *is unilaterally connected, and (b) for each arc* vw*, if there are at least two directed paths from* v *to* w*, then there is also a directed path from* w *to* v*;*

(iii) $\vec{L}(D)$ *is connected if and only if (a)* D *is connected and (b) there is no separating set of vertices consisting only of sources and sinks.*

Now, we consider the connectivity and edge-connectivity of P_k-graphs, $k \geq 3$.

It is known that $\kappa(L(G)) \geq \lambda(G)$. Motivated by this result, Li [37] proved the following: If each component of G contains at least 3 vertices, then $P_3(G)$ is connected if and only if G is connected and each vertex of G is adjacent to at most one vertex of degree one. Moreover, if $P_3(G)$ is connected, then $\kappa(P_3(G)) \geq \lambda(G)$. The theorem below also gave the conditions of P_3-graphs disconnected.

Theorem 6.6 (Knor and Niepel [28])**.** *Let* G *be a connected graph. Then* $P_3(G)$ *is disconnected if and only if* G *contains two distinct paths* A *and* B *of length two, such that the degrees of both endvertices of* A *are 1 in* G*.*

Hence, restricted to the graphs with minimum degree at least 3, the following result holds.

Theorem 6.7 (Knor, Niepel and Malah [29])**.**

(i) Let G *be a connected graph with* $\delta(G) \geq 3$*. Then* $P_3(G)$ *is* $(\delta - 1)$*-connected.*

(ii) Let G *be 2-connected graph with* $\delta(G) \geq 3$*. Then* $P_3(G)$ *is* $(2\delta - 2)$*-connected.*

As a straightforward consequence of Theorem 6.7, Knor, Niepel and Malah also got

Theorem 6.8 ([29])**.** *Let* G *be a connected* δ*-regular graph,* $\delta \geq 3$*. Then for all* i*,* $i \geq 2$*, the connectivity of* $P_3^i(G)$ *equals to the degree of* $P_3^i(G)$*.*

For the case $k \geq 4$, Knor and Niepel [27] generalized Theorem 6.6 to P_k-graphs if G does not contain a cycle of length smaller than k. Moreover, they completely solved the case of P_4-graphs.

We follow the notations in [27]. Let G be a graph, $k \geq 3$, $0 \leq t \leq k-3$, and let A be a path of length $k-1$ in G. Denote by $P_{k,t}^*$ an induced subgraph of G which is a tree of diameter $k+t-1$ with a diametric path $x_t x_{t-1} \cdots x_1 v_0 v_1 \cdots v_{k-t-1} y_1 y_2 \cdots y_t$, such that all endvertices of $P_{k,t}^*$ have distance $\leq t$ either to v_0 or to v_{k-t-1} and the degrees of $v_1, v_2, \cdots, v_{k-t-2}$ are 2 in $P_{k,t}^*$. Moreover, no vertex of $V(P_{k,t}^*) - \{v_1, v_2, \cdots, v_{k-t-2}\}$ is joined by an edge to a vertex in $V(G) - V(P_{k,t}^*)$. The path $v_0 v_1 \cdots v_{k-t-1}$ is a *base* of $P_{k,t}^*$, and A *lies in* $P_{k,t}^*$, $A \in P_{k,t}^*$, if and only if the base of $P_{k,t}^*$ is a subpath of A.

Theorem 6.9 ([27])**.** *Let* G *be a connected graph without cycles of length smaller than* k*. Then* $P_k(G)$ *is disconnected if and only if* G *contains* $P_{k,t}^*$*,* $0 \leq t \leq k-3$*, and a path* A *of length* $k-1$ *such that* $A \notin P_{k,t}^*$*.*

From Theorem 6.6, it follows that if G is a connected graph with at most one vertex of degree one, then $P_3(G)$ is also connected. Thus Balbuena and Ferrero [8] considered the connected graph G with minimum degree at least 2, and presented the lower bounds on the edge-connectivity of $P_3(G)$ stated in the following.

Theorem 6.10 ([8])**.** *Let G be a connected graph with $\delta(G) \geq 2$. Then*

(i) $\lambda(P_3(G)) \geq \delta(G) - 1$;

(ii) $\lambda(P_3(G)) \geq 2\delta(G) - 2$ *if* $\lambda(G) \geq 2$.

Corollary 6.11 ([8])**.** *Let G be a connected graph with $\delta(G) \geq 3$. Then* $\lambda(P_3^2(G)) \geq 4\delta(G) - 6$.

From Theorem 6.9, it follows that if G has girth $g(G) \geq k$ and minimum degree at least 2, then $P_k(G)$ is connected. Hence, Theorem 6.10 was generalized for all $k \geq 3$ with G having minimum degree $\delta(G) \geq 2$ and girth $g(G) \geq k$.

Theorem 6.12 (Balbuena and García-Vázquez [9])**.**

(i) Let $k \geq 3$ be an integer. Assume that G is a connected graph with $\delta(G) \geq 2$ and girth $g(G) \geq k$. Then $\lambda(P_k(G)) \geq \delta(G) - 1$.

(ii) Let $k \geq 4$ be an integer. Let G be a connected graph with $\delta(G) \geq 3$ and girth $g(G) \geq k$. Then $\lambda(P_k(G)) \geq 2\delta(G) - 2$.

For directed $\vec{P}_3$-graphs, there is only a result as a corollary of Theorem 6.5. With respect to (undirected) graphs, we could do much more on the connectedness of directed path graphs.

Corollary 6.13 ([13])**.** *For any digraph D containing at least one $\vec{P}_3$, $\vec{P}_3(D)$ is strongly connected if and only if $Asym(\vec{L}(D))$ is strongly connected.*

References

[1] M. Aigner, *On the line graph of a directed graph*, Math. Z. **102** (1967), 56-61.

[2] R.E.L. Aldred, M.N. Ellingham, R.L. Hemminger and P. Jipsen, *P_3-isomorphisms for graphs*, J. Graph Theory **26** (1997), 35-51.

[3] J.S. Bagga, *Old and new generalizations of line graphs*, Int. J. Math. Sci. (2004), no. 29-32, 1509-1521.

[4] J.S. Bagga, L.W. Beineke and B.N. Varma, *Super line graphs*, Graph theory, combinatorics, and algorithms, Vol. 1, 2 (Kalamazoo, MI, 1992), Wiley-Intersci. Publ., Wiley, New York, (1995), 35-46.

[5] J.S. Bagga, L.W. Beineke and B.N. Varma, *Super line graphs and their properties*, Combinatorics, graph theory, algorithms and applications (Beijing, 1993), World Sci. Publ., River Edge, NJ, 1994, 1-6.

[6] J.S. Bagga, L.W. Beineke and B.N. Varma, *Independence and cycles in super line graphs*, Australas. J. Combin. **19** (1999), 171-178.

[7] J.S. Bagga, L.W. Beineke and B.N. Varma, *The super line graph* $\mathscr{L}_2$, Discrete Math. **206** (1999), no. 1-3, 51-61.

[8] C. Balbuena and D. Ferrero, *Edge-connectivity and super edge-connectivity of P_2-path graphs*, Discrete Math. **269** (2003), 13-20.

[9] C. Balbuena and P. García-Vázquez, *Edge-connectivity in P_k-path graphs*, Discrete Math. **286** (2004), 213-218.

[10] G. Balconi, *Sui singrammi a commutato periodico*, Ist. Lombardo Accad. Sci. Lett. Rend. (A) **102** (1968), 427-436.

[11] L.W. Beineke, *Characterizations of derived graphs*, J. Combin. Theory **9** (1970), 129-135.

[12] H.J. Broersma and C. Hoede, *Path graphs*, J. Graph Theory **13** (1989), 427-444.

[13] H.J. Broersma and X. Li, *Isomorphisms and traversability of directed path graphs*, Discuss. Math. Graph Theory **22** (2002), 215-228.

[14] J.C. Bermond and P. Rosenstiehl, *Pancyclisme du carré du graphe aux arêtes d'un graphe*, Cahiers Centre Études Recherche Opér. **15** (1973), 285-286.

[15] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications, MacMillan/Elesevier, London/New York, 1976.

[16] G. Chartrand, *On hamiltonian line-graphs*, Trans. Amer. Math. Soc. **134** (1968), 559-566.

[17] G. Chartrand and M.J. Stewart, *The connectivity of line-grphs*, Math. Ann. **182** (1969), 170-174.

[18] B. Grünbaum, *Incidence patterns of graphs and complexes*, Lecture Notes in Mathematics (ed. G. Chartrand and S.F. Kapoor) **110** (1969), 115-128.

[19] F. Harary and C.St.J.A. Nash-Williams, *On eulerian and hamiltonian graphs and line graphs*, Canad. Math. Bull. **8** (1965), 701-709.

[20] F. Harary and R.Z. Norman, *Some properties of line digraphs*, Rend. Circ. Mat. Palermo (2) **9** (1960), 161-168.

[21] R.L. Hemminger, *Digraphs with periodic line digraphs*, Studia Sci. Math. Hungar. **9** (1974), 27-31.

[22] R.L. Hemminger and L.W. Beineke, *Line graphs and line digraphs*, in: L.W. Beineke and R.J. Wilson, eds, Selected Topics in Graph Theory, Academic Press, London, New York, San Francisco, 1978.

[23] C. Heuchenne, *Sur une certaine correspondance entre graphs*, Bull. Soc. Roy. Sci. Liège **33** (1964), 743-753.

[24] H.A. Jung, *Zu einem isomorphiesatz von H. Whitney für graphen*, Math. Ann. **164** (1966), 270-271.

[25] P.W. Kasteleyn, *Graph theory and crystal physics*, in Graph Theory and Theoretical Physics (ed. F. Harary), Academic Press, London and New York, 1967, 43-110.

[26] M. Knor and L'. Niepel, *Centers in path graphs*, J. Combin. Inform. System Sci. **24** (1999), no. 2-4, 79-86.

[27] M. Knor and L'. Niepel, *Connectivity of path graphs*, Discuss. Math. Graph Theory **20** (2000), no. 2, 181-195.

[28] M. Knor and L'. Niepel, *Diameter in iterated path graphs*, Discrete Math. **233** (2001), no. 1-3, 151-161.

[29] M. Knor and L'. Niepel and M. Malah, *Connectivity of path graphs*, Australas. J. Combin. **25** (2002), 175-184.

[30] M. Knor and L'. Niepel, *Graphs isomorphic to their path graphs*, Math. Bohem. **127** (2002), no. 3, 473-480.

[31] M. Knor and L'. Niepel, *Connectivity of iterated line graphs*, Discrete Appl. Math. **125** (2003), no. 2-3, 255-266.

[32] M. Knor and L'. Niepel, *Independence number in path graphs*, Comput. Inform. **23** (2004), no. 2, 179-187.

[33] A. Kotzig, *Aus der Theorie der endlichen regulären Graphen dritten und vierten Grades (Slovak with Russian and German summaries)*, Časopis Pěst. Mat. **82** (1957), 76-92.

[34] J. Krausz, *Démonstration nouvelle d'une théorème de Whitney sur les réseaux (Hungarian with French summary)*, Mat. Fiz. Lapok **50** (1943), 75-85.

[35] Van Bang Le and E. Prisner, *Iterated k-line graphs*, Graphs Combin. **10** (1994), 193-203.

[36] H. Li and Y. Lin, *On the charaterization of path graphs*, J. Graph Theory **17** (1993), 463-466.

[37] X. Li, *The connectivity of path graphs*, Combinatorics, Graph Theory, Algorithms and Applications, Beijing 1993, 187-192.

[38] X. Li, *Isomorphisms of P_3-graphs*, J. Graph Theory **21** (1996), 81-85.

[39] X. Li, *On the determination problem for P_3-transformation of graphs*, Ars Combin. **49** (1998), 296-302.

[40] X. Li and Y. Liu, *Nonexistence of triples of nonisomorphic connected graphs with isomorphic connected P_3-graphs*, preprint 2005.

[41] X. Li and Y. Liu, *On the determination problem for P_4-transformation of graphs*, preprint 2006.

[42] X. Li, Y. Liu and B. Zhao, *Isomorphisms of directed $\vec{P}_3$-graphs*, preprint 2005.

[43] X. Li and B. Zhao, *Isomorphisms of P_4-graphs*, Australas. J. Combin. **15** (1997), 135-143.

[44] X. Li and B. Zhao, *Isomorphisms of P_k-graphs for $k \geq 4$*, Discrete Math. **289** (2004), 71-80.

[45] P. Martin, *Cycles hamiltoniens dans les graphes 4-réguliers 4-connexes*, Aequationes Math. **14** (1976), 37-40.

[46] L. Nebeský, *A note on line graphs*, Comment. Math. Univ. Carolinae **15** (1974), 567-570.

[47] K.Y. Ouyang and K.Z. Ouyang, *Digraphs whose line digraphs are isomorphic*, J. Lanzhou University (Natural Sciences) **23** (3) (1987) 6-9.

[48] E. Prisner, *The dynamics of the line and path graph operators*, Graphs Combin. **9** (1993), 335-352.

[49] E. Prisner, *A common generalization of line graphs and clique graphs*, J. Graph Theory **18** (1994), 301-313.

[50] E. Prisner, *Graph Dynamics*, Pitman Research Notes in Mathematics Series, vol.338, Longman, Harlow, 1995.

[51] E. Prisner, *Line graphs and generalizations — a survey*, in: Surveys in Graph Theory (G. Chartrand, M. Jacobson eds.) Congr. Numer. **116** (1996) 193-229.

[52] E. Prisner, *Bicliques in graphs. II. Recognizing k-path graphs and underlying graphs of line digraphs*, Graph-Theoretic Concepts in Computer Science (Berlin, 1997), 273–287, Lecture Notes in Comput. Sci. No.1335, Springer, Berlin, 1997.

[53] E. Prisner, *Bicliques in graphs I: Bounds on their number*, Combinatorica **20** (2000), no.1, 109-117.

[54] E. Prisner, *Recognizing k-path graphs*, Proceedings of the 5th Twente Workshop on Graphs and Combinatorial Optimization (Enschede, 1997), Discrete Appl. Math. **99** (2000), no.1-3, 169–181.

[55] P.I. Richards, *Precedence constraints and arrow diagrams*, SIAM Rev. **9** (1967), 548-553.

[56] A.C.M. van Rooij and H.S. Wilf, *The interchange graph of a finite graph*, Acta Math. Acad. Sci. Hungar. **16** (1965), 263-269.

[57] H. Whitney, *Congruent graphs and connectivity of graphs*, Amer. J. Math. **54** (1932), 150-168.

[58] X. Yu, *Trees and unicyclic graphs with hamiltonian path graphs*, J. Graph Theory **14** (1990), 705-708.

[59] J. Yuan and Y. Lin, *Two results on hamiltonian path graph $P_3(G)$*, Chinese Sci. Bull. **36** (20) (1991), 1759-1760.

[60] T. Zamfirescu, *On the line-connectivity of line-graphs*, Math. Ann. **187** (1970), 305-309.

[61] B. Zhao and X. Li, *Kernels semikernels and quasi-kernels in super line digraphs*, Graph Theory Notes of New York, **XXX** (1996), 11-13.