Factor-Critical Property in 3-Dominating-Critical Graphs *

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Abstract

A vertex subset S of a graph G is a *dominating set* if every vertex of G either belongs to S or is adjacent to a vertex of S. The cardinality of a smallest dominating set is called the *dominating number* of G and is denoted by $\gamma(G)$. A graph G is said to be γ*-vertex-critical* if $\gamma(G-v) < \gamma(G)$, for every vertex v in G.

Let G be a 2-connected $K_{1,5}$-free 3-vertex-critical graph of odd order. For any vertex $v \in V(G)$, we show that $G - v$ has a perfect matching (except two graphs), which solves a conjecture posed by Ananchuen and Plummer [2].

Key words: matching, factor-critical, dominating set, 3-vertex-critical graphs
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1 Introduction

Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. A vertex subset S of G is a *dominating set* if every vertex of G either belongs to S or is adjacent to a vertex of S. The minimum size of such a set is called the *dominating number* of G and is denoted by $\gamma(G)$. A graph G is *vertex domination-critical*, or γ*-vertex-critical*, if for any vertex v of G, $\gamma(G-v) < \gamma(G)$. We use $G[S]$ to denote the subgraph induced by S for some $S \subseteq V(G)$. The minimum degree of G is denoted by $\delta(G)$. A graph is called $K_{1,k}$*-free* if it has no induced subgraph isomorphic to the complete bipartite graph $K_{1,k}$.

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A matching is *perfect* if it is incident with every vertex of G. If $G-v$ has a perfect matching, for every choice of $v \in V(G)$, G is said to be *factor-critical.* The concept of factor-critical graphs was first introduced by Gallai in 1963 and it plays an important role in the study of matching theory. To be contrary to its apparent strong property, such graphs form a relatively rich family for study. It is the essential "building block" for the so-called Gallai-Edmonds structure of the graphs with matchings.

The subject of γ-vertex-critical graphs was studied first by Brigham, Chinn and Dutton [3, 4] and continued by Fulman *et al.* [5, 6]. Clearly, the only 1-vertex-critical graph is K_1 (a single vertex). Brigham, Chinn and Dutton [3] pointed out that the 2-vertex-critical graphs are precisely the family of graphs obtained from the complete graphs K_{2n} with a perfect matching removed. For $\gamma > 2$, however, much remains unknown about the structure of γ-vertex-critical graphs. Recently, Ananchuen and Plummer [1, 2] began to study matchings in 3-vertex-critical graphs. They showed that a $K_{1,5}$-free 3-vertex-critical graph of even order has a perfect matching (see [1]) and a $K_{1,4}$-free 3-vertex-critical graph of odd order is factor-critical (see [2]). Furthermore, they posed the following conjecture.

Conjecture 1. *If G is a $K_{1,5}$-free 3-vertex-critical 2-connected graph of odd order with $\delta(G) \geq 3$, then G is factor-critical.*

In this paper, we show that the conjecture holds for almost all graphs and there are only two counterexamples.

If $v \in V(G)$, we denote a minimum dominating set of $G-v$ by D_v. The following facts about D_v follow immediately from the definition of 3-vertex-criticality and we shall use it frequently in the proof of the main theorem.

Facts: If G is 3-vertex-critical, then the followings hold

(1) For every vertex v of G, $|D_v| = 2$.

(2) If $D_v = \{x, y\}$, then x and y are not adjacent to v.

(3) For every pair of distinct vertices v and w, $D_v \neq D_w$.

The readers are referred to [7] for other terminology not specified in this paper.

2 Main Result

By Tutte's well-known 1-Factor Theorem, if a graph G has no perfect matching, then there exits a set $S \subseteq V(G)$ such that the number of components in $G-S$ having odd order is greater than the order of S. If $S \subseteq V(G)$, we shall denote by $\omega(G-S)$, the number of components of $G-S$ and by $c_o(G-S)$, the number of odd components of $G-S$. A criterion similar to 1-Factor Theorem for factor-critical graphs is as follows.

Lemma 2.1. *(see [7]) A graph G is factor-critical if and only if $c_o(G-S) \leq |S|-1$, for every nonempty set $S \subseteq V(G)$.*

Lemma 2.2. *Let G be 3-vertex-critical and S be a cutset in G with $|S| \geq 4$. If $D_u \subseteq S$ for each vertex $u \in S$, then there exists no vertex of degree 1 in $G[S]$.*

Proof. Suppose to the contrary that there exists some $v \in S$ such that v is of degree 1 in $G[S]$. Without loss of generality, let $vw \in E(G)$, where $w \in S$. By Fact 2, $v \notin D_w$. Since $D_w \subseteq S$, D_w does not dominate v, a contradiction. ∎

The following two lemmas, proved by Ananchuen and Plummer [2], will be used in our proof of the main theorem.

Lemma 2.3. *If G is 3-vertex-critical and S is a cutset in G such that $\omega(G-S) \geq 4$ or $\omega(G-S) = 3$, but each component has at least 2 vertices, then each vertex of $G-S$ is not adjacent to at least one vertex of S.*

Lemma 2.4. *Let G be a 3-vertex-critical graph and suppose that S is a cutset of order 2 in G, then $\omega(G-S) \leq 3$. Furthermore, if $\omega(G-S) = 3$, then $G-S$ must contain at least one singleton component.*

Before giving our main result, we note that the graphs G_1 and G_2 in Figure 1 are $K_{1,5}$-free 3-vertex-critical 2-connected graph of order 11 with $\delta(G) = 3$, but are not factor-critical, since $G_i - v_i$ has no perfect matching for $i = 1, 2$. We shall show that these two graphs are the only two counterexamples to Conjecture 1.

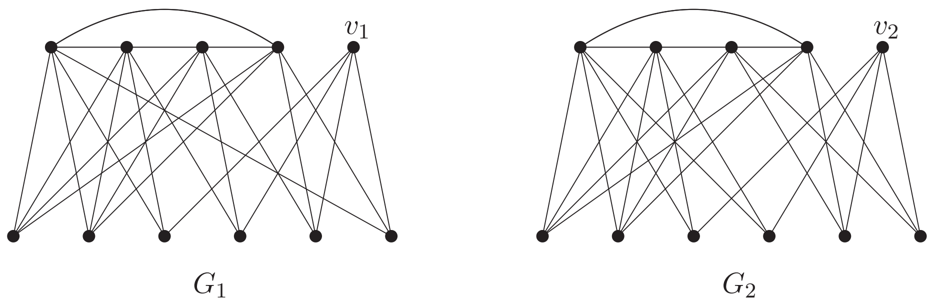

Figure 1: The graphs G_1 and G_2.

Theorem 2.1. *If G is a $K_{1,5}$-free 3-vertex-critical 2-connected graph of odd order with $\delta(G) \geq 3$, except the graphs G_1 and G_2 shown in Figure 1, then G is factor-critical.*

Proof. Suppose that G is not factor-critical. By Lemma 2.1 and the parity, there exists a nonempty set $S \subseteq V(G)$ such that $c_o(G-S) \geq |S|+1$. Without loss of generality, let S be a minimal such set with $|S| = k$. Then $k \geq 2$ as G is 2-connected. Let $C_1, C_2, \ldots, C_t$ be the odd components of $G-S$ and $E_1, E_2, \ldots, E_n$ the even components of $G-S$. We consider the following cases.

Case 1. $k = 2$.

By Lemma 2.4, then $t = 3$ and $G - S$ has no even components. Since $\delta(G) \geq 3$ and $k = 2$, each odd component of $G - S$ has at least three vertices, which contradicts to Lemma 2.4.

Case 2. $k = 3$.

Thus, $t \geq 4$. By Lemma 2.3, each vertex of $G - S$ is not adjacent to at least one vertex of S. Since $\delta(G) \geq 3$ and $k = 3$, we have $|V(C_i)| \geq 3$ for $i = 1, 2, \dots, t$. By Fact 3, there must exist a vertex x in some odd component of $G - S$ such that $D_x \nsubseteq S$. Clearly, $D_x \cap S \neq \emptyset$. Without loss of generality, let $x \in V(C_1)$ and $D_x = \{u, y\}$, where $u \in S$ and $y \in V(G) - S$. Since G is $K_{1,5}$-free, by the parity, so $t = 4$ and $G - S$ has at most one even component.

Claim 1. There exists an odd component C_j $(j \geq 2)$ such that C_j is a complete graph and u is adjacent to every vertex of $V(C_j)$.

If $y \in V(C_1) - \{x\}$, then u is adjacent to every vertex of $\bigcup_{i=2}^{4} V(C_i)$. Since G is $K_{1,5}$-free, at least two of C_2, C_3 and C_4 are complete. If $y \in \bigcup_{i=2}^{4} V(C_i)$, say $y \in V(C_2)$, then u dominates all vertices of $(V(C_1) \cup V(C_3) \cup V(C_4)) - \{x\}$, and at least one of C_3 and C_4 is complete, by $K_{1,5}$-freeness in G again. If $G - S$ has an even component E_1 and $y \in V(E_1)$, then u is adjacent to every vertex of $\bigcup_{i=1}^{4} V(C_i) - \{x\}$. Since G is $K_{1,5}$-free, C_2, C_3 and C_4 are all complete. So Claim 1 is proved.

Without loss of generality, assume that C_4 is complete and u is adjacent to every vertex of $V(C_4)$.

Claim 2. Each vertex of $S - \{u\}$ is not adjacent to any vertex of $V(C_4)$.

Suppose to the contrary that $va_4 \in E(G)$ for some $v \in S - \{u\}$ and $a_4 \in V(C_4)$. Then $D_{a_4} \cap (\{u, v\} \cup V(C_4)) = \emptyset$, as C_4 is complete and $ua_4 \in E(G)$. Let $S - \{u, v\} = \{w\}$. Clearly, $w \in D_{a_4}$. Then $wa_4 \notin E(G)$ and w dominates $V(C_4) - \{a_4\}$. Let $b_4 \in V(C_4) - \{a_4\}$. Then $ub_4 \in E(G)$ and $wb_4 \in E(G)$. Consequently, $D_{b_4} \cap (\{u, w\} \cup V(C_4)) = \emptyset$ and $v \in D_{b_4}$. So $vb_4 \notin E(G)$ and v dominates $V(C_4) - \{b_4\}$. Now let $c_4 \in V(C_4) - \{a_4, b_4\}$, then c_4 is adjacent to every vertex of S, which contradicts to Lemma 2.3.

From Claim 2, u is a cut-vertex in G, which is against the fact that G is 2-connected.

Case 3. $k = 4$.

Thus, $t \geq 5$. We first show that there exists some $a \in S$ such that $D_a \nsubseteq S$. Otherwise, $D_b \subseteq S$ for each vertex $b \in S$. By Lemma 2.2 and Fact 2, every vertex of S in $G[S]$ has degree 0. It is easy to check that this is impossible.

So let $u \in S$ such that $D_u \nsubseteq S$. Clearly, $D_u \cap S \neq \emptyset$. Let $D_u = \{v, x\}$, where $v \in S$ and $x \in V(G) - S$. Since G is $K_{1,5}$-free, so $t = 5$ and $G - S$ has no even components. Without loss of generality, let $x \in V(C_1)$, then v dominates all vertices of $\bigcup_{i=2}^{5} V(C_i)$. Moreover, by $K_{1,5}$-freeness again, C_2, C_3, C_4 and C_5 are all complete, v is not adjacent to any vertex of $V(C_1)$.

Claim 3. Each vertex of S is adjacent to at least three odd components of $G-S$.

Otherwise, there exists a vertex $c \in S$ such that c is adjacent to at most two odd components of $G-S$. Let $S' = S - \{c\}$. It is easy to see that S' is a nonempty set which satisfies the condition that $c_o(G-S') \geq |S'|+1$, contradicting to the minimality of S.

Let $S - \{u, v\} = \{w, z\}$. By Claim 3, w is adjacent to at least two of C_2, C_3, C_4 and C_5. Without loss of generality, let $wc_i \in E(G)$, where $c_i \in V(C_i)$ for $i = 2, 3$. Then $z \in D_{c_2}$. Otherwise, $u \in D_{c_2}$ and $D_{c_2} \cap V(C_1) \neq \emptyset$ since $ux \notin E(G)$. But then D_{c_2} can not dominate v, a contradiction. Similarly, $z \in D_{c_3}$. Thus, $zc_i \notin E(G)$ for $i = 2, 3$. By Fact 3, then either $D_{c_2} \neq \{u, z\}$ or $D_{c_3} \neq \{u, z\}$, say $D_{c_2} \neq \{u, z\}$. Since $zc_3 \notin E(G)$, it follows that $D_{c_2} \cap V(C_3) \neq \emptyset$ and z dominates every vertex of $V(C_1) \cup V(C_4) \cup V(C_5)$. By similar arguments, $w \in D_{c_4}$, $w \in D_{c_5}$ for some $c_4 \in V(C_4)$ and $c_5 \in V(C_5)$. Furthermore, $wc_i \notin E(G)$ for $i = 4, 5$, and w is adjacent to all vertices of $V(C_1) \cup V(C_2) \cup V(C_3)$.

We next show that C_2 is a singleton. Otherwise, $|V(C_2)| \geq 3$ and let a_2, $b_2 \in V(C_2) - \{c_2\}$. By similar arguments as the above, $z \in D_{a_2}$, $z \in D_{b_2}$ and either $D_{a_2} \neq \{u, z\}$ or $D_{b_2} \neq \{u, z\}$. Assume that $D_{a_2} \neq \{u, z\}$. Then $D_{a_2} \cap V(C_3) \neq \emptyset$, since $zc_3 \notin E(G)$. But then z is adjacent to all vertices of $V(C_2) - \{a_2\}$ and this contradicts to the fact that $zc_2 \notin E(G)$. Similarly, C_3, C_4 and C_5 are all singletons of $G-S$. Since $\delta(G) \geq 3$, $uc_i \in E(G)$ for $i = 2, 3, 4, 5$. As G is $K_{1,5}$-free, u is not adjacent to any vertex of $V(C_1)$.

Because $\delta(G) \geq 3$ and u, v are not adjacent to any vertex of $V(C_1)$, we have $|V(C_1)| \geq 3$. Moreover, $D_x \cap (V(C_1) - \{x\}) \neq \emptyset$ and $D_x \cap \{u, v\} \neq \emptyset$ (say, $u \in D_x$). Recall that $uv \notin E(G)$ and v is not adjacent to any vertex of $V(C_1)$, thus v is not dominated by D_x, a contradiction.

Case 4. $k = 5$.

Claim 4. For every vertex $x \in S$, $D_x \subseteq S$.

Otherwise, $D_u \nsubseteq S$ for some $u \in S$. Clearly, $D_u \cap S \neq \emptyset$. Let $D_u = \{y, z\}$, where $y \in S$ and $z \in V(G) - S$. Since $t \geq 6$, y must dominate at least 5 odd components of $G-S$, which contradicts to the fact that G is $K_{1,5}$-free.

Let $S = \{s_1, s_2, s_3, s_4, s_5\}$. By Fact 3, there are $\binom{5}{2} = 10$ distinct pairs of vertices in S and at least 11 vertices in G. So there must exist a vertex $x \in V(G) - S$ such that $D_x \nsubseteq S$. Assume that $x \in V(C_1)$. Clearly, $D_x \cap S \neq \emptyset$. Since G is $K_{1,5}$-free, we have $t = 6$ and $G-S$ has no even components. By Claim 4 and Lemma 2.2, each vertex of S in $G[S]$ has degree 0 or 2. It is not hard to see that $G[S]$ can only be a 5-cycle or a union of a 4-cycle and an isolated vertex.

Case 4.1. $G[S]$ is a 5-cycle.

Let $s_1s_2s_3s_4s_5s_1$ be the 5-cycle in the counterclockwise order and $D_x = \{s_1, w\}$, where $w \in V(G) - S$. Since G is $K_{1,5}$-free, $w \notin V(C_1)$. Assume that $w \in V(C_2)$. Then s_1 is adjacent to all vertices of $\bigcup_{i=3}^{6} V(C_i)$ and w dominates s_3, s_4. Moreover, $K_{1,5}$-freeness of G implies that C_3, C_4, C_5 and C_6 are all complete, C_1 is a singleton and s_1 is not adjacent to any vertex of $V(C_1) \cup V(C_2)$.

Since $D_{s_3} = \{s_1, s_5\}$, s_5 is adjacent to each vertex of $V(C_1) \cup V(C_2)$. Similarly, since $D_{s_4} = \{s_1, s_2\}$, s_2 is adjacent to each vertex of $V(C_1) \cup V(C_2)$. Therefore, w is adjacent to all vertices of $S - \{s_1\}$. Now consider D_w. Since $D_w \cap S = \{s_1\}$ and $s_1x \notin E(G)$, it follows $D_w = \{s_1, x\}$. Hence, x dominates s_3, s_4 and $V(C_2) = \{w\}$. But then $\{s_1, s_3\}$ is a dominating set in G, contradicting the assumption that $\gamma(G) = 3$.

Case 4.2. $G[S]$ is a union of a 4-cycle and an isolated vertex.

Let $s_1s_2s_3s_4s_1$ be the 4-cycle in the counterclockwise order and s_5 the isolated vertex in $G[S]$. Then $D_{s_1} = \{s_3, s_5\}$, $D_{s_2} = \{s_4, s_5\}$, $D_{s_3} = \{s_1, s_5\}$, and $D_{s_4} = \{s_2, s_5\}$.

Since G is $K_{1,5}$-free, s_5 is adjacent to at most 4 odd components of $G-S$. Without loss of generality, let $C_1, \ldots, C_r$ be the components which are not adjacent to s_5. Then $t = 6$ implies $r \geq 2$. Thus s_i is adjacent to every vertex of $\bigcup_{j=1}^{r} V(C_j)$ for $i = 1, 2, 3, 4$. Now consider D_y, $y \in V(C_1)$. Clearly, $D_y \cap S = \{s_5\}$. Since s_5 can not dominate $V(C_2)$, $D_y \cap V(C_2) \neq \emptyset$. Therefore, $r = 2$ and s_5 is adjacent to every vertex of $\bigcup_{i=3}^{6} V(C_i)$. Moreover, $V(C_1) = \{y\}$. By a similar argument, C_2 is also a singleton. For each vertex $v \in \bigcup_{i=3}^{6} V(C_i)$, we have $D_v \cap S \neq \emptyset$ and $D_v \nsubseteq S$, since $s_5 \notin D_v$ and the vertices in $S - \{s_5\}$ do not dominate s_5. From $K_{1,5}$-freeness of G, C_3, C_4, C_5 and C_6 are all singletons, say $V(C_i) = \{c_i\}$ for $i = 3, 4, 5, 6$.

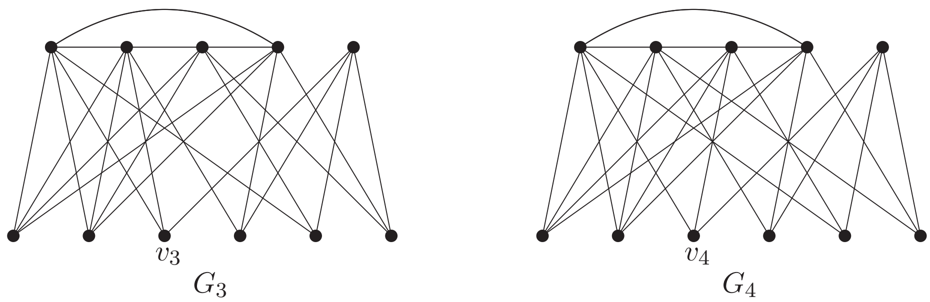

Figure 2: The graphs G_3 and G_4.

Let H be the induced subgraph in G with vertex set $\{s_i, c_j \mid 1 \leq i \leq 4, 3 \leq j \leq 6\}$ by deleting the edges in $G[S]$. For $3 \leq j \leq 6$, since $\delta(G) \geq 3$, c_j is adjacent to at least two vertices of $S - \{s_5\}$. On the other hand, since G is $K_{1,5}$-free, each vertex of $S - \{s_5\}$ is adjacent to at most two vertices of $\bigcup_{i=3}^{6}\{c_i\}$. Thus H is a 2-regular bipartite graph and hence consists of either a 8-cycle or a union of two 4-cycles.

However, there are only four such graphs under the isomorphism (see Figure 1 and Figure 2). It is easy to see that G_3 and G_4 are not 3-vertex-critical, since $|D_{v_i}| > 2$ in G_i for $i = 3, 4$. Therefore, G_1 and G_2 are two counterexamples to Conjecture 1.

Case 5. $k \geq 6$.

Claim 5. For every vertex $x \in V(G)$, $D_x \subseteq S$.

Suppose that $D_x \nsubseteq S$ for some $x \in V(G)$. Clearly, $D_x \cap S \neq \emptyset$. Let $D_x = \{y, z\}$, where $y \in S$ and $z \in V(G) - S$. Since $t \geq 7$, y must dominate at least 5 odd components of $G - S$, a contradiction to $K_{1,5}$-freeness.

Let w be any vertex in S, then $D_w \subseteq S$ by Claim 5. Since G is $K_{1,5}$-free, each vertex of D_w can dominate at most 4 components of $G - S$, which implies that the number of components of $G - S$ is at most 8 or $t \leq 8$. That is, $6 \leq k \leq 7$.

Let $S_i \subseteq S$ be the set of vertices in S which are adjacent to some vertex in C_i for $i = 1, 2, \ldots, t$, and let $d = \min\{|S_i|\}$. Without loss of generality, assume that $|S_1| = d$. Note that for any vertex $v \in V(G) - V(C_1)$, $D_v \cap S_1 \neq \emptyset$. We call such a set D_v *normal* 2*-set associated with* v *and* S_1, or *normal set* in short . By a simple counting, we see that there are at most $\binom{k}{2}$-$\binom{k-d}{2}$ normal sets. Since $|V(G) - V(C_1)| \geq 2k$, Facts 3 implies $\binom{k}{2}$-$\binom{k-d}{2} \geq 2k$ or $d \geq 3$. On the other hand, since G is $K_{1,5}$-free, each vertex of S is adjacent to at most 4 components of $G - S$, that is, $d \leq \frac{4k}{k+1}$ or $d \leq 3$. Hence $d = 3$.

Case 5.1. $k = 6$.

Thus $t = 7$ and $G - S$ has at most one even component. By Claim 5, there are $\binom{6}{2} = 15$ distinct pairs of vertices in S and at least 13 vertices in G. So by Fact 3, $|V(G)| = 13$ or 15, and $G - S$ has at least 6 singletons.

It is not hard to see that there exists at least four odd components whose corresponding S_i's having the order exactly 3, and at least two of them are singletons. Without loss of generality, let $C_1 = \{c_1\}$ and $C_2 = \{c_2\}$ be two singletons. Then, for every vertex $v \in V(G) - \{c_1\}$, $D_v \cap S_1 \neq \emptyset$. There are 12 normal sets associated with S_1 in S, and thus $|V(G)| = 13$. Next consider S_2. If $S_2 = S_1$, then D_{c_2} can not dominate c_1, a contradiction. If $|S_2 \cap S_1| \leq 2$, however, there must exist 2 normal sets associated S_1 which are not adjacent to c_2, at most one can be realized as D_{c_2}, and the other can not dominate c_2, a contradiction again.

Case 5.2. $k = 7$.

Thus $t = 8$ and $G - S$ has no even components. By a similar argument that used in the proof of Case 5.1, one reaches the same contradiction.

This completes the proof of our theorem. ∎

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