2007/05/23

FIX-MAHONIAN CALCULUS III; A QUADRUPLE DISTRIBUTION

Dominique Foata and Guo-Niu Han

Abstract

A four-variable distribution on permutations is derived, with two dual combinatorial interpretations. The first one includes the number of fixed points "fix", the second the so-called "pix" statistic. This shows that the duality between derangements and desarrangements can be extended to the case of multivariable statistics. Several specializations are obtained, including the joint distribution of (des, exc), where "des" and "exc" stand for the number of descents and excedances, respectively.

1. Introduction

Let

$$(a;q)_n := \begin{cases} 1, & \text{if } n=0;\\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n\ge 1;\end{cases}$$
$$(a;q)_\infty := \prod_{n\ge 0}(1-aq^n),$$

be the traditional notation for the q-ascending factorial. For each $r\ge 0$ form the rational fraction

$$C(r;u,s,q,Y) := \frac{(1-sq)\,(u;q)_r\,(usq;q)_r}{((u;q)_r - sq(usq;q)_r)(uY;q)_{r+1}} \tag{1.1}$$

in four variables u, s, q, Y and expand it as a formal power series in u:

$$C(r;u,s,q,Y) = \sum_{n\ge 0} u^n C_n(r;s,q,Y). \tag{1.2}$$

It can be verified that each coefficient $C_n(r;s,q,Y)$ is actually a polynomial in three variables with nonnegative integral coefficients. For $r,n\ge 0$ consider the set $W_n(r)=[0,r]^n$ of all finite words of length n, whose letters

2000 *Mathematics Subject Classification.* Primary 05A15, 05A30, 05E15.

Key words and phrases. major index, excedance number, descent number, permutations, fixed points, pixed points, derangements, desarrangements, Lyndon factorization, H-factorization, V-decomposition, decreases.

are taken from the alphabet $[0,r]=\{0,1,\dots,r\}$. The first purpose of this paper is to show that $C_n(r;s,q,Y)$ is the generating polynomial for $W_n(r)$ by two three-variable statistics $(\mathrm{dec},\mathrm{tot},\mathrm{single})$ and $(\mathrm{wlec},\mathrm{tot},\mathrm{wpix})$, respectively, defined by means of two classical *word factorizations*, the *Lyndon factorization* and the *H-factorization*. See Theorems 2.1 and 2.3 thereafter and their corollaries.

The second purpose of this paper is to consider the formal power series

$$(1.3)\quad \sum_{r\ge 0} t^r \frac{(1-sq)\,(u;q)_r\,(usq;q)_r}{((u;q)_r - sq(usq;q)_r)(uY;q)_{r+1}} = \sum_{r\ge 0} t^r C(r;u,s,q,Y) = \sum_{r\ge 0} t^r \sum_{n\ge 0} u^n C_n(r;s,q,Y),$$

expand it as a formal power series in u, but normalized by denominators of the form $(t;q)_{n+1}$, that is,

$$(1.4)\quad \sum_{r\ge 0} t^r \frac{(1-sq)\,(u;q)_r\,(usq;q)_r}{((u;q)_r - sq(usq;q)_r)(uY;q)_{r+1}} = \sum_{n\ge 0} A_n(s,t,q,Y)\frac{u^n}{(t;q)_{n+1}},$$

and show that each $A_n(s,t,q,Y)$ is actually the *generating polynomial* for the symmetric group $\mathfrak{S}_n$ by two four-variable statistics $(\mathrm{exc},\mathrm{des},\mathrm{maj},\mathrm{fix})$ and $(\mathrm{lec},\mathrm{ides},\mathrm{imaj},\mathrm{pix})$, respectively. The first (resp. second) statistic involves the number of fixed points "fix" (resp. the variable "pix") and is referred to as the *fix-version* (resp. the *pix-version*). Several specializations of the polynomials $A_n(s,t,q,Y)$ are then derived with their combinatorial interpretations. In particular, the *joint* distribution of the two classical Eulerian statistics "des" and "exc" is explicitly calculated.

The *fix-version* statistic on $\mathfrak{S}_n$, denoted by $(\mathrm{exc},\mathrm{des},\mathrm{maj},\mathrm{fix})$, contains the following classical integral-valued statistics: the *number of excedances* "exc," the *number of descents* "des," the *major index* "maj," the *number of fixed points* "fix," defined for each permutation $\sigma=\sigma(1)\sigma(2)\cdots\sigma(n)$ from $\mathfrak{S}_n$ by

$$\begin{aligned}
\mathrm{exc}\,\sigma &:= \#\{i : 1\le i\le n-1,\ \sigma(i)>i\};\\
\mathrm{des}\,\sigma &:= \#\{i : 1\le i\le n-1,\ \sigma(i)>\sigma(i+1)\};\\
\mathrm{maj}\,\sigma &:= \sum_i i \quad (1\le i\le n-1,\ \sigma(i)>\sigma(i+1));\\
\mathrm{fix}\,\sigma &:= \#\{i : 1\le i\le n,\ \sigma(i)=i\}.
\end{aligned}$$

As was introduced by Désarménien [5], a *desarrangement* is defined to be a word $w=x_1x_2\cdots x_n$, whose letters are *distinct* positive integers such that the inequalities $x_1>x_2>\cdots>x_{2j}$ and $x_{2j}<x_{2j+1}$ hold

for some j with $1 \le j \le n/2$ (by convention: $x_{n+1} = +\infty$). There is no desarrangement of length 1. Each desarrangement $w = x_1 x_2 \cdots x_n$ is called a *hook*, if $x_1 > x_2$ and either $n = 2$, or $n \ge 3$ and $x_2 < x_3 < \cdots < x_n$. As proved by Gessel [12], each permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ admits a unique factorization, called its *hook factorization*, $p\tau_1\tau_2\cdots\tau_k$, where p is an *increasing* word and each factor $\tau_1, \tau_2, \ldots, \tau_k$ is a hook. To derive the hook factorization of a permutation, it suffices to start from the right and at each step determine the right factor which is a hook, or equivalently, the shortest right factor which is a desarrangement.

The *pix-version* statistic is denoted by (lec, ides, imaj, pix). The second and third components are classical: if σ^{-1} denotes the inverse of the permutation σ, they are simply defined by

$$\operatorname{ides}\sigma := \operatorname{des}\sigma^{-1};$$
$$\operatorname{imaj}\sigma := \operatorname{maj}\sigma^{-1}.$$

The first and fourth components refer to the hook factorization $p\tau_1\tau_2\cdots\tau_k$ of σ. For each i let $\operatorname{inv}\tau_i$ denote the *number of inversions* of τ_i. Then, we define:

$$\operatorname{lec}\sigma := \sum_{1\le i\le k} \operatorname{inv}\tau_i;$$
$$\operatorname{pix}\sigma := \text{length of the factor } p.$$

For instance, the hook factorization of the following permutation of order 14 is indicated by vertical bars.

$$\sigma = 1\ 3\ 4\ 14 \mid 12\ 2\ 5\ 11\ 15 \mid 8\ 6\ 7 \mid 13\ 9\ 10$$

We have $p = 1\ 3\ 4\ 14$, so that $\operatorname{pix}\sigma = 4$. Also $\operatorname{inv}(12\ 2\ 5\ 11\ 15) = 3$, $\operatorname{inv}(8\ 6\ 7) = 2$, $\operatorname{inv}(13\ 9\ 10) = 2$, so that $\operatorname{lec}\sigma = 7$. Our main two theorems are the following.

Theorem 1.1 (The fix-version). *Let $A_n(s,t,q,Y)$ $(n \ge 0)$ be the sequence of polynomials in four variables, whose factorial generating function is given by* (1.4). *Then, the generating polynomial for $\mathfrak{S}_n$ by the four-variable statistic* (exc, des, maj, fix) *is equal to $A_n(s,t,q,Y)$. In other words,*

$$\sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{exc}\sigma} t^{\operatorname{des}\sigma} q^{\operatorname{maj}\sigma} Y^{\operatorname{fix}\sigma} = A_n(s,t,q,Y). \tag{1.5}$$

Theorem 1.2 (The pix-version). *Let $A_n(s,t,q,Y)$ $(n \ge 0)$ be the sequence of polynomials in four variables, whose factorial generating function is given by* (1.4). *Then, the generating polynomial for $\mathfrak{S}_n$ by* *the four-variable statistic* (lec, ides, imaj, pix) *is equal to* $A_n(s,t,q,Y)$. *In other words,*

$$\sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{lec}\sigma} t^{\operatorname{ides}\sigma} q^{\operatorname{imaj}\sigma} Y^{\operatorname{pix}\sigma} = A_n(s,t,q,Y). \tag{1.6}$$

The *ligne of route*, $\operatorname{Ligne}\sigma$, of a permutation $\sigma=\sigma(1)\sigma(2)\cdots\sigma(n)$ (also called *descent set*) is defined to be the *set* of all i such that $1\le i\le n-1$ and $\sigma(i)>\sigma(i+1)$. In particular, $\operatorname{des}\sigma=\#\operatorname{Ligne}\sigma$ and $\operatorname{maj}\sigma$ is the sum of all i such that $i\in\operatorname{Ligne}\sigma$. Also, let the *inverse ligne of route* of σ be defined by $\operatorname{Iligne}\sigma:=\operatorname{Ligne}\sigma^{-1}$, so that $\operatorname{ides}\sigma=\#\operatorname{Iligne}\sigma$ and $\operatorname{imaj}\sigma=\sum_i i\ (i\in\operatorname{Iligne}\sigma)$. Finally, let $\operatorname{iexc}\sigma:=\operatorname{exc}\sigma^{-1}$.

It follows from Theorem 1.1 and Theorem 1.2 that the two four-variable statistics (iexc, ides, imaj, fix) and (lec, ides, imaj, pix) are equidistributed on each symmetric group $\mathfrak{S}_n$. The third goal of this paper is to prove the following stronger result.

Theorem 1.3. *The two three-variable statistics*

(iexc, fix, Iligne) *and* (lec, pix, Iligne)

are equidistributed on each symmetric group $\mathfrak{S}_n$.

Note that the third component in each of the previous triples is a *set-valued* statistic. So far, it was known that the two pairs (fix, Iligne) and (pix, Iligne) were equidistributed, a result derived by Désarménien and Wachs [6, 7], so that Theorem 1.3 may be regarded as an extension of their result. In the following table we reproduce the nine derangements (resp. desarrangements) σ from $\mathfrak{S}_4$, which are such that $\operatorname{fix}\sigma=0$ (resp. $\operatorname{pix}\sigma=0$), together with the values of the pairs $(\operatorname{iexc}\sigma,\operatorname{Iligne}\sigma)$ (resp. $(\operatorname{lec}\sigma,\operatorname{Iligne}\sigma)$).

<table>
<tr><th>lec</th><th>Iligne</th><th>Desarrangements</th><th>Derangements</th><th>Iligne</th><th>iexc</th></tr>
<tr><td>1</td><td>1</td><td>2 1 3 4</td><td>2 3 4 1</td><td>1</td><td>1</td></tr>
<tr><td rowspan="6">2</td><td>1, 2</td><td>3 2 4 1</td><td>3 4 2 1</td><td>1, 2</td><td rowspan="6">2</td></tr>
<tr><td>1, 3</td><td>4 2 3 1</td><td>2 4 1 3</td><td>1, 3</td></tr>
<tr><td>2</td><td>3 1 2 4
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3 4 1 2</td><td>2</td></tr>
<tr><td>1, 3</td><td>2 1 4 3</td><td>2 1 4 3</td><td>1, 3</td></tr>
<tr><td>2, 3</td><td>4 1 3 2</td><td>4 3 1 2</td><td>2, 3</td></tr>
<tr><td>1, 2, 3</td><td>4 3 2 1</td><td>4 3 2 1</td><td>1, 2, 3</td></tr>
<tr><td>3</td><td>3</td><td>4 1 2 3</td><td>4 1 2 3</td><td>3</td><td>3</td></tr>
</table>

Theorem 1.3 has been recently used by Han and Xin [14] to set up a relation between the generating polynomial for derangements by number of excedances and the corresponding polynomial for permutations with one fixed point.

In our previous papers [9, 10] we have introduced three statistics "dez," "maz" and "maf" on $\mathfrak{S}_n$. If σ is a permutation, let $i_1, i_2, \ldots, i_h$ be the increasing sequence of its fixed points. Let $D\sigma$ (resp. $Z\sigma$) be the word derived from $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ by *deleting* all the fixed points (resp. by *replacing* all those fixed points by 0). Then those three statistics are simply defined by: $\operatorname{dez}\sigma := \operatorname{des} Z\sigma$, $\operatorname{maz}\sigma := \operatorname{maj} Z\sigma$ and $\operatorname{maf}\sigma := (i_1 - 1) + (i_2 - 2) + \cdots + (i_j - h) + \operatorname{maj} D\sigma$. For instance, with $\sigma = 8\,2\,1\,3\,5\,6\,4\,9\,7$ we have $(i_1, \ldots, i_h) = (2, 5, 6)$, $Z\sigma = 8\,0\,1\,3\,0\,0\,4\,9\,7$, $D\sigma = 8\,1\,3\,4\,9\,7$ and $\operatorname{dez}\sigma = 3$, $\operatorname{maz}\sigma = 1 + 4 + 8 = 13$, $\operatorname{maf}\sigma = (2-1) + (5-2) + (6-3) + \operatorname{maj}(8\,1\,3\,4\,9\,7) = 13$. Theorem 1.4 in [9] and Theorem 1.1 above provide another combinatorial interpretation for $A_n(s, t, q, Y)$, namely

$$\sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{exc}\sigma} t^{\operatorname{dez}\sigma} q^{\operatorname{maz}\sigma} Y^{\operatorname{fix}\sigma} = A_n(s, t, q, Y). \tag{1.7}$$

In the sequel we need the notations for the q-multinomial coefficients

$$\begin{bmatrix} n \\ m_1, \ldots, m_k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_{m_1} \cdots (q;q)_{m_k}} \qquad (m_1 + \cdots + m_k = n);$$

and the first q-exponential

$$e_q(u) = \sum_{n\ge 0} \frac{u^n}{(q;q)_n} = \frac{1}{(u;q)_\infty}.$$

Multiply both sides of (1.4) by $1-t$ and let $t = 1$. We obtain the factorial generating function for a sequence of polynomials $(A_n(s, 1, q, Y))$ $(n \ge 0)$ in three variables:

$$\sum_{n\ge 0} A_n(s, 1, q, Y) \frac{u^n}{(q;q)_n} = \frac{(1 - sq)e_q(Yu)}{e_q(squ) - sq e_q(u)}. \tag{1.8}$$

It follows from Theorem 1.1 that

$$\sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{exc}\sigma} q^{\operatorname{maj}\sigma} Y^{\operatorname{fix}\sigma} = A_n(s, 1, q, Y) \tag{1.9}$$

holds for every $n \ge 0$, a result stated and proved by Shareshian and Wachs [19] by means of a symmetric function argument, so that identity (1.8) with the interpretation (1.9) belongs to those two authors. Identity (1.4) can be regarded as a graded form of (1.8). The interest of the graded form also lies in the fact that it provides the joint distribution of (exc, des), as shown in (1.15) below.

Of course, Theorem 1.2 yields a second combinatorial interpretation for the polynomials $A_n(s,1,q,Y)$ in the form

$$\sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{lec}\sigma} q^{\operatorname{imaj}\sigma} Y^{\operatorname{pix}\sigma} = A_n(s,1,q,Y). \tag{1.10}$$

However we have a third combinatorial interpretation, where the statistic "imaj" is replaced by the number of inversions "inv." We state it as our fourth main theorem.

Theorem 1.4. *Let $A_n(s,1,q,Y)$ $(n\ge 0)$ be the sequence of polynomials in three variables, whose factorial generating function is given by* (1.8). *Then, the generating polynomial for $\mathfrak{S}_n$ by the three-variable statistic* $(\operatorname{lec},\operatorname{inv},\operatorname{pix})$ *is equal to $A_n(s,1,q,Y)$. In other words,*

$$\sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{lec}\sigma} q^{\operatorname{inv}\sigma} Y^{\operatorname{pix}\sigma} = A_n(s,1,q,Y). \tag{1.11}$$

Again, Theorem 1.4 in [9] and Theorem 1.1 provide a fourth combinatorial interpretation of $A_n(s,1,q,Y)$, namely

$$\sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{exc}\sigma} q^{\operatorname{maf}\sigma} Y^{\operatorname{fix}\sigma} = A_n(s,1,q,Y). \tag{1.12}$$

Note that the statistic "maf" was introduced and studied in [4].

Let $s=1$ in identity (1.4). We get:

$$\sum_{n\ge 0} A_n(1,t,q,Y)\frac{u^n}{(t;q)_{n+1}} = \sum_{r\ge 0} t^r \Bigl(1-u\sum_{i=0}^{r} q^i\Bigr)^{-1} \frac{(u;q)_{r+1}}{(uY;q)_{r+1}}, \tag{1.13}$$

so that Theorem 1.1 implies

$$\sum_{\sigma\in\mathfrak{S}_n} t^{\operatorname{des}\sigma} q^{\operatorname{maj}\sigma} Y^{\operatorname{fix}\sigma} = A_n(1,t,q,Y), \tag{1.14}$$

an identity derived by Gessel and Reutenauer [13].

Finally, by letting $q=Y:=1$ we get the generating function for polynomials in two variables $A_n(s,t,1,1)$ $(n\ge 0)$ in the form

$$\sum_{n\ge 0} A_n(s,t,1,1)\frac{u^n}{(1-t)^{n+1}} = \sum_{r\ge 0} t^r \frac{1-s}{(1-u)^{r+1}(1-us)^{-r}-s(1-u)}. \tag{1.15}$$

It then follows from Theorems 1.1 and 1.2 that

$$\sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{exc}\sigma} t^{\operatorname{des}\sigma} = \sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{lec}\sigma} t^{\operatorname{ides}\sigma} = A_n(s,t,1,1). \tag{1.16}$$

As is well-known (see, *e.g.*, [11]) "exc" and "des" are equidistributed over $\mathfrak{S}_n$, their common generating polynomial being the Eulerian polynomial $A_n(t) := A_n(t,1,1,1) = A_n(1,t,1,1)$, which satifies the identity

$$\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{r\geq 0} t^r (r+1)^n, \tag{1.17}$$

easily deduced from (1.15).

The polynomials $A_n(s,t,1,1)$ do not have any particular symmetries. This is perhaps the reason why their generating function has never been calculated before, to the best of the authors' knowledge. However, with $q=1$ and $Y=0$ we obtain

$$\sum_{n\geq 0} A_n(s,t,1,0)\frac{u^n}{(1-t)^{n+1}} = \sum_{r\geq 0} t^r \frac{1-s}{(1-us)^{-r} - s(1-u)^{-r}}. \tag{1.18}$$

The right-hand side is invariant under the change of variables $u \leftarrow us$, $s \leftarrow s^{-1}$, so that the polynomials $A_n(s,t,1,0)$, which are the generating polynomials for the set of all *derangements* by the pair $(\mathrm{exc}, \mathrm{des})$, satisfy $A_n(s,t,1,0) = s^n A_n(s^{-1},t,1,0)$. This means that $(\mathrm{exc}, \mathrm{des})$ and $(\mathrm{iexc}, \mathrm{des})$ are equidistributed on the set of all derangements. There is a stronger combinatorial result that can be derived as follows. Let $\mathbf{c}$ be the *complement of* $(n+1)$ and $\mathbf{r}$ the *reverse image*, which map each permutation $\sigma = \sigma(1)\ldots\sigma(n)$ onto $\mathbf{c}\,\sigma := (n+1-\sigma(1))(n+1-\sigma(2))\ldots(n+1-\sigma(n))$ and $\mathbf{r}\,\sigma := \sigma(n)\,\ldots\,\sigma(2)\sigma(1)$, respectively. Then

$$(\mathrm{exc}, \mathrm{fix}, \mathrm{des}, \mathrm{ides})\,\sigma = (\mathrm{iexc}, \mathrm{fix}, \mathrm{des}, \mathrm{ides})\,\mathbf{c}\,\mathbf{r}\,\sigma. \tag{1.19}$$

The paper is organized as follows. In order to prove that $C(r;u,s,q,Y)$ is the generating polynomial for $W_n(r)$ by two multivariable statistics, we show in the next section that it suffices to construct two explicit bijections ϕ^{fix} and ϕ^{pix}. The first bijection ϕ^{fix}, defined in Section 3, relates to the algebra of *Lyndon words*, first introduced by Chen, Fox and Lyndon [3], popularized in Combinatorics by Schützenberger [18] and now set in common usage in Lothaire [17]. It is based on the techniques introduced by Kim and Zeng [15]. In particular, we show that the V-cycle decomposition introduced by those two authors, which is attached to each permutation, can be extended to the case of words. This is the content of Theorem 3.4, which may be regarded as our fifth main result.

The second bijection ϕ^{pix}, constructed in Section 4, relates to the less classical H-factorization, the analog for words of the hook factorization introduced by Gessel [12].

In Section 5 we complete the proofs of Theorems 1.1 and 1.2. By combining the two bijections $\phi^{\rm fix}$ and $\phi^{\rm pix}$ we obtain a transformation on words serving to prove that two bivariable statistics are equidistributed on the same rearrangement class. This is done in Section 6, as well as the proof of Theorem 1.3. Finally, Theorem 1.4 is proved in Section 7 by means of a new property of the second fundamental transformation.

2. Two multivariable generating functions for words

As $1/(u;q)_r = \sum\limits_{n\ge 0} \left[{r+n-1 \atop n}\right]_q u^n$ (see, *e.g.*, [2, chap. 3]), we may rewrite the fraction $C(r;u,s,q,Y) = \dfrac{(1-sq)\,(u;q)_r\,(usq;q)_r}{((u;q)_r - sq(usq;q)_r)(uY;q)_{r+1}}$ as

(2.1) $C(r;u,s,q,Y)$
$$= \Bigl(1-\sum_{n\ge 2}\begin{bmatrix} r+n-1\\ n\end{bmatrix}_q u^n((sq)+(sq)^2+\cdots+(sq)^{n-1})\Bigr)^{-1}\frac{1}{(uY;q)_{r+1}}.$$

If $c = c_1c_2\cdots c_n$ is a word, whose letters are nonnegative integers, let $\lambda\, c := n$ be the *length* of c and $\mathrm{tot}\, c := c_1 + c_2 + \cdots + c_n$ the *sum* of its letters. Furthermore, NIW_n (resp. $\mathrm{NIW}_n(r)$) designates the set of all *monotonic nonincreasing* words $c = c_1c_2\cdots c_n$ of length n, whose letters are nonnegative integers (resp. nonnegative integers at most equal to r): $c_1 \ge c_2 \ge \cdots \ge c_1 \ge 0$ (resp. $r \ge c_1 \ge c_2 \ge \cdots \ge c_1 \ge 0$). Also let $\mathrm{NIW}(r)$ be the union of all $\mathrm{NIW}_n(r)$ for $n \ge 0$. It is q-routine (see, *e.g.*, [2, chap. 3]) to prove
$$\begin{bmatrix} r+n-1\\ n\end{bmatrix}_q = \sum_{w\in \mathrm{NIW}_n(r-1)} q^{\mathrm{tot}\, w}.$$

The sum $\sum\limits_{n\ge 2}\left[{r+n-1 \atop n}\right]_q u^n((sq)+(sq)^2+\cdots+(sq)^{n-1})$ can then be rewritten as $\sum\limits_{(w,i)} s^i q^{i+\mathrm{tot}\, w}u^{\lambda w}$, where the sum is over all pairs (w,i) such that $w \in \mathrm{NIW}(r-1)$, $\lambda w \ge 2$ and i is an integer satisfying $1 \le i \le \lambda w - 1$. Let $D(r)$ (resp. $D_n(r)$) denote the set of all those pairs (w,i) (resp. those pairs such that $\lambda w = n$). Therefore, equation (2.1) can also be expressed as
$$C(r;u,s,q,Y) = (1-\sum_{(w,i)\in D(r)} s^i q^{i+\mathrm{tot}\, w}u^{\lambda w})^{-1}\sum_{n\ge 0}u^n \sum_{w\in \mathrm{NIW}_n(r)} q^{\mathrm{tot}\, w}Y^{\lambda w},$$
and the coefficient $C_n(r;s,q,Y)$ of u^n defined in (1.3) as

(2.2) $C_n(r;s,q,Y) = \sum s^{i_1+\cdots+i_m}\, q^{i_1+\cdots+i_m+\mathrm{tot}\, w_0+\mathrm{tot}\, w_1+\cdots+\mathrm{tot}\, w_m}\, Y^{\lambda w_0}$,

the sum being over all sequences $(w_0,(w_1,i_1),\ldots,(w_m,i_m))$ such that $w_0 \in \mathrm{NIW}(r)$, each of the pairs (w_1,i_1), … , (w_m,i_m) belongs to $D(r)$, and $\lambda w_0 + \lambda w_1 + \cdots + \lambda w_m = n$. Denote the set of those sequences by $D_n^*(r)$. The next step is to construct the two bijections $\phi^{\rm fix}$ and $\phi^{\rm pix}$ of $D_n^*(r)$ onto $W_n(r)$ enabling us to calculate certain multivariable statistical distributions *on words*.

Let $l = x_1 x_2 \cdots x_n$ be a nonempty word, whose letters are nonnegative integers. Then l is said to be a *Lyndon word*, if either $n = 1$, or if $n \geq 2$ and, with respect to the lexicographic order, the inequality $x_1 x_2 \cdots x_n > x_i x_{i+1} \cdots x_n x_1 \cdots x_{i-1}$ holds for every i such that $2 \leq i \leq n$. When $n \geq 2$, we always have $x_1 \geq x_i$ for all $i = 2, \ldots, n$ and $x_i > x_{i+1}$ for at least one integer i $(1 \leq i \leq n-1)$, so that it makes sense to define the *rightmost minimal letter* of l, denoted by $\operatorname{rmin} l$, as the unique letter x_{i+1} satisfying the inequalities $x_i > x_{i+1}$, $x_{i+1} \leq x_{i+2} \leq \cdots \leq x_n$.

Let w, w' be two nonempty primitive words (none of them can be expressed as v^b, where v is a word and b an integer greater than or equal to 2). We write $w \preceq w'$ if and only if $w^b \leq w'^b$, with respect to the lexicographic order, when b is large enough. As shown for instance in [17, Theorem 5.1.5] each nonempty word w, whose letters are nonnegative integers, can be written uniquely as a product $l_1 l_2 \cdots l_k$, where each l_i is a Lyndon word and $l_1 \preceq l_2 \preceq \cdots \preceq l_k$. Classically, each Lyndon word is defined to be the minimum within its class of cyclic rearrangements, so that the sequence $l_1 \preceq l_2 \preceq \cdots$ is replaced by $l_1 \geq l_2 \geq \cdots$ The modification made here is for convenience.

For instance, the factorization of the following word as a nondecreasing product of Lyndon words with respect to "$\preceq$" [in short, *Lyndon word factorization*] is indicated by vertical bars:

$$w = \mid 2 \mid 3\,2\,1\,1 \mid 3 \mid 5 \mid 6\,4\,2\,1\,3\,2\,3 \mid 6\,6\,3\,1\,6\,6\,2 \mid 6 \mid .$$

Now let $w = x_1 x_2 \cdots x_n$ be an *arbitrary* word. We say that a positive integer i is a *decrease* of w if $1 \leq i \leq n-1$ and $x_i \geq x_{i+1} \geq \cdots \geq x_j > x_{j+1}$ for some j such that $i \leq j \leq n-1$. In particular, i is a decrease if $x_i > x_{i+1}$. The letter x_i is said to be a *decrease value* of w. If $1 \leq i_1 < i_2 < \cdots < i_m \leq n-1$ is the increasing sequence of the decreases of w, the subword $x_{i_1} x_{i_2} \cdots x_{i_m}$ is called the *decrease value subword* of w. It will be denoted by $\operatorname{decval} w$. The *number of decreases* itself of w is denoted by $\operatorname{dec}(w)$. We have $\operatorname{dec}(w) = 0$ if all letters of w are equal. Also $\operatorname{dec}(w) \geq 1$ if w is a Lyndon word having at least two letters. In the previous example we have $\operatorname{decval} w = 3\,2\,6\,4\,2\,3\,6\,6\,3\,6\,6$, of length 11, so that $\operatorname{dec} w = 11$.

Let $l_1 l_2 \ldots l_k$ be the Lyndon word factorization of a word w and let $(l_{i_1}, l_{i_2}, \ldots, l_{i_h})$ $(1 \leq i_1 < i_2 < \cdots < i_h \leq k)$ be the sequence of all the *one-letter* factors in its Lyndon word factorization. Form the nonincreasing word $\operatorname{Single} w$ defined by $\operatorname{Single} w := l_{i_h} \cdots l_{i_2} l_{i_1}$ and let $\operatorname{single} w = h$

be the number of letters of $\mathrm{Single}\,w$. In the previous example we have: $\mathrm{Single}\,w = 6\,5\,3\,2$ and $\mathrm{single}\,w = 4$.

Theorem 2.1. *The map* $\phi^{\mathrm{fix}} : (w_0, (w_1, i_1), \ldots, (w_m, i_m)) \mapsto w$ *of* $D_n^*(r)$ *onto* $W_n(r)$*, defined in Section 3, is a bijection having the properties:*

$$
\begin{aligned}
&i_1 + \cdots + i_m = \mathrm{dec}\, w;\\
&i_1 + \cdots + i_m + \mathrm{tot}\, w_0 + \mathrm{tot}\, w_1 + \cdots + \mathrm{tot}\, w_m = \mathrm{tot}\, w;\\
&\lambda w_0 = \mathrm{single}\, w.
\end{aligned}
\tag{2.3}
$$

The next Corollary is then a consequence of (2.2) and the above theorem.

Corollary 2.2. *The sum* $C_n(r; s, q, Y)$ *defined in* (2.2) *is also equal to*

$$C_n(r; s, q, Y) = \sum_w s^{\mathrm{dec}\, w} q^{\mathrm{tot}\, w} Y^{\mathrm{single}\, w}, \tag{2.4}$$

where the sum is over all words $w \in W_n(r)$*.*

To define the second bijection $\phi^{\mathrm{pix}} : D_n^*(r) \to W_n(r)$ another class of words is in use. We call them *H-words*. They are defined as follows: let $h = x_1 x_2 \cdots x_n$ be a word of length $\lambda h \geq 2$, whose letters are nonnegative integers. Say that h is a *H-word*, if $x_1 < x_2$, and either $n = 2$, or $n \geq 3$ and $x_2 \geq x_3 \geq \cdots \geq x_n$.

Each nonnempty word w, whose letters are nonnegative integers, can be written uniquely as a product $u h_1 h_2 \cdots h_k$, where u is a monotonic *nonincreasing* word (possibly empty) and each h_i a H-word. This factorization is called the *H-factorization* of w. Unless w is monotonic nonincreasing, it ends with a H-word, so that its H-factorization is obtained by removing that H-word and determining the next rightmost H-word. Note the discrepancy between the hook factorization for *permutations* mentioned in the introduction and the present H-factorization used for *words*.

For instance, the H-factorization of the following word is indicated by vertical bars:

$$w = |\, 6\,5\,3\,2 \,|\, 1\,3\,2\,1 \,|\, 3\,6\,4 \,|\, 1\,2 \,|\, 2\,3 \,|\, 1\,6\,6\,3 \,|\, 2\,6\,6 \,|\,.$$

Three statistics are now defined that relate to the H-factorization $u h_1 h_2 \cdots h_k$ of each *arbitrary* word w. First, let $\mathrm{wpix}(w)$ be the length λu of u. Then, if $\mathbf{r}$ denotes the *reverse image*, which maps each word $x_1 x_2 \ldots x_n$ onto $x_n \ldots x_2 x_1$, define the statistic $\mathrm{wlec}(w)$ by

$$\mathrm{wlec}(w) := \sum_{i=1}^{k} \mathrm{rinv}(h_i),$$

where $\mathrm{rinv}(w) = \mathrm{inv}(\mathbf{r}(w))$. In the previous example, $\mathrm{wpix}\, w = \lambda(6532) = 4$ and $\mathrm{wlec}\, w = \mathrm{inv}(1231) + \mathrm{inv}(463) + \mathrm{inv}(21) + \mathrm{inv}(32) + \mathrm{inv}(3661) + \mathrm{inv}(662) = 2 + 2 + 1 + 1 + 3 + 2 = 11$.

Theorem 2.3. *The map* $\phi^{\rm pix} : (w_0, (w_1, i_1), \ldots, (w_m, i_m)) \mapsto w$ *of* $D^*_n(r)$ *onto* $W_n(r)$*, defined in Section 4, is a bijection having the properties:*

$$\begin{aligned} & i_1 + \cdots + i_m = \operatorname{wlec} w; \\ (2.5) \qquad & i_1 + \cdots + i_m + \operatorname{tot} w_0 + \operatorname{tot} w_1 + \cdots + \operatorname{tot} w_m = \operatorname{tot} w; \\ & \lambda w_0 = \operatorname{wpix} w. \end{aligned}$$

Corollary 2.4. *The sum* $C_n(r; s, q, Y)$ *defined in* (2.2) *is also equal to*

$$C_n(r; s, q, Y) = \sum_w s^{\operatorname{wlec} w} q^{\operatorname{tot} w} Y^{\operatorname{wpix} w},$$

where the sum is over all words $w \in W_n(r)$.

3. The bijection $\phi^{\rm fix}$

The construction of the bijection $\phi^{\rm fix}$ of $D^*_n(r)$ onto $W_n(r)$ proceeds in four steps and involves three subclasses of Lyndon words: the V-words, U-words and L-words. We can say that V- and U-words are the word analogs of the V- and U-cycles introduced by Kim and Zeng [15] for permutations. The present construction is directly inspired by their work.

Each word $w = x_1 x_2 \cdots x_n$ is said to be a *V-word* (resp. a *U-word*), if it is of length $n \ge 2$ and its letters satisfy the following inequalities

(3.1) $x_1 \ge x_2 \ge \cdots \ge x_i > x_{i+1}$ and $x_{i+1} \le x_{i+2} \le \cdots \le x_n < x_i$,
(resp.
(3.2) $x_1 \ge x_2 \ge \cdots \ge x_i > x_{i+1}$ and $x_{i+1} \le x_{i+2} \le \cdots \le x_n < x_1$)

for some i such that $1 \le i \le n-1$. Note that if (3.1) or (3.2) holds, then $\operatorname{dec}(w) = i$. Also $\max w$ (the maximum letter of w) $= x_1 > x_n$. For example, $v = 5\,5\,4\,\underline{1}\,1\,2$ is a V-word and $u = 8\,7\,\underline{5}\,7\,7$ is a U-word, but not a V-word. Their rightmost minimal letters have been underlined.

Now w is said to be a *L-word*, if it is a Lyndon word of length at least equal to 2 and whenever $x_1 = x_i$ for some i such that $2 \le i \le n$, then $x_1 = x_2 = \cdots = x_i$. For instance, 6 6 3 1 6 6 2 is a Lyndon word, but not a L-word, but 6 6 3 1 2 1 is a L-word.

Let $V_n(r)$ (resp. $U_n(r)$, resp. $L_n(r)$, $\operatorname{Lyndon}_n(r)$) be the set of V-words (resp. U-words, resp. L-words, resp. Lyndon words), of length n, whose letters are at most equal to r. Also, let $V(r)$ (resp. $U(r)$, resp. $L(r)$, resp. $\operatorname{Lyndon}(r)$) be the union of the $V_n(r)$'s (resp. the $U_n(r)$'s, resp. the $L_n(r)$'s, resp. the $\operatorname{Lyndon}_n(r)$'s) for $n \ge 2$. Clearly, $V_n(r) \subset U_n(r) \subset L_n(r) \subset$

$\text{Lyndon}_n(r)$. Parallel to $D_n^*(r)$, whose definition was given in (2.2), we introduce three sets $V_n^*(r)$, $U_n^*(r)$, $L_n^*(r)$ of sequences $(w_0, w_1, \ldots, w_k)$ of words from $W(r)$ such that $w_0 \in \text{NIW}(r)$, $\lambda w_0 + \lambda w_1 + \cdots + \lambda w_k = n$ and

(i) for $V_n^*(r)$ the components w_i $(1 \le i \le k)$ belong to $V(r)$;

(ii) for $U_n^*(r)$ the components w_i $(1 \le i \le k)$ belong to $U(r)$ and are such that: $\text{rmin}\, w_1 < \max w_2$, $\text{rmin}\, w_2 < \max w_3$, ... , $\text{rmin}\, w_{k-1} < \max w_k$;

(iii) for $L_n^*(r)$ the components w_i $(1 \le i \le k)$ belong to $L(r)$ and are such that: $\max w_1 \le \max w_2 \le \cdots \le \max w_k$.

The first step consists of mapping the set $D_n(r)$ onto $V_n(r)$. This is made by means of a very simple bijection, defined as follows: let $w = x_1 x_2 \cdots x_n$ be a nonincreasing word and let (w, i) belong to $D_n(r)$, so that $n \ge 2$ and $1 \le i \le n-1$. Let

$$\begin{gathered} y_1 := x_1 + 1,\ y_2 := x_2 + 1,\ \ldots,\ y_i := x_i + 1, \\ y_{i+1} := x_n,\ y_{i+2} := x_{n-1},\ \ldots,\ y_n := x_{i+1}; \\ v := y_1 y_2 \ldots y_n. \end{gathered}$$

The following proposition is evident.

Proposition 3.1. *The mapping $(w, i) \mapsto v$ is a bijection of $D_n(r)$ onto $V_n(r)$ satisfying* $\text{dec}(v) = i$ *and* $\text{tot}\, v = \text{tot}\, w + i$.

For instance, the image of $(w = 4\,4\,3\,2\,1\,1, i = 3)$ is the V-word $v = 5\,5\,4\,1\,1\,2$ under the above bijection and $\text{dec}(v) = 3$.

Let $(w_0, (w_1, i_1), (w_2, i_2), \ldots, (w_k, i_k))$ belong to $D_n^*(r)$ and, using the bijection of Proposition 3.1, let $(w_1, i_1) \mapsto v_1$, $(w_2, i_2) \mapsto v_2$, ... , $(w_k, i_k) \mapsto v_k$. Then

$$(w_0, (w_1, i_1), (w_2, i_2), \ldots, (w_k, i_k)) \mapsto (w_0, v_1, v_2, \ldots, v_k) \tag{3.3}$$

is a bijection of $D_n^*(r)$ onto $V_n^*(r)$ having the property that

$$\begin{aligned} &i_1 + i_2 + \cdots + i_k = \text{dec}(v_1) + \text{dec}(v_2) + \cdots + \text{dec}(v_k); \\ &i_1 + i_2 + \cdots + i_k + \text{tot}\, w_0 + \text{tot}\, w_1 + \text{tot}\, w_2 + \cdots + \text{tot}\, w_k \\ &\qquad\qquad = \text{tot}\, w_0 + \text{tot}\, v_1 + \text{tot}\, v_2 + \cdots + \text{tot}\, v_k. \end{aligned} \tag{3.4}$$

For instance, the sequence

$$\big(6\,5\,3\,2,\ (2\,1\,1\,1,\ 2), (5\,3\,3,\ 2), (1\,1,\ 1), (2\,2,\ 1), (5\,5\,2\,1,\ 3), (5\,5\,2,\ 2)\big)$$

from $D_{22}^*(6)$ is mapped under (3.3) onto the sequence

$$(6\,5\,3\,2,\ 3\,2\,1\,1,\ 6\,4\,3,\ 2\,1,\ 3\,2,\ 6\,6\,3\,1,\ 6\,6\,2) \in V_{22}^*(6).$$

Also $\text{dec}(3\,2\,1\,1) + \text{dec}(6\,4\,3) + \text{dec}(2\,1) + \text{dec}(3\,2) + \text{dec}(6\,6\,3\,1) + \text{dec}(6\,6\,2) = 2 + 2 + 1 + 1 + 3 + 2 = 11$.

The second step is to map $V_n^*(r)$ onto $U_n^*(r)$. Let $u = y_1 y_2 \cdots y_k \in U(r)$ and $v = z_1 z_2 \cdots z_l \in V(r)$. Suppose that $\operatorname{rmin} u$ is the $(i+1)$-st leftmost letter of u and $\operatorname{rmin} v$ is the $(j+1)$-st letter of v. Also assume that $\operatorname{rmin} u \ge \max v$. Then, the word $[u,v] := y_1 \cdots y_i z_1 \cdots z_j z_{j+1} \cdots z_l y_{i+1} \cdots y_k$ belongs to $U(r)$. Furthermore, $\operatorname{rmin}[u,v]$ is the $(i+j+1)$-st leftmost letter of $[u,v]$ and its value is z_{j+1}. We also have the inequalities: $y_i > y_{i+1}$ (by definition of $\operatorname{rmin} u$), $y_{i+1} \ge z_1$ (since $\min u \ge \max v$) and $z_j > z_l$ (since v is a V-word). These properties allow us to get back the pair (u,v) from $[u,v]$ by *successively* determining the *critical* letters z_{j+1}, z_j, z_l, y_{i+1}, z_1. The mapping $(u,v) \mapsto [u,v]$ is perfectly reversible.

For example, with $u = 8\,7\,\underline{5}\,7\,7$ and $v = \mathbf{5}\,\underline{\mathbf{2}}\,\mathbf{2}$ we have $[u,v] = 8\,7\,\mathbf{5}\,\underline{\mathbf{2}}\,\mathbf{2}\,5\,7\,7$.

Now let $(w_0, v_1, v_2, \ldots, v_k) \in V_n^*(r)$. If $k=1$, let $(w_0, u_1) := (w_0, v_1) \in U_n^*(r)$. If $k \ge 2$, let $(1, 2, \ldots, a)$ be the longest sequence of integers such that $\operatorname{rmin} v_1 \ge \max v_2 > \operatorname{rmin} v_2 \ge \max v_2 > \cdots \ge \max v_a > \operatorname{rmin} v_a$ and, either $a = k$, or $a \le k-1$ and $\operatorname{rmin} v_a < \max v_{a+1}$. Let

$$u_1 := \begin{cases} v_1, & \text{if } a = 1; \\ [\cdots[[v_1, v_2], v_3], \cdots, v_a], & \text{if } a \ge 2. \end{cases} \tag{3.5}$$

We have $u_1 \in U(r)$ and $(w_0, u_1) \in U_n^*(r)$ if $a = k$. Otherwise, $\operatorname{rmin} u_1 < \max v_{a+1}$. We can then apply the procedure described in (3.5) to the sequence $(v_{a+1}, v_{a+2}, \ldots, v_k)$. When reaching v_k we obtain a sequence $(w_0, u_1, \ldots, u_h) \in U_n^*(r)$. The whole procedure is perfectly reversible. We have then the following proposition.

Proposition 3.2. *The mapping*

$$(w_0, v_1, v_2, \ldots, v_k) \mapsto (w_0, u_1, u_2, \ldots, u_h) \tag{3.6}$$

described in (3.5) *is a bijection of* $V_n^*(r)$ *onto* $U_n^*(r)$ *having the following properties:*
(i) $u_1 u_2 \cdots u_h$ *is a rearrangement of* $v_1 v_2 \cdots v_k$, *so that* $\operatorname{tot} u_1 + \operatorname{tot} u_2 + \cdots + \operatorname{tot} u_h = \operatorname{tot} v_1 + \operatorname{tot} v_2 + \cdots + \operatorname{tot} v_k$;
(ii) $\operatorname{dec}(u_1) + \operatorname{dec}(u_2) + \cdots + \operatorname{dec}(u_h) = \operatorname{dec}(v_1) + \operatorname{dec}(v_2) + \cdots + \operatorname{dec}(v_k)$.

For example, the above sequence

$$(6\,5\,3\,2,\ 3\,2\,1\,1,\ 6\,4\,3,\ 2\,1,\ 3\,2,\ 6\,6\,3\,1,\ 6\,6\,2) \in V_{22}^*(6)$$

is mapped onto the sequence

$$(6\,5\,3\,2,\ 3\,2\,1\,1,\ 6\,4\,2\,1\,3,\ 3\,2,\ 6\,6\,3\,1,\ 6\,6\,2) \in U_{22}^*(6).$$

where $[6\,4\,3,\ 2\,1] = 6\,4\,2\,1\,3$. Also $\operatorname{dec}(3\,2\,1\,1) + \operatorname{dec}(6\,4\,2\,1\,3) + \operatorname{dec}(3\,2) + \operatorname{dec}(6\,6\,3\,1) + \operatorname{dec}(6\,6\,2) = 11$.

The third step is to map $U_n^*(r)$ onto $L_n^*(r)$. Let $l = x_1x_2\cdots x_j \in L(r)$ and $u = y_1y_2\cdots y_{j'} \in U(r)$. Suppose that $\mathrm{rmin}\, l$ is the $(i+1)$-st leftmost letter of l and $\mathrm{rmin}\, u$ is the $(i'+1)$-st leftmost letter of u. Also assume that $\mathrm{rmin}\, l < \max u$ and $\max l > \max u$. If $x_j < y_1$, let $\langle l, u\rangle := lu$. If $x_j \ge y_1$, there is a unique integer $a \ge i+1$ such that $x_a < y_1 = \max u \le x_{a+1}$. Then, let

$$\langle l, u\rangle := x_1\cdots x_i x_{i+1}\cdots x_a y_1\cdots y_{i'} y_{i'+1}\cdots y_{j'} x_{a+1}\cdots x_j.$$

The word $\langle l, u\rangle$ belongs to $L(r)$ and $\mathrm{rmin}\langle l, u\rangle$ is the $(a+i'+1)$-st leftmost letter of $\langle l, u\rangle$, its value being $y_{i'+1}$. Now y_1 is the rightmost letter of $\langle l, u\rangle$ to the left of $\mathrm{rmin}\langle l, u\rangle = y_{i'+1}$ such that the letter preceding it, that is x_a, satisfies $x_a < y_1$ and the letter following it, that is y_2, is such that $y_1 \ge y_2$. On the other hand, $y_{j'}$ is the unique letter in the nondecreasing factor $y_{i'+1}\cdots y_{j'} x_{a+1}\cdots x_j$ that satisfies $y_{j'} < y_1 = \max u \le x_{a+1}$. Hence the mapping $(l, u) \mapsto \langle l, u\rangle$ is completely reversible.

For example, with $l = 8\,2\,5\,\underline{3}\,3\,5\,7\,7$ and $u = \mathbf{7\,6\,\underline{1}\,2}$ we have $\langle l, u\rangle = 8\,2\,5\,3\,3\,5\,\mathbf{7\,6\,\underline{1}\,2}\,7\,7$ (the leftmost minimal letters have been underlined). The letter $\mathbf{7}$ is the rightmost letter in $\langle l, u\rangle$ greater than its predecessor 5 and greater than or equal to its successor $\mathbf{6}$. Also $\mathbf{2}$ is the unique letter in the factor $1\,\mathbf{2}\,7\,7$ that satisfies $\mathbf{2} < \max u = \mathbf{7} \le 7$.

Let $(w_0, u_1, u_2, \ldots, u_h) \in U_n^*(r)$. If $h = 1$, let $(w_0, l_1) := (w_0, u_1) \in L_n^*(r)$. If $h \ge 2$, let $(1, 2, \ldots, a)$ be the longest sequence of integers such that $\max u_1 > \max u_j$ for all $j = 2, \ldots, a$ and, and either $a = h$, or $a \le h-1$ and $\max u_1 \le \max u_{a+1}$. Let

$$l_1 := \begin{cases} u_1, & \text{if } a = 1;\\ \langle\cdots\langle\,\langle u_1, u_2\rangle, u_3\rangle, \,\cdots\,, u_a\rangle, & \text{if } a \ge 2.\end{cases} \tag{3.7}$$

We have $l_1 \in L(r)$ and $(w_0, l_1) \in L_n^*(r)$ if $a = h$. Otherwise, $\max l_1 \le \max u_{a+1}$. We then apply the procedure described in (3.7) to the sequence $(u_{a+1}, u_{a+2}, \ldots, u_h)$. When reaching u_h we obtain a sequence $(w_0, l_1, \ldots, l_m) \in L_n^*(r)$. The whole procedure is perfectly reversible. We have then the following proposition.

Proposition 3.3. *The mapping*

$$(w_0, u_1, u_2, \ldots, u_h) \mapsto (w_0, l_1, l_2, \ldots, l_m) \tag{3.8}$$

described in (3.7) *is a bijection of* $U_n^*(r)$ *onto* $L_n^*(r)$ *having the following properties:*
(i) $l_1l_2\cdots l_m$ *is a rearrangement of* $u_1u_2\cdots u_h$, *so that* $\mathrm{tot}\, l_1 + \mathrm{tot}\, l_2 + \cdots + \mathrm{tot}\, l_m = \mathrm{tot}\, u_1 + \mathrm{tot}\, u_2 + \cdots + \mathrm{tot}\, u_h$;
(ii) $\mathrm{dec}(l_1) + \mathrm{dec}(l_2) + \cdots + \mathrm{dec}(l_m) = \mathrm{dec}(u_1) + \mathrm{dec}(u_2) + \cdots + \mathrm{dec}(u_h)$.

For example the above sequence

$$(6\,5\,3\,2,\ 3\,2\,1\,1,\ 6\,4\,2\,1\,3,\ 3\,2,\ 6\,6\,3\,1,\ 6\,6\,2)) \in U^*_{22}(6).$$

is mapped onto the sequence

$$(6\,5\,3\,2,\ 3\,2\,1\,1,\ 6\,4\,2\,1\,3\,2\,3,\ 6\,6\,3\,1,\ 6\,6\,2) \in L^*_{22}(6),$$

where $\langle 6\,4\,2\,1\,3,\ 3\,2\rangle = 6\,4\,2\,1\,3\,2\,3$. Also $\mathrm{dec}(3\,2\,1\,1) + \mathrm{dec}(6\,4\,2\,1\,3\,2\,3) + \mathrm{dec}(6\,6\,3\,1) + \mathrm{dec}(6\,6\,2) = 11$.

The fourth step is to map $L^*_n(r)$ onto $W_n(r)$. Let $(w_0, l_1, l_2, \ldots, l_m) \in L^*_n(r)$. If w_0 is nonempty, of length b, denote by $f_1,\ f_2,\ \ldots\ ,\ f_b$ its b letters from left to right, so that $r \geq f_1 \geq f_2 \geq \cdots \geq f_b \geq 0$. If $m = 1$, let $\sigma_1 := l_1$. If $m \geq 2$, let a be the greatest integer such that $l_1 \succ l_2$, $l_1 l_2 \succ l_3,\ \ldots\ ,\ l_1 \cdots l_{a-1} \succ l_a$. If $a \leq h-1$, let $a' > a$ be the greatest integer such that $l_{a+1} \succ l_{a+2}$, $l_{a+1}l_{a+2} \succ l_{a+3},\ \ldots\ ,\ l_{a+1}\cdots l_{a'-1} \succ l_{a'}$, etc. Form $\sigma_1 := l_1 l_2 \cdots l_a$, $\sigma_2 := l_{a+1} \cdots l_{a'}$, etc. The sequence $(\sigma_1, \sigma_2, \ldots)$ is a *nonincreasing* sequence of Lyndon words. Let $(\tau_1, \tau_2, \ldots, \tau_p)$ be the *nonincreasing* rearrangement of the sequence $(\sigma_1, \sigma_2, \ldots, f_1, f_2, \ldots, f_b)$ if w_0 is nonempty, and of $(\sigma_1, \sigma_2, \ldots)$ otherwise. Then, $(\tau_1, \tau_2, \ldots, \tau_p)$ is the *Lyndon word factorization* of a unique word $w \in W_n(r)$. The mapping

$$(w_0, l_1, l_2, \ldots, l_m) \mapsto w \tag{3.9}$$

is perfectly reversible. Also the verification of $\mathrm{dec}(w) = \mathrm{dec}(l_1) + \mathrm{dec}(l_2) + \cdots + \mathrm{dec}(l_m)$ is immediate.

For example, the above sequence

$$(6\,5\,3\,2,\ 3\,2\,1\,1,\ 6\,4\,2\,1\,3\,2\,3,\ 6\,6\,3\,1,\ 6\,6\,2) \in L^*_{22}(6)$$

is mapped onto the Lyndon word factorization

$$w = \mid 2 \mid 3\,2\,1\,1 \mid 3 \mid 5 \mid 6\,4\,2\,1\,3\,2\,3 \mid 6\,6\,3\,1\,6\,6\,2 \mid 6 \mid . \tag{3.10}$$

The map ϕ^{fix} is then defined as being the composition product of

$$\begin{aligned} (w_0, (w_1, i_1), (w_2, i_2), \ldots, (w_k, i_k)) &\mapsto (w_0, v_1, v_2, \ldots, v_k) && \text{in (3.3)}\\ (w_0, v_1, v_2, \ldots, v_k) &\mapsto (w_0, u_1, u_2, \ldots, u_h) && \text{in (3.6)}\\ (w_0, u_1, u_2, \ldots, u_h) &\mapsto (w_0, l_1, l_2, \ldots, l_m) && \text{in (3.8)}\\ (w_0, l_1, l_2, \ldots, l_m) &\mapsto w. && \text{in (3.9)} \end{aligned}$$

Therefore, ϕ^{fix} is a bijection of $D^*_n(r)$ onto $W_n(r)$ having the properties stated in Theorem 2.1. The latter theorem is then proved.

From the property of the bijection $w \mapsto (w_0, v_1, v_2, \ldots, v_k)$ of $W_n(r)$ onto $V^*_n(r)$ we deduce the following theorem, which may be regarded as a *word analog* of Theorem 2.4 in Kim-Zeng's paper [15].

Theorem 3.4 (V-word decomposition)**.** *To each word $w = x_1x_2\cdots x_n$ whose letters are nonnegative integers there corresponds a unique sequence $(w_0, v_1, v_2, \ldots, v_k)$, where w_0 is a nondecreasing word and $v_1, v_2, \ldots, v_k$ are V-words with the further property that $w_0v_1v_2\cdots v_k$ is a rearrangement of w and* $\operatorname{decval} w$ *is the juxtaposition product of the* $\operatorname{decval} v_i$*'s:*

$$\operatorname{decval} w = (\operatorname{decval} v_1)(\operatorname{decval} v_2)\cdots(\operatorname{decval} v_k).$$

In particular,

$$\operatorname{dec} w = \operatorname{dec} v_1 + \operatorname{dec} v_2 + \cdots + \operatorname{dec} v_k.$$

For instance, the decrease values of the word w below and of the V-factors of its V-decomposition are reproduced in boldface:

$$w = 2\,\mathbf{3}\,\mathbf{2}\,1\,1\,3\,5\,\mathbf{6}\,\mathbf{4}\,\mathbf{2}\,1\,\mathbf{3}\,2\,3\,\mathbf{6}\,\mathbf{6}\,\mathbf{3}\,1\,\mathbf{6}\,\mathbf{6}\,2\,6;$$

$$(6\,5\,3\,2,\ \mathbf{3}\,\mathbf{2}\,1\,1,\ \mathbf{6}\,\mathbf{4}\,3,\ \mathbf{2}\,1,\ \mathbf{3}\,2,\ \mathbf{6}\,\mathbf{6}\,\mathbf{3}\,1,\ \mathbf{6}\,\mathbf{6}\,2).$$

4. The bijection ϕ^{pix}

In the introduction a hook was defined to be a word $x_1x_2\cdots x_n$ with distinct letters such that $x_1 > x_2$ and either $n = 2$, or $n \geq 3$ and $x_2 < x_3 < \cdots < x_n$. In the next definition the letters can be repeated and the inequalities are reversed. Let $h = x_1x_2\cdots x_n$ be a word of length $\lambda h \geq 2$, whose letters are nonnegative integers. Say that h is a *H-word*, if $x_1 < x_2$ and either $n = 2$, or $n \geq 3$ and $x_2 \geq x_3 \geq \cdots \geq x_n$.

Each nonnempty word w, whose letters are nonnegative integers, can be written uniquely as a product $uh_1h_2\cdots h_k$, where u is a monotonic *nonincreasing* word (possibly empty) and each h_i is a H-word. This factorization is called the *H-factorization* of w. Unless w is monotonic nonincreasing, it ends with a H-word, so that its H-factorization is easily obtained by removing that H-word and determining the next rightmost H-word. For instance, the H-factorization of the following word is indicated by vertical bars:

$$w = \mid 6\,5\,3\,2 \mid 1\,3\,2\,1 \mid 3\,6\,4 \mid 1\,2 \mid 2\,3 \mid 1\,6\,6\,3 \mid 2\,6\,6 \mid .$$

Three statistics are now defined that relate to the H-factorization $uh_1h_2\cdots h_k$ of each *arbitrary* word w. First, let $\operatorname{wpix}(w)$ be the length λu of u. Then, if $\mathbf{r}$ denotes the *reverse image*, which maps each word $x_1x_2\ldots x_n$ onto $x_n\ldots x_2x_1$, let $\operatorname{rinv} := \operatorname{inv}\circ\mathbf{r}$ and define the statistic $\operatorname{wlec}(w)$ by

$$\operatorname{wlec}(w) := \sum_{i=1}^{k} \operatorname{rinv}(h_i).$$

In the previous example, $\operatorname{wpix} w = \lambda(6532) = 4$ and $\operatorname{wlec} w = \operatorname{inv}(1231) + \operatorname{inv}(463) + \operatorname{inv}(21) + \operatorname{inv}(32) + \operatorname{inv}(3661) + \operatorname{inv}(662) = 2+2+1+1+3+2 = 11.$

The bijection $\phi^{\rm pix} : D_n^*(r) \rightarrow W_n(r)$ whose properties were stated in Theorem 3.2 is easy to construct. Let $H_n(r)$ be the set of all H-words of length n, whose letters are at most equal to r and $H(r)$ be the union of all $H_n(r)$'s for $n \geq 2$. We first map $D_n(r)$ onto $H_n(r)$ as follows. Let $w = x_1x_2\cdots x_n$ be a nonincreasing word and let (w,i) belong to $D_n(r)$, so that $n \geq 2$ and $1 \leq i \leq n-1$. Define:

$$h := x_{i+1}(x_1+1)(x_2+1)\ldots(x_i+1)x_{i+2}x_{i+3}\ldots x_n.$$

The following proposition is evident.

Proposition 4.1. *The mapping* $(w,i) \mapsto h$ *is a bijection of* $D_n(r)$ *onto* $H_n(r)$ *satisfying* $\operatorname{rinv}(h) = i$ *and* $\operatorname{tot} h = \operatorname{tot} w + i$.

For instance, the image of $(w = 4\,4\,3\,2\,2\,1,\ i = 3)$ is the H-word $h = 2\,5\,5\,4\,2\,1$ under the above bijection and $\operatorname{rinv}(h) = 3$.

Let $(w_0,(w_1,i_1),(w_2,i_2),\ldots,(w_k,i_k))$ belong to $D_n^*(r)$ and, using the bijection of Proposition 4.1, let $(w_1,i_1) \mapsto h_1$, $(w_2,i_2) \mapsto h_2$, $\ldots$, $(w_k,i_k) \mapsto h_k$. Then $w_0h_1h_2\cdots h_k$ is the H-factorization of a word $w \in W_n(r)$. Accordingly,

$$\phi^{\rm pix} : (w_0,(w_1,i_1),(w_2,i_2),\ldots,(w_k,i_k)) \mapsto w := w_0h_1h_2\cdots h_k$$

is a bijection of $D_n^*(r)$ onto $W_n(r)$ having the properties listed in (2.5). This completes the proof of Theorem 2.3.

For instance, the sequence

$$\big(6\,5\,3\,2,\ (2\,1\,1\,1,\ 2),(5\,3\,3,\ 2),(1\,1,\ 1),(2\,2,\ 1),(5\,5\,2\,1,\ 3),(5\,5\,2,\ 2)\big)$$

from $D_{22}^*(6)$ is mapped under $\phi^{\rm pix}$ onto the word

$$6\,5\,3\,2 \mid 1\,3\,2\,1 \mid 3\,6\,4 \mid 1\,2 \mid 2\,3 \mid 1\,6\,6\,3 \mid 2\,6\,6 \in W_{22}(6).$$

Also $(\operatorname{wlec},\operatorname{tot},\operatorname{wpix})\,w = (11,74,4)$.

5. From words to permutations

We are now in a position to prove Theorems 1.1 and 1.2. Suppose that identity (1.5) holds. As

$$\frac{1}{(t;q)_{n+1}} = \sum_{j\geq 0} t^j \sum_{w\in {\rm NIW}_n(j)} q^{\operatorname{tot} w},$$

the right-hand side of (1.4) can then be written as $\sum\limits_{r\geq 0} t^r \sum\limits_{n\geq 0} B_n^{\rm fix}(r;s,q,Y)$, where

$$B_n^{\rm fix}(r;s,q,Y) := \sum_{(\sigma,c)} s^{\operatorname{exc}\sigma}\, q^{\operatorname{maj}\sigma+\operatorname{tot} c}\, Y^{\operatorname{fix}\sigma}, \tag{5.1}$$

the sum being over all pairs (σ,c) such that $\sigma \in \mathfrak{S}_n$, $\operatorname{des}\sigma \leq r$ and $c \in {\rm NIW}_n(r-\operatorname{des}\sigma)$. Denote the set of all those pairs by $\mathfrak{S}_n(r,\operatorname{des})$.

In the same manner, let $\mathfrak{S}_n(r, \mathrm{ides})$ denote the set of all pairs (σ, c) such that $\sigma \in \mathfrak{S}_n$, $\mathrm{ides}\,\sigma \le r$ and $c \in \mathrm{NIW}_n(r - \mathrm{ides}\,\sigma)$ and let

$$B_n^{\mathrm{pix}}(r; s, q, Y) := \sum_{(\sigma,c)} s^{\mathrm{lec}\,\sigma}\, q^{\mathrm{imaj}\,\sigma + \mathrm{tot}\,c}\, Y^{\mathrm{pix}\,\sigma}, \tag{5.2}$$

where the sum is over all $(\sigma, c) \in \mathfrak{S}_n(r, \mathrm{ides})$. If (1.6) holds, the right-hand side of (1.4) is equal to $\sum\limits_{r\ge0} t^r \sum\limits_{n\ge0} B_n^{\mathrm{pix}}(r; s, q, Y)$.

Accordingly, for proving identity (1.5) (resp. (1.6)) it suffices to show that $C_n(r; s, q, Y) = B_n^{\mathrm{fix}}(r; s, q, Y)$ (resp. $C_n(r; s, q, Y) = B_n^{\mathrm{pix}}(r; s, q, Y)$) holds for all pairs (r, n). Referring to Corollaries 2.2 and 2.4 it suffices to construct a bijection

$$\psi^{\mathrm{fix}} : w \mapsto (\sigma, c)$$

of $W_n(r)$ onto $\mathfrak{S}_n(r, \mathrm{des})$ having the following properties

$$\begin{aligned} \mathrm{dec}\,w &= \mathrm{exc}\,\sigma;\\ \mathrm{tot}\,w &= \mathrm{maj}\,\sigma + \mathrm{tot}\,c;\\ \mathrm{single}\,w &= \mathrm{fix}\,\sigma; \end{aligned} \tag{5.3}$$

and a bijection

$$\psi^{\mathrm{pix}} : w \mapsto (\sigma, c)$$

of $W_n(r)$ onto $\mathfrak{S}_n(r, \mathrm{ides})$ having the following properties

$$\begin{aligned} \mathrm{wlec}\,w &= \mathrm{lec}\,\sigma;\\ \mathrm{tot}\,w &= \mathrm{imaj}\,\sigma + \mathrm{tot}\,c;\\ \mathrm{wpix}\,w &= \mathrm{pix}\,\sigma. \end{aligned} \tag{5.4}$$

The construction of ψ^{fix} is achieved by adapting a classical bijection used by Gessel-Reutenauer [13] and Désarménien-Wachs [6, 7]. Start with the Lyndon word factorization $(\tau_1, \tau_2, \ldots, \tau_p)$ of a word $w \in W_n(r)$. If x is a letter of the factor $\tau_i = y_1 \cdots y_{j-1} x y_{j+1} \cdots y_h$, form the cyclic rearrangement $\mathrm{cyc}(x) := x y_{j+1} \cdots y_h y_1 \cdots y_{j-1}$. If x, y are two letters of w, we say that x *precedes* y, if $\mathrm{cyc}\,x \succ \mathrm{cyc}\,y$, or if $\mathrm{cyc}\,x = \mathrm{cyc}\,y$ and the letter x is to the right of the letter y in the word w. Accordingly, to each letter x of w there corresponds a unique integer $p(x)$, which is the number of letters *preceding* x plus one.

When replacing each letter x in the Lyndon word factorization of w by $p(x)$, we obtain a *cycle decomposition* of a permutation σ. Furthermore, the cycles start with their minima and when reading the word from left to right the cycle minima are in *decreasing* order.

When this replacement is applied to the Lyndon word factorization displayed in (3.10), we obtain:

$$\begin{array}{rl} w = & 2 \mid 3\ 2\ 1\ 1 \mid 3 \mid 5 \mid 6\ 4\ 2\ 1\ 3\ 2\ 3 \mid 6\ 6\ 3\ 1\ 6\ 6\ 2 \mid 6 \\ \sigma = & \mathbf{16} \mid 12\ 18\ 22\ 21 \mid \mathbf{10} \mid \mathbf{7} \mid 4\ 8\ 17\ 20\ 11\ 15\ 9 \mid 2\ 5\ 13\ 19\ 3\ 6\ 14 \mid \mathbf{1} \end{array}$$

Let $\overline{c}_i := p^{-1}(i)$ for $i = 1, 2, \ldots, n$. As the permutation σ is expressed as the product of its disjoint cycles, we can form the three-row matrix

$$\begin{array}{rcccc} \mathrm{Id} = & 1 & 2 & \cdots & n \\ \sigma = & \sigma(1) & \sigma(2) & \cdots & \sigma(n) \\ \overline{c} = & \overline{c}_1 & \overline{c}_2 & \cdots & \overline{c}_n \end{array}$$

The essential feature is that the word $\overline{c}$ just defined is the monotonic *nonincreasing* rearrangement of w and it has the property that

$$\sigma(i) > \sigma(i+1) \Rightarrow \overline{c}_i > \overline{c}_{i+1}. \tag{5.5}$$

See [13, 7] for a detailed proof. The rest of the proof is routine. Let $z = z_1 z_2 \cdots z_n$ be the word defined by

$$z_i := \#\{j : i \le j \le n-1,\ \sigma(j) > \sigma(j+1)\}.$$

In other words, z_i is the number of descents of σ within the right factor $\sigma(i)\sigma(i+1)\cdots\sigma(n)$. In particular, $z_1 = \operatorname{des}\sigma$. Because of (5.5) the word $c = c_1 c_2 \cdots c_n$ defined by $c_i := \overline{c}_i - z_i$ for $i = 1, 2, \ldots, n$ belongs to $\mathrm{NIW}(r - \operatorname{des}\sigma)$ and $\operatorname{des}\sigma \le r$.

Finally, the verification of the three properties (5.3) is straightforward. Thus, we have constructed the desired bijection $\psi^{\mathrm{fix}} : w \mapsto (\sigma, c)$, as the reverse construction requires no further development.

With the above example we have:

$$\begin{array}{rcccccccccccccccccccccc} \mathrm{Id} = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 \\ \sigma = & \mathbf{1} & \underline{5} & \underline{6} & \underline{8} & \underline{13} & \underline{14} & \mathbf{7} & \underline{17} & 4 & \mathbf{10} & \underline{15} & \underline{18} & \underline{19} & 2 & 9 & \mathbf{16} & \underline{20} & \underline{22} & 3 & 11 & 12 & 21 \\ \overline{c} = & 6 & 6 & 6 & 6 & 6 & 6 & 5 & 4 & 3 & 3 & 3 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ z = & 4 & 4 & 4 & 4 & 4 & 4 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ c = & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

The excedances of σ have been underlined ($\operatorname{exc}\sigma = 11$). As $\operatorname{tot} z = \operatorname{maj}\sigma$, we have $74 = \operatorname{tot} w = \operatorname{maj}\sigma + \operatorname{tot} c = 45 + 29$. The fixed points are written in boldface ($\operatorname{fix}\sigma = 4$).

The bijection $\psi^{\mathrm{pix}} : w \mapsto (\sigma, c)$ of $W_n(r)$ onto $\mathfrak{S}_n(r, \mathrm{ides})$ is constructed by means of the classical *standardisation* of words. Read w from left to right and label 1, 2, … all the maximal letters. If there are m such letters, restart the reading from left to right and label $m+1$, $m+2$, ... the second greatest letters. Pursue this reading method until reaching the minimal letters. Call $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ the permutation derived by reading those labels from left to right.

The permutation σ and the word w have the same hook-factorization *type*. This means that if $ah_1h_2\ldots h_s$ (resp. $bp_1p_2\ldots p_k$) is the hook-factorization of σ (resp. H-factorization of w), then $k = s$ and $\lambda a = \lambda b$. For each $1 \le i \le k$ we have $\lambda h_i = \lambda p_i$ and $\operatorname{inv}(h_i) = \operatorname{rinv}(p_i)$. Hence $\operatorname{wlec} w = \operatorname{lec}\sigma$ and $\operatorname{wpix} w = \operatorname{pix}\sigma$.

Now define the word $z = z_1z_2\ldots z_n$ as follows. If $\sigma(j) = n$ is the maximal letter, then $z_j := 0$; if $\sigma(j) = \sigma(k) - 1$ and $j < k$, then $z_j := z_k$; if $\sigma(j) = \sigma(k) - 1$ and $j > k$, then $z_j := z_k + 1$. We can verify that $\operatorname{imaj}\sigma = \operatorname{tot} z$. With $w = x_1x_2\cdots x_n$ define the word $d = d_1d_2\cdots d_n$ by $d_i := x_i - z_i$ $(1 \le i \le n)$. As $z_j = z_k + 1 \Rightarrow x_j \ge x_k + 1$, the letters of d are all nonnegative. The final word c is just defined to be the monotonic nonincreasing rearrangement of d. Finally, properties (5.4) are easily verified.

For defining the reverse of ψ^{pix} we just have to remember that the following inequality holds: $\sigma(j) < \sigma(k) \Rightarrow d_j \ge d_k$. This achieves the proofs of Theorems 1.1 and 1.2. For example,

Id =	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
w =	6	5	3	2	1	3	2	1	3	6	4	1	2	2	3	1	6	6	3	2	6	6
σ =	1	7	9	14	19	10	15	20	11	2	8	21	16	17	12	22	3	4	13	18	5	6
z =	4	3	2	1	0	2	1	0	2	4	3	0	1	1	2	0	4	4	2	1	4	4
d =	2	2	1	1	1	1	1	1	1	2	1	1	1	1	1	1	2	2	1	1	2	2
c =	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

6. A bijection on words and the proof of Theorem 1.3

Consider the two bijections ϕ^{fix} and ϕ^{pix} that have been constructed in Sections 3 and 4 and consider the bijection F defined by the following diagram:

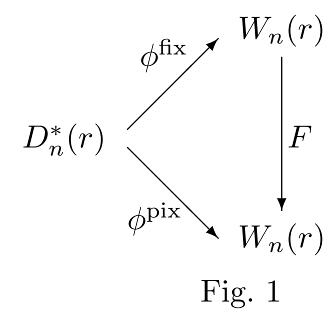

Fig. 1

On the other hand, go back to the definition of the bijection $(w,i) \mapsto v$ (resp. $(w,i) \mapsto h$) given in Proposition 3.1 (resp. in Proposition 4.1).

If $w = x_1 x_2 \cdots x_n$, then *both* v and h are *rearrangements* of the word $(x_1+1)(x_2+1)\cdots(x_i+1)x_{i+1}\cdots x_n$. Now consider the two bijections

$$\begin{aligned}\phi^{\mathrm{fix}} &: (w_0,(w_1,i_1),\ldots,(w_m,i_m)) \mapsto w;\\ \phi^{\mathrm{pix}} &: (w_0,(w_1,i_1),\ldots,(w_m,i_m)) \mapsto w'.\end{aligned}$$

It then follows from Proposition 3.2, Proposition 3.3 and (3.9), on the one hand, and from the very definition of ϕ^{pix}, on the other hand, that the words w and w' are *rearrangements of each other*. Finally, Theorems 2.1 and 2.3 imply the following result.

Theorem 6.1. *The transformation F defined by the diagram of Fig.* 1 *maps each word whose letters are nonnegative integers on another word $F(w)$ and has the following properties:*

(i) $F(w)$ is a rearrangement of w and the restriction of F to each rearrangement class is a bijection of that class onto itself;

(ii) $(\mathrm{dec},\mathrm{single})\, w = (\mathrm{wlec},\mathrm{wpix})\, F(w)$.

Let $\mathbf{c}$ be the *complement to* $(n+1)$ that maps each permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ onto $\mathbf{c}\,\sigma := (n+1-\sigma(1))(n+1-\sigma(2))\cdots(n+1-\sigma(n))$. When restricted to the symmetric group $\mathfrak{S}_n$ the mapping $F\circ\mathbf{c}$ maps $\mathfrak{S}_n$ onto itself and has the property

$$(\mathrm{des},\mathrm{single})\,\sigma = (\mathrm{lec},\mathrm{pix})\,(F\circ\mathbf{c})(\sigma).$$

Note that "dec" was replaced by "des", as all the decreases in a permutation are descents. Finally, the so-called first fundamental transformation (see [11]) $\sigma \mapsto \hat\sigma$ maps $\mathfrak{S}_n$ onto itself and is such that

$$(\mathrm{exc},\mathrm{fix})\,\sigma = (\mathrm{des},\mathrm{single})\,\hat\sigma.$$

Hence

$$(\mathrm{exc},\mathrm{fix})\,\sigma = (\mathrm{lec},\mathrm{pix})\,(F\circ\mathbf{c})(\hat\sigma).$$

As announced in the introduction we have a stronger result stated in Theorem 1.3. Its proof is as follows.

Proof of Theorem 1.3. For each composition $J = j_1 j_2 \cdots j_m$ (word with positive letters) define the set $L(J)$ and the monotonic nonincreasing word $c(J)$ by

$$\begin{aligned}L(J) &:= \{j_m, j_m + j_{m-1}, \ldots, j_m + j_{m-1} + \cdots + j_2 + j_1\};\\ c(J) &:= m^{j_m}(m-1)^{j_{m-1}}\cdots 2^{j_2}1^{j_1}.\end{aligned}$$

For example, with $J = 455116$ we have $L(J) = \{6,7,8,13,18,22\}$ and $c(J) = 6666665433333222221111$.

Fix a composition J of n (i.e., $\operatorname{tot} J = n$) and let $\mathfrak{S}^J$ be the set of all permutations σ of order n such that $\operatorname{Iligne} \sigma \subset L(J)$. Using the bijection $\psi^{\rm pix}$ given in Section 5, define a bijection $w_1 \mapsto \sigma_1$ between the set R_J of all rearrangements of $c(J)$ and $\mathfrak{S}^J$ by

$$(\sigma_1, *) = \psi^{\rm pix}(w_1).$$

For defining the reverse $\sigma_1 \mapsto w_1$ we only have to take the multiplicity of $w_1 \in R_J$ into account. This is well-defined because $\operatorname{Iligne} \sigma_1 \subset L(J)$. For example, take the same example used in Section 5 for $\psi^{\rm pix}$:

$$\begin{array}{rcllllllllllllllllllllll}
\mathrm{Id} &=& 1&2&3&4&5&6&7&8&9&10&11&12&13&14&15&16&17&18&19&20&21&22\\
w_1 &=& 6&5&3&2&1&3&2&1&3&6&4&1&2&2&3&1&6&6&3&2&6&6\\
\sigma_1 &=& 1&7&9&14&19&10&15&20&11&2&8&21&16&17&12&22&3&4&13&18&5&6
\end{array}$$

Then $\operatorname{Iligne} \sigma_1 = \{6, 8, 13, 18\} \subset \mathrm{L}(J)$ and the basic properties of this bijection are

$$\operatorname{wlec} w_1 = \operatorname{lec} \sigma_1, \quad \operatorname{wpix} w_1 = \operatorname{pix} \sigma_1.$$

On the other hand the bijection $\psi^{\rm fix}$ given in Section 5 defines a bijection $w_2 \mapsto \sigma_2$ between R_J and $\mathfrak{S}^J$ by

$$(\sigma_2^{-1}, *) = \psi^{\rm fix}(w_2).$$

Again, for the reverse $\sigma_2 \mapsto w_2$ the multiplicity of $w_2 \in R_J$ is to be taken into account. This is also well-defined, since $\operatorname{Iligne} \sigma_2 \subset L(J)$. With the example used in Section 5 for $\psi^{\rm fix}$ we have:

$$\begin{array}{rcllllllllllllllllllllll}
\mathrm{Id} &=& 1&2&3&4&5&6&7&8&9&10&11&12&13&14&15&16&17&18&19&20&21&22\\
w_2 &=& 2&3&2&1&1&3&5&6&4&2&1&3&2&3&6&6&3&1&6&6&2&6\\
\sigma_2^{-1} &=& \mathbf{1}&5&6&8&13&14&\mathbf{7}&17&4&\mathbf{10}&15&18&19&2&9&\mathbf{16}&20&22&3&11&12&21\\
\sigma_2 &=& \mathbf{1}&14&19&9&2&3&\mathbf{7}&4&15&\mathbf{10}&20&21&5&6&11&\mathbf{16}&8&12&13&17&22&18
\end{array}$$

Also $\operatorname{Iligne} \sigma_2 = \operatorname{Ligne} \sigma_2^{-1} = \{6, 8, 13, 18\} \subset \mathrm{L}(J)$.

The basic properties of this bijection are

$$\operatorname{dec} w_2 = \operatorname{iexc} \sigma_2, \quad \operatorname{single} w_2 = \operatorname{fix} \sigma_2.$$

We can use those two bijections and the bijection F defined in Fig. 1 to form the chain

$$\sigma \mapsto w_1 \overset{F}{\mapsto} w_2 \mapsto \sigma_2,$$

and therefore obtain a bijection $\sigma_1 \mapsto \sigma_2$ of $\mathfrak{S}^J$ onto itself having the following properties

$$\operatorname{iexc} \sigma_2 = \operatorname{lec} \sigma_1, \quad \operatorname{fix} \sigma_2 = \operatorname{pix} \sigma_1.$$

In other words, the pairs (iexc, fix) and (lec, pix) are equidistributed on $\{\sigma \in \mathfrak{S}_n, \mathrm{Iligne}\,\sigma \subset J\}$ for all compositions J of n. By the inclusion-exclusion principle those pairs are also equidistributed on each set $\{\sigma \in \mathfrak{S}_n, \mathrm{Iligne}\,\sigma = J\}$. Hence the triples (iexc, fix, Iligne) and (lec, pix, Iligne) are equidistributed on $\mathfrak{S}_n$. □

7. Proof of Theorem 1.4

If $\mathbf{m} = (m_1, m_2, \dots, m_n)$ is a sequence of n nonnegative integers, the rearrangement class of the nondecreasing word $1^{m_1}2^{m_2}\dots r^{m_n}$, that is, the class of all the words than can be derived from $1^{m_1}2^{m_2}\dots r^{m_n}$ by permutation of the letters, is denoted by $R_{\mathbf{m}}$. The definitions of "des," "maj" and "inv" used so far for permutations are also valid for words. The *second fundamental transformation*, as it was called later on (see [8], [17], §10.6 or [16], ex. 5.1.1.19) denoted by Φ, maps each word w on another word $\Phi(w)$ and has the following properties:

(a) $\mathrm{maj}\, w = \mathrm{inv}\, \Phi(w)$;

(b) $\Phi(w)$ is a rearrangement of w, and the restriction of Φ to each rearrangement class $R_{\mathbf{m}}$ is a bijection of $R_{\mathbf{m}}$ onto itself.

Further properties were further proved by Foata, Schützenberger [11] and Björner, Wachs [1], in particular, when the transformation is restricted to act on rearrangement classes $R_{\mathbf{m}}$ such that $m_1 = \cdots = m_n = 1$, that is, on symmetric groups $\mathfrak{S}_n$.

Ligne and inverse ligne of route have been defined in the Introduction. As was proved in [11], the transformation Φ preserves the inverse ligne of route, so that the pairs (Iligne, maj) and (Iligne, inv) are equidistributed on $\mathfrak{S}_n$, a result that we express as

$$(\mathrm{Iligne}, \mathrm{maj}) \simeq (\mathrm{Iligne}, \mathrm{inv}); \tag{7.1}$$

or as

$$(\mathrm{Ligne}, \mathrm{imaj}) \simeq (\mathrm{Ligne}, \mathrm{inv}). \tag{7.2}$$

The refinement of (7.2) we now derive (see Proposition 7.1 and Theorem 7.2 below) is based on the properties of a new statistic called LAC.

For each permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ and each integer i such that $1 \le i \le n$ define $\ell_i := 0$ if $\sigma(i) < \sigma(i+1)$ and $\ell_i := k$ if $\sigma(i)$ is greater than all the letters $\sigma(i+1), \sigma(i+2), \dots, \sigma(i+k)$, but $\sigma(i) < \sigma(i+k+1)$. [By convention, $\sigma(n+1) = +\infty$.]

Definition. The statistic $\mathrm{LAC}\,\sigma$ attached to each permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ is defined to be the word $\mathrm{LAC}\,\sigma = \ell_1\ell_2\dots\ell_n$.

Example. We have

id	=	1	2	3	4	5	6	7	8	9	10	11	12
σ	=	3	4	8	1	9	2	5	10	12	7	6	11
$\mathrm{LAC}\,\sigma$	=	0	0	1	0	2	0	0	0	3	1	0	0

In the above table $\ell_5 = 2$ because $\sigma = \ldots \mathbf{9}\ 2\ 5\ \mathbf{10} \cdots$ and $\ell_9 = 3$ because $\sigma = \ldots \mathbf{12}\ 7\ 6\ 11$.

Proposition 7.1. *Let $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ be a permutation and let $\mathrm{LAC}\,\sigma = \ell_1\ell_2\ldots\ell_n$. Then $i \in \mathrm{Ligne}\,\sigma$ if and only if $\ell_i \geq 1$.*

Theorem 7.2. *We have*

$$(\mathrm{LAC}, \mathrm{imaj}) \simeq (\mathrm{LAC}, \mathrm{inv}). \tag{7.3}$$

Proof. Define $\mathrm{ILAC}\,\sigma := \mathrm{LAC}\,\sigma^{-1}$. Since Φ maps "maj" to "inv," property (7.3) will be proved if we show that Φ preserves "ILAC", that is,

$$\mathrm{ILAC}\,\Phi(\sigma) = \mathrm{ILAC}\,\sigma. \tag{7.4}$$

A direct description of $\mathrm{ILAC}\,\sigma$ can be given as follows. Let $\mathrm{ILAC}\,\sigma = f_1f_2\ldots f_n$. Then $f_i = j$ if and only if within the word $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ the integer j is such that the letters of σ equal to $i+1, i+2, \ldots, i+j$ are on the left of the letter equal to i and either $(i+j+1)$ is on the right of i, or $i+j=n$.

As can be seen in ([17], chap. 10), the second fundamental transformation Φ is defined by induction: $\Phi(x) = x$ for each letter x and $\Phi(wx) = \gamma_x(\Phi(w))x$ for each word w and each letter x, where γ_x is a well-defined bijection. See the above reference for an explicit description of γ_x. Identity (7.4) is then a simple consequence of the following property of γ_x (we omit its proof): *Let w be a word and x a letter. If u is a subword of w such that all letters of u are smaller (resp. greater) than x, then u is also a subword of $\gamma_x(w)$.* □

Proposition 7.3. *Let σ and τ be two permutations of order n. If $\mathrm{LAC}\,\sigma = \mathrm{LAC}\,\tau$, then*
(i) $\mathrm{Ligne}\,\sigma = \mathrm{Ligne}\,\tau$;
(ii) $(\mathrm{des}, \mathrm{maj})\sigma = (\mathrm{des}, \mathrm{maj})\tau$;
(iii) $\mathrm{pix}\,\sigma = \mathrm{pix}\,\tau$;
(iv) $\mathrm{lec}\,\sigma = \mathrm{lec}\,\tau$.

Proof. (i) follows from Proposition 7.1. (ii) follows from (i). By (i) we see that σ and τ have the same hook-factorization *type*. That means that if $ah_1h_2\ldots h_s$ (resp. $bp_1p_2\ldots p_k$) is the hook-factorization of σ (resp. of

τ), then $k = s$ and $\lambda\, a = \lambda\, b$, $\lambda\, h_i = \lambda\, p_i$ for $1 \le i \le k$. Hence (iii) holds. For proving (iv) it suffices to prove that $\mathrm{inv}(h_i) = \mathrm{inv}(p_i)$ for $1 \le i \le k$. This is true since $\mathrm{LAC}\,\sigma = \mathrm{LAC}\,\tau$ by hypothesis. □

It follows from Proposition 7.3 that

$$(\mathrm{lec}, \mathrm{imaj}, \mathrm{pix}) \simeq (\mathrm{lec}, \mathrm{inv}, \mathrm{pix}) \tag{7.5}$$

and this is all we need to prove Theorem 1.4.

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Dominique Foata,
Institut Lothaire, 1 rue Murner, F-67000 Strasbourg, France.
Email address: foata@math.u-strasbg.fr

Guo-Niu Han,
Center for Combinatorics, LPMC, Nankai University,
Tianjin 300071, P. R. China.

I.R.M.A. UMR 7501, Université Louis Pasteur et CNRS,
7, rue René-Descartes, F-67084 Strasbourg, France.
Email address: guoniu@math.u-strasbg.fr