Conflict-free connection of trees*

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Abstract

An edge-colored graph G is *conflict-free connected* if, between each pair of distinct vertices, there exists a path containing a color used on exactly one of its edges. The *conflict-free connection number* of a connected graph G, denoted by $cfc(G)$, is defined as the smallest number of colors that are required in order to make G conflict-free connected. A coloring of vertices of a hypergraph $H = (\mathcal{V}, \mathcal{E})$ is called *conflict-free* if each hyperedge e of H has a vertex of unique color that does not get repeated in e. The smallest number of colors required for such a coloring is called the *conflict-free chromatic number* of H, and is denoted by $\chi_{cf}(H)$. In this paper, we study the conflict-free connection coloring of trees, which is also the conflict-free coloring of edge-path hypergraphs of trees. We first prove that for a tree T of order n, $cfc(T) \geq cfc(P_n) = \lceil \log_2 n \rceil$, and this completely confirms the conjecture of Li and Wu. We then present a sharp upper bound for the conflict-free connection number of trees by a simple algorithm. Furthermore, we show that the conflict-free connection number of the binomial tree with 2^{k-1} vertices is $k-1$. At last, we construct some tree classes which are k-cfc-critical for every positive integer k.

Keywords: tree; conflict-free connection coloring; conflict-free coloring; k-cfc-critical

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1 Introduction

All graphs in this paper are undirected, simple and nontirivial. We follow [3] for graph theoretical notation and terminology not described here. Let G be a graph. We use $V(G), E(G), n(G), m(G)$, and $\Delta(G)$ to denote the vertex set, edge set, number of vertices (order of G), number of edges (size of G), and maximum degree of G, respectively. Let $N(v)$ denote the neighborhood of v in G. Given two graphs G_1 and G_2, the *union* of G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. A hypergraph H is a pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the vertex set of H and $\mathcal{E}$ is the hyperedge set which is a family of nonempty subsets of $\mathcal{V}$. The concept of hypergraphs is a generalization of graphs, because a graph is a hypergraph in which each hyperedge is a pair of vertices.

A vertex-coloring of a hypergraph $H = (\mathcal{V}, \mathcal{E})$ is called *conflict-free* if each hyperedge e of H has a vertex of unique color that does not get repeated in e. The smallest number of colors required for such a coloring is called the *conflict-free chromatic number* of H, and is denoted by $\chi_{cf}(H)$. This parameter was first introduced by Even, Lotker, Ron and Smorodinsky [11] (FOCS 2002), with an emphasis on hypergraphs induced by geometric shapes. The main application of a conflict-free coloring is that it models a frequency assignment for cellular networks. A cellular network consists of two kinds of nodes: base stations and mobile agents. Base stations have fixed positions and provide the backbone of the network; they are represented by vertices in $\mathcal{V}$. Mobile agents are the clients of the network and are served by base stations. This is done as follows: every base station has a fixed frequency; this is represented by the coloring c; i.e., colors represent frequencies. If an agent wants to establish a link with a base station, it has to tune itself to this base station's frequency. Since agents are mobile, they can be in the range of many different base stations. To avoid interference, the system must assign frequencies to base stations in the following way: for any range, there must be a base station in the range with a frequency that is not used by some other base station in the range. One can solve the problem by assigning n different frequencies to the n base stations. However, using many frequencies is expensive, and therefore a scheme that reuses frequencies, where possible, is preferable. Conflict-free coloring problems have been the subject of many recent papers due to their practical and theoretical interest. One can find many results on conflict-free coloring, see [2, 5, 12, 14, 20, 21].

Other hypergraphs that have been studied with respect to the conflict-free coloring are ones which are induced by a graph $G = (V, E)$ and its neighborhoods or its paths:

(i) The vertex-neighborhood hypergraph $H_N(G) = (\mathcal{V}, \mathcal{E})$ is a hypergraph with

$\mathcal{V}(H_N) = V(G)$ and $\mathcal{E}(H_N) = \{N_G(x)|x \in V(G)\}$, which has been studied in [4, 19].

(ii) The vertex-path hypergraph $H_{VP}(G) = (\mathcal{V}, \mathcal{E})$ is a hypergraph with $\mathcal{V}(H_{VP}) = V(G)$ and $\mathcal{E}(H_{VP}) = \{V(P)|\ P$ is a path of $G\ \}$. A conflict-free coloring of $H_{VP}(G)$ is called a *conflict-free coloring of G with respect to paths*; we also define the corresponding graph chromatic number, $\chi_{cf}^{P}(G) = \chi_{cf}(H_{VP}(G))$. In [9], the authors proved that it is coNP-complete to decide whether a given vertex-coloring of a graph is conflict-free with respect to paths. And in [8], the authors studied the conflict-free coloring of tree graphs with respect to paths, and they showed that $\chi_{cf}^{P}(P_n) = \lceil \log_2(n+1) \rceil$ for a path P_n on n vertices, and $\chi_{cf}^{P}(B^{*}_{2(r+1)+3r}) \leq 4r+2$ for the complete binary tree with $5r+2$ levels.

(iii) The edge-path hypergraph $H_{EP}(G) = (\mathcal{V}, \mathcal{E})$ is a hypergraph with $\mathcal{V}(H_{EP}) = E(G)$ and $\mathcal{E}(H_{EP}) = \{E(P)|\ P$ is a path of $G\ \}$, which is studied in this paper.

Inspired by rainbow connection colorings [16, 17] and proper connection colorings [15] of graphs and conflict-free colorings of hypergraphs, Czap et al. [7] introduced the concept of conflict-free connection colorings of graphs. An edge-colored graph G is called *conflict-free connected* if each pair of distinct vertices is connected by a path which contains at least one color used on exactly one of its edges. This path is called a *conflict-free path*, and this coloring is called a *conflict-free connection coloring* of G. The *conflict-free connection number* of a connected graph G, denoted by $cfc(G)$, is the smallest number of colors required to color the edges of G so that G is conflict-free connected. It is easy to see that for a tree T, each of a proper connection coloring and a conflict-free connection coloring of T is a proper edge-coloring, and $cfc(T) \geq \chi'(T) = \Delta(T) = pc(G)$. Note that a conflict-free coloring of the edge-path hypergraph H of a graph G is a conflict-free connection coloring of G. The other way round is not true in general, since some pairs of vertices in a general graph G may have more than one path between them. However, for any tree T, the conflict-free coloring of the edge-path hypergraph of T is equivalent to the conflict-free connection coloring of T, i.e., $\chi_{cf}(H_{EP}(T)) = cfc(T)$, since every pair of vertices in T has a unique path between them. In this paper, we study the conflict-free connection coloring of trees, which is also the conflict-free coloring of edge-path hypergraphs of trees. At first, we present some results on the conflict-free connection for general graphs.

Lemma 1.1 [6] *Let G be a connected graph of order n. Then $1 \leq cfc(G) \leq n-1$. Moreover, $cfc(G) = 1$ if and only if $G = K_n$, and $cfc(G) = n-1$ if and only if $G = K_{1,n-1}$.*

Theorem 1.2 [7] *If G is a noncomplete 2-connected graph, then $cfc(G) = 2$.*

In [6], the authors weaken the condition of the above theorem and got the following result.

Theorem 1.3 [6, 10] *Let G be a noncomplete 2-edge-connected graph. Then $cfc(G) = 2$.*

Let $C(G)$ be the subgraph of G induced by the set of cut-edges of G. It is easy to see that every component of $C(G)$ is a tree and hence $C(G)$ is a forest. Let $h(G) = \max\{cfc(T) : \text{T is a component of } C(G)\}$. For a graph G with cut-edges, the authors of [7] gave lower and upper bounds of $cfc(G)$ in terms of $h(G)$.

Theorem 1.4 [7] *If G is a connected graph with cut-edges, then $h(G) \leq cfc(G) \leq h(G) + 1$. Moreover, the bounds are sharp.*

Recently, the authors in [6] gave a sufficient condition such that the lower bound is sharp for $h(G) \geq 2$.

Theorem 1.5 [6] *Let G be a connected graph with $h(G) \geq 2$. If there exists a unique component T of $C(G)$ such that $cfc(G) = h(G)$, then $cfc(G) = h(G)$.*

So, the problem of determining the value of $cfc(G)$ for graphs G without bridges or cut-edges is completely solved. The rest graphs all have cut-edges. The extremal such graphs are trees for which every edge is a cut-edge. And by the above theorem, to determine the conflict-free connection number of general graphs relies on determining the conflict-free connection number of trees, with an error of only one. Next, we present some known results on the conflict-free connection number of trees.

Lemma 1.6 [7] *If P_n is a path on n vertices, then $cfc(P_n) = \lceil \log_2 n \rceil$.*

Lemma 1.7 [7] *If T is a tree on n vertices with maximum degree $\Delta(T) \geq 3$ and diameter $diam(T)$, then*

$$\max\{\Delta(T), \log_2 diam(T)\} \leq cfc(T) \leq \frac{(\Delta(T) - 2)\log_2 n}{\log_2 \Delta(T) - 1}.$$

The following result indicates that when the maximum degree of a tree is large, the conflict-free connection number is immediately determined by its maximum degree.

Theorem 1.8 [6] *Let T be a tree of order n, and let t be a natural number such that $t \geq 1$ and $n \geq 2t + 2$. Then $cfc(T) = n - t$ if and only if $\Delta(T) = n - t$.*

It is easy to obtain the following result for trees with diameter 3.

Lemma 1.9 *Let $S_{a,b}$ be a tree with diameter 3 such that the two non-leaf vertices have degrees a and b, Then $cfc(S_{a,b}) = \Delta(S_{a,b})$.*

Proof. By Lemma 1.7, we have $cfc(S_{a,b}) \geq \Delta(S_{a,b}) = \max\{a, b\}$. It remains to prove the matching lower bound. Without loss of generality, we assume that the non-leaf vertex u has maximum degree a and the other non-leaf vertex v has degree b. We provide an edge-coloring of $S_{a,b}$ with a colors: assign a distinct colors to the edges incident with u, then for the remaining edges, assign $b-1$ distinct used colors which do not contain the color on the edge uv. It is obvious that this is a conflict-free connection coloring of $S_{a,b}$. Thus, $cfc(S_{a,b}) \leq \Delta(S_{a,b}) = \max\{a, b\}$. □

Recently, Li and Wu proposed the following conjecture in [18].

Conjecture 1.10 [18] *For a tree T of order n, $cfc(T) \geq cfc(P_n) = \lceil \log_2 n \rceil$.*

Definition 1.11 *A tree T is called k-cfc-*critical *if $cfc(T) = k$, and for every proper subtree T' of T, $cfc(T') < k$, which means that for every edge e of T, any one of the (two) nontrivial components in $T - e$ has a conflict-free connection number less than k.*

Let us give an overview of the results of this paper. In Section 2, we present a sharp lower bound for the conflict-free connection number of trees, this completely confirms Conjecture 1.10. In Section 3, we give a sharp upper bound for the conflict-free connection number of trees by a simple algorithm we develop. Furthermore, we show that the conflict-free connection number of the binomial tree with 2^{k-1} vertices is $k-1$. In Section 4, we construct some classes of trees which are k-cfc-critical for every positive integer k.

2 The lower bound

In order to obtain a lower bound for the conflict-free connection number of trees, we define a new kind of edge-colorings of graphs, which is regarded as the generalization of the conflict-free connection colorings of graphs.

Definition 2.1 *Let G be a nontrivial connected graph with an edge-coloring. A path in G is called an* odd path *if there is a color that occurs an odd number of times* *on the edges of the path. An edge-colored graph G is called* odd connected *if any two distinct vertices of G are connected by an odd path, and this coloring is called an* odd connection coloring *of G. For a connected graph G, the minimum number of colors that are required in order to make G odd connected is called the* odd connection number *of G, denoted by $oc(G)$.*

It is easy to see that every conflict-free connection coloring of G is an odd connection coloring, which implies the following easy result.

Proposition 2.2 *Let G be a connected graph, then $oc(G) \leq cfc(G)$.*

The parity vector below is a very useful tool to study the odd connection coloring of graphs.

Definition 2.3 *Given an edge-coloring $c : E \rightarrow \{1, \cdots, k\}$ of G and a path P of G, the parity vector of P is an element of $\{0, 1\}^k$ in which the i-th coordinate equals the parity (0 for even, or 1 for odd) of the number of edges in P with color i.*

It is clear that an edge-colored graph G is odd connected if and only if any two distinct vertices of G are connected by a path whose parity vector is not the all-zero vector. Now we are ready to give the proof of our main result of this section.

Lemma 2.4 *Let T be a tree of order n. Then $oc(T) \geq \lceil \log_2 n \rceil$.*

Proof. Let c be an odd connection coloring of T with $oc(T)$ colors. Take $n-1$ distinct paths each starting from a fixed leaf vertex of T to the other $n-1$ vertices. First, we claim that any two of these $n-1$ paths have distinct parity vectors. Assume, to the contrary, two of them, say P_1 and P_2, have the same parity vector $\mathbf{a}$. Let E' be the symmetric difference of $E(P_1)$ and $E(P_2)$. Then the subgraph induced by E' is also a path P'. Notice that the parity vector $\mathbf{a}'$ of P' is $\mathbf{a} + \mathbf{a} = \mathbf{0}$, as the colors in $E(P_1) \cap E(P_2)$ are counted twice, which cannot change the value of $\mathbf{a}'$. So, on the path P' each color appears an even number of times, a contradiction. Thus, there are $n-1$ distinct parity vectors, none of which is the all-zero vector. On the other hand, the number of nonzero parity vectors is at most $2^{oc(T)} - 1$, which implies that $2^{oc(T)} - 1 \geq n - 1$, and hence $oc(T) \geq \lceil \log_2 n \rceil$. □

The following result is an immediate consequence of Lemmas 1.6 and 2.4 and Proposition 2.2, which completely confirms Conjecture 1.10.

Theorem 2.5 *Let T be a tree of order n. Then $cfc(T) \geq cfc(P_n) = \lceil \log_2 n \rceil$.*

3 An algorithm for the upper bound

At the very beginning of this section, we give the following concept.

Definition 3.1 *Let T be a tree. An edge e of T is called* balanced *if the difference of the sizes of two resulting subtrees of $T - e$ is minimum.*

We below present an algorithm for constructing a conflict-free connection coloring of a given tree. This algorithm starts from the single connected component T, and in each iteration, removes one balanced edge from each generated subtree of T to split it into two generated subtrees, until all the generated subtrees become singletons as illustrated in Fig. 1.

For convenience, the *depth* of a tree T, denoted by $d(T)$, is defined as the number of the iterations of the following algorithm; the *depth* of an edge e in T, denoted by $d(e)$, is defined as the sequence number of the iteration that deletes e. That is, the edges deleted in d-th iteration are the ones with depth d.

Algorithm 1 **for conflict-free connection coloring of a tree**

Input: A tree $T = (V, E)$.
Output: A conflict-free connection coloring c of T.
Step 1: Set $F = T$, $c : E \to \{0\}$.
Step 2: Determine whether there exists a component which has more than one vertex in F. If so, go to Step 3; otherwise, go to Step 4.
Step 3: Choose all the components which have more than one vertex, and then delete a balanced edge e from each of such components to get new components which form a forest F'. Replace F by F'. Update c: color edges e with $d(e)$. Go to Step 2.
Step 4: Return c.

Lemma 3.2 *Let T be a tree of order n. Then* $\max\{\Delta(T), \lceil \log_2 n \rceil\} \leq d(T) \leq n - 1$.

Proof. It is obvious that $d(T) \leq n - 1$. And since we remove only one balanced edge from a generated subtree of T in Algorithm 1, it follows that edges in the same depth are not adjacent, which means $d(T) \geq \Delta(T)$. Thus, we only need to prove $d(T) \geq \lceil \log_2 n \rceil$. Note that the number of edges with depth i is at most 2^{i-1}. Algorithm 1 is not terminated until all the generated subtrees become singletons. It follows that $m(T) \leq 2^0 + 2^1 + \cdots + 2^{d(T)-1} = 2^{d(T)} - 1$, which implies $d(T) \geq \lceil \log_2 n \rceil$. □

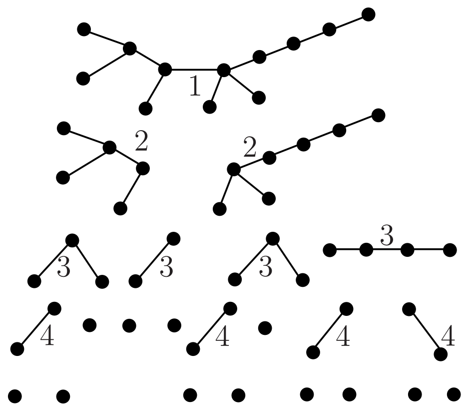

Figure 1: A tree of depth 4

Note that the choice of the balanced edge of each generated subtree is unique for $k_{1,n-1}$ and $S_{a,n-a}$ by symmetry in Algorithm 1, and it is easy to check that $d(k_{1,n-1}) = n-1$ and $d(S_{a,n-a}) = \Delta(S_{a,n-a})$. Next, we study the depths of other trees. Obviously, the path P_n has the same property as $k_{1,n-1}$ and $S_{a,n-a}$.

Lemma 3.3 *Let P_n be a path on n vertices. Then $d(P_n) = \lceil \log_2 n \rceil$.*

Proof. By Lemma 3.2, we have $d(P_n) \geq \lceil \log_2 n \rceil$. Thus, it remains to verify the matching lower bound. We use induction on n. The statement is evidently true for $n = 1$ and $n = 2$. Let P_n be a path on n vertices. Then we delete a central edge from P_n. The resulting paths P and P' have at most $\lceil \frac{n}{2} \rceil$ vertices. Therefore, by the induction hypothesis, $\max\{d(P), d(P')\} \leq \lceil \log_2 \frac{n}{2} \rceil$. Thus, $d(P_n) \leq 1 + \max\{d(P), d(P')\} \leq 1 + \lceil \log_2 \frac{n}{2} \rceil \leq \lceil \log_2 n \rceil$. $\square$

The (rooted) binomial tree B_k with 2^{k-1} vertices is defined as follows: B_1 is a single vertex; for $k > 1$, B_k consists of two disjoint copies of B_{k-1} and an edge between their two roots, where the root of B_k is the root of the first copy. These trees are used in [1, 13]. The binomial tree B_k is another tree class for which the choice of the balanced edge of each generated subtree is unique in Algorithm 1.

Lemma 3.4 *Let B_k be the binomial tree with 2^{k-1} vertices for $k \geq 2$. Then $d(B_k) = k-1$.*

Proof. By Lemma 3.2, we have $d(B_k) \geq \lceil \log_2 n \rceil = k-1$. For the converse, we use induction on k. Noticing that B_2 is a path of order 2, it follows that the result holds trivially for $k = 2$. Suppose that the result holds for B_{k-1} for $k \geq 3$. Let B_k be the binomial tree with 2^{k-1} vertices, it follows that B_k consists of two disjoint copies of B_{k-1} and an edge e_0 between their two roots. We delete the edge e_0 from B_k. Therefore, by the induction hypothesis, $d(B_{k-1}) \leq k-2$. Thus, $d(B_k) \leq 1 + d(B_{k-1}) \leq k-1$. $\square$

The correctness of Algorithm 1 is confirmed by the following theorem, and this gives a sharp upper bound for the conflict-free connection number of trees. Note that for general trees, the choice of the balanced edge of each generated subtree may not be unique in Algorithm 1, which implies that there exist many priorities of removing edges of T.

Theorem 3.5 *Algorithm* 1 *constructs a conflict-free connection coloring of a given tree T. Moreover, $cfc(T) \leq \min\{d(T)|\ d(T)$ is the depth of T by a certain priority of removal edges of T in Algorithm* 1$\}$*, where all possible priorities of removing edges of T are taken.*

Proof. It is sufficient to show that there exists a conflict-free path for each pair of distinct vertices u, v of T under the coloring given by Algorithm 1. Since adjacent edges are colored with distinct colors in Algorithm 1, we may assume that u and v are not adjacent. Let e_0 be the balanced edge of the first generated subtree that simultaneously contains u and v starting from u and v, respectively. Since the color of e_0 on the path between u and v is minimal and unique, it follows that the path between u and v is conflict-free. Thus, this is a conflict-free connection coloring of T, which implies that the result holds. $\square$

Remark: Algorithm 1 provides an optimal conflict-free connection coloring for the path, the star, the tree with diameter 3, and the binomial tree which is proved below. These imply that the upper bound in Theorem 3.5 is sharp.

Let $D(T) = \min\{d(T)|\ d(T)$ is the depth of T by a certain priority of removing edges of T in Algorithm 1$\}$, where all possible priorities of removing edges of T are taken. For general trees, we propose the following conjecture.

Conjecture 3.6 *For a given tree T, $D(T) \leq 2cfc(T)$.*

Next, we determine the conflict-free connection number of the binomial tree. Combining Theorems 2.5 and 3.5 and Lemma 3.4, we get the following conclusion.

Theorem 3.7 *Let B_k be the binomial tree with 2^{k-1} vertices, then $cfc(B_k) = k-1$.*

Recall that the complete binary tree B_k^* has k levels and $2^k - 1$ vertices. It is not difficult to prove by induction that $B_k^* \subseteq B_{2k-1}$, and so we get the following corollary.

Corollary 3.8 *Let B_k^* be the complete binary tree with k levels. Then $k \leq cfc(B_k^*) \leq 2k-2$.*

Proof. Since B_k^* has $2^k - 1$ vertices, it follows that $cfc(B_k^*) \geq \lceil \log_2(2^k - 1) \rceil = k$ by Theorem 2.5. We only need to prove the upper bound. Since $B_k^* \subseteq B_{2k-1}$, it follows that a conflict-free connection coloring of B_{2k-1} restricted on the edges of B_k^* is conflict-free connected. Thus, $cfc(B_k^*) \leq cfc(B_{2k-1}) = 2k-2$ by Theorem 3.7. □

4 k-cfc-critical

In this section, we study trees which are k-cfc-critical for every positive integer k. The following two results are immediate corollaries of Lemmas 1.1 and 1.6.

Proposition 4.1 *The star $K_{1,k}$ of order $k+1$ is k-cfc-critical.*

Proposition 4.2 *The path $P_{2^{k-1}+1}$ on $2^{k-1}+1$ vertices is k-cfc-critical.*

Next, we construct two other kinds of trees which are k-cfc-critical for every integer $k \geq 2$.

Theorem 4.3 *Let Q_k be the graph obtained from two copies of $K_{1,k-1}$ with $k \geq 2$ by identifying a leaf vertex in one copy with a leaf vertex in the other copy. Then Q_k is k-cfc-critical.*

Proof. Since Q_k is a path on $2^{k-1}+1$ vertices for $k = 2, 3$, it follows that the result holds by Proposition 4.2. Next, we may assume that $k \geq 4$. Note that Q_k is a tree of order $2k-1$ with diameter 4. Let w be the only vertex of degree 2, which is adjacent to u and v, and let $N(u) = \{w, u_1, \cdots, u_{k-2}\}$ and $N(v) = \{w, v_1, \cdots, v_{k-2}\}$.

We first show that $cfc(Q_k) = k$. Suppose that there exists a conflict-free connection coloring of Q_k with at most $k-1$ colors. It follows that all the edges incident with each of u and v are assigned distinct colors. Without loss of generality, we assume that the color of the edge uw is c_i and the color of the edge vw is c_j. Thus, there exists a path of length 4 having the color sequence c_i, c_j, c_i, c_j, a contradiction. Thus, $cfc(Q_k) \geq k$. And we define an edge-coloring of Q_k with k colors: assign k distinct colors to vw and all the edges incident with u, then for the remaining edges assign $k-2$ distinct used colors, none of which is the color on the edge vw. It is obvious that Q_k is conflict-free connected under the coloring. Thus, $cfc(Q_k) = k$.

Next, we prove that Q_k is k-cfc-critical. It suffices to show that for every edge e of Q_k, each nontrivial component in $Q_k - e$ has a conflict-free connection number less than k. Suppose that e is one of the edges uw and vw. Then the resulting subtrees of $Q_k - e$ are $K_{1,k-2}$ and $K_{1,k-1}$. It follows that the result holds by Lemma 1.1. Suppose that e is a pendent edge of Q_k. We may assume that $e = uu_i$ (for some i with $1 \le i \le k-2$) is incident with u by symmetry. Notice that the resulting subtrees in $Q_k - e$ are a singleton $\{u_i\}$ and $Q_k - u_i$. We provide an edge-coloring of $Q_k - u_i$ as follows: color edges incident with v using $k-1$ distinct colors, and color edges incident with u using $k-2$ distinct used colors, which do not contain the color on the edge vw. It can be checked that this is a conflict-free connection coloring of $Q_k - u_i$ with $k-1$ colors. Thus, $cfc(Q_k - u_i) \le k-1$, and so each nontrivial component in $Q_k - e$ has a conflict-free connection number less than k for every edge e of Q_k. □

Lemma 4.4 *Let R_k be the graph obtained from $K_{1,k-1}$ and $P_{2^{k-2}+1}$ with $k \ge 2$ by identifying a leaf vertex in $K_{1,k-1}$ with an end vertex in $P_{2^{k-2}+1}$. Then $cfc(R_k) = k$.*

Proof. Firstly, we prove that $cfc(R_k) \ge k$. Note that R_2 is a path on 3 vertices, and $cfc(R_2) = 2 > 1$. Then we focus on $k \ge 3$. Assume to the contrary that there is a R_k such that $cfc(R_k) \le k-1$, and let k_0 be the minimum k with such property. Suppose that T_1 is the copy of K_{1,k_0-1} in R_{k_0}, T_2 is the copy of $P_{2^{k_0-2}+1}$ in R_{k_0} that has one common leaf vertex with T_1. Let $V(T_1) = \{u, w, u_1, \cdots, u_{k_0-2}\}$ and $V(T_2) = \{w = v_0, v_1 \cdots, v_{2^{k_0-2}}\}$, where u is the only vertex of maximum degree $k_0 - 1$ in T_1, $v_i v_{i+1}$ is an edge of T_2 for $i = 0, \cdots, 2^{k_0-2} - 1$, and $V(T_1) \cap V(T_2) = w$. Let c be a conflict-free connection coloring of R_{k_0} with $k_0 - 1$ colors. Then we present the following claim.

Claim 1. There exist exactly two colors c_1, c_2 each of which is used on exactly one of the edges of T_2.

Proof of Claim 1: To make T_2 conflict-free connected, it needs $k_0 - 1$ colors by Lemma 1.6, and there exists one color c_1 used on exactly one of edges of T_2. Suppose that there exists a unique color c_1 used on exactly one of edges of T_2. Note that the edges of T_1 are colored with $k_0 - 1$ colors. We assume that uu_i (or uw) is colored with c_1 for some i with $1 \le i \le k_0 - 2$. It follows that there is no conflict-free path from $v_{2^{k_0-2}}$ to u_i (or u), which is a contradiction. Thus, there exist at least two colors c_1, c_2 each of which is used on exactly one of edges of T_2. Suppose that there exists another color c_3 used on exactly one of the edges of T_2. Without loss of generality, assume that the edge colored with c_1 appears between the edges colored with c_2, c_3. Then we can get a conflict-free connection coloring of T_2 with $k_0 - 2$ colors obtained from the above coloring c by replacing c_3 with c_2, which is impossible. Thus, there exist exactly two colors c_1, c_2 each of which is used on exactly one of the edges of T_2.

Claim 2. The color on the edge uw is neither c_1 nor c_2.

Proof of Claim 2: By contradiction. Without loss of generality, assume that the color on the edge uw is c_1, and the color on the edge uu_ℓ is c_2 for some ℓ with $1 \leq \ell \leq k_0 - 2$. Then there is no conflict-free path from $v_{2^{k_0-2}}$ to u_ℓ, which is a contradiction.

Let T_2' be the subpath resulting from the removal of the two edges with the colors c_1, c_2 in T_2 such that $v_{2^{k_0-2}} \in T_2'$. Since the edges of T_2' are colored with at most $k_0 - 3$ colors, it follows that $n(T_2') \leq 2^{k_0-3}$ by Theorem 2.5. Let T_2'' be the subpath starting from w with order $2^{k_0-3} + 1$ in $T_2 - V(T_2')$, and T_1' be the subtree obtained from T_1 by deleting the edge with color c_2, where the edge with color c_1 is closer to w than the edge with color c_2. It is obtained that $T' = T_1' \cup T_2''$ is exactly R_{k_0-1}, and the coloring c restricted on the edges of T' is a conflict-free connection coloring with k_0-2 colors. It follows that $cfc(R_{k_0-1}) \leq k_0-2$, which contradicts the minimality of k_0. Thus, $cfc(R_k) \geq k$. And we provide an edge-coloring of R_k with k colors: color the edges of $T_2 - w$ with $k-2$ distinct colors $1, \cdots, k-2$ such that it is conflict-free connected, and color the remaining edges of R_k with k distinct colors $1, \cdots, k$ such that the color on the edge wv_1 is k. It is easy to see that R_k is conflict-free connected under the coloring. Thus, $cfc(R_k) = k$. $\square$

Theorem 4.5 *Let R_k be the graph obtained from $K_{1,k-1}$ and $P_{2^{k-2}+1}$ with $k \geq 2$ by identifying a leaf vertex in $K_{1,k-1}$ with an end vertex in $P_{2^{k-2}+1}$. Then R_k is k-cfc-critical.*

Proof. By Lemma 4.4, we have $cfc(R_k) = k$. We only need to prove that for every edge e of R_k, each nontrivial component in $R_k - e$ has a conflict-free connection number less than k. If $k = 2$, then R_2 is a path on 3 vertices, and the result holds by Proposition 4.2. Thus, we assume that $k \geq 3$. Suppose that T_1 is the copy of $K_{1,k-1}$ in R_k, T_2 is the copy of $P_{2^{k-2}+1}$ in R_k that has one common leaf vertex with T_1. Let $V(T_1) = \{u, w, u_1, \cdots, u_{k-2}\}$ and $V(T_2) = \{w = v_0, v_1 \cdots, v_{2^{k-2}}\}$, where u is the only vertex of maximum degree $k-1$ in T_1, v_iv_{i+1} is an edge of T_2 for $i = 0, \cdots, 2^{k-2} - 1$, and $V(T_1) \cap V(T_2) = w$. In order to obtain our result, We distinguish the following three cases.

Case 1. $e = uw$.

Note that the resulting subtrees in $R_k - e$ are $K_{1,k-2}$ and $P_{2^{k-2}+1}$. Thus, R_k is k-cfc-critical by Lemmas 1.1 and 1.6.

Case 2. $e = uu_i$ $(1 \leq i \leq k-2)$.

It is obtained that the resulting subtrees in $R_k - e$ are a singleton $\{u_i\}$ and $R_k - u_i$. We provide an edge-coloring of $R_k - u_i$: color the edges of $T_2 - w$ with $k-2$ distinct colors $1, \cdots, k-2$ such that it is conflict-free connected, and color the remaining edges with $k-1$ distinct colors $1, \cdots, k-1$ such that the color on the edge wv_1 is $k-1$. It can be checked that $R_k - u_i$ is conflict-free connected under the coloring. Thus, $cfc(R_k - u_i) \leq k-1$, which implies that R_k is k-cfc-critical.

Case 3. $e = v_i v_{i+1}$ $(0 \leq i \leq 2^{k-2} - 1)$.

Let T' and T'' be the resulting subtrees in $R_k - e$, where $u \in T'$. Obviously, $cfc(T'') \leq k-2$. For T', we define an edge-coloring as follows: color the edges on the subpath from w to v_i with $k-2$ colors $1, \cdots, k-2$ such that the subpath is conflict-free connected, and color the remaining edges of T' with $k-1$ distinct colors $1, \cdots, k-1$ such that the color on the edge uw is $k-1$. It is obtained that this is a conflict-free connection coloring of T'. Thus, $cfc(T') \leq k-1$, and so R_k is k-cfc-critical. $\square$

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