Group Connectivity under 3-Edge-Connectivity

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Abstract

Let S, T be two distinct finite Abelian groups with $|S| = |T|$. A fundamental theorem of Tutte shows that a graph admits a nowhere-zero S-flow if and only if it admits a nowhere-zero T-flow. Jaeger et al. in 1992 introduced group connectivity as an extension of flow theory, and they asked whether such a relation holds for group connectivity analogy. It was negatively answered by Hušek et al. in 2017 for graphs with edge-connectivity 2 when the groups $S = \mathbb{Z}_4$ and $T = \mathbb{Z}_2^2$. In this paper, we extend their results to 3-edge-connected graphs (including both cubic and general graphs), which answers open problems proposed by Hušek et al.(2017) and Lai et al.(2011). Combining some previous results, this characterizes all the equivalence of group connectivity under 3-edge-connectivity, showing that every 3-edge-connected S-connected graph is T-connected if and only if $\{S, T\} \neq \{\mathbb{Z}_4, \mathbb{Z}_2^2\}$.

Keywords: nowhere-zero flows; group connectivity; group flows

AMS Subject Classification (2010): 05C21, 05C40, 05C15

1 Introduction

Graphs considered in this paper are finite and loopless, with possible parallel edges. Throughout this paper, let S, T be (additive) Abelian groups, and $\mathbb{Z}_k$ the cyclic group of order k. We follow [1] for undefined notation and terminology. Fix an orientation D of a graph G. For any $x \in V(G)$, let $E_D^+(x)$ ($E_D^-(x)$, resp.) denote the set of all edges directed away from (into, resp.) x. Given a mapping $\varphi : E(G) \mapsto S$, define, for every vertex $u \in V(G)$,

$$\partial\varphi(u) = \sum_{e \in E_D^+(u)} \varphi(e) \; - \sum_{e \in E_D^-(u)} \varphi(e).$$

Evidently, we have $\sum_{u\in V(G)} \partial\varphi(u) = 0$ since each directed edge is counted exactly once in both its head and tail. A **zero-sum boundary function** is a mapping $\gamma : V(G) \mapsto S$ satisfying $\sum_{u\in V(G)} \gamma(u) = 0$, which is necessary for the existence of such mapping φ with $\partial\varphi = \gamma$. Let $\mathcal{Z}(G, S)$ denote the collection of all zero-sum boundary functions of G. A group flow, S-**flow**, of G is a mapping $\varphi : E(G) \mapsto S$ with $\partial\varphi = \mathbf{0}$, where $\mathbf{0} \in \mathcal{Z}(G, S)$ denotes the constant zero mapping. If $\varphi(e) \neq 0$ for each edge $e \in E(G)$, then φ is called a **nowhere-zero** S-**flow**, abbreviated as S-NZF. When $S = \mathbb{Z}$ and $0 < |\varphi(e)| < k, \forall e \in E(G)$, it is known as a **nowhere-zero** k-**flow**, abbreviated as k-NZF.

The flow theory was initiated by Tutte [16] in studying face coloring problems of graphs on the plane and other surfaces. Tutte [16] proposed some flow conjectures, which are considered as core problems in graph theory. Tutte's 3-flow and 5-flow conjectures predict the existence of flow for given edge-connectivity 4 and 2, respectively, regardless the topological embedding structures of graphs. The 4-flow conjecture [17], generalizing the celebrated Four Coloring Theorem, asserts every Peterson-minor-free graph admits a 4-NZF. Those problems are widely studied and remain open, while significant progresses have been made by Jaeger [5], Seymour [14], Thomassen [15], and Lovász et al. [13]. We refer to [11] for a recent survey on those topics. One of the critical tools in studying nowhere-zero flows is the following fundamental theorem of Tutte [17], converting group flows into integer flows.

Theorem 1.1 *[17] A graph admits a k-NZF if and only if it admits an S-NZF for some Abelian group S with $|S| = k$.*

The advantage of group flows is to provide much more flexibility in proving related integer flow theorems, which allows to use certain contraction operations and local adjustments on graphs. To facilitate this approach, Jaeger et al. [6] introduced group connectivity concept as a generalization of S-flow. If for every $\gamma \in \mathcal{Z}(G, S)$, there is a mapping $\varphi : E(G) \mapsto S \setminus \{0\}$ such that $\partial\varphi = \gamma$, then G is called S-**connected**. Due to certain stronger conditions in group connectivity, some nice properties of flows can not be easily extended to group connectivity. For example, the monotonicity fails for group connectivity. It follows from definition that every k-NZF admissible graph has a $(k + 1)$-NZF, and so by Theorem 1.1 every T-NZF admissible graph has an S-NZF for any finite Abelian groups S, T with $|S| \geq |T|$. However, Jaeger et al. [6] showed that there exist $\mathbb{Z}_5$-connected graphs which are not $\mathbb{Z}_6$-connected, and similar examples were exhibited for some other large groups of prime order. On the positive side, an unusual monotonicity of group connectivity was proved in [12] that every $\mathbb{Z}_3$-connected graph is S-connected for $|S| \geq 4$.

For two distinct finite Abelian groups S, T with the same order, Jaeger et al. [6] asked whether S-connectivity and T-connectivity are equivalent, similar as Theorem 1.1, and they remarked that it is even unknown for the first case concerning $\mathbb{Z}_4$ and $\mathbb{Z}_2^2$. Lai et al. [10] further proposed the problem below for 3-edge-connected graphs.

Problem 1.2 (Problem 1.8 in Lai et al. [10]) *Let $\mathcal{F}(S)$ be the family of all 3-edge-connected S-connected graphs. Is it true that for two Abelian groups S_1 and S_2, if $|S_1| = |S_2|$, then*

$$\mathcal{F}(S_1) = \mathcal{F}(S_2)?$$

With a computer-aided approach, Hušek, Mohelníková and Šámal [4] constructed 2-edge-connected graphs to show that $\mathbb{Z}_4$-connectivity and $\mathbb{Z}_2^2$-connectivity are not equivalent and obtained the following theorem, which provides a negative answer to the question of Jaeger et al. [6].

Theorem 1.3 *[4] Denote by H_1, H_2 as the graphs depicted in Figure 1.*

(1) The graph H_1 is $\mathbb{Z}_2^2$-connected but not $\mathbb{Z}_4$-connected.

(2) The graph H_2 is $\mathbb{Z}_4$-connected but not $\mathbb{Z}_2^2$-connected.

Furthermore, infinitely many such examples can be constructed by replacing some vertices with triangles repeatedly.

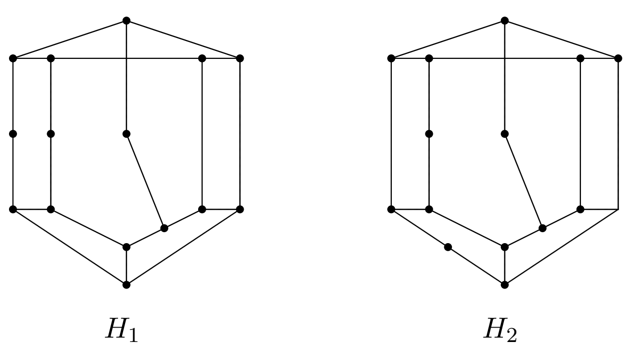

Figure 1: *The graphs for Theorem 1.3.*

By developing a 2-sum operation for group connectivity (as defined below), we extend Theorem 1.3 to 3-edge-connected graphs.

Theorem 1.4

(1) There exists a 3-edge-connected graph which is $\mathbb{Z}_4$-connected but not $\mathbb{Z}_2^2$-connected.

(2) There exists a 3-edge-connected graph which is $\mathbb{Z}_2^2$-connected but not $\mathbb{Z}_4$-connected.

Furthermore, infinitely many such graphs can be generated by a number of 2*-sum operations.*

It is worth noting that our proof of Theorem 1.4 is theoretical, although it assumes the truth of Theorem 1.3 (whose proof is computer-aided).

Extending Jaeger's 4-flow theorem and Seymour's 6-flow theorem, Jaeger et al. [6] obtained the following group connectivity analogy.

Theorem 1.5 *[6] (i) Every* 4*-edge-connected graph is* S*-connected for* $|S| \geq 4$*.*
(ii) Every 3*-edge-connected graph is* S*-connected for* $|S| \geq 6$*.*

Combining Theorems 1.4 and 1.5, we immediately have the following corollary, characterizing the equivalence of group connectivity for all 3-edge-connected graphs completely. This answers Problem 1.2.

Corollary 1.6 *Let* S, T *be two distinct Abelian groups with* $|S| = |T|$*. Then every* 3*-edge-connected* S*-connected graph is* T*-connected if and only if* $\{S, T\} \neq \{\mathbb{Z}_4, \mathbb{Z}_2^2\}$*.*

In [4], Hušek et al. also asked whether such 3-edge-connected cubic graphs exist. In fact, Theorem 1.4 was obtained in early 2018, and the second author communicated with Robert Šámal in SIAM Conference on Discrete Mathematics, Denver, June 2018. The existence of such 3-edge-connected cubic graphs was still open for a while, see Section 5 in Hušek et al. [4]. Now we are able to solve it by a new construction method.

Theorem 1.7

(1) There exists a 3*-edge-connected cubic graph which is* $\mathbb{Z}_4$*-connected but not* $\mathbb{Z}_2^2$*-connected.*

(2) There exists a 3*-edge-connected cubic graph which is* $\mathbb{Z}_2^2$*-connected but not* $\mathbb{Z}_4$*-connected.*

Moreover, infinitely many such graphs can be constructed by substituting some vertices with triangles repeatedly.

The paper is organized as follows. In Section 2 we first develop a 2-sum operation for group connectivity and use it to prove Theorem 1.4. Then in Section 3 we apply a new method to construct such cubic graphs through flow properties of two special graphs. In Section 4, we end this paper with a few concluding remarks.

2 Constructions via 2-sum operations

For $1 \le i \le 2$, let Γ_i be a graph with two distinct vertices $u_i, v_i \in V(\Gamma_i)$. If $u_1v_1 \in E(\Gamma_1)$, then we define $\Gamma = \Gamma_1(u_1v_1) \oplus \Gamma_2(u_2, v_2)$, called the **2-sum** of Γ_1 and Γ_2, as the graph obtained from Γ_1 and Γ_2 by removing the edge u_1v_1 in Γ_1, and then identifying u_1 and u_2 as a new vertex u, and identifying v_1 and v_2 as a new vertex v (see Figure 2).

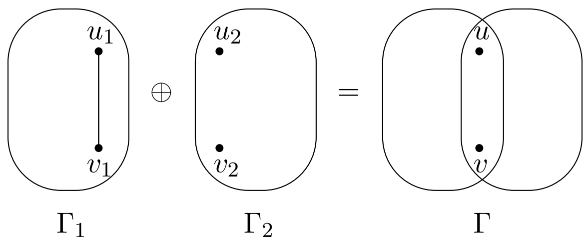

Figure 2: *The* 2*-sum* $\Gamma = \Gamma_1(u_1v_1) \oplus \Gamma_2(u_2, v_2)$.

This 2-sum operation can be viewed as a dual operation of Hajós join on graph coloring. It was first developed by Kochol [7] in studying 3-flow problem, and later generalized to $\mathbb{Z}_3$-connectivity in [3]. Here we extend this 2-sum property to group connectivity of arbitrary finite Abelian groups.

Lemma 2.1 *Let S be a finite Abelian group with $|S| \ge 3$. If neither Γ_1 nor Γ_2 is S-connected, then $\Gamma = \Gamma_1 \oplus \Gamma_2$ is not S-connected.*

Proof. Let $u, v \in V(\Gamma)$ and $u_i, v_i \in V(\Gamma_i)$ where $i = 1, 2$ as defined above. That is, $\Gamma = \Gamma_1(u_1v_1) \oplus \Gamma_2(u_2, v_2)$. Since Γ_i is not S-connected for each $i \in \{1, 2\}$, there exists a $\beta_i \in \mathcal{Z}(\Gamma_i, S)$ such that for any orientation of Γ_i and any mapping $\varphi_i : E(\Gamma_i) \mapsto S \setminus \{0\}$, we have $\partial\varphi_i \ne \beta_i$.

For each $z \in V(\Gamma)$, define

$$\varepsilon(z) = \begin{cases} \beta_1(u_1) + \beta_2(u_2) & \text{if } z = u; \\ \beta_1(v_1) + \beta_2(v_2) & \text{if } z = v; \\ \beta_1(z) & \text{if } z \in V(\Gamma_1) \setminus \{u_1, v_1\}; \\ \beta_2(z) & \text{otherwise.} \end{cases}$$

It is routine to check that $\sum_{z \in V(\Gamma)} \varepsilon(z) = \sum_{x \in V(\Gamma_1)} \beta_1(x) + \sum_{y \in V(\Gamma_2)} \beta_2(y) = 0$, and so $\varepsilon \in \mathcal{Z}(\Gamma, S)$.

Suppose, on the contrary, that Γ is S-connected. Fix an orientation D of Γ. Then there exists a mapping $\eta : E(\Gamma) \mapsto S \setminus \{0\}$ such that $\partial\eta = \varepsilon$. In particular, we have

$$\sum_{e \in E_D^+(u)} \eta(e) - \sum_{e \in E_D^-(u)} \eta(e) = \partial\eta(u) = \varepsilon(u)$$

and

$$\sum_{e \in E_D^+(v)} \eta(e) - \sum_{e \in E_D^-(v)} \eta(e) = \partial\eta(v) = \varepsilon(v).$$

Let D_2 be the restriction of D in Γ_2. Consider D_2 and η on Γ_2. As $\partial\eta(z) = \beta_2(z), \forall z \in V(\Gamma_2) \setminus \{u_2, v_2\}$, we have

$$\begin{aligned} \partial\eta(u_2) + \partial\eta(v_2) &= 0 - \sum_{z \in V(\Gamma_2)\setminus\{u_2,v_2\}} \partial\eta(z) \\ &= 0 - \sum_{z \in V(\Gamma_2)\setminus\{u_2,v_2\}} \beta_2(z) \\ &= \beta_2(u_2) + \beta_2(v_2). \end{aligned}$$

Since $\partial\varphi \neq \beta_2$ for any mapping $\varphi : E(\Gamma_2) \mapsto S \setminus \{0\}$, it follows that $\partial\eta \neq \beta_2$, and so $\partial\eta(u_2) \neq \beta_2(u_2)$ from the above equation. Thus there exists a nonzero element $b \in S$ such that $\partial\eta(u_2) = \beta_2(u_2) + b$ and $\partial\eta(v_2) = \beta_2(v_2) - b$ in Γ_2.

Now consider η and D_1, the restriction of D on $\Gamma_1 - u_1v_1$. We have

$$\partial\eta(u_1) = \varepsilon(u) - [\beta_2(u_2) + b] = [\beta_1(u_1) + \beta_2(u_2)] - [\beta_2(u_2) + b] = \beta_1(u_1) - b$$

and

$$\partial\eta(v_1) = \varepsilon(v) - [\beta_2(v_2) - b] = \beta_1(v_1) + b.$$

We orient the edge u_1v_1 from u_1 to v_1 in Γ_1. Together with D_1, this gives an orientation D_1' of Γ_1. Define a mapping $\omega : E(\Gamma_1) \mapsto S \setminus \{0\}$ such that, for every $e \in E(\Gamma_1)$,

$$\omega(e) = \begin{cases} b & \text{if } e = u_1v_1; \\ \eta(e) & \text{otherwise.} \end{cases}$$

Then $\partial\omega(z) = \partial\eta(z) = \beta_1(z)$, $\forall z \in V(\Gamma_1)\setminus\{u_1, v_1\}$. Moreover, $\partial\omega(u_1) = \partial\eta(u_1) + \omega(u_1v_1) = \beta_1(u_1)$ and $\partial\omega(v_1) = \partial\eta(v_1) - \omega(u_1v_1) = \beta_1(v_1)$. Conclude that $\partial\omega = \beta_1$, which is a contradiction. ■

For $X \subseteq E(G)$, the **contraction** G/X is the graph obtained by identifying the two ends of each edge in X and then deleting the resulting loops from G. If H is a subgraph of G, G/H is used to represent $G/E(H)$ for short. For proving S-connectivity, the following lemma would be helpful.

Lemma 2.2 *[9] (1) A cycle C_n of length n is S-connected if and only if $|S| \geq n+1$.*
(2) If H is an S-connected subgraph of a graph G, then G is S-connected if and only if G/H is S-connected.

A vertex of degree k is called a k-vertex. Let C_4 be a 4-cycle with $V(C_4) = \{v_4, v_3, v_2, v_1\}$. Fix $i \in \{1, 2\}$. In Figure 1, observe that there are exactly three 2-vertices, denoted by x_i, y_i, z_i in H_i. Attach two copies of H_i, namely H_i and H_i' (whose corresponding 2-vertices are x_i', y_i', z_i'). Let H_i^1 be the graph obtained from C_4 and H_i by the 2-sum operation on v_1v_2 and x_i, y_i, namely $H_i^1 = C_4(v_1v_2) \oplus H_i(x_i, y_i)$. Construct a graph H_i^2 from H_i^1 and H_i' by the 2-sum operation on v_4v_3 and x_i', y_i', that is, $H_i^2 = H_i^1(v_4v_3) \oplus H_i'(x_i', y_i')$.

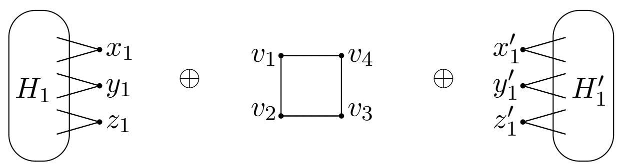

Figure 3: *The graph H_1^2 in Lemma 2.3.*

Lemma 2.3 *(1) The graph H_1^2 is $\mathbb{Z}_2^2$-connected, but not $\mathbb{Z}_4$-connected.*
(2) The graph H_2^2 is $\mathbb{Z}_4$-connected, but not $\mathbb{Z}_2^2$-connected.

Proof. (1) By Theorem 1.3, H_1 and H_1' are $\mathbb{Z}_2^2$-connected. Notice that

$$(H_1^2/H_1)/H_1' = (C_4/v_1v_2)/v_3v_4 = C_2,$$

which is $\mathbb{Z}_2^2$-connected. By Lemma 2.2 we see that H_1^2/H_1 is $\mathbb{Z}_2^2$-connected. As H_1 is $\mathbb{Z}_2^2$-connected and by Lemma 2.2 again, H_1^2 is $\mathbb{Z}_2^2$-connected as desired. Since $H_1^1 = C_4(v_1v_2) \oplus H_1(x_1, y_1)$ is obtained from the 2-sum of two non-$\mathbb{Z}_2^2$-connected graphs C_4 and H_1, we know that H_1^1 is not $\mathbb{Z}_2^2$-connected by Lemma 2.1. Similarly, as $H_1^2 = H_1^1(v_4v_3) \oplus H_1'(x_1', y_1')$, where neither H_1^1 nor H_1' is $\mathbb{Z}_2^2$-connected, it follows from Lemma 2.1 that H_1^2 is not $\mathbb{Z}_2^2$-connected either.

(2) The proof is very similar to (1). Since H_2 is $\mathbb{Z}_4$-connected, but not $\mathbb{Z}_2^2$-connected, after applying the 2-sum operation twice, the resulting graph H_2^2 is $\mathbb{Z}_4$-connected by Lemma 2.2, but not $\mathbb{Z}_2^2$-connected by Lemma 2.1. ■

Note that, by the construction above, the graph H_i^2, for each $i \in \{1, 2\}$, has precisely two vertices z_i and z_i' of degree two. Now we would construct $H_i^3 = C_4 \oplus H_i^2 \oplus H_i^2 \oplus H_i^2$,

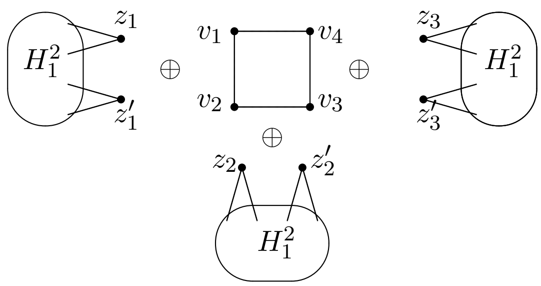

Figure 4: H_1^3 : *Graph of Theorem 2.4 (1).*

$i \in \{1, 2\}$, that would be used in the following theorem. The way to construct H_2^3 from H_2^2 is the same as constructing H_1^3 from H_1^2. So we take H_1^3 as an example. Attach three copies of H_1^2, whose 2-vertices are denoted by z_1, z_1', z_2, z_2' and z_3, z_3', respectively. Apply the 2-sum operation three times on C_4 and the copies of H_1^2. Specifically, we first apply 2-sum on the edge v_1v_2 with z_1, z_1' in the first copy of H_1^2, then apply 2-sum on the edge v_2v_3 with z_2, z_2' in the second copy, and apply the last 2-sum on the edge v_3v_4 with z_3, z_3' in the third copy, as demonstrated in Figure 4. H_1^3 is the resulting graph.

Theorem 2.4 *(1) The graph H_1^3 is 3-edge-connected, $\mathbb{Z}_2^2$-connected, but not $\mathbb{Z}_4$-connected. (2) The graph H_2^3 is 3-edge-connected, $\mathbb{Z}_4$-connected, but not $\mathbb{Z}_2^2$-connected.*

Proof. (1) As H_1^2 is $\mathbb{Z}_2^2$-connected and, after contracting copies of H_1^2 in H_1^3, the resulting graph is a singleton which is $\mathbb{Z}_2^2$-connected, we conclude by Lemma 2.2 that H_1^3 is $\mathbb{Z}_2^2$-connected. Since H_1^3 is obtained from 2-sum operation of non-$\mathbb{Z}_4$-connected graphs, Lemma 2.1 shows that it is not $\mathbb{Z}_4$-connected.

It is also very straightforward to verify that H_1^3 is 3-edge-connected. Firstly, one can easily check that H_1 has only three trivial 2-edge-cuts. Secondly, the graph H_1^2, obtained from 2-sum of C_4 and two copies of H_1, has exactly three 2-edge-cuts, each of which separates z_1 and z_1'. At last, we can use these facts to show that H_1^3 is 3-edge-connected as follows. Specifically, the minimal degree of H_1^3 is three, so we only look at nontrivial edge-cuts. If an edge-cut separates z_k and z_k' for some $k \in \{1, 2, 3\}$ in a copy of H_1^2, then it has a size at least 3 since we need at least two edges to separate z_k and z_k' in the copy of H_1^2 and there is a z_kz_k'-path outside that copy. Assume instead, an edge-cut does not separate z_k and z_k' for any $k \in \{1, 2, 3\}$. Then either it lies in the edges incident to $V(C_4)$, or it separates a copy of H_1^2 (where z_k and z_k' are in one component). In each case, the edge-cut must have a size at least 3. This proves that H_1^3 is 3-edge-connected.

(2) The proof applies the same argument as (1) and thus omitted.

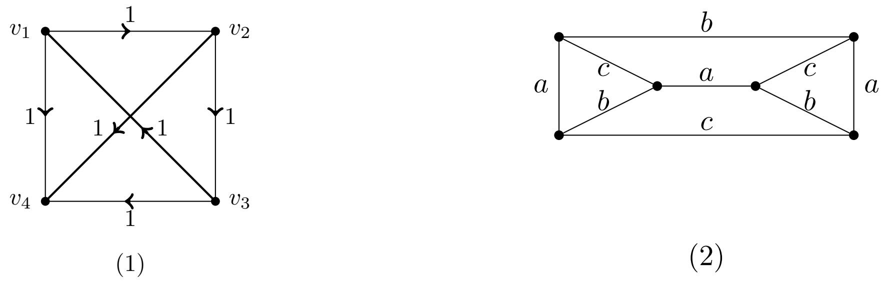

Figure 5: A $\mathbb{Z}_4$-flow of K_4 with boundary 1 and a 3-prism.

Now Theorem 1.4 follows from Theorem 2.4 and Lemma 2.1. ■

3 Constructions of cubic graphs

The constructions in this section rely on some basic properties of K_4 and 3-prism (see Figure 5), as shown in the following lemmas.

Lemma 3.1 *Let G be the complete graph K_4 with an orientation D. Define $\beta : V(G) \mapsto \{1\}$, which is a zero-sum boundary function in $\mathcal{Z}(G, \mathbb{Z}_4)$. Then for any mapping $\varphi : E(G) \to \mathbb{Z}_4 \setminus \{0\}$ with $\partial\varphi = \beta$, there exists a vertex v of G such that each edge $e = uv \in E(G)$ is either directed into v with flow value $\varphi(e) = 1$ or directed away from v with flow value $\varphi(e) = 3$.*

Proof. Since $3 = -1 (\mathrm{mod}\ 4)$, for convenience we may assign the flow value of edges in $\{1, 2\}$ and adapt an appropriate orientation from D. By contradiction, suppose that there exists an orientation of G and a mapping $\varphi : E(G) \to \{1, 2\}$ with $\partial\varphi = \beta$ such that no vertex satisfies that all incident edges are directed into it and with flow value 1. Since $\forall v \in V(G)$, the degree of v is 3 and $\beta(v) = 1$, there is at least one edge e assigned with flow value $\varphi(e) = 1$. By symmetry, assume $\varphi(v_1v_2) = 1$ and the orientation is from v_1 to v_2 as in Figure 5 (1). Since $\beta(v_2) = 1$, we must have $\varphi(v_2v_3) = \varphi(v_2v_4) = 1$ and v_2v_3, v_2v_4 are all directed away from v_2. The similar assignments are applied for v_3v_1 and v_3v_4. At last, we need only to assign the orientation and flow value of v_1v_4 to satisfy $\beta = \partial\varphi$. We shall find that all the edges incident to v_4 are directed into v_4 with flow value 1, a contradiction. ■

Lemma 3.2 *The 3-prism graph is unique 3-edge-colorable. (That is, all proper 3-edge-colorings $\phi : E(G) \to \{a, b, c\}$ are isomorphic. See Figure 5(2).)*

Proof. This fact is easy to observe and thus omitted. ■

Now we shall prove Theorem 1.7 with the following constructions.

Theorem 3.3 *Construct a graph G by replacing every vertex of K_4 with a copy of H_1, where every 2-vertex in each copy is incident with an edge of K_4 (see Figure 6). Then the 3-edge-connected cubic graph G is $\mathbb{Z}_2^2$-connected, but not $\mathbb{Z}_4$-connected.*

Proof. Clearly, G is 3-edge-connected. It follows from Lemma 2.2 that G is $\mathbb{Z}_2^2$-connected since both H_1 and K_4 are $\mathbb{Z}_2^2$-connected. We shall prove below that G is not $\mathbb{Z}_4$-connected. For $1 \le i \le 4$, let A_i be a copy of H_1, where the 2-vertices of A_i are x_i, y_i and z_i (see Figure 6). Since H_1 is not $\mathbb{Z}_4$-connected, there is a failed zero-sum boundary $\beta_1 \in \mathcal{Z}(H_1, \mathbb{Z}_4)$ such that

$$\begin{array}{l}\text{for any orientation of } H_1,\\ \text{there is no mapping } \varphi : E(H_1) \to \mathbb{Z}_4 \setminus \{0\} \text{ such that } \partial\varphi = \beta_1.\end{array} \tag{1}$$

Suppose, on the contrary, that G is $\mathbb{Z}_4$-connected. Define $\beta : V(G) \to \mathbb{Z}_4$ by

$$\beta(v) = \begin{cases} \beta_1(v) - 1 & \text{if } v \in \{x_i, y_i, z_i | 1 \le i \le 4\}; \\ \beta_1(v) & \textit{otherwise.} \end{cases}$$

Since $\sum_{v \in V(A_i)} \beta_1(v) \equiv 0 \pmod 4$ for each i, we have

$$\sum_{v \in V(G)} \beta(v) = 4 \sum_{v \in V(A_1)} \beta_1(v) - 12 \equiv 0 \pmod 4,$$

and so $\beta \in \mathcal{Z}(G, \mathbb{Z}_4)$. Hence there is an orientation of G and a mapping $f : E(G) \to \mathbb{Z}_4 \setminus \{0\}$ such that $\partial f = \beta$.

Consider the graph $F = G/\{\bigcup_{1 \le i \le 4} A_i\}$, which is a K_4. Suppose w_i of $V(F)$ is the vertex corresponds to A_i. Let

$$\beta'(w_i) = \sum_{v \in V(A_i)} \beta(v) = \sum_{v \in V(A_i)} \beta_1(v) - 3 = 1 (\text{mod } 4).$$

Denote f' as the restriction of f on F. Obviously, β' is a zero-sum boundary of F and $\partial f' = \beta'$. By Lemma 3.1, there is a vertex u in F such that each incident edge of u is either directed into u with flow value 1 or directed away from u with flow value 3. Assume, without loss of generality, that the vertex u corresponds to A_1 in G.

This implies that $\varphi = f|_{A_1}$, f restricted to A_1, is a mapping such that $\partial\varphi = \beta_1$ by the definition of β, which contradicts to (1). Hence G is not $\mathbb{Z}_4$-connected. ■

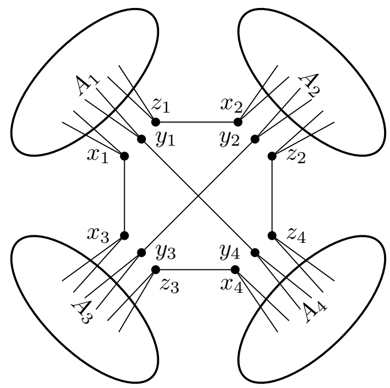

Figure 6: *A* 3*-edge-connected cubic graph that is* $\mathbb{Z}_2^2$*-connected, but not* $\mathbb{Z}_4$*-connected.*

In the proof of Theorem 3.3, one may observe that the key ingredient is to apply Lemma 3.1 to show that the flow values outside a copy A_i are uniquely determined, and so the flow restricted to A_i satisfies the failed zero-sum boundary, yielding a contradiction. The next construction is based on the same motivation, for which we apply the property of 3-prism in Lemma 3.2 instead.

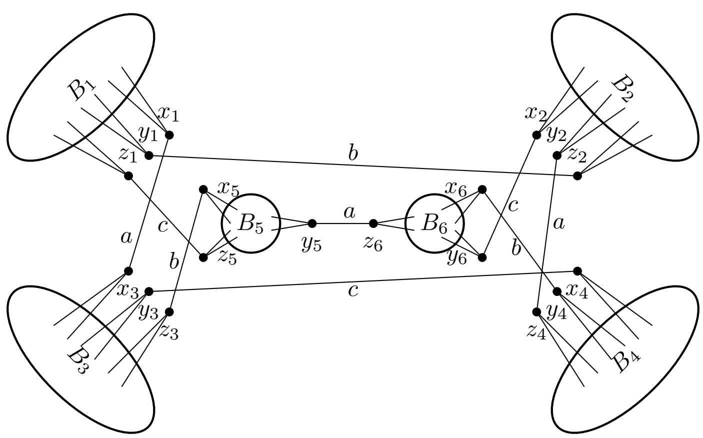

Figure 7: *A* 3*-edge-connected cubic graph that is* $\mathbb{Z}_4$*-connected, but not* $\mathbb{Z}_2^2$*-connected.*

Let $B_i (1 \leq i \leq 6)$ be a copy of H_2, where the 2-vertices of B_i are x_i, y_i and z_i.

Theorem 3.4 *Assume that the* 3*-prism is* 3*-edge-colored with colors* a, b, c*. Let* (p_i, q_i, r_i) *(*$1 \leq i \leq 6$*) be all the permutations of* a, b, c*. Replace each vertex of the* 3*-prism with a copy* B_i *of* H_1*, where the vertex-triple* (x_i, y_i, z_i) *is identified with edges incident to that vertex* *with color-triple (p_i, q_i, r_i) for each $1 \leq i \leq 6$. Let G be the resulting graph. See Figure 7. Then G is $\mathbb{Z}_4$-connected, but not $\mathbb{Z}_2^2$-connected.*

Proof. Since both H_2 and the 3-prism are $\mathbb{Z}_4$-connected, the graph G is $\mathbb{Z}_4$-connected by Lemma 2.2. We shall show below that G is not $\mathbb{Z}_2^2$-connected. Note that for $\mathbb{Z}_2^2$-group connectivity, the orientation is irrelevant since each element is self-inverse. Thus we will omit the statements of orientations. As H_2 is not $\mathbb{Z}_2^2$-connected, there is a failed boundary $\beta_1 \in \mathcal{Z}(H_2, \mathbb{Z}_2^2)$ such that

$$\text{there is no mapping } \varphi : E(H_2) \to \{(0,1), (1,0), (1,1)\} \text{ with } \partial\varphi = \beta_1. \tag{2}$$

Define a function $\beta : V(G) \mapsto \mathbb{Z}_2^2$ as follows:

$$\beta(v) = \begin{cases} \beta_1(v) - (0,1) & \text{if } v \in \{x_i | 1 \leq i \leq 6\}; \\ \beta_1(v) - (1,0) & \text{if } v \in \{y_i | 1 \leq i \leq 6\}; \\ \beta_1(v) - (1,1) & \text{if } v \in \{z_i | 1 \leq i \leq 6\}; \\ \beta_1(v) & \textit{otherwise}. \end{cases}$$

Since $\sum_{v \in V(B_i)} \beta_1(v) = (0,0)$ in $\mathbb{Z}_2^2$ for each $1 \leq i \leq 6$, we have

$$\sum_{v \in V(G)} \beta(v) = \sum_{i=1}^{6} \left(\sum_{v \in V(B_i)} \beta_1(v) \right) - 6[(0,1) - (1,0) - (1,1)] = (0,0) \text{ in } \mathbb{Z}_2^2,$$

and thus $\beta \in \mathcal{Z}(G, \mathbb{Z}_2^2)$.

By contradiction, suppose that G is $\mathbb{Z}_2^2$-connected. So there is a mapping $f : E(G) \to \{(0,1), (1,0), (1,1)\}$ such that $\partial f = \beta$.

Consider the graph $F = G/\{\bigcup_{1 \leq i \leq 6} B_i\}$, which is a 3-prism. The flow f restricted to it provides a nowhere-zero $\mathbb{Z}_2^2$-flow, which is indeed a proper 3-edge-coloring and the color-classes are precisely the edges with values $(0,1), (1,0), (1,1)$, respectively. Hence the color-triple (a, b, c) is a permutation of $(0,1), (1,0), (1,1)$. Notice that edges incident to the triples of $\{x_i, y_i, z_i | 1 \leq i \leq 6\}$ for different i are colored with different permutation of color-set $\{a, b, c\}$. So each of the six permutations appears on exactly one vertex. Hence there exists a triple (x_k, y_k, z_k) corresponding to $((0,1), (1,0), (1,1))$, say $k = 1$ without loss of generality. That is $f(x_1x_3) = (0,1)$, $f(y_1z_2) = (1,0)$ and $f(z_1z_5) = (1,1)$. Now by definition of β, the mapping f restricted to B_1, $\varphi = f|_{B_1}$, is a mapping of H_2 such that $\partial\varphi = \beta_1$, a contradiction to (2). Therefore, G is not $\mathbb{Z}_2^2$-connected. ■

4 Concluding Remarks

Theorem 1.5 of Jaeger et al. [6] says that every 4-edge-connected graph is S-connected for $|S| \geq 4$. This particularly shows that group connectivity is equivalent for distinct groups of a same size for 4-edge connected graphs. In fact, the graphs constructed in Theorems 1.4 and 1.7 are far from being 4-edge-connected and contain a lot of 3-edge-cuts. It would be curious that whether lowing down the number of 3-edge-cuts could guarantee the equivalence relation of group connectivity.

Problem 4.1 *What is the maximum number k such that, for all 3-edge-connected graphs with at most k 3-edge-cuts, $\mathbb{Z}_2^2$-connectivity and $\mathbb{Z}_4$-connectivity are equivalent?*

Note that, using a smaller $\mathbb{Z}_4$-connected non-$\mathbb{Z}_2^2$-connected graph obtained in Section 2 of [4] (Figure 2 in that paper), the smallest such 3-edge-connected graphs that we can construct in Theorem 1.7 have 48 edge-cuts of size three, which shows $k < 48$.

On the other hand, we provide a partial positive result from some known results on collapsible graphs (which are contractible graphs for Eulerian subgraph problem). A graph G is *collapsible* if for any $N \subseteq V(G)$ of even order, there is a spanning connected subgraph of G whose vertices have degree exactly odd in N and even otherwise. Lai [8] showed that every collapsible graph is both $\mathbb{Z}_4$-connected and $\mathbb{Z}_2^2$-connected. Moreover, it was proved in [2] that every 3-edge-connected graph with at most nine 3-edge-cuts is collapsible, and therefore, both $\mathbb{Z}_4$-connected and $\mathbb{Z}_2^2$-connected. Hence, we conclude that

$$9 \leq k \leq 47.$$

It would also be interesting to find the smallest $\mathbb{Z}_4$-connected non-$\mathbb{Z}_2^2$-connected graphs (with edge-connectivity 3), and the other way around. This may help to solve Problem 4.1.

In this paper, Corollary 1.6 completely answers the equivalence of group connectivity for 3-edge-connected graphs. The dual problem on graph coloring is still open, see [10]. Is it true that for distinct groups S and T with a same order, S-group-colorability and T-group-colorability are equivalent (for simple graphs)?

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