Available online at www.sciencedirect.com
ScienceDirect
Computers and Mathematics with Applications 54 (2007) 679–686
An International Journal
computers & mathematics with applications
www.elsevier.com/locate/camwa

The neighbour-scattering number can be computed in polynomial time for interval graphs☆

Fengwei Li, Xueliang Li*

Centre for Combinatorics and LPMC, Nankai University, Tianjin 300071, PR China

Received 15 March 2006; received in revised form 5 January 2007; accepted 20 February 2007

Abstract

Neighbour-scattering number is a useful measure for graph vulnerability. For some special kinds of graphs, explicit formulas are given for this number. However, for general graphs it is shown that to compute this number is NP-complete. In this paper, we prove that for interval graphs this number can be computed in polynomial time.

Keywords: Neighbour-scattering number; Interval graph; Consecutive clique arrangement

1. Introduction

Throughout this paper, we use Bondy and Murty [1] for terminology and notations not defined here and consider finite simple undirected graphs only. The vertex set of a graph G is denoted by V and the edge set of G is denoted by E. We always denote the number of vertices of G by n and the number of edges of G by m. By $\omega(G)$ we denote the number of components of G. $\deg(v)$ denotes the degree of a vertex v in G. If S is a vertex subset of V, we use $G[S]$ to denote the subgraph of G induced by S.

The scattering number of a graph was introduced by Jung [2] as an alternative measure of the vulnerability of graphs to disruption caused by the removal of vertices.

In [3–5] Gunther and Hartnell introduced the idea of modeling a spy network by a graph whose vertices represent the agents and whose edges represent lines of communication. Clearly, if a spy is discovered or arrested, the espionage agency can no longer trust any of the spies with whom he or she was in direct communication, and so the betrayed agents become effectively useless to the network as a whole. Such betrayals are clearly equivalent to the removal of the closed neighbourhood of v in the modelling graph, where v is the vertex representing the particular agent who has been subverted.

Therefore, instead of considering the scattering number of a communication network, we discuss the (vertex) neighbour-scattering number of graphs—disruption caused by the removal of vertices and their adjacent vertices.

☆ Supported by NSFC, PCSIRT and the "973" programme.

* Corresponding author.
E-mail addresses: fengwei.li@eyou.com (F. Li), x.li@eyou.com (X. Li).

0898-1221/$ - see front matter
doi:10.1016/j.camwa.2007.02.006

Let $G = (V, E)$ be a graph and u a vertex in G. The *open neighbourhood* of u is $N(u) = \{v \in V(G) \mid (u, v) \in E(G)\}$, and the *closed neighbourhood* of u is $N[u] = \{u\} \cup N(u)$. We define analogously for any $S \subseteq V(G)$ the open neighbourhood $N(S) = \cup_{u \in S} N(u)$ and the closed neighbourhood $N[S] = \cup_{u \in S} N[u]$. A vertex $u \in V(G)$ is said to be *subverted* when the closed neighbourhood $N[u]$ is deleted from G. A *vertex subversion strategy* of G, X, is a set of vertices whose closed neighbourhood is deleted from G. The survival-subgraph, G/X, is defined to be the subgraph left after the subversion strategy X is applied to G, i.e. $G/X = G - N[X]$. X is called a *cut-strategy* of G if the survival subgraph G/X is disconnected, or a clique, or $\emptyset$.

Definition 1.1 (*[6]*). The *(vertex) neighbour-scattering number* of a graph G is defined as

$$S(G) = \max\{\omega(G/X) - |X| : X \textit{ is cut} - \textit{strategy of } G, \omega(G/X) \geq 1\},$$

where the maximum is taken over all the cut-strategies of G, $\omega(G/X)$ is the number of connected components in the graph G/X. Especially, we define $S(K_n) = 1$.

Definition 1.2. A cut-strategy X of G is called an S-set of G if $S(G) = \omega(G/X) - |X|$.

In [7], F. Li and X. Li proved that, in general, the problem of computing the neighbour-scattering number of a graph is NP-complete. So, it is interesting to compute the neighbour-scattering number of special graphs, and some results of this kind were obtained in [6]. In Section 3, we prove that for interval graphs the neighbour-scattering number can be computed in polynomial time. Before proving this, in Section 2, we need to set up relationship between neighbour-scattering number and minimal cut-strategy of a graph and give a formula for calculating the neighbour-scattering number.

2. Minimal cut-strategy and neighbour-scattering number

In this section, we characterize the property of minimal cut-strategy X of a graph G with $\omega(G/X) \geq 1$, and give a formula to calculate neighbour-scattering number via minimal cut-strategy X of a graph G with $\omega(G/X) \geq 1$. First, we give the definition of the minimal cut-strategy of a graph G as follows.

Definition 2.1. A subset $X \subset V$ is a *cut-strategy* of a graph $G = (V, E)$ if G/X is disconnected, a clique. If no proper subset of X is a *cut-strategy* of graph G, then X is called a *minimal cut-strategy* of G.

Remark. From the above definition we know that if X is a minimal cut-strategy of graph G, then the removal of closed neighbourhood of any vertex set $X' \subset X$ neither disconnects G nor results in the remaining subgraph being a clique.

Lemma 2.1. *Let $X = \{v_1, v_2, \ldots, v_t\}$, $t \geq 1$, be a cut-strategy of graph G with $\omega(G/X) \geq 1$. Then X is a minimal cut-strategy of G if and only if one of the following conditions is satisfied:*

(a) *There are at least two different connected components, say $C_1, C_2, \ldots, C_k (k \geq 2)$, in G/X. For every vertex $v_i \in X$ and every connected component C_j $(j = 1, 2, \ldots, k)$ of G/X, v_i has a neighbour set B_{ij} in $N(C_j)$. And if $|X| \geq 2$, for distinct vertices v_s and v_t in X, neither $B_{si} \subseteq B_{ti}$ nor $B_{ti} \subseteq B_{si}$ for C_i $(i = 1, 2, \ldots, k)$. Furthermore, for any vertex $v_i \in X$, and any connected component C_j, $B_{ij} \nsubseteq \cup_{s=1}^{t} B_{sj}$, $s \neq i$. For every vertex $v \in X$, if v is adjacent to some vertices in X, then, there exists at least one vertex $v_i \in N(v) - X$ which is also in $\cap_{j=1}^{k} N(C_j)$, and v_i is not adjacent to any vertices in $X - v$. For every vertex $v \in X$, if $v_j \in N[v]$ and there doesn't exist any edge joining v_j with any component C_i $(i = 1, 2, \ldots, k)$, then there exists no edge joining v with other vertex in X if $|X| \geq 2$.*
(b) *G/X is a maximal clique C and every vertex $v_i \in X$ has a neighbour B_i in $N(C)$, and if $|X| \geq 2$, for distinct vertices v_i and v_j in X, neither $B_i \subseteq B_j$ nor $B_j \subseteq B_i$ for C. Furthermore, for any vertex $v_i \in X$, and the maximal clique C, $B_i \nsubseteq \cup_{j=1}^{t} B_j$, $j \neq i$. Furthermore, for $v \in X$, if $v_j \in N[v]$ and there doesn't exist any edge join v_j with this clique, then there exists no edge joining v with other vertex in X if $|X| \geq 2$.*

Proof. We prove the necessity first. If X is a minimal cut-strategy of G, then (a) or (b) must hold. We distinguish two cases:

Case 1. If (a) does not hold, we assume there exists a vertex v in X which does not have any neighbour in the open neighbourhood of one of these components. Without loss of generality, we assume that for component C_i ($i = 1, 2, \dots, k-1$), v has a neighbour in the open neighbourhood of these components but v does not have any neighbour in the open neighbourhood of component C_k. It is easy to see that under this condition $X' = X - v$ is also a cut-strategy of G with $\omega(G/X') \geq 2$, contradicting the minimality of X. Thus for every vertex $v \in X$ and every connected component C_i ($i = 1, 2, \dots, k$) of G/X, v has at least one neighbour in $N(C_i)$.

When $|X| \geq 2$, for distinct vertices v_s and v_t in X, if either $B_{si} \subseteq B_{ti}$ or $B_{ti} \subseteq B_{si}$ for a same component C_i ($1 \leq i \leq k$). Without loss of generality, we suppose that $B_{s1} \subseteq B_{t1}$ or $B_{t1} \subseteq B_{s1}$ for C_1, then it is easily seen that $X' = X - v_s$ or $X' = X - v_t$ is also a cut-strategy of G with $\omega(G/X') \geq 2$, contradicting the minimality of X. Furthermore, if for any vertex $v_i \in X$, and any connected component C_j, $B_{ij} \subseteq \cup_{s=1}^{t} B_{sj}$, $s \neq i$. Then vertex set $X' = X - v_i$ is obvious a cut-strategy of G with $\omega(G/X') \geq 2$, contradicting the minimality of X.

When $|X| \geq 2$, for any vertex $v \in X$ which is adjacent to some vertices in X, if there exists no vertex in $N(v) - X$ which is also in $\cap_{j=1}^{k} N(C_j)$. Then, it is easily checked that $X' = X - v$ is also a cut-strategy of G with $\omega(G/X') \geq 2$, contradicting the minimality of X.

When $|X| \geq 2$, if $v_j \in N[v]$ and there does not exist any edge joining v_j with any component C_i ($i = 1, 2, \dots, k$), then there exists no edge joining v with other vertex in X. Otherwise, if there exists an edge joining v with a vertex $v' \in X$, then it is easily checked that $X' = X - v$ is a cut-strategy of G with $\omega(G/X') \geq 2$, contradicting the minimality of X. So, there exists no edge joining v with other vertex in X.

Case 2. If (b) does not hold, there must exist a vertex v in X which does not have any neighbour in the open neighbourhood of the only clique of $C = G/X$. It is obvious that under this condition there must exist an edge (v, u) joining v with a vertex $u \in X$, otherwise contradicts the fact that G is connected. It is easily checked that $X' = X - v$ is also a cut-strategy of G with $\omega(G/X') \geq 1$, contradicting the minimality of X.

If $|X| \geq 2$, then for distinct vertices v_s and v_t in X, if either $B_s \subseteq B_t$ or $B_t \subseteq B_s$ for C, it is easily seen that $X' = X - v_s$ or $X' = X - v_t$ is also a cut-strategy of G with $\omega(G/X') = 2$ when there exists no edge joining v_s with v_t, or with G/X' is the clique C when v_s is adjacent to v_t, contradicting the minimality of X. Furthermore, if for any vertex $v_i \in X$, and the maximal clique C, $B_i \subseteq \cup_{j=1}^{t} B_j$, $j \neq i$. Then vertex set $X' = X - v_i$ is obvious a cut-strategy of G with $\omega(G/X') \geq 2$ when $v_i \in X$ is isolated from all the other vertices in X, or with G/X' is the clique C when $v_i \in X$ is adjacent to some vertices in X, contradicting the minimality of X.

When $|X| \geq 2$, if $v_j \in N[v]$ and there doesn't exist any edge joining v_j with clique C, then there exists no edge joining v with other vertex in X. Otherwise, if there exists a vertex $v' \in X$ and $(v, v') \in E(G[X])$, it is easy to see that $X' = X - v$ is a cut-strategy of G with $\omega(G/X') \geq 2$, contradicting the minimality of X. So, there exists no edge joining v with other vertex in X.

The proof of the sufficiency proceeds in the following two cases:

Case 1. If (a) holds, then X must be a minimal cut-strategy of graph G. Otherwise, there exists a subset $X' \subset X$ which is a cut-strategy of graph G. Then, for every vertex $v_i \in X - X'$, v_i has a neighbour in each neighbourhoods of these components, and we know that for every component C_i and other connected components C_j, $B_{ij} \nsubseteq \cup_{s=1}^{t} B_{sj}$, $s \neq i$., so, in this case, when each v_i is isolated from other vertices in X, then, the graph G/X' is connected. On the other hand, if v_i is adjacent to some vertices in X, and there exists no vertex in $N(v_i) - X$ which is also in $\cap_{j=1}^{k} N(C_j)$, then, the graph G/X' is connected. And under this condition G/X' is not a clique, for there exists no edge joining v with any components of G/X'. This leads to a contradiction to the hypothesis that X' is a cut-strategy of graph G.

Case 2. If (b) holds, then X must be a minimal cut-strategy of graph G. Otherwise, there exists a subset $X' \subset X$ which is a cut-strategy of graph G. Then, for every vertex $v \in X - X'$, v has at least one neighbour in the neighbourhood of this clique, so, the graph G/X' is connected, and under this condition G/X' is not a clique, otherwise contradicts the fact that for distinct vertices v_i and v_j in X, neither $B_i \subseteq B_j$ nor $B_j \subseteq B_i$ for C. This leads to a contradiction to the hypothesis that X' is a cut-strategy of graph G. Thus the proof is completed. ■

Theorem 2.2. *Let G be a non-complete graph. Then*

$$S(G) = \max_{X^*} \left\{ \sum_{i=1}^{k} \max\{S(G[C_i]), 1\} - |X^*| \right\} \tag{1}$$

where the maximum is taken over all minimal cut-strategies X^* of the graph G with $\omega(G/X^*) \geq 1$ and the $C_1, C_2, \dots, C_k$ are the connected components of G/X^*.

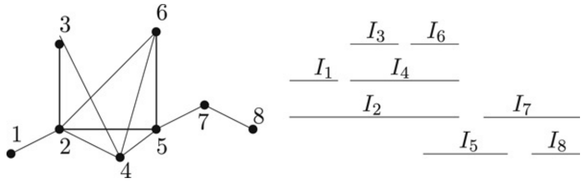

Fig. 1. An interval graph G and an interval representation for it.

Proof. First let X be an S-set of G, i.e., $S(G) = \omega(G/X) - |X|$ and $\omega(G/X) \geq 1$. Let X^* be a minimal cut-strategy of G with $\omega(G/X^*) \geq 1$ that is a subset of X and let $C_1, C_2, \ldots, C_k$ be the connected components of G/X^*. We consider the sets $X_i = X \cap C_i$, $i \in \{1, 2, \ldots, k\}$. The proof proceeds in the following two cases:

Case 1. If we assume $X_i = \emptyset$, i.e. $X \subset N[X^*]$, then we know that $N(X_i) = \emptyset$ is not a cut-set of C_i, i.e., X_i is not a cut-strategy of $G[C_i]$. Then, $\omega(C_i/X_i) = 1$, hence, $\omega(C_i/X_i) - |X_i| = 1$.

Case 2. Now assume $X_i \neq \emptyset$. Suppose that X_i is not a cut-strategy of $G[C_i]$. Then we have $\omega(G/(X - X_i)) = \omega(G/X)$. Furthermore, it is obvious that $\omega(G/(X-X_i)) - |X-X_i| = \omega(G/X) - |X| + |X_i| > \omega(G/X) - |X| = S(G)$, a contradiction to the definition of neighbour-scattering number of graphs.

Hence $X \neq \emptyset$ implies that X_i is a cut-strategy of C_i. Thus, $S(G[C_i]) \geq \omega(C_i/X_i) - |X_i|$.

Summing up the values of $\omega(C_i/X_i) - |X_i|$ over all components C_i of G/X^* will achieve the value of $\omega(G/X) - |X| = S(G)$. Thus we have $S(G) = \omega(G/X) - |X| = \sum_{i=1}^{k}\{\omega(C_i/X_i) - |X_i|\} - |X^*| \leq \sum_{i=1}^{k} \max\{S(G[C_i]), 1\} - |X^*|$.

On the other hand, let X^* be a minimal cut-strategy of G with $\omega(G/X^*) \geq 1$. Furthermore let $C_1, C_2, \ldots, C_k$ be the connected components of G/X^*. Then we construct a cut-strategy of G such that $X = X^* \cup \cup_{i=1}^{k} X_i$ with $X_i \subset C_i$ for every $i \in \{1, 2, \ldots, k\}$. For $i \in \{1, 2, \ldots, k\}$ we set $X_i = \emptyset$ if $S(G[C_i]) \leq 1$. Otherwise if $S(G[C_i]) > 1$, we choose a cut-strategy X_i of $G[C_i]$ with $\omega(C_i/X_i) \geq 1$ such that $S(G[C_i]) = \omega(C_i/X_i) - |X_i|$. Then $X \supset X^*$ is a cut-strategy of G and we have $S(G) \geq \omega(G/X) - |X| = \sum_{i=1}^{k}\{\omega(C_i/X_i) - |X_i|\} - |X^*|$.

Without loss of generality, let $C_1, C_2, \ldots, C_r$, $0 \leq r \leq k$, be the connected components of G with $S(G[C_i]) \leq 1$. Consequently, $S(G) = \sum_{i=1}^{k}\{\omega(C_i/X_i) - |X_i|\} - |X^*| = \sum_{i=1}^{r} 1 + \sum_{i=r+1}^{k} S(G[C_i]) - |X^*| = \sum_{i=1}^{k} \max\{S(G[C_i]), 1\} - |X^*|$. This completes the proof. ■

3. Neighbour-scattering number for interval graphs

Interval graphs are a large class of graphs and important modelling for useful networks. In this section we try to compute the neighbour-scattering number for interval graphs, and prove that neighbour-scattering number can be computed in polynomial time for interval graphs. First, we give the definition of an interval graph.

Definition 3.1 (*[8]*). An undirected graph G is called an *interval graph* if its vertices can be put into one to one correspondence with a set of intervals ℓ of a linearly ordered set (like the real line) such that two vertices are connected by an edge if and only if their corresponding intervals have nonempty intersection. We call ℓ an interval representation for G.

Example 1. In Fig. 1, we give an interval graph G and its interval representation:

Interval graphs are a well-known family of perfect graphs [8] with plenty of nice structural properties. The following characterizations were given by Gilmore and Hoffman [9]:

Lemma 3.1 (*[9]*). *Any induced subgraph of an interval graph is an interval graph.*

Lemma 3.2 (*Booth and Leuker (1976) [10]*). *Interval graphs can be recognized in $O(m + n)$ time.*

Lemma 3.3 (*Fulkerson and Gross (1965) [11]*). *A triangulated graph on n vertices has at most n maximal cliques, with equality if and only if the graph has no edges.*

Lemma 3.4 (*[9]*). *A graph G is an interval graph if and only if the maximal cliques of G can be linearly ordered, such that, for every vertex v of G, the maximal cliques containing v occur consecutively.*

Such a linear ordering of the maximal cliques of an interval graph is said to be a *consecutive clique arrangement*. Notice that interval graphs are triangulated graphs, and by Lemma 3.3 we know that an interval graph with n vertices has at most n maximal cliques [11]. Booth and Lueker [10] give a linear time recognition algorithm for interval graphs and the algorithm also computes a consecutive clique arrangement of the input graph if it is an interval graph.

Using Lemma 3.1, we can easily identify the minimal cut-strategy of an interval graph G. And it is easy to see that any minimal cut-strategy of an interval graph G consists of only one vertex. When there exist at least three maximal cliques in G, if we assume that vertex v is a minimal cut-strategy of G with $\omega(G/v) = 1$, i.e. G/v is a clique, then by Theorem 2.2, we know that v contributes zero to (1). And under this condition, we can easily find a minimal cut-strategy u with $\omega(G/u) \geq 2$ and it is easily checked that $\{\sum_{i=1}^{k} \max\{S(G[C_i]), 1\} - |u|\} > \{\max\{S(G/v), 1\} - |v|\} = 0$. So, when there are at least three maximal cliques in G, we only consider the minimal cut-strategy v with $\omega(G/v) \geq 2$.

Theorem 3.5. *Let G be an interval graph and let A_i, $1 \leq i \leq 2$, be a consecutive clique arrangement of G. Then, $S(G) = 0$.*

Proof. Under the condition of Theorem 3.5, the minimal cut-strategy, say v, of G with $\omega(G/v) = 1$ consists of vertex $v \in X = \{v : v \in A_1 - S_1 \text{ and } N(v) \cap (A_2 - S_1) = \emptyset, \text{ or } v \in A_2 - S_1 \text{ and } N(v) \cap (A_1 - S_1) = \emptyset\}$, where $S_1 = A_1 \cap A_2$. Therefore, by Theorem 2.2 we know that

$$S(G) = \max_{X^*} \left\{ \sum_{i=1}^{k} \max\{S(G[C_i]), 1\} - |X^*| \right\} = \max_{v}\{\max\{S(G/v), 1\} - |v|\} = 0. \quad \blacksquare$$

Theorem 3.6. *Let G be an interval graph and let A_1, A_2, A_3 be a consecutive clique arrangement of G. Then*

$$S(G) = \begin{cases} 1, & \text{if there exists vertex } v \in A_i - (S_1 \cup S_2) - (A_j \cup A_k), \text{ and } v \text{ is adjacent to all} \\ & \text{vertices in } S_1 \cup S_2, \{i, j, k\} = \{1, 2, 3\} \\ 0, & \text{otherwise} \end{cases}$$

where $S_i = A_i \cap A_{i+1}$, $i = 1, 2$.

Proof. If there exists a vertex $v \in A_i - (S_1 \cup S_2) - (A_j \cup A_k)$, $\{i, j, k\} = \{1, 2, 3\}$, such that it is adjacent to all vertices in $S_1 \cup S_2$, then it is obvious that v is a minimal cut-strategy of G with $\omega(G/v) = 2$ and the components of G/v are all cliques, thus $\{\max\{S(G/v), 1\} - |v|\} = 1$. Otherwise, if there is no vertex $v \in A_i - (S_1 \cup S_2) - (A_j \cup A_k)$ such that it is adjacent to all vertices in $S_1 \cup S_2$, then any vertex $v \in (A_1 \cup A_2 \cup A_3) - (A_1 \cap A_2 \cap A_3)$ is a minimal cut-strategy of G with $\omega(G/v) = 1$, i.e. G/v is a clique, and then $\{\max\{S(G/v), 1\} - |v|\} = 0$. Hence the proof is completed. ■

Lemma 3.7. *Let G be an interval graph and let $A_1, A_2, \ldots, A_t$, $t \leq n$, be a consecutive clique arrangement of G. Let $S_p = A_p \cap A_{p+1}$ for $p \in \{1, 2, \ldots, t-1\}$. If $t \geq 4$, then the minimal cut-strategy, say X, of G with $\omega(G/X) \geq 2$ consists of vertex $v \in \{v : 2 \leq p \leq t-1, v \in A_p - (S_{p-1} \cup S_p \cup (\cup_{i \neq p} A_i))$,* or $v \in S_p - S$, where $2 \leq p \leq t-2$, $S = S_1 \cup S_{t-1} \cup (A_1 \cap A_2 \cap A_3) \cup (A_{t-2} \cap A_{t-1} \cap A_t)$, or $v \in A_1 - (S_1 \cup S_2 \cup (\cup_{i=2}^{t} A_i))$ *and it is adjacent to all vertices in $S_1 \cup S_2$,* or $v \in A_t - (S_{t-2} \cup S_{t-1} \cup (\cup_{i=1}^{t-1} A_i))$ *and it is adjacent to all vertices in $S_{t-2} \cup S_{t-1}\}$ if there exists no S_i and S_j, $i \neq j$, such that $S_i \subseteq S_j$. Otherwise, if $t \geq 4$ and there exist S_i and S_j, $i \neq j$, such that $S_i \subseteq S_j$, then $v \in \{v : 1 \leq p \leq t, v \in A_p - S_j$, and it is adjacent to all vertices in $S_j\}$.*

Proof. By Lemmas 2.1 and 3.4, it is easily checked that this Lemma holds. ■

From above we know that an interval graph $G = (V, E)$ on n vertices has at most n minimal cut-strategies.

Definition 3.2 (*[12]*). Let G be an interval graph with consecutive clique arrangement $A_1, A_2, \ldots, A_t$. We define $A_0 = A_{t+1} = \emptyset$. For all l, r with $1 \leq l \leq r \leq t$ we define $\mathcal{P}(l, r) = (\cup_{i=l}^{r} A_i) - (A_{l-1} \cup A_{r+1})$. A set $\mathcal{P}(l, r)$, $1 \leq l \leq r \leq t$, is said to be a piece of G if $\mathcal{P}(l, r) \neq \emptyset$ and $G[\mathcal{P}(l, r)]$ is connected. Furthermore, $V = \mathcal{P}(1, t)$ is a piece of G (even if G is disconnected).

Remark. It is obvious that cliques in $\mathcal{P}(l, r)$ are listed in the same order as that they are listed in graph G.

Lemma 3.8. *Let X be a minimal cut-strategy of connected subgraph $G[\mathcal{P}(l, r)]$, $1 \le l \le r \le t$ with $\omega(G[\mathcal{P}(l, r)]/X) \ge 1$, especially, $\omega(G[\mathcal{P}(l, r)]/X) \ge 2$, $1 \le l \le r \le t$, when $G[\mathcal{P}(l, r)]$, has at least four cliques. Then there exists a minimal cut-strategy X' of G, such that $X = X' \cap \mathcal{P}(l, r) = X' - (A_{l-1} \cup A_{r+1})$. Moreover, every connected component of $G[\mathcal{P}(l, r)/X']$ is a piece of G.*

Proof. By Lemma 3.1, we know that piece $\mathcal{P}(l, r)$ is an interval graph. And it is obvious that the linear arrangement $A_l - (A_{l-1} \cup A_{r+1})$, $A_{l+1} - (A_{l-1} \cup A_{r+1}), \ldots, A_r - (A_{l-1} \cup A_{r+1})$ has all properties of a consecutive clique arrangement for $\mathcal{P}(l, r)$, except that cliques may occur more than once. We use the notation S_p defined in Lemma 3.7. We distinguish three cases:

Case 1. $\mathcal{P}(l, r)$ has two maximal cliques, say A_1, A_2.

Then applying Theorem 3.5 to $\mathcal{P}(l, r)$ implies that all minimal cut-strategies of $\mathcal{P}(l, r)$ with $\omega(G[\mathcal{P}(l, r)]/X) = 1$ are sets of the form:

When $l \ne 1$ and $r \ne t$, $X' - (A_{l-1} \cup A_{r+1}) = \{v : v \in A_1 - S_1 - (A_{l-1} \cup A_{r+1})$ and $N(v) \cap (A_2 - S_1 - (A_{l-1} \cup A_{r+1})) = \emptyset$, or $v \in A_2 - S_1 - (A_{l-1} \cup A_{r+1})$ and $N(v) \cap (A_1 - S_1 - (A_{l-1} \cup A_{r+1})) = \emptyset\}$. Especially, when $l = 1$, then $X' - (A_{l-1} \cup A_{r+1}) = \{v : v \in A_2 - S_1 - (A_{l-1} \cup A_{r+1})$ and $N(v) \cap (A_1 - S_1 - (A_{l-1} \cup A_{r+1})) = \emptyset\}$. When $r = t$, then $X' - (A_{l-1} \cup A_{r+1}) = \{v : v \in A_1 - S_1 - (A_{l-1} \cup A_{r+1})$ and $N(v) \cap (A_2 - S_1 - (A_{l-1} \cup A_{r+1})) = \emptyset\}$.

Case 2. $\mathcal{P}(l, r)$ has three maximal cliques, say A_1, A_2, A_3. By applying Theorem 3.6 to $\mathcal{P}(l, r)$ we get all minimal cut-strategies of $\mathcal{P}(l, r)$ with $\omega(G[\mathcal{P}(l, r)]/X) \ge 1$.

Subcase 2.1. There exists a vertex $v \in A_i - (S_{l+1} \cup S_{l+2}) - (A_j \cup A_k) - (A_{l-1} \cup A_{r+1})$, and v is adjacent to all vertices in $S_{l+1} \cup S_{l+2}$, $\{i, j, k\} = \{1, 2, 3\}$, then the minimal cut-strategies of $\mathcal{P}(l, r)$ with $\omega(G[\mathcal{P}(l, r)]/X) = 2$ are sets of the form $X' - (A_{l-1} \cup A_{r+1}) = \{v : v \in A_i - (S_1 \cup S_2) - (A_j \cup A_k) - (A_{l-1} \cup A_{r+1})$, and v is adjacent to all vertices in $S_1 \cup S_2 - (A_{l-1} \cup A_{r+1})$, $\{i, j, k\} = \{1, 2, 3\}\}$.

Subcase 2.2. There exists no vertex $v \in A_i - (S_{l+1} \cup S_{l+2}) - (A_j \cup A_k) - (A_{l-1} \cup A_{r+1})$, and v is adjacent to all vertices in $S_1 \cup S_2 - (A_{l-1} \cup A_{r+1})$, $\{i, j, k\} = \{1, 2, 3\}$, then the minimal cut-strategies of $\mathcal{P}(l, r)$ with $\omega(G[\mathcal{P}(l, r)]/X) = 1$ are sets of the form $X' - (A_{l-1} \cup A_{r+1}) = \{v : v \in (A_1 \cup A_2 \cup A_3) - (A_1 \cap A_2 \cap A_3) - (A_{l-1} \cup A_{r+1})\}$

Case 3. $\mathcal{P}(l, r)$ has at least four maximal cliques,say $A_1, A_2, A_3, A_4, \ldots, A_k$, $k \ge 4$.

Application of Theorem 3.6 to $\mathcal{P}(l, r)$ implies that all minimal cut-strategies of $\mathcal{P}(l, r)$ with $\omega(G[\mathcal{P}(l, r)]/X) \ge 2$ are sets of the form $X' - (A_{l-1} \cup A_{r+1}) = \{v : 2 \le p \le k - 1, v \in A_p - (S_{p-1} \cup S_p) - (A_{l-1} \cup A_{r+1})$, or $v \in S_p - X - (A_{l-1} \cup A_{r+1})$ where $2 \le p \le k - 1$, $X = S_1 \cup S_{k-1} \cup (A_1 \cap A_2 \cap A_3) \cup (A_{k-2} \cap A_{k-1} \cap A_k)$, or $v \in A_1 - (S_1 \cup S_2) - (A_{l-1} \cup A_{r+1})$ and it is adjacent to all vertices in $S_1 \cup S_2$, or $v \in A_k - (S_{k-2} \cup S_{k-1}) - (A_{l-1} \cup A_{r+1})$ and it is adjacent to all vertices in $S_{k-2} \cup S_{k-1}\}$, if there exists no S_i and S_j, $1 \le i \ne j \le k - 1$, such that $S_i \subseteq S_j$. Otherwise, if there exist S_i and S_j, $1 \le i \ne j \le k - 1$, such that $S_i \subseteq S_j$, then $X' = \{v : 1 \le p \le k, v \in A_p - S_j - (A_{l-1} \cup A_{r+1})$, and it is adjacent to all vertices in $S_j\}$.

For every $v \in V$ we define $l(v) = \min\{k : v \in A_k\}$ and $r(v) = \max\{k : v \in A_k\}$. Then for all l, r with $1 \le l \le r \le t$ and for every component C of $\mathcal{P}(l, r)$ holds $C = \mathcal{P}(l(C), r(C))$ with $l(C) = \min\{l(v) : v \in C\}$ and $r(C) = \max\{r(v) : v \in C\}$, i.e. C is a piece.

Now let $X = X' \cap \mathcal{P}(l, r)$ be a minimal cut-strategy of $\mathcal{P}(l, r)$, $1 \le l \le r \le t$. Then it is easy to see that graph $G[\mathcal{P}(l, r)/X] = G[\mathcal{P}(l, r)/X']$ is either the disjoint union of $G[\mathcal{P}(l, p - 1)]$ and $G[\mathcal{P}(p + 1, r)]$, or is the disjoint union of $G[\mathcal{P}(l + 1, l + 1)]$ and $G[\mathcal{P}(l + 2, r)]$ or is the disjoint union of $G[\mathcal{P}(l, p)]$ and $G[\mathcal{P}(p + 2, r)]$, or is the disjoint union of $G[\mathcal{P}(l, p - 1)]$, $G[\mathcal{P}(l + 1, p)]$ and $G[\mathcal{P}(p + 1, r)]$, or is equal to one of them (in case that $\mathcal{P}(l, p) = \emptyset$ or $\mathcal{P}(p + 1, r) = \emptyset$) or is $\emptyset$. Hence the set of components of $G[\mathcal{P}(l, r)/X']$ is equal to the union of the set of components of $G[\mathcal{P}(l, p)]$ and of the set of components of $G[\mathcal{P}(p + 1, r)]$ or is equal to one of these sets. Therefore, all components of $G[\mathcal{P}(l, r)/X']$ are pieces. ■

From the definition of piece of G, we know that there have essentially two different types of pieces in an interval graph. A piece is called *complete* if it induces a complete graph and it is called a *non-complete* otherwise. It is obvious that pieces $\mathcal{P}(l, l)$ are complete or $\emptyset$. Furthermore, a piece $\mathcal{P}(l, r)$, $l < r$, may also be complete. And for every complete piece induced graph $G[\mathcal{P}(l, r)]$, $l < r$, holds

$$S(G[\mathcal{P}(l, r)]) = 1 \quad (2)$$

Furthermore, when piece $\mathcal{P}(l,r)$, $l < r$, has two or three maximal cliques, we know that $S(G[\mathcal{P}(l,r)]) = 0$ or one by Theorems 3.5 and 3.6.

If there are at least four maximal cliques in it, the induced subgraph $G[\mathcal{P}(l,r)]$, $1 \le l \le r \le t$, has minimal cut-strategy X with $\omega(G[\mathcal{P}(l,r)]/X) \ge 2$. So, for every non-complete piece $G[\mathcal{P}(l,r)]$, $1 \le l \le r \le t$, having at least four maximal cliques, holds

$$S(G[\mathcal{P}(l,r)]) = \max\left\{\sum_{i=1}^{k} \max\{S(G[P_i]), 1\} - |X' \cap \mathcal{P}(l,r)|\right\} \tag{3}$$

where the maximum is taken over all minimal cut-strategies $X' \cap \mathcal{P}(l,r)$, with $\omega(G[\mathcal{P}(l,r)]/X) \ge 2$, of graph $G[\mathcal{P}(l,r)]$ and X' is a minimal cut-strategy of G, $P_1, P_2, \ldots, P_k$ are the connected components of $G[\mathcal{P}(l,r)/X]$.

Let G be an interval graph. If G is complete, then $S(G) = 1$. Otherwise the 'dynamic programming on pieces' works as follows:

Step 1. Compute a consecutive clique arrangement $A_1, A_2, \ldots, A_t$ of G, then compute $l(v) = \min\{k : v \in A_k\}$ and $r(v) = \max\{k : v \in A_k\}$ for every $v \in V$, and then compute all minimal cut-strategies.

(a) When $t = 2$, $v \in X = \{v : v \in A_1 - S_1$ and $N(v) \cap (A_2 - S_1) = \emptyset$, or $v \in A_2 - S_1$ and $N(v) \cap (A_1 - S_1) = \emptyset\}$.

(b) When $t = 3$, $v \in X = \{v : v \in A_i - (S_1 \cup S_2) - (A_j \cup A_k), i \ne j \ne k \in \{1,2,3\}$, and it is adjacent to all vertices in $S_1 \cup S_2\}$, or $v \in X = \{v : v \in (A_1 \cup A_2 \cup A_3) - (A_1 \cap A_2 \cap A_3)\}$.

(c) When $t \ge 4$, $v \in X = \{v : 2 \le p \le t-1, v \in A_p - (S_{p-1} \cup S_p)$, or $v \in S_p - X$ where $2 \le p \le t-2$, $X = S_1 \cup S_{t-1} \cup (A_1 \cap A_2 \cap A_3) \cup (A_{t-2} \cap A_{t-1} \cap A_t)$, or $v \in A_1 - (S_1 \cup S_2)$ and it is adjacent to all vertices in $S_1 \cup S_2$, or $v \in A_t - (S_{t-2} \cup S_{t-1})$ and it is adjacent to all vertices in $S_{t-2} \cup S_{t-1}\}$, if there exists no S_i and S_j, $i \ne j$, such that $S_i \subseteq S_j$. Otherwise, if there exist S_i and S_j, $i \ne j$, such that $S_i \subseteq S_j$, then $v \in X = \{v : 1 \le p \le t, v \in A_p - S_j$, and it is adjacent to all vertices in $S_j\}$.

Step 2. For all l, r with $1 \le l \le r \le t$ compute the vertex set $\mathcal{P}(l,r)$, mark (l,r) 'empty' if $\mathcal{P}(l,r) = \emptyset$ and mark (l,r) 'complete' if $\mathcal{P}(l,r) \ne \emptyset$ and $G[\mathcal{P}(l,r)]$ is a complete induced graph.

Step 3. For all non-marked tuples (l,r) check whether $G[\mathcal{P}(l,r)]$ is connected. If so, mark (l,r) 'non-complete'. Else, mark (l,r) 'disconnected', and then compute the components $P_j = \mathcal{P}(l_j, r_j)$, $1 \le j \le k$, of $G[\mathcal{P}(l,r)]$ and store $(l_1, r_1), (l_2, r_2), \ldots, (l_k, r_k)$ in a linked list with a pointer from (l,r) to the head of this list.

Step 4. For all marked 'non-complete' tuples (l,r), $1 \le l \le r \le t$, compute the components $P_j = \mathcal{P}(l_j, r_j)$, $1 \le j \le k$, of $G[\mathcal{P}(l,r)/v]$, where v is a cut-strategy of $G[\mathcal{P}(l,r)]$, and then check whether $\{v\} \cap \mathcal{P}(l,r)$, is a minimal cut-strategy of $G[\mathcal{P}(l,r)]$, and if so, mark (v,l,r) 'minimal', store $(l_1, r_1), (l_2, r_2), \ldots, (l_k, r_k)$ in a linked list with a pointer from (v,l,r) to the head of this list and it is obvious that $|\{v\} \cap \mathcal{P}(l,r)| = 1$.

Step 5. For every pair (l,r) marked 'complete' compute $S(G[\mathcal{P}(l,r)])$ according to (2).

Step 6. For $d := 1$ to t for $l := 1$ to $t - d$, if $(l, l+d)$ is marked 'non-complete', compute $S(G[\mathcal{P}(l, l+d)])$ according to Lemma 3.4 if $G[\mathcal{P}(l, l+d)]$ has two maximal cliques, according to Theorem 3.6 if $G[\mathcal{P}(l, l+d)]$ has three maximal cliques, and according to (3) when $G[\mathcal{P}(l, l+d)]$ has at least four maximal cliques.

Step 7. Output $S(G) = S(G[\mathcal{P}(1,t)])$.

Theorem 3.9. *The above algorithm can compute the neighbour-scattering number for interval graphs with time complexity* $O(n^4)$.

Proof. The correctness of this algorithm follows from Theorem 2.2 and Theorem 3.6. It is easy to see that steps 1, 2, 5, 7 can be done in time $O(n^4)$ in a straightforward manner. In step 3, testing connectedness and computing the components can be done by an $O(n+m)$ algorithm for at most n^2 graphs $G[\mathcal{P}(l,r)]$. If $G[\mathcal{P}(l,r)]$ is disconnected and P_j is a component, then $P_j = \mathcal{P}(l_j, r_j)$, $1 \le j \le k$, with $l_j = \min\{l(v) : v \in P_j\}$ and $r_j = \max\{r(v) : v \in P_j\}$ which can be computed in time $O(n)$. Hence, step 3 can be done in time $O(n^4)$.

Step 4 has to be executed for at most n^3 triples (v,l,r) with $v \in V(G[\mathcal{P}(l,r)])$. If $\mathcal{P}(l,r)/v \ne \emptyset$, then the components of $G[\mathcal{P}(l,r)/v]$ are computed as indicated in the proof of Lemma 3.7 by using the marks of $(l, p-1)$ and $(p+1, r)$, or $(l+1, l+1)$ and $(l+2, r)$, etc. namely, if the mark is 'complete' or 'non-complete', then $(l, p-1)$ and $(p+1, r)$, or $(l+1, l+1)$ and $(l+2, r)$, etc. respectively, are stored and if the mark is 'disconnected', then the corresponding linked list is added. Thus the linked list of (v,l,r) can be computed in time $O(n)$. As we know that $\{v\} \cap \mathcal{P}(l,r)$ is a minimal cut-strategy of $G[\mathcal{P}(l,r)]$ if and only if (a) or (b) in Lemma 2.1 holds. Because of the properties of a consecutive clique arrangement it suffices to check that two components P_j of $G[\mathcal{P}(l,p)]$ with the two largest values of r_j and the two components of P_j of $G[\mathcal{P}(p+1,r)]$ with the two smaller values of l_j (if they exist). This can be done in time $O(n)$. Hence step 4 needs time $O(n^4)$.

Step 6 requires the evaluation of the right-hand side of (3) for at most n^2 pairs $(l, l+d)$. For every $v \in V(G[\mathcal{P}(l, l+d)])$ and $(v, l, l+d)$ marked 'minimal' the components P_j of $G[\mathcal{P}(l, l+d)/v]$ can be obtained in time $O(n)$ from the linked list of $(v, l, l+d)$. Each of the at most n values $S(G[P_i])$ can be determined in constant time by table look-up since the neighbour-scattering numbers of smaller pieces are already known. Thus $\sum_{i=1}^{k} \max\{S(G[G[P_i]]), 1\} - |\{v\} \cap \mathcal{P}(l, l+d)|$ can be evaluated in time $O(n)$. Consequently, step 6 of the algorithm can be done in time $O(n^4)$. ■

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan, London; Elsevier, New York, 1976.

[2] H.A. Jung, On maximal circuits in finite graphs, Ann. Discrete Math. 3 (1978) 129–144.

[3] G. Gunther, B.L. Hartnell, On minimizing the effects of betrayals in a resistance movement, in: Proc. Eighth Manitoba Conference on Numerical Mathematics and Computing, 1978, pp. 285–306.

[4] G. Gunther, B.L. Hartnell, Optimal K-secure graphs, Discrete Appl. Math. 2 (1980) 225–231.

[5] G. Gunther, On the existence of neighbour-connected graphs, Congr. Numer. 54 (1986) 105–110.

[6] Z. Wei, (supervisor X. Li), On the reliability parameters of networks, M.S.Thesis, Northwestern Polytechnical University, 2003, pp. 30–40.

[7] F. Li, X. Li, Computational complexity and bounds for neighbour-scattering number of graphs, in: Proc. ISPAN 2005, IEEE Computer Society, Nevade, Las Vegas, USA, 2005.

[8] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs, Academic Press, 1980.

[9] P.C. Gilmore, A.J. Hoffman, A characterization of comparability graphs and of interval graphs, Canad. J. Math. 16 (99) (1964) 539–548.

[10] K.S. Booth, G.S. Lueker, Testing for the consecutive ones property, interval graphs, and graph planarity using PQ-tree algorithms, J. Comput. System Sci. 13 (3) (1976) 335–379.

[11] D.R. Fulkerson, O.A. Gross, Incidence matrices and interval graphs, Pacific J. Math. 15 (1965) 835–855.

[12] D. Kratsch, T. Klocks, H. Müller, Computing the toughness and the scattering number for interval and other graphs, IRISA Research Report, France, 1994.