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Rainbow monochromatic k-edge-connection colorings of graphs

Ping Li · **Xueliang Li**

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Abstract A path in an edge-colored graph is called a monochromatic path if all edges of the path have a same color. We call k paths $P_1, \cdots, P_k$ rainbow monochromatic paths if every P_i is monochromatic and for any two $i \neq j$, P_i and P_j have different colors. An edge-coloring of a graph G is said to be a rainbow monochromatic k-edge-connection coloring (or RMC_k-coloring for short) if every two distinct vertices of G are connected by at least k rainbow monochromatic paths. We use $rmc_k(G)$ to denote the maximum number of colors that ensures G has an RMC_k-coloring, and this number is called the rainbow monochromatic k-edge-connection number. We prove the existence of RMC_k-colorings of graphs, and then give some bounds of $rmc_k(G)$ and present some graphs whose $rmc_k(G)$ reaches the lower bound. We also obtain the threshold function for $rmc_k(G(n,p)) \geq f(n)$, where $\lfloor \frac{n}{2} \rfloor > k \geq 1$.

Keywords Monochromatic path · Rainbow monochromatic paths · Rainbow monochromatic k-edge-connection coloring (number) · Threshold function

1 Introduction

The monochromatic connection coloring of a graph, introduced in [4], allows that any two vertices are connected by a monochromatic path. In order to generalize this concept, we consider an edge-coloring of a given graph G with any two vertices are connected by at least k (a fixed integer) edge-disjoint monochromatic paths. If we allow some of those k monochromatic paths to have different colors, then the

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Ping Li
Center for Combinatorics and LPMC, Nankai University, Tianjin 300071, People's Republic of China
E-mail: wjlpqdxs@163.com

Xueliang Li*
Center for Combinatorics and LPMC, Nankai University, Tianjin 300071, People's Republic of China
E-mail: lxl@nankai.edu.cn
* Corresponding author.

edge-coloring is called MC_k-coloring of G. If we require that those k monochromatic paths have the same color, then the edge-coloring is called UMC_k-coloring of G. The two generalized concepts are introduced in [12]. In this paper, we discuss the third generalized concept, RMC_k-coloring, which requires that the colors of those k monochromatic paths are pairwise differently. We will introduce the above four concepts systematically, and also introduce some notations and previous work below.

For a graph G, let $\Gamma: E(G) \to [k]$ be an edge-coloring of G that allows a same color to be assigned to adjacent edges, here and in what follows $[k]$ denotes the set $\{1,2,\cdots,k\}$ of integers for a positive integer k. For an edge e of G, we use $\Gamma(e)$ to denote the color of e. If H is a subgraph of G, we also use $\Gamma(H)$ to denote the set of colors on the edges of H and use $|\Gamma(H)|$ to denote the number of colors in $\Gamma(H)$. For all other terminology and notation not defined here we follow Bondy and Murty [2].

A *monochromatic uv-path* is a uv-path of G whose edges are colored with a same color, and G is *monochromatically connected* if for any two vertices of G, G has a monochromatic path connecting them. An edge-coloring Γ of G is a *monochromatic connection coloring (or MC-coloring for short)* if it makes G monochromatically connected. The *monochromatic connection number* of a connected graph G, denoted by $mc(G)$, is the maximum number of colors that are allowed in order to make G monochromatically connected. An *extremal MC-coloring* of G is an MC-coloring that uses $mc(G)$ colors.

The notion monochromatic connection coloring was introduced by Caro and Yuster [4]. Huang and Li [10] recently showed that it is NP-hard to compute the monochromatic connection number for a given graph. Some results were obtained in [3,9,11,14,13]. Later, Gonzaléz-Moreno et al. in [8] generalized the above concept to digraphs.

We list the main results in [4] below.

Theorem 1 ([4]) *Let G be a connected graph with $n \geq 3$. If G satisfies any of the following properties, then $mc(G) = m - n + 2$.*

1. *$\overline{G}$ (the complement of G) is a 4-connected graph;*
2. *G is triangle-free;*
3. *$\Delta(G) < n - \frac{2m-3(n-1)}{n-3}$;*
4. *$diam(G) \geq 3$;*
5. *G has a cut vertex.*

The Erdös-Rényi random graph model $G(n,p)$ will be studied in this paper. The graph $G(n,p)$ is defined on n labeled vertices (informally, we use $[n]$ to denote the n labeled vertices) in which each edge is chosen independently and randomly with probability p. A *property* of graphs is a subset of the set of all graphs on $[n]$ (such as connectivity, minimum degree, et al). If a property Q has $Pr[G \sim G(n,p)$ satisfies $Q] \to 1$ when $n \to +\infty$, then we call the property Q *almost surely*. A property Q is *monotone increasing* if whenever H is a graph obtained from H' by adding some edges and H' has property Q, then H also has the property Q.

A function $h(n)$ is a *threshold function* for an increasing property Q, if for any two functions $h_1(n) = o(h(n))$ and $h(n) = o(h_2(n))$, $G(n,h_1(n))$ does not have property Q almost surely and $G(n,h_2(n))$ has property Q almost surely. Moreover, $h(n)$ is called a *sharp threshold function* of Q if there exist two positive constants c_1 and c_2 such that $G(n,p(n))$ does not have property Q almost surely when $p(n) \leq c_1 h(n)$ and $G(n,p(n))$ has property Q almost surely when $p(n) \geq c_2 h(n)$. It was proved in [6] that every monotone increasing graph property has a sharp threshold function. The property monochromatic connection coloring of a graph (and also the properties monochromatic k-edge-connection coloring, uniformly monochromatic k-edge-connection coloring and rainbow monochromatic k-edge-connection coloring of graphs which are defined later) is monotone increasing, and therefore it has a sharp threshold function.

Theorem 2 **([9])** *Let $f(n)$ be a function satisfying $1 \leq f(n) < \binom{n}{2}$. Then*

$$p = \begin{cases} \frac{f(n)+n\log\log n}{n^2}, & \textit{if } f(n) = \Omega(n\log n) \textit{ and } f(n) < \binom{n}{2}; \\ \frac{\log n}{n}, & \textit{if } f(n) = o(n\log n). \end{cases}$$

is a sharp threshold function for the property $mc(G(n,p)) \geq f(n)$.

Now we generalize the concept monochromatic connection coloring of graphs. There are three ways to generalize this concept.

The first generalized concept is called the *monochromatic k-edge-connection coloring* (or *MC_k-coloring* for short) of G, which requires that every two distinct vertices of G are connected by at least k edge-disjoint monochromatic paths (allow some of the paths to have different colors). The *monochromatically k-edge-connection number* of a connected G, denoted by $mc_k(G)$, is the maximum number of colors that are allowed in order to make G monochromatically k-edge-connected.

The second generalized concept is called the *uniformly monochromatic k-edge-connection coloring* (or *UMC_k-coloring* for short) of G, which requires that every two distinct vertices of G are connected by at least k edge-disjoint monochromatic paths such that all these k paths have the same color (note that for different pairs of vertices the paths may have different colors). The *uniformly monochromatically k-edge-connection number* of a connected G, denoted by $umc_k(G)$, is the maximum number of colors that are allowed in order to make G uniformly monochromatically k-edge-connected. These two concepts were studied in [12].

It is obvious that a graph has an MC_k-coloring (or UMC_k-coloring) if and only if G is k-edge-connected. We mainly study the third generalized concept in this paper, which is called the *rainbow monochromatic k-edge-connection coloring* (or *RMC_k-coloring* for short) of a connected graph. One can see later, compare the results for MC-colorings, MC_k-colorings, UMC_k-colorings and RMC_k-colorings of graphs, the concept RMC_k-coloring has the best form among all the generalized concepts of the MC-coloring.

The definition of the third generalized concept goes as follows. For an edge-colored simple graph G (if G has parallel edges but no loops, the following notions are also reasonable), if for any two distinct vertices u and v of G, G has k edge-disjoint monochromatic paths connecting them, and the colors of these k paths are pairwise differently, then we call such k monochromatic paths *k rainbow monochromatic uv-paths*. An edge-colored graph is *rainbow monochromatically k-edge-connected* if every two vertices of the graph are connected by at least k rainbow monochromatic paths in the graph. An edge-coloring Γ of a connected graph G is a *rainbow monochromatic k-edge-connection coloring* (or *RMC_k-coloring* for short) if it makes G rainbow monochromatically k-edge-connected. The *rainbow monochromatically k-edge-connection number* of a connected graph G, denoted by $rmc_k(G)$, is the maximum number of colors that are allowed in order to make G rainbow monochromatically k-edge-connected. An *extremal RMC_k-coloring* of G is an RMC_k-coloring that uses $rmc_k(G)$ colors.

If $k = 1$, then an RMC_k-coloring (also MC_k-coloring and UMC_k-coloring) is reduced to a monochromatic connection coloring for any connected graph.

In an edge-colored graph G, if a color i only colors one edge of $E(G)$, then we call the color i a *trivial color*, and call the edge (tree) a *trivial edge* (*trivial tree*). Otherwise we call the edges (colors, trees) *nontrivial*. A subgraph H of G is called an *i-induced subgraph* if H is induced by all the edges of G with the same color i. Sometimes, we also call H a *color-induced subgraph*.

If Γ is an extremal RMC_k-coloring of G, then each color-induced subgraph is connected. Otherwise we can recolor the edges in one of its components by a fresh color, then the new edge-coloring is also an RMC_k-coloring of G, but the number of colors is increased by one, which contradicts that Γ is extremal. Furthermore, each color-induced subgraph does not have cycles; otherwise we can recolor one edge in a cycle by a fresh color. Then the new edge-coloring is also an RMC_k-coloring of G, but the number of colors is increased, a contradiction. Therefore, we have the following result.

Proposition 1 *If Γ is an extremal RMC_k-coloring of G, then each color-induced subgraph is a tree.*

If Γ is an extremal RMC_k-coloring of G for $i \in \Gamma(G)$, we call an i-induced subgraph of G an *i-induced tree* or a *color-induced tree*. We also call it a tree sometimes if there is no confusion.

The paper is organized as follows. Section 2 will give some preliminary results. In Section 3, we study the existence of RMC_k-colorings of graphs. In Section 4, we give some bounds of $rmc_k(G)$, and present some graphs whose $rmc_k(G)$ reaches the lower bound. In Section 5, we obtain the threshold function for $rmc_k(G) \geq f(n)$, where $\lfloor \frac{n}{2} \rfloor > k \geq 1$.

2 Preliminaries

Suppose that $a = (a_1, \cdots, a_q)$ and $b = (b_1, \cdots, b_p)$ are two positive integer sequences whose lengths p and q may be different. Let $\prec$ be the *lexicographic order* for integer sequences, i.e., $a \prec b$ if for some $h \geq 1$, $a_j = b_j$ for $j < h$ and $a_h < b_h$, or $p > q$ and $a_j = b_j$ for $j \leq q$.

Let D, n, s be integers with $n \geq 5$ and $1 \leq s \leq n-4$. Let r be an integer satisfying $D < r\binom{n-s}{2}$. For an integer $t \geq r$, suppose $f(\mathbf{x}_t) = f(x_1, \cdots, x_t) = \sum_{i\in[t]} \binom{x_i-1}{2}$ and $g(\mathbf{x}_t) = g(x_1, \cdots, x_t) = \sum_{i\in[t]}(x_i - 2)$, where $x_i \in \{3, 4, \cdots, n-s\}$. We use $\mathscr{S}_t$ to denote the set of optimum solutions of the following problem:

$$\begin{array}{ll} \min & g(\mathbf{x}_t) \\ s.t. & f(\mathbf{x}_t) \geq D \text{ and } x_i \in \{3, \cdots, n-s\} \text{ for each } i \in [t]. \end{array}$$

Lemma 1 *There are integers r, x with $r \leq t$ and $3 \leq x < n-s$, such that the above problem has a solution $\mathbf{x}_t = (x_1, \cdots, x_t)$ in $\mathscr{S}_t$ satisfying that $x_i = n-s$ for $i \in [r-1]$, $x_r = x$ and $x_j = 3$ for $j \in \{r+1, \cdots, t\}$.*

Proof Let $\mathbf{c}_t = (c_1, \cdots, c_t)$ be a maximum integer sequence of $\mathscr{S}_t$. Then $c_i \geq c_{i+1}$ for $i \in [t-1]$. Since $D < t\binom{n-s}{2}$, there is an integer $r \leq t$ such that $c_i = n-s$ for $i \leq r-1$ and $3 \leq c_i < n-s$ for $i \in \{r, \cdots, t\}$. Let $x = c_r$. Then $3 \leq x < n-s$. We need to show $c_i = 3$ for each $i \in \{r+1, \cdots, t\}$. Otherwise, suppose j is the maximum integer of $\{r+1, \cdots, t\}$ with $n-s > c_j > 3$. Let $\mathbf{d}_t = (d_1, \cdots, d_t)$, where $d_i = c_i$ when $i \notin \{r, j\}$, $d_r = c_r + 1$ and $d_j = c_j - 1$. Then $f(\mathbf{d}_t) \geq f(\mathbf{c}_t) \geq D$, $3 \leq d_i < n-s$ for each $i \in [t]$, and $g(\mathbf{c}_t) = g(\mathbf{d}_t)$. i.e., $\mathbf{d}_t \in \mathscr{S}_t$. However, $\mathbf{c}_t \prec \mathbf{d}_t$, which contradicts that $\mathbf{c}_t$ is a maximum integer sequence of $\mathscr{S}_t$. $\square$

Lemma 2 *Suppose $t \geq r$, $\mathbf{a}_t \in \mathscr{S}_t$ and $\mathbf{b}_r \in \mathscr{S}_r$. Then $g(\mathbf{b}_r) \leq g(\mathbf{a}_t)$.*

Proof The result holds for $t = r$, so let $t > r$. W.l.o.g., suppose $\mathbf{a}_t = (a_1, \cdots, a_t)$, where $a_1 = \cdots = a_{l-1} = n-s$, $3 \leq a_l < n-s$ and $a_{l+1} = \cdots = x_t = 3$. Since $t > r$ and $D < r\binom{n-s}{2}$, $l < t$ and $a_t = 3$. Let $\mathbf{c}_{t-1} = (c_1, \cdots, c_{t-1})$, where $c_1 = \cdots = c_{l-1} = n-s$, $c_l = a_l + 1$ and $c_{l+1} = \cdots = x_{t-1} = 3$. Then $f(\mathbf{c}_{t-1}) \geq D$ and $g(\mathbf{c}_{t-1}) = g(\mathbf{a}_t)$. Let $\mathbf{d}_{t-1} \in \mathscr{S}_{t-1}$. Then $g(\mathbf{c}_{t-1}) \geq g(\mathbf{d}_{t-1})$. By induction on $t-r$, $g(\mathbf{b}_r) \leq g(\mathbf{d}_{t-1})$. Thus $g(\mathbf{b}_r) \leq g(\mathbf{a}_t)$. $\square$

The following result is easily seen.

Lemma 3 *If a, b, c are positive integers with $c + a - 1 \geq 2$ and $a + b = c$, then $\binom{c}{2} - \binom{a}{2} \geq b$.*

Suppose X is a proper vertex set of G. We use $E(X)$ to denote the set of edges whose ends are in X. For a graph G and $X \subseteq V(G)$, to shrink X is to delete $E(X)$ and then merge the vertices of X into a single vertex. A partition of the vertex set V is to divide V into some mutual disjoint nonempty sets. Suppose $\mathscr{P} = \{V_1, \cdots, V_s\}$ is a partition of $V(G)$. Then $G/\mathscr{P}$ is a graph obtained from G by shrinking every V_i into a single vertex.

The spanning tree packing number (STP number) of a graph is the maximum number of edge-disjoint spanning trees contained in the graph. We use $T(G)$ to denote the number of edge-disjoint spanning trees of G. The following theorem was proved by Nash-Williams and Tutte independently.

Theorem 3 ([15] [16]) *A graph G has at least k edge-disjoint spanning trees if and only if $e(G/\mathscr{P}) \geq k(|G/\mathscr{P}| - 1)$ for any vertex-partition $\mathscr{P}$ of $V(G)$.*

We denote $\tau(G) = \min_{|\mathscr{P}| \geq 2} \frac{e(G/\mathscr{P})}{|G/\mathscr{P}|-1}$. Then Nash-Williams-Tutte Theorem can be restated as follows.

Theorem 4 *$T(G) = k$ if and only if $\lfloor \tau(G) \rfloor = k$.*

If Γ is an extremal RMC_k-coloring of G, then we say that Γ *wastes* $\omega = \sum_{i\in[r]}(|T_i| - 2)$ colors, where $T_1, \cdots, T_r$ are all the nontrivial color-induced trees of G. Thus $rmc_k(G) = m - \omega$.

Suppose that Γ is an edge-coloring of G and v is a vertex of G. The *nontrivial color degree* of v under Γ is denoted by $d^n(v)$, that is, the number of nontrivial colors appearing on the edges incident with v.

Lemma 4 *Suppose that Γ is an RMC_k-coloring of G with $k \geq 2$. Then $d^n(v) \geq k$ for every vertex v of G.*

Proof Since every two vertices have $k \geq 2$ rainbow monochromatic paths connecting them and G is simple, every two vertices have at least one nontrivial monochromatic path connecting them, i.e., $d^n(v) \geq 1$ for each $v \in V(G)$. Let $e = vu$ be a nontrivial edge. Then there are $k-1$ rainbow monochromatic paths of order at least three connecting u and v. Since these $k-1$ rainbow monochromatic paths are nontrivial, $d^n(v) \geq k$ for each $v \in V(G)$. □

3 Existence of RMC_k-colorings

We knew that there exists an MC_k-coloring or a UMC_k-coloring of G if and only if G is k-edge-connected. It is natural to ask how about RMC_k-colorings ? It is obvious that any cycle of order at least 3 is 2-edge-connected, but it does not have an RMC_2-coloring.

We mainly think about simple graphs in this paper, but in the following result, all graphs may have parallel edges but no loops.

Theorem 5 *A graph G has an RMC_k-coloring if and only if $\tau(G) \geq k$.*

Proof If G has k edge-disjoint spanning trees $T_1, \cdots, T_k$, then we can color the edges of each T_i by i and color the other edges of G by colors in $[k]$ arbitrarily. Then the coloring is an RMC_k-coloring of G. Therefore, G has an RMC_k-coloring when $\tau(G) \geq k$.

We will prove that if there exists an RMC_k-coloring of G, then G has k edge-disjoint spanning trees, i.e., $\tau(G) \geq k$. Before proceeding to the proof, we need a critical claim as follows.

Claim If G has an RMC_k-coloring, then $e(G) \geq k(n-1)$.

Proof Suppose that Γ is an extremal RMC_k-coloring of G and $G_1, \cdots, G_t$ are all the color-induced trees of G (say G_i is the i-induced tree). If there are two color-induced trees G_i and G_j satisfying that all the three sets $V(G_i) - V(G_j)$, $V(G_j) - V(G_i)$ and $V(G_i) \cap V(G_j)$ are nonempty, then we use $P(G, \Gamma, i, j)$ to denote the graph $(G - E(G_i \cup G_j)) \cup T_1 \cup T_2$, where T_1 and T_2 are two new trees with $V(T_1) = V(G_i) \cup V(G_j)$ and $V(T_2) = V(G_i) \cap V(G_j)$ (note that T_1, T_2 and $G - E(G_i \cup G_j)$ are mutually edge disjoint, then $P(G, \Gamma, i, j)$ may have parallel edges); we also use $\Upsilon(G, \Gamma, i, j)$ to denote the edge-coloring of $P(G, \Gamma, i, j)$, which is obtained from Γ by coloring $E(T_1)$ with i and coloring $E(T_2)$ with j, respectively. Then $|G| = |P(G, \Gamma, i, j)|$ and $e(G) = e(P(G, \Gamma, i, j))$.

We claim that $\Upsilon(G,\Gamma,i,j)$ is an RMC_k-coloring of $P(G,\Gamma,i,j)$, and we prove it below. For any two vertices u,v of G, if at least one of them is in $V(G)-V(G_i\cup G_j)$, or one is in $V(G_i)-V(G_j)$ and the other is in $v\in V(G_j)-V(G_i)$, then none of rainbow monochromatic uv-paths of G are colored by i or j, these rainbow monochromatic uv-paths of G are kept unchanged. Thus there are at least k rainbow monochromatic uv-paths in $P(G,\Gamma,i,j)$ under $\Upsilon(G,\Gamma,i,j)$; if both of u,v are in $V(G_i)\cap V(G_j)$, then there are at least $k-2$ rainbow monochromatic uv-paths of G with colors different from i and j, and these rainbow monochromatic uv-paths are kept unchanged. Since T_1 and T_2 provide two rainbow monochromatic uv-paths, one is colored by i and the other is colored by j, there are at least k rainbow monochromatic uv-paths in $P(G,\Gamma,i,j)$ under $\Upsilon(G,\Gamma,i,j)$; if, by symmetry, u and v are in G_i and at most one of them is in $V(G_i)\cap V(G_j)$, then there are at least $k-1$ rainbow monochromatic uv-paths with colors different from i and j, and these rainbow monochromatic uv-paths are kept unchanged. Since T_1 provides a monochromatic uv-path with color i, there are at least k rainbow monochromatic uv-paths in $P(G,\Gamma,i,j)$ under $\Upsilon(G,\Gamma,i,j)$.

We now introduce a simple algorithm on G. Setting $H:=G$ and $\Gamma^*:=\Gamma$. If there are two color-induced subgraphs H_i and H_j of H satisfying that all the three sets $V(H_i)-V(H_j)$, $V(H_j)-V(H_i)$ and $V(H_i)\cap V(H_j)$ are nonempty, then replace H by $P(H,\Gamma^*,i,j)$ and replace Γ^* by $\Upsilon(H,\Gamma^*,i,j)$.

We now show that the algorithm will terminate in a finite steps. In the ith step, let $H=H_i$ and $\Gamma^*=\Gamma_i$, and let $G_1^i,\cdots,G_{t_i}^i$ be all the color-induced subgraphs of H_i such that $|G_1^i|\geq|G_2^i|\geq\cdots\geq|G_{t_i}^i|$ (in fact, in each step, each color-induced subgraph is a tree), and let $l_i=(|G_1^i|,|G_2^i|,\cdots,|G_{t_i}^i|)$ be an integer sequence. Suppose $H_{i+1}=P(H_i,\Gamma_i,s,t)$, i.e., $H_{i+1}=H_i-E(G_s^i\cup G_t^i)\cup T_1\cup T_2$, where $V(T_1)=V(G_s^i)\cup V(G_t^i)$ and $V(T_2)=V(G_s^i)\cap V(G_t^i)$. Then $|T_1|>\max\{|G_s^i|,|G_t^i|\}$. Therefore, $l_i\prec l_{i+1}$. Since G is a finite graph and $e(H_i)=e(G)$ in each step, the algorithm will terminate in a finite step.

Let H' be the resulting graph and Γ' be the resulting RMC_k-coloring of H', and $T_1',\cdots,T_r'$ be the color-induced trees of H' with $|T_1'|\geq\cdots\geq|T_r'|$. Then T_k' is a spanning tree of H'; otherwise, there is al least one vertex w in $V(G)-V(T_k)$. Suppose $u\in V(T_k)$. Since $T_1',\cdots,T_{k-1}'$ provide at most $k-1$ rainbow monochromatic uw-paths, there is a tree of $\{T_{k+1}',\cdots,T_r'\}$, say T_a', containing u and w. Then $V(T_k')-V(T_a')\neq\emptyset$; otherwise $|T_k'|<|T_a'|$, a contradiction. Thus $V(T_k')-V(T_a')$, $V(T_a')\cap V(T_k')$ and $V(T_a')-V(T_k')$ are nonempty sets, which contradicts that H' is the resulting graph of the algorithm. Therefore, there are at least k spanning trees of H', i.e., $e(G)=e(H')\geq k(n-1)$. □

Now, we are ready to prove $\tau(G)\geq k$ by contradiction. Suppose that Γ is an RMC_k-coloring of G but $\tau(G)<k$. By Theorem 3, there exists a partition $\mathscr{P}=\{V_1,\cdots,V_t\}$ of $V(G)$ ($|\mathscr{P}|=t\geq 2$), such that $e(G/\mathscr{P})<k(|\mathscr{P}|-1)$. Let $G^*=G/\mathscr{P}$ be the graph obtained from G by shrinking each V_i into a single vertex v_i, $1\leq i\leq t$.

Suppose that Γ^* is an edge-coloring of G^* obtained from Γ by keeping the color of every edge of G not being deleted (we only delete edges contained in each V_i). It is obvious that Γ^* is an RMC_k-coloring of G^*. However, $e(G^*)<k(|G^*|-1)$, a contradiction to Claim 3. So, $\tau(G)\geq k$. □

We will turn to discuss simple graphs below. Because a simple graph is also a loopless graph, Theorem 5 holds for simple graphs. For a connected simple graph G, since $1 \leq \tau(G) \leq \tau(K_n) = \left\lfloor \frac{e(K_n)}{n-1} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor$, we have the following result.

Corollary 1 *If G is a simple graph of order n and G has an RMC_k-coloring, then $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$.*

By Theorem 5, if $\tau(G) \geq k$, a trivial RMC_k-coloring of a graph G is a coloring that colors the edges of the k edge-disjoint spanning trees of G by colors in $[k]$, respectively, and then colors the other edges trivial. Since the edge-coloring wastes $k(n-2)$ colors, $rmc_k(G) \geq m - k(n-2)$. Thus, $m - k(n-2)$ is a lower bound of $rmc_k(G)$ if G has an RMC_k-coloring.

Corollary 2 *If G is a graph with $\tau(G) \geq k$, then $rmc_k(G) \geq m - k(n-2)$.*

4 Some graphs with rainbow monochromatic k-edge-connection number $m-k(n-2)$

In this section, we mainly study the graphs with rainbow monochromatic k-edge-connection number $m-k(n-2)$ (graphs in the following theorem).

Theorem 6 *Let G be a graph with $\tau(G) \geq k$. If G satisfies any of the following properties, then $rmc_k(G) = m - k(n-2)$.*

1. *G is triangle-free;*
2. *$diam(G) \geq 3$;*
3. *G has a cut vertex;*
4. *G is not $k+1$-edge-connected.*

We will prove this theorem separately by four propositions below (the second result is a corollary of Proposition 3).

Proposition 2 *If G is a triangle-free graph with $\tau(G) \geq k$, then $rmc_k(G) = m - k(n-2)$.*

Proof By Theorem 1, the result holds for $k = 1$. Therefore, let $k \geq 2$ (this requires $n \geq 4$). Since G is a triangle-free graph, by Turán's Theorem, $e(G) \leq \frac{n^2}{4}$. Then

$$k \leq \tau(G) \leq \frac{e(G)}{|G|-1} \leq \frac{n+1}{4} + \frac{1}{4(n-1)}.$$

So, $n \geq 4k - 1 - \frac{1}{n-1}$, i.e., $n \geq 4k-1$.

Suppose Γ is an extremal RMC_k-coloring of G. If there is a color-induced tree, say T, that forms a spanning tree of G, then Γ is an extremal RMC_{k-1}-coloring restricted on $G-E(T)$. Otherwise, suppose Γ is not an extremal RMC_{k-1}-coloring restricted on $G-E(T)$. Since Γ is obviously an RMC_{k-1}-coloring restricted on $G-E(T)$, there is an RMC_{k-1}-coloring Γ' of $G-E(T)$ such that $|\Gamma(G-E(T))| < |\Gamma'(G-E(T))|$. Let Γ'' be an edge-coloring of G obtained from Γ' by assigning $E(T)$ with a new color.

Then Γ'' is an RMC_k-coloring of G. However, $|\Gamma(G)| < |\Gamma''(G)|$, a contradiction. Since $G-E(T)$ is triangle-free, by induction on k,

$$rmc_{k-1}(G-E(T)) = e(G-E(T)) - (k-1)(n-2) = m-k(n-2)-1.$$

Therefore,

$$rmc_k(G) = 1+|\Gamma(G-E(T))| = 1+rmc_{k-1}(G-E(T)) = m-k(n-2).$$

Now, suppose that each color-induced tree is not a spanning tree. We use $\mathscr{S}$ to denote the set of nontrivial color-induced trees of G. We will prove that Γ wastes at least $k(n-2)$ colors below.

Case 1. There is a vertex v of G such that $d^n(v)=k$.

Suppose that $\mathscr{T}=\{T_1,\cdots,T_k\}$ is the set of the k nontrivial color-induced trees containing v. Since each vertex connects v by at least $k-1\geq 1$ nontrivial rainbow monochromatic paths, $V(G)=\bigcup_{i\in[k]}V(T_i)$. Let $S=\bigcap_{i\in[k]}V(T_i)$ and $S_i=V(T_i)-S$.

For any $i,j\in[k]$, both S_i-S_j and S_j-S_i are nonempty. Otherwise, suppose $S_i\subseteq S_j$. Since T_j is not a spanning tree, there is a vertex $u'\in V(G)-V(T_j)$. Then there are at most $k-2$ nontrivial rainbow monochromatic $u'v$-paths, a contradiction.

According to the above discussion, $S,S_1,\cdots,S_k$ are all nonempty sets. Moreover, since $k\geq 2$, $|V(G)-S|\geq 2$.

For each $i\in[k]$ and a vertex u in S_i, there is an $i_u\in[k]$ such that $u\notin V(T_{i_u})$. Furthermore, $u\in V(T_j)$ for each $j\in[k]-\{i_u\}$; for otherwise, there are at most $k-2$ nontrivial rainbow monochromatic uv-paths, which contradicts that Γ is an RMC_k-coloring of G. Therefore, there are exactly $k-1$ nontrivial rainbow monochromatic uv-paths. This implies that uv is a trivial edge of G. Thus, v connects each vertex of $V(G)-S$ by a trivial edge. Since G is triangle-free, $V(G)-S$ is an independent set. It is easy to verify that $\mathscr{T}$ wastes

$$\sum_{i\in[k]}(|T_i|-2) = \sum_{i\in[k]}|T_i|-2k = k|S|+(k-1)(n-|S|)-2k = k(n-2)+|S|-n$$

colors.

Let $\mathscr{F}=\mathscr{S}-\mathscr{T}$ (recall that $\mathscr{S}$ is the set of nontrivial trees of G). Since each two vertices of $V(G)-S$ are in at most $k-1$ trees of $\mathscr{T}$ and $V(G)-S$ is an independent set, there is at least one tree of $\mathscr{F}$ containing them. Moreover, such a tree contains at least one vertex of S. Suppose that $F_1,\cdots,F_t$ are trees of $\mathscr{F}$ with $|V(F_i)\cap(V(G)-S)|=x_i\geq 2$ and $x_1\geq x_2\geq\cdots\geq x_t$. Let $w_i\in V(F_i)\cap S$ and $W_i=V(F_i)\cap(V(G)-S)\cup\{w_i\}$. Then $3\leq|W_i|\leq n-|S|+1$ for each $i\in[t]$, and

$$\sum_{i\in[t]}\binom{|W_i|-1}{2}\geq\binom{n-|S|}{2}. \tag{1}$$

$\mathscr{F}$ wastes at least $\sum_{i\in[t]}(|F_i|-2)\geq\sum_{i\in[t]}(|W_i|-2)$ colors.

For any $i,j\in[k]$, since both S_i-S_j and S_j-S_i are nonempty, there are at most $k-2$ rainbow monochromatic paths connecting every vertex of S_i-S_j and every vertex of S_j-S_i in $\mathscr{T}$. Thus there are at least two trees of $\mathscr{F}$ containing the two vertices, i.e., $t\geq 2$.

If $k=2$ and $|S|-1=3$, then $\mathscr{F}$ wastes at least two colors, and thus Γ wastes at least $k(n-2)$ colors. Otherwise, $|S|-1\geq 4$. Then by Lemma 1, the expression $\sum_{i\in[t]}(|W_i|-2)$, subjects to (1), $n-|S|+1\geq |W_i|\geq 3$ and $t\geq 2$, is minimum when $|W_1|=n-|S|+1$, and $|W_i|=3$ for $i=2,3\cdots,t$. Then $\mathscr{F}$ wastes at least $n-|S|$ colors, and thus Γ wastes at least $k(n-2)$ colors.

Case 2. each vertex v of G has $d^n(v)\geq k+1$.

Suppose $\mathscr{S}=\{T_1,\cdots,T_r\}$ and $|T_i|\geq |T_{i+1}|$ for $i\in[r-1]$. Since $d^n(v)\geq k+1$ for each vertex v of G, $\sum_{i\in[r]}|T_i|\geq (k+1)n$.

If $r\leq \frac{n}{2}+k$, then $\sum_{i\in[r]}(|T_i|-2)\geq k(n-2)$. This implies that Γ wastes at least $k(n-2)$ colors. Thus, we consider $r>\frac{n}{2}+k$.

Since each pair of non-adjacent vertices are connected by at least k rainbow monochromatic paths of order at least three, and each pair of adjacent vertices are connected by at least $k-1$ rainbow monochromatic paths of order at least three, there are at least $k[\binom{n}{2}-e(G)]+(k-1)e(G)=k\binom{n}{2}-e(G)$ such paths. Since each T_i of $\mathscr{S}$ provides $\binom{|T_i|-1}{2}$ paths of order at least three, we have

$$\sum_{i\in[r]}\binom{|T_i|-1}{2}\geq k\binom{n}{2}-e(G).$$

Since $e(G)\leq \frac{n^2}{4}$,

$$\sum_{i\in[r]}\binom{|T_i|-1}{2}\geq k\binom{n}{2}-\frac{n^2}{4}. \tag{2}$$

If $|T_i|=n-1$ for each $i\in[r]$, since $r>\frac{n}{2}+k$, Γ wastes $r(n-3)>k(n-2)$ colors. Thus, we assume that there are some trees of $\mathscr{S}$ with order less than $n-1$. By Lemma 1, there are integers t,x with $t<r$ and $3\leq x\leq n-2$, such that the expression $\sum_{i\in[r]}(|T_i|-2)$, subject to (2) and $3\leq |T_i|\leq n-1$, is minimum when $|T_i|=n-1$ for $i\in[t]$, $|T_{t+1}|=x$ and $|T_j|=3$ for $j\in\{t+1,\cdots,r\}$. By (2),

$$t\binom{n-2}{2}+\binom{x-1}{2}+r-t-1\geq k\binom{n}{2}-\frac{n^2}{4}. \tag{3}$$

This implies that Γ wastes at least

$$w(\Gamma)=t(n-3)+x-2+r-t-1 \tag{4}$$

colors.

If $t\geq k$, or $t=k-1$ and $x\geq \frac{n}{2}+k-1$, then Γ wastes at least

$$(k-1)(n-3)+x-2+r-k=k(n-2)+(r+x+1-2k-n)\geq k(n-2)$$

colors.

If $t=k-1$ and $x<\frac{n}{2}+k-1$, then suppose y is a positive integer such that $x+y=\lceil\frac{n}{2}+k-1\rceil$. Let $z=\lceil\frac{n}{2}+k-1\rceil$. Recall that $n\geq 4k-1$ and $x\geq 3$, and then

$x+z-3 \geq 7$. By Lemma 3, $\binom{z-1}{2}-\binom{x-1}{2} \geq y-1$. We have

$$\begin{aligned}
\sum_{i\in[r]} \binom{|T_i|-1}{2} &= (k-1)\binom{n-2}{2}+\binom{x-1}{2}+r-k \\
&\leq (k-1)\binom{n-2}{2}+\binom{z-1}{2}-y+1+r-k \\
&\leq (k-1)\binom{n-2}{2}+\binom{\frac{n}{2}+k-1}{2}-y+1+r-k \\
&= \frac{4k-3}{8}n^2-\frac{8k-7}{4}n+\frac{(k-1)(k+2)}{2}+r-y \\
&= k\binom{n}{2}-\frac{n^2}{4}-(\frac{n^2}{8}+\frac{6k-7}{4}n-\frac{(k+2)(k-1)}{2})+r-y.
\end{aligned}$$

By (2), we have

$$-(\frac{n^2}{8}+\frac{6k-7}{4}n-\frac{(k+2)(k-1)}{2})+r-y \geq 0,$$

i.e., $r \geq \varepsilon + y$, where $\varepsilon = \frac{n^2}{8}+\frac{6k-7}{4}n-\frac{(k+2)(k-1)}{2}$. Then Γ wastes

$$\begin{aligned}
\sum_{i\in[r]} (|T_i|-2) &\geq (k-1)(n-3)+x-2+r-k \\
&\geq k(n-2)+(x+y-k+1)-n-k+\varepsilon \\
&\geq k(n-2)-\frac{n}{2}-k+\varepsilon
\end{aligned}$$

colors. Let

$$h(n) = -\frac{n}{2}-k+\varepsilon = \frac{1}{8}[n^2+(12k-18)n-4(k^2+3k-2)].$$

Then $h(n) \geq 0$ when $n \geq \frac{1}{2}(\sqrt{160k^2-384k+292}-12k+18)$. Thus $h(n) \geq 0$ when $n \geq \frac{k}{2}+9$. Recall that $n \geq 4k-1$, and then $n \geq \frac{k}{2}+9$ holds for $k \geq 3$. So Γ wastes at least $k(n-2)$ colors if $k \geq 3$. If $k=2$, then $h(n) = \frac{1}{8}(n^2+6n-32)$. Since $n \geq 4k-1 = 7$, $h(n) \geq 0$. Therefore, Γ wastes at least $k(n-2)$ colors when $k=2$.

If $t \leq k-2$, then the number of trees of order 3 is at least $r-t-1$. Recall that $n \geq 4k-1 \geq 7$ and $k \geq 2$. By (3),

$$\begin{aligned}
r-t-1 &\geq k\binom{n}{2}-\frac{n^2}{4}-t\binom{n-2}{2}-\binom{x-1}{2} \\
&\geq k\binom{n}{2}-\frac{n^2}{4}-(k-1)\binom{n-2}{2} \\
&\geq k(2n-3)+\frac{1}{4}(n^2-10n+12) \\
&\geq k(2n-3)-\frac{9}{4} \geq k(n-2).
\end{aligned}$$

Thus, Γ wastes at least $k(n-2)$ colors. $\square$

For a graph G, we use N_{uv} to denote the set of common neighbors of u and v, and let $n_{uv} = |N_{uv}|$, $n_G = \min\{n_{uv} : u, v \in V(G) \text{ and } u \neq v\}$.

Proposition 3 *If G is a graph with $\tau(G) \geq k$, then $rmc_k(G) \leq m - k(n-2) + n_G$.*

Proof Suppose Γ is an extremal RMC_k-coloring of G. Let u, v be two vertices of G with $n_{uv} = n_G$. Let $V(G) - N[v] - \{u\} = A$, $N_{uv} = C$ and $N(v) - \{u\} = B$. Then $C \subseteq B$. Suppose that $\mathscr{T}$ is the set of nontrivial trees containing u and v, $\mathscr{F}$ is the set of nontrivial trees containing u and at least one vertex of B but not v, and $\mathscr{H}$ is the set of nontrivial trees containing v and at least one vertex of A but not u. Thus, $\mathscr{T}, \mathscr{F}$ and $\mathscr{H}$ are pairwise disjoint.

The vertex set A is partitioned into $k+1$ pairwise disjoint subsets $A_0, \cdots, A_k$ (some sets may be empty) such that every vertex of A_i is in exactly i nontrivial trees of $\mathscr{T}$ for $i \in \{0, \cdots, k-1\}$ and every vertex of A_k is in at least k nontrivial trees of $\mathscr{T}$. The vertex set B can also be partitioned into $k+1$ pairwise disjoint subsets $B_0, \cdots, B_k$ (some sets may be empty) such that every vertex of B_i is in exactly i nontrivial trees of $\mathscr{T}$ for $i \in \{0, \cdots, k-1\}$ and every vertex of B_k is in at least k nontrivial trees of $\mathscr{T}$. Then $\mathscr{T}$ wastes

$$w_1 = \Sigma_{T \in \mathscr{T}}(|T| - 2) \geq \Sigma_{i=0}^{k} i(|A_i| + |B_i|)$$

colors.

For every vertex w of A_i, since $N(v) \cap A = \emptyset$, there are at least k nontrivial trees containing v and w. Since there are i such trees in $\mathscr{T}$ for $i \neq k$, there are at least $k - i$ nontrivial trees connecting v and w in $\mathscr{H}$. Since every nontrivial tree of $\mathscr{H}$ must contain v and a vertex of B, $\mathscr{H}$ wastes

$$w_2 = \Sigma_{H \in \mathscr{H}}(|H| - 2) \geq \Sigma_{i=0}^{k}(k - i)|A_i|$$

colors.

Let $C_i = \{w : w \in B_i \cap C \text{ and } uw \text{ is a trivial edge}\}$. For each vertex w of B, if $w \in B_i - C_i$, then there are at least k nontrivial trees containing u and w; if $w \in C_i$, there are at least $k - 1$ nontrivial trees containing u and w. This implies that each vertex of $B_i - C_i$, $i \in \{0, \cdots, k-1\}$, is in at least $k - i$ nontrivial trees of $\mathscr{F}$, and each vertex of C_i is in at least $k - i - 1$ nontrivial trees of $\mathscr{F}$. Now we partition $\mathscr{F}$ into two parts, $\mathscr{F}_1$ and $\mathscr{F}_2$, such that

$$\mathscr{F}_1 = \{F \in \mathscr{F} : V(F) \subseteq B \cup \{u\}\}$$

and

$$\mathscr{F}_2 = \mathscr{F} - \mathscr{F}_1.$$

Then for every F of $\mathscr{F}_1$, u connects a vertex of C in F. Thus, there are at most $|C| - \sum_{i=0}^{k}|C_i|$ trees in $\mathscr{F}_1$. Therefore, $\mathscr{F}$ wastes

$$\begin{aligned} w_3 &= \Sigma_{F \in \mathscr{F}}(|F| - 2) \\ &\geq \Sigma_{i=0}^{k}(k - i)|B_i - C_i| + \Sigma_{i=0}^{k-1}(k - i - 1)|C_i| - (|C| - \sum_{i=0}^{k-1}|C_i|) \\ &= -|C| + \Sigma_{i=0}^{k}(k - i)|B_i| \end{aligned}$$

colors.

According to the above discussion, Γ wastes at least

$$w_1 + w_2 + w_3 \geq -|C| + \Sigma_{i=0}^{k}[k(|A_i| + |B_i|)] = k(n-2) - n_G$$

colors. Therefore, $rmc_k(G) \leq m - k(n-2) + n_G$. $\square$

If G is not an $s+1$-connected graph, then $n_G \leq s$. Thus, we have the following result.

Corollary 3 *If G is a graph with $\tau(G) \geq k$ and G is not $s+1$-connected, then $rmc_k(G) \leq m - k(n-2) + s$.*

The next theorem decreases this upper bound by one when $s = 1$.

Proposition 4 *If G has a cut vertex and $\tau(G) \geq k \geq 2$, then $rmc_k(G) = m - k(n-2)$.*

Proof Let Γ be an extremal RMC_k-coloring of G. Suppose that a is a vertex cut of G and $A_1, \cdots, A_t$ are components of $G - \{a\}$. Let w be a vertex of A_1, and let $\mathscr{T} = \{T_1, \cdots, T_r\}$ be the set of nontrivial trees connecting w and some vertices of $\bigcup_{i=2}^{t} A_i$. Then each T_i contains a. Suppose $\{S_0, S_1, \cdots, S_k\}$ is a vertex partition of $A_1 - w$ such that each vertex of S_i is in exactly i nontrivial trees of $\mathscr{T}$ for $i = 0, 1 \cdots, k-1$ and each vertex of S_k is in at least k nontrivial trees of $\mathscr{T}$. Since each vertex of $\bigcup_{i=2}^{t} A_i$ connects w by at least k trees of $\mathscr{T}$, $\mathscr{T}$ wastes

$$\sum_{i \in [r]} (|T_i| - 2) \geq k \sum_{i=2}^{t} |A_i| + \sum_{i=0}^{k} i|S_i|$$

colors.

Let $\mathscr{F} = \{F_1, \cdots, F_l\}$ be the set of nontrivial trees connecting at least one vertex of $\bigcup_{i=2}^{t} A_i$ and at least one vertex of A_1 but not w. Then $\mathscr{T} \cap \mathscr{F} = \emptyset$. Since a is a cut vertex of G, each F_i of $\mathscr{F}$ contains a. Since $\mathscr{T}$ provides at most i rainbow monochromatic paths connecting every vertex of S_i and every vertex of $\bigcup_{i=2}^{t} A_i$, each vertex of S_i is in at least $k - i$ trees of $\mathscr{F}$. Then $\mathscr{F}$ wastes at least

$$\sum_{i \in [l]} (|F_i| - 2) \geq \sum_{i=0}^{k} (k - i)|S_i|$$

colors. Thus, Γ wastes at least

$$\sum_{i \in [r]} (|T_i| - 2) + \sum_{i \in [l]} (|F_i| - 2) \geq k(\sum_{i=2}^{t} |A_i| + \sum_{i=0}^{k} |S_i|) = k(n-2)$$

colors, $rmc_k(G) = m - k(n-2)$. $\square$

Proposition 5 *If G is not a $k+1$-edge-connected graph and $\tau(G) \geq k \geq 2$, then $rmc_k(G) = m - k(n-2)$.*

Proof Since $\tau(G) \geq k$, G is k-edge-connected. Thus, G has an edge cut S such that $|S| = k$. Then $G - S$ has two components, say D_1 and D_2. Let $x \in V(D_1)$ and $y \in V(D_2)$. For an extremal RMC_k-coloring of G, there are k color-induced trees (say $T_1, \cdots, T_k$) containing x and y, i.e., each T_i contains exactly one edge of S. For each $u \in V(D_1)$, since there are k rainbow monochromatic uy-paths, each path contains exactly one edge of S. Thus each T_i contains u. By the same reason, each T_i contains each vertex of V_2. Therefore, each T_i is a spanning tree of G, and so $rmc_k(G) = m - k(n-2)$. □

Proposition 6 **([4])** *If G is a cycle of order n, then $mc(\overline{G}) \geq e(\overline{G}) - \lceil \frac{2n}{3} \rceil$.*

By Proposition 6, if P is a Hamiltonian path of K_n with $n \geq 4$, then $mc(G \backslash P) \geq e(G \backslash P) - \lceil \frac{2n}{3} \rceil$. The following result is obvious.

Corollary 4 $rmc_2(K_n) \geq \left\lfloor \frac{3n^2 - 13n}{6} \right\rfloor + 2$*, $n \geq 4$.*

Remark 1: The above corollary implies that there are indeed some graphs with rainbow monochromatic k-edge-connection number greater that the lower bound. In fact, for any $k \geq 2$ and $s \geq 2$, there exist graphs with rainbow monochromatic k-edge-connection number greater than or equal to $m - k(n-2) + s - 1$. We construct the (k,s)-*perfectly-connected* graphs below. A graph G is called a (k,s)-perfectly-connected graph if $V(G)$ can be partitioned into $s+1$ parts $\{v\}, V_1, \cdots, V_s$, such that $\tau(G[V_i]) \geq k$, $V_1, \cdots, V_s$ induces a corresponding complete s-partite graph (call it K^s), and v has precisely k neighbors in each V_i. Since $\tau(G[V_i]) \geq k$, each $G[V_i]$ has k edge-disjoint spanning trees (say $T_1^i, \cdots, T_k^i$). Let the k neighbors of v in V_i be $u_1^i, \cdots, u_k^i$ and let $e_1^i = vu_1^i, \cdots, e_k^i = vu_k^i$. Let $T_j = \bigcup_{i \in [s]} e_j^i \cup \bigcup_{i \in [s]} T_j^i$ for $j \in \{2, \cdots, k\}$. Let Γ be an edge-coloring of G such that $\Gamma(T_1^i \cup e_1^i) = i$ for $i \in [s]$, $\Gamma(T_j) = s + j - 1$ for $j \in \{2, \cdots, k\}$, and the other edges are trivial. Then Γ is an RMC_k-coloring of G and $|\Gamma(G)| = m - k(n-2) + s - 1$, and thus $rmc_k(G) \geq m - k(n-2) + s - 1$. □

We propose an open problem below. If the answer for the problem is true, then it will cover our main Theorem 6.

Problem 1 For an integer $k \geq 2$ and a graph G with $\tau(G) \geq k$, does $rmc_k(G) \leq mc(G) - (k-1)(n-2)$ hold ? More generally, does $rmc_k(G) \leq rmc_t(G) - (k-t)(n-2)$ hold for any integer $1 \leq t < k$?

5 Random results

The following result can be found in text books.

Lemma 5 **([1], Chernoff Bound)** *If X is a binomial random variable with expectation μ, and $0 < \delta < 1$, then*

$$Pr[X < (1-\delta)\mu] \leq \exp(-\frac{\delta^2 \mu}{2})$$

and if $\delta > 0$,

$$Pr[X > (1+\delta)\mu] \leq \exp(-\frac{\delta^2 \mu}{2+\delta}).$$

Let $p = \frac{\log n + a}{n}$. The authors in [5] proved that

$$Pr[G(n,p) \text{ is connected}] \to \begin{cases} 1, & a \longrightarrow +\infty; \\ e^{-e^{-a}}, & |a| = O(1); \\ 0, & a \longrightarrow -\infty. \end{cases}$$

Thus, $p = \frac{\log n}{n}$ is the threshold function for $G(n,p)$ being connected.

A sufficient condition for $G(n,p)$ to have an RMC_k-coloring almost surely is that $T(G(n,p)) \geq k$ almost surely. For the STP number problem of $G(n,p)$, Gao et al. proved the following results.

Lemma 6 ([7]) *For every $p \in [0,1]$, we have*

$$T(G(n,p)) = \min\{\delta(G(n,p)), \left\lfloor \frac{e(G(n,p))}{n-1} \right\rfloor\}$$

almost surely.

In this section, we denote $\beta = \frac{2}{\log e - \log 2} \approx 6.51778$.

Lemma 7 ([7]) *If*

$$p \geq \frac{\beta(\log n - \log\log n/2) + \omega(1)}{n-1},$$

then $T(G(n,p)) = \left\lfloor \frac{e(G(n,p))}{n-1} \right\rfloor$ almost surely; if

$$p \leq \frac{\beta(\log n - \log\log n/2) - \omega(1)}{n-1},$$

then $T(G(n,p)) = \delta(G(n,p))$ almost surely.

We knew that $m - k(n-2)$ is a lower bound of $rmc_k(G)$. Next is an upper bound of $rmc_k(G)$. Although the upper bound is rough, it is useful for the subsequent proof.

Proposition 7 *If G is a graph with $\tau(G) \geq k$, then $rmc_k(G) \leq m - (k-1)(n-2)$.*

Proof Since the result holds for $k = 1$, we only consider $k \geq 2$. Suppose Γ is an extremal RMC_k-coloring of G and $\mathscr{T} = \{T_1, \cdots, T_r\}$ is the set of nontrivial color-induced trees with $|T_1| \geq \cdots \geq |T_r|$. Then

$$k\binom{n}{2} - e(G) \leq \sum_{i\in[r]} \binom{|T_i|-1}{2}. \tag{5}$$

Case 1. T_1 is a spanning tree of G.

Then Γ is an extremal RMC_{k-1}-coloring restricted on $G' = G - E(T_1)$ (this result has been proved in Theorem 2). By induction on k,

$$|\Gamma(G')| = rmc_{k-1}(G') \leq e(G') - (k-2)(n-2).$$

Then

$$rmc_k(G)=1+|\Gamma(G')|=1+rmc_{k-1}(G')\le 1+e(G')-(k-2)(n-2)\le m-(k-1)(n-2).$$

Case 2. $|T_i|\le n-1$ for each $i\in[r]$.

By Lemmas 1 and 2, the expression $\sum_{i\in[r]}(|T_i|-2)$, subjects to (5) and $3\le |T_i|\le n-1$, is minimum when $|T_1|=\cdots=|T_{r-1}|=n-1$ and $|T_r|=x+1$, where x is an integer with $3\le x+1\le n-2$.

If $r\le k-1$, then $\sum_{i\in[r]}\binom{|T_i|-1}{2}<(k-1)\binom{n-2}{2}<k\binom{n}{2}-e(G)$, a contradiction to (5).

If $r>k$, then Γ wastes at least $k(n-3)\ge (k-1)(n-2)$ colors. Thus $rmc_k(G)\le m-(k-1)(n-2)$.

If $r=k$, then

$$(k-1)\binom{n-2}{2}+\binom{x}{2}\ge k\binom{n}{2}-e(G).$$

So, $x^2-x-\alpha\ge 0$, where

$$\alpha=2[\binom{n}{2}+(2n-3)(k-1)-e(G)]=2[(2n-3)(k-1)+e(\overline{G})].$$

The inequality holds when $x\ge\frac{1+\sqrt{1+4\alpha}}{2}\ge\sqrt{\alpha}$. Thus, Γ wastes at least

$$\Sigma_{i\in[k]}(|T_i|-2)=(k-1)(n-2)+x-1\ge (k-1)(n-2)+\sqrt{\alpha}-1.$$

Since $k\ge 2$, $\sqrt{\alpha}\ge 1$. Thus $rmc_k(G)\le m-(k-1)(n-2)$. $\square$

Theorem 7 *Let $k=k(n)$ be an integer such that $\lfloor\frac{n}{2}\rfloor>k\ge 1$ and let $rmc_k(K_n)>f(n)\ge k(n-1)$. Then*

$$p=\begin{cases}\frac{f(n)+kn}{n^2}, & f(n)\ge O(n\log n) \text{ and } k=o(n);\\ \min\{\frac{k}{n},\frac{\log n}{n}\}, & f(n)=o(n\log n) \text{ and } k=o(n);\\ 1, & k=O(n) \text{ and } f(n)<rmc_k(K_n).\end{cases}$$

is a sharp threshold function for the property $rmc_k(G(n,p))\ge f(n)$.

Proof Let c be a positive constant and let $E(||G(n,cp)||)$ be the expectation of the number of edges in $G(n,cp)$. Then

$$E(||G(n,cp)||)=\begin{cases}\frac{c(n-1)}{2n}f(n)+\frac{c\cdot k(n-1)}{2}, & f(n)\ge O(n\log n) \text{ and } k=o(n);\\ \frac{c\cdot k(n-1)}{2}, & f(n)=o(n\log n), k=o(n) \text{ and } k>\log n;\\ \frac{c\log n(n-1)}{2}, & f(n)=o(n\log n), k=o(n) \text{ and } k\le\log n;\\ c\binom{n}{2}, & k=O(n) \text{ and } f(n)<rmc_k(K_n).\end{cases}$$

By Lemma 5, both inequalities

$$Pr[||G(n,cp)||<\frac{1}{2}E(||G(n,cp)||)]\le\exp(-\frac{1}{8}E(||G(n,cp)||))=o(1)$$

and

$$Pr[||G(n,cp)|| > \frac{3}{2}E(||G(n,cp)||)] \leq \exp(-\frac{1}{10}E(||G(n,cp)||)) = o(1)$$

hold for each p.

Case 1. $k = O(n)$, i.e., there is an $l \in \mathbb{R}^+$ such that $l \cdot n \leq k < \left\lfloor \frac{n}{2} \right\rfloor$.

Since $G(n,p) = K_n$, $rmc_k(G(n,p)) \geq f(n)$ always holds. On the other hand, we have

$$||G(n,l\cdot p)|| \leq \frac{3}{2}E(||G(n,l\cdot p)||) = \frac{3l}{2} \cdot \binom{n}{2} < k(n-2)$$

almost surely. By Claim 3, $G(n,l\cdot p)$ does not have RMC_k-colorings almost surely.

Case 2. $k = o(n)$.

Case 2.1. $f(n) \geq O(n\log n)$.

Then, there is an $s \in \mathbb{R}^+$ and $f(n) \geq s\cdot n\log n$. Let

$$c_1 = \begin{cases} \beta+1, & s \geq 1; \\ \frac{\beta+1}{s}, & 0 < s < 1. \end{cases}$$

Since $f(n) \geq s \cdot n\log n$, we have

$$c_1 p \geq \frac{(\beta+1)(\log n + kn)}{n} \geq \frac{\beta(\log n - \log\log n/2) + \omega(1)}{n-1}.$$

Since

$$||G(n,c_1p)|| \geq \frac{1}{2}E(||G(n,c_1p)||) = \frac{\beta+1}{2} \cdot \frac{n-1}{2n} f(n) + \frac{k(n-1)(\beta+1)}{4}$$

almost surely, by Lemma 7, $T(G(n,c_1p)) = \left\lfloor \frac{||G(n,c_1p)||}{n-1} \right\rfloor > k$ almost surely, i.e., $G(n,c_1p)$ has RMC_k-colorings almost surely. Therefore,

$$\begin{aligned} rmc_k(G(n,c_1p)) &\geq ||G(n,c_1p)|| - k(n-2) \\ &\geq \frac{\beta+1}{2} \cdot \frac{n-1}{2n} f(n) + \frac{k(n-1)(\beta+1)}{4} - k(n-2) \\ &> \frac{(\beta+1)(n-1)}{4n} f(n) \\ &> f(n) \end{aligned}$$

almost surely.

Let $c_2 = \frac{2}{3}$. Then

$$\begin{aligned} ||G(n,c_2p)|| &\leq \frac{3}{2}E(||G(n,c_2p)||) \\ &\leq \frac{3c_2}{2} \cdot \frac{n-1}{2n} f(n) + \frac{3c_2}{2} \cdot \frac{k(n-1)}{2} \\ &< \frac{1}{2}[f(n) + k(n-1)] \end{aligned}$$

almost surely. Thus, either $G(n,c_2p)$ does not have RMC_k-colorings almost surely, or

$$rmc_k(G(n,c_2p)) < ||G(n,c_2p)|| - (k-1)(n-2) < \frac{1}{2}f(n)$$

almost surely (recall that $rmc_k(G) \leq m-(k-1)(n-2)$ by Proposition 7).

Case 2.2. $f(n) = o(n\log n)$.

If $k \leq \log n$, then $p = \frac{\log n}{n}$. Let $c_1 = \beta+1$ and $c_2 = \frac{1}{2}$ be two constants. Since

$$c_1p > \frac{(\beta+1)\log n}{n} \geq \frac{\beta(\log n - \log\log n/2) + \omega(1)}{n-1},$$

by Lemma 7, $T(G(n,c_1p)) = \left\lfloor \frac{||G(n,c_1p)||}{n-1} \right\rfloor$ almost surely. Since

$$||G(n,c_1p)|| \geq \frac{1}{2}E(||G(n,c_1p)||) = \frac{\log n(n-1)(\beta+1)}{4}$$

almost surely, $T(G(n,c_1p)) \geq \log n \geq k$ almost surely, i.e., $G(n,c_1p)$ has RMC_k-coloring almost surely. Therefore,

$$\begin{aligned} rmc_k(G(n,c_1p)) &\geq ||G(n,c_1p)|| - k(n-2) \\ &\geq \frac{\log n(n-1)(\beta+1)}{4} - k(n-2) \\ &\geq \frac{3\log n(n-1)}{4} > f(n) \end{aligned}$$

almost surely. For $G(n,c_2p)$, since $c_2p = \frac{\log n}{2n}$, $G(n,c_2p)$ is not connected almost surely, i.e., $G(n,c_2p)$ does not have RMC_k-colorings almost surely.

If $k > \log n$ and $k = o(n)$, then $p = \frac{k}{n}$. Let $c_1 = \beta+1$ and $c_2 = 1$. Then

$$c_1p = \frac{(\beta+1)k}{n} > \frac{(\beta+1)\log n}{n} \geq \frac{\beta(\log n - \log\log n/2) + \omega(1)}{n-1},$$

i.e., $T(G(n,c_1p)) = \left\lfloor \frac{||G(n,c_1p)||}{n-1} \right\rfloor$ almost surely. Since

$$||G(n,c_1p)|| \geq \frac{1}{2}E(||G(n,c_1p)||) = \frac{k(n-1)(\beta+1)}{4}$$

almost surely, $T(G(n,c_1p)) \geq k$ almost surely, i.e., $G(n,c_1p)$ has RMC_k-colorings almost surely. Thus

$$rmc_k(G(n,c_1p)) \geq ||G(n,c_1p)|| - k(n-2) > \frac{3}{4}k(n-1) > \frac{3}{4}(n-1)\log n > f(n)$$

almost surely. For $G(n,c_2p)$, since

$$||G(n,c_2p)|| \leq \frac{3}{2}E(||G(n,c_2p)||) = \frac{3}{4}k(n-1) < k(n-2)$$

almost surely. By Claim 3, $G(n,c_2p)$ does not have RMC_k-colorings almost surely.

$\square$

Remark 2. Since $rmc_k(G) = rmc_k(K_n)$ if and only if $G = K_n$, we only concentrate on the case $1 \leq f(n) < rmc_k(K_n)$. If n is odd, then G has $RMC_{\lfloor \frac{n}{2} \rfloor}$-colorings if and only if $G = K_n$. So, we are not going to consider the case $k = \lfloor \frac{n}{2} \rfloor$. $\square$

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