k-Critical Graphs in P_5-Free Graphs[*]

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Abstract

Given two graphs H_1 and H_2, a graph G is (H_1, H_2)-free if it contains no induced subgraph isomorphic to H_1 or H_2. Let P_t be the path on t vertices. A graph G is k-vertex-critical if G has chromatic number k but every proper induced subgraph of G has chromatic number less than k. The study of k-vertex-critical graphs for graph classes is an important topic in algorithmic graph theory because if the number of such graphs that are in a given hereditary graph class is finite, then there is a polynomial-time algorithm to decide if a graph in the class is $(k-1)$-colorable.

In this paper, we initiate a systematic study of the finiteness of k-vertex-critical graphs in subclasses of P_5-free graphs. Our main result is a complete classification of the finiteness of k-vertex-critical graphs in the class of (P_5, H)-free graphs for all graphs H on 4 vertices. To obtain the complete dichotomy, we prove the finiteness for four new graphs H using various techniques – such as Ramsey-type arguments and the dual of Dilworth's Theorem – that may be of independent interest.

Keywords. Graph coloring; k-critical graphs; Dilworth's Theorem; forbidden induced subgraphs.

1 Introduction

All graphs in this paper are finite and simple. We say that a graph G *contains* a graph H if H is isomorphic to an induced subgraph of G. A graph G is H*-free* if it does not contain H. For a family of graphs $\mathcal{H}$, G is $\mathcal{H}$*-free* if G is H-free for every $H \in \mathcal{H}$. When $\mathcal{H}$ consists of two graphs, we write (H_1, H_2)-free instead of $\{H_1, H_2\}$-free. As

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usual, P_t and C_s denote the path on t vertices and the cycle on s vertices, respectively. The complete graph on n vertices is denoted by K_n. The graph K_3 is also referred to as the *triangle*. For two graphs G and H, we use $G + H$ to denote the *disjoint union* of G and H. For a positive integer r, we use rG to denote the disjoint union of r copies of G. The *complement* of G is denoted by $\overline{G}$. A *clique* (resp. *independent set*) in a graph is a set of pairwise adjacent (resp. nonadjacent) vertices. If a graph G can be partitioned into k independent sets $S_1, \ldots, S_k$ such that there is an edge between every vertex in S_i and every vertex in S_j for all $1 \leq i < j \leq k$, G is called a *complete k-partite graph*; each S_i is called a *part* of G. If we do not specify the number of parts in G, we simply say that G is a *complete multipartite graph*. We denote by $K_{n_1,\ldots,n_k}$ the complete k-partite graph such that the ith part S_i has size n_i, for each $1 \leq i \leq k$. A *q-coloring* of a graph G is a function $\phi : V(G) \longrightarrow \{1, \ldots, q\}$ such that $\phi(u) \neq \phi(v)$ whenever u and v are adjacent in G. Equivalently, a q-coloring of G is a partition of $V(G)$ into q independent sets. A graph is *q-colorable* if it admits a q-coloring. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number q for which G is q-colorable. The *clique number* of G, denoted by $\omega(G)$, is the size of a largest clique in G.

A graph G is *k-chromatic* if $\chi(G) = k$. We say that G is *k-critical* if it is k-chromatic and $\chi(G - e) < \chi(G)$ for any edge $e \in E(G)$. For instance, K_2 is the only 2-critical graph and odd cycles are the only 3-critical graphs. A graph is *critical* if it is k-critical for some integer $k \geq 1$. Critical graphs were first defined and studied by Dirac [12, 13, 14] in the early 1950s, and then by Gallai and Ore [16, 17, 30] among many others, and more recently by Kostochka and Yancey [25].

A weaker notion of criticality is the so-called vertex-criticality. A graph G is *k-vertex-critical* if $\chi(G) = k$ and $\chi(G - v) < k$ for any $v \in V(G)$. For a set $\mathcal{H}$ of graphs and a graph G, we say that G is *k-vertex-critical $\mathcal{H}$-free* if it is k-vertex-critical and $\mathcal{H}$-free. We are mainly interested in the following question.

The meta question. Given a set $\mathcal{H}$ of graphs and an integer $k \geq 1$, are there only finitely many k-vertex-critical $\mathcal{H}$-free graphs?

This question is important in the study of algorithmic graph theory because of the following theorem.

Theorem 1 (Folklore)**.** *Given a set $\mathcal{H}$ of graphs and an integer $k \geq 1$, if the set of all k-vertex-critical $\mathcal{H}$-free graphs is finite, then there is a polynomial-time algorithm to determine whether an $\mathcal{H}$-free graph is $(k-1)$-colorable.*

In this paper, we study k-vertex-critical graphs in the class of P_5-free graphs. Our research is mainly motivated by the following two results.

Theorem 2 ([22])**.** *For any fixed $k \geq 5$, there are infinitely many k-vertex-critical P_5-free graphs.*

Theorem 3 ([4, 27])**.** *There are exactly 12 4-vertex-critical P_5-free graphs.*

In light of Theorem 2 and Theorem 3, it is natural to ask which subclasses of P_5-free graphs have finitely many k-vertex-critical graphs for $k \geq 5$. For example, it was known that there are exactly 13 5-vertex-critical (P_5, C_5)-free graphs [22], and that there are finitely many 5-vertex-critical (P_5,banner)-free graphs [5, 23], and finitely many k-vertex-critical $(P_5, \overline{P_5})$-free graphs for every fixed k [10]. Hell and Huang proved that there are finitely many k-vertex-critical (P_6, C_4)-free graphs [20]. This was later generalized to $(P_t, K_{r,s})$-free graphs in the context of H-coloring [24]. Apart from these, there seem to be very few results on the finiteness of k-vertex-critical graphs for $k \geq 5$. The reason for this, we think, is largely because of the lack of a good characterization of k-vertex-critical graphs. In this paper, we introduce new techniques into the problem and prove some new results beyond 5-vertex-criticality.

1.1 Our Contributions

We initiate a systematic study on the subclasses of P_5-free graphs. In particular, we focus on (P_5, H)-free graphs when H has small number of vertices. If H has at most three vertices, the answer is either trivial or can be easily deduced from known results. So we study the problem for graphs H when H has four vertices. There are 11 graphs on four vertices up to isomorphism:

- K_4 and $\overline{K_4} = 4P_1$;
- $P_2 + 2P_1$ and $\overline{P_2 + 2P_1}$;
- C_4 and $\overline{C_4} = 2P_2$;
- $P_1 + P_3$ and $\overline{P_1 + P_3}$;
- $K_{1,3}$ and $\overline{K_{1,3}} = P_1 + K_3$;
- $P_4 = \overline{P_4}$.

The graphs $\overline{P_2 + 2P_1}$, $\overline{P_1 + P_3}$ and $K_{1,3}$ are usually called *diamond*, *paw* and *claw*, respectively.

One can easily answer our meta question for some graphs H using known results, e.g., Ramsey's Theorem for $4P_1$-free graphs: any k-vertex-critical $(P_5, 4P_1)$-free graph is either K_k or has at most $R(k, 4) - 1$ vertices, where $R(s, t)$ is the Ramsey number, namely the minimum positive integer n such that every graph of order n contains either a clique of size s or an independent set of size t. However, the answer for certain graphs H cannot be directly deduced from known results. In this paper, we prove that there are only finitely many k-vertex-critical (P_5, H)-free graphs for every fixed $k \geq 1$ when H is K_4, or $\overline{P_2 + 2P_1}$, or $P_2 + 2P_1$, or $P_1 + P_3$. (Note that these results do not follow from the finiteness of k-vertex-critical $(P_5, \overline{P_5})$-free graphs proved in [10].) By combining our new results with known results, we obtain a complete classification of the finiteness of k-vertex-critical (P_5, H)-free graphs when H has 4 vertices.

Theorem 4. *Let H be a graph of order 4 and $k \geq 5$ be a fixed integer. Then there are infinitely many k-vertex-critical (P_5, H)-free graphs if and only if H is $2P_2$ or $P_1 + K_3$.*

To obtain the complete classification, we employ various techniques, some of which have not been used before to the best of our knowledge. For $H = K_4$, we used a hybrid approach combining the power of a computer algorithm and mathematical analysis. For $P_1 + P_3$ and $P_2 + 2P_1$, we used the idea of fixed sets (that was first used in [21] to give a polynomial-time algorithm for k-coloring P_5-free graphs for every fixed k) combined with Ramsey-type arguments and the dual of Dilworth's Theorem. We hope that these techniques could be helpful for attacking other related problems.

The remainder of the paper is organized as follows. We present some preliminaries in Section 2 and prove our new results in Section 3. Finally, we give the proof of Theorem 4 in Section 4.

2 Preliminaries

For general graph theory notation we follow [1]. Let $G = (V, E)$ be a graph. If $uv \in E$, we say that u and v are *neighbors* or *adjacent*; otherwise u and v are *nonneighbors* or *nonadjacent*. The *neighborhood* of a vertex v, denoted by $N_G(v)$, is the set of neighbors of v. For a set $X \subseteq V(G)$, let $N_G(X) = \bigcup_{v \in X} N_G(v) \setminus X$. We shall omit the subscript whenever the context is clear. For $X, Y \subseteq V$, we say that X is *complete* (resp. *anticomplete*) to Y if every vertex in X is adjacent (resp. nonadjacent) to every vertex in Y. If $X = \{x\}$, we write "x is complete (resp. anticomplete) to Y" instead of "$\{x\}$ is complete (resp. anticomplete) to Y". If a vertex v is neither complete nor anticomplete to a set S, we say that v is *mixed* on S. We say that H is a *homogeneous set* if no vertex in $V - H$ is mixed on H. A vertex is *universal* in G if it is adjacent to all other vertices. A vertex subset $K \subseteq V$ is a *clique cutset* if $G - K$ has more components than G and K induces a clique. For $S \subseteq V$, the subgraph *induced* by S, is denoted by $G[S]$. A *k-hole* in a graph is an induced cycle H of length $k \geq 4$. If k is odd, we say that H is an *odd hole*. A *k-antihole* in G is a k-hole in $\overline{G}$. Odd antiholes are defined analogously. The graph obtained from C_k by adding a universal vertex, denoted by W_k, is called the *k-wheel*.

List coloring. Let $[k]$ denote the set $\{1, 2, \dots, k\}$. A *k-list assignment* of a graph G is a function $L : V(G) \to 2^{[k]}$. The set $L(v)$, for a vertex v in G, is called the *list* of v. In the *list k-coloring* problem, we are given a graph G with a k-list assignment L and asked whether G has an *L-coloring*, i.e., a k-coloring of G such that every vertex is assigned a color from its list. We say that G is *L-colorable* if G has an L-coloring. If the list of every vertex is $[k]$, then the list k-coloring problem is precisely the k-coloring problem.

A common technique in the study of graph coloring is called *propagation*. If a vertex v has its color forced to be $i \in [k]$, then no neighbor of v can be colored with color i. This motivates the following definition.

Let (G, L) be an instance of the list k-coloring problem. The color of a vertex v is said to be *forced* if $|L(v)| = 1$. A *propagation from a vertex* v with $L(v) = \{i\}$ is the procedure of removing i from the list of every neighbor of v. If we denote the resulting k-list assignment by L', then G is L-colorable if and only if $G - v$ is L'-colorable. A propagation from v could make the color of other vertices forced; if we continue to propagate from those vertices until no propagation is possible, we call the procedure "*exhaustive propagation from* v". It is worth mentioning that the idea of propagation is featured in many recent studies on coloring P_t-free graphs and related problems, see [2, 6] for example.

An example of propagation. Let G be a 4-vertex path w, x, y, z with $L(w) = \{1\}$, $L(x) = \{1, 2\}$, $L(y) = \{2, 3\}$, and $L(z) = \{1, 2\}$. Then propagation from w results in the new list assignment L' where $L'(x) = \{2\}$ and $L'(v) = L(v)$ for $v \neq x$. On the other hand, exhaustive propagation from w results in the new list assignment L'' where $L''(w) = \{1\}$, $L''(x) = \{2\}$, $L''(y) = \{3\}$, $L''(z) = \{1, 2\}$.

We proceed with a few useful results that will be needed later. The first one is a folklore property of k-vertex-critical graphs.

Lemma 1 (Folklore). *Any k-vertex-critical graph cannot contain clique cutsets.*

Another folklore property of vertex-critical graphs is that such graph cannot contain two nonadjacent vertices u, v such that $N(v) \subseteq N(u)$. We generalize this property to anticomplete subsets.

Lemma 2. *Let G be a k-vertex-critical graph. Then G has no two nonempty disjoint subsets X and Y of $V(G)$ that satisfy all the following conditions.*

- *X and Y are anticomplete to each other.*
- *$\chi(G[X]) \leq \chi(G[Y])$.*
- *Y is complete to $N(X)$.*

Proof. Suppose that G has a pair of nonempty subsets X and Y that satisfy all three conditions. Since G is k-vertex-critical, $G - X$ has a $(k-1)$-coloring ϕ. Let $t = \chi(G[Y])$. Since Y is complete to $N(X)$, at least t colors do not appear on any vertex in $N(X)$ under ϕ. So we can obtain a $(k-1)$-coloring of G by coloring $G[X]$ with those t colors. This contradicts that G is k-chromatic. $\square$

A graph G is *perfect* if $\chi(H) = \omega(H)$ for each induced subgraph H of G. An *imperfect* graph is a graph that is not perfect. A classical theorem of Dilworth [11] states that the largest size of an antichain in a partially ordered set is equal to the minimum number of chains that partition the set. We will use the dual of Dilworth's Theorem which says that the largest size of a chain in a partially ordered set is equal to the minimum number of antichains that partition the set. This was first proved by Mirsky [28] and it has an equivalent graph-theoretic interpretation via comparability graphs. A graph is a *comparability graph* if the vertices of the graph are elements of a partially ordered set and two vertices are connected by an edge if and only if the corresponding elements are comparable.

Theorem 5 (Dual Dilworth Theorem [28])**.** *Every comparability graph is perfect.*

We conclude this section with the celebrated Strong Perfect Graph Theorem [8].

Theorem 6 (Strong Perfect Graph Theorem [8])**.** *A graph is perfect if and only if it contains no odd holes or odd antiholes.*

3 New Results

In this section, we prove four new results: there are finitely many k-vertex-critical (P_5, H)-free graphs when $H \in \{K_4, \overline{P_2 + 2P_1}, P_2 + 2P_1, P_1 + P_3\}$.

3.1 K_4-Free Graphs

Let G_1 be the 13-vertex graph with vertex set $\{0, 1, \ldots, 12\}$ and the following edges:

- $\{3, 4, 5, 6, 7\}$ and $\{0, 1, 2, 8, 9\}$ induce two disjoint 5-holes Q and Q';
- 12 is complete to $Q \cup Q'$;
- 11 is complete to Q and 10 is complete to Q' with 10 and 11 being connected by an edge.

Let G_2 be the 14-vertex graph with vertex set $\{0, 1, \ldots, 13\}$ and the following edges:

- $\{12, 13\}$ is a cutset of G_2 such that 12 and 13 are not adjacent and $G_2 - \{12, 13\}$ has exactly two components;

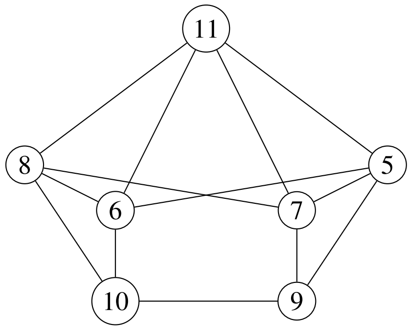

Figure 1: One component of $G_2 - \{12, 13\}$.

- One component of $G_2 - \{12, 13\}$ is a 5-hole induced by $\{0, 1, 2, 3, 4\}$, and this 5-hole is complete to $\{12, 13\}$;
- The other component, induced by $\{5, 6, 7, 8, 9, 10, 11\}$, is the graph in Figure 1, and 12 is complete to $\{5, 8, 9, 10, 11\}$ and 13 is complete to $\{6, 7, 9, 10, 11\}$.

The adjacency lists of G_1 and G_2 are given in the Appendix. It is routine to verify that G_1 and G_2 are 5-vertex-critical (P_5, K_4)-free graphs. The main result in this subsection is that they are the only 5-vertex-critical (P_5, K_4)-free graphs.

Theorem 7. *Let G be a 5-vertex-critical (P_5, K_4)-free graph. Then G is isomorphic to either G_1 or G_2.*

We will prove Theorem 7 in a series of intermediate steps. We will need the following result.

Theorem 8 ([9])**.** *Any K_4-free graph with no odd holes is 4-colorable.*

The next two lemmas are based on a computer generation approach to exhaustively generate all k-vertex-critical graphs in a given class of $\mathcal{H}$-free graphs via a recursive algorithm. The idea of computer generation was first used in [22], and later developed extensively by Goedgebeur and Schaudt [19] and Chudnovsky et al. [7].

We say that G' is a *1-vertex extension* of G if G can be obtained from G' by deleting a vertex in G'. Roughly speaking, the generation algorithm starts with some small substructure which must occur in any k-vertex-critical graph, and then exhaustively searches for all 1-vertex extensions of the substructure. The algorithm stores those extensions that are k-vertex-critical and $\mathcal{H}$-free in the output list $\mathcal{F}$. Then it recursively repeats the procedure for all $(k-1)$-colorable substructures found in the previous iterations. The pesudocode of the generation algorithm is given in Algorithm 1 and Algorithm 2.

It should be noted that with a naive implementation the algorithm may not terminate. For instance, if we extend a graph G by repeatedly adding vertices that have the same neighborhood as a vertex in G, the program will never terminate. So one has to design certain pruning rules to make the algorithm terminate. For instance, if G contains two nonadjacent vertices u, v such that $N(u) \subseteq N(v)$, then we only need to consider all 1-vertex extensions G' such that the unique vertex in $V(G') \setminus V(G)$ is adjacent to u but not adjacent to v (by Lemma 2). In [22], the authors designed two pruning rules like this so that the algorithm terminates with 13 5-vertex-critical (P_5, C_5)-free graphs.

Later, the technique was extensively developed by Goedgebeur and Schaudt [19] who introduced many more useful pruning rules that are essential for generating all critical graphs in certain classes of graphs, e.g., 4-vertex-critical (P_7, C_4)-free graphs and 4-vertex-critical (P_8, C_4)-free graphs. The word "valid" in Algorithm 2 is used precisely to quantify those extensions that survive a specific set of pruning rules.

The algorithm we use in this paper is exactly the one developed in [19]. Hence, the valid extensions on line 8 in Extend(G) are with respect to all pruning rules given in Algorithm 2 in [19] (since we only use those rules as a black box, we do not define them here).

Theorem 9 ([19]). *If Algorithm 1 terminates and returns the list $\mathcal{F}$, then $\mathcal{F}$ is exactly the set of all k-vertex-critical $\mathcal{H}$-free graphs containing S.*

Algorithm 1: Generate(k, $\mathcal{H}$, S)

Input: An integer k, a set $\mathcal{H}$ of forbidden induced subgraphs, and a graph S.
Output: A list $\mathcal{F}$ of all k-vertex-critical $\mathcal{H}$-free graphs containing S.

```
1  Let F be an empty list.
2  Extend(S).
3  Return F.
```

Algorithm 2: Extend(G)

```
1  if G is H-free and is not generated before then
2      if χ(G) ≥ k then
3          if G is k-vertex-critical then
4              add G to F
5          end
6      end
7      else
8          for each valid 1-vertex extension G' of G do
9              Extend(G')
10         end
11     end
12 end
```

Let F be the graph obtained from a 5-hole by adding a new vertex and making it adjacent to four vertices on the hole.

Lemma 3. *Let G be a 5-vertex-critical (P_5, K_4)-free graph. If G contains an induced W_5 or F, then Then G is isomorphic to either G_1 or G_2.*

Proof. We run Algorithm 1 with the following inputs:

- $k = 5$;
- $\mathcal{H} = \{P_5, K_4\}$;
- $S = W_5$ or $S = F$.

If $S = W_5$, then the algorithm terminates with the graphs G_1 and G_2, and if $S = F$, then it terminates with only the graph G_2. The correctness of the algorithm follows from Theorem 9. □

Lemma 4. *Let G be a 5-vertex-critical (P_5, K_4)-free graph. If G is 7-antihole-free, then G is isomorphic to G_1.*

Proof. By Theorem 7, G must contain a 5-hole. We run Algorithm 1 with the following inputs:

- $k = 5$;
- $\mathcal{H} = \{P_5, K_4, \overline{C_7}\}$;
- $S = C_5$

The algorithm terminates and outputs G_1 as the only critical graph. The correctness of the algorithm follows from Theorem 9. □

Lemma 5. *Let G be a (P_5, K_4, W_5, F)-free graph. If G contains an 7-antihole, then G is 4-colorable.*

Proof. Let $C = v_1, v_2, \ldots, v_7$ be a 7-antihole with v_iv_{i+1} being a nonedge. For each $1 \leq i \leq 7$, let T_i be the set of vertices in $V \setminus V(C)$ that are adjacent to v_{i-1}, v_i, v_{i+1}, and F_i be the set of vertices in $V \setminus V(C)$ that are adjacent to $V(C) \setminus \{v_{i-1}, v_i, v_{i+1}\}$.

Claim 1. $V \setminus V(C) = \cup_{1 \leq i \leq 7}(F_i \cup T_i)$.

Proof. Let $x \in V \setminus V(C)$ that has at least one neighbor in C. Since G is K_4-free, x has at most four neighbors on C. Suppose first that x has at most two neighbors on C. If x is adjacent to v_4 and v_5, then $\{v_3, v_4, v_5, v_6, x\}$ induces a 5-hole and v_1 is adjacent to four vertices on the hole. This contradicts that G is F-free. So x cannot be adjacent only to v_i and v_{i+1} for some i. Thus, we may assume by symmetry that x is adjacent to v_1 and possibly to v_3 or v_4 (but not both). Then x, v_1, v_6, v_2, v_7 is an induced P_5, a contradiction. Now suppose that x has three neighbors on C. Since G is K_4-free, x has at least two consecutive neighbors, say v_1, v_2 by symmetry. If x is adjacent to v_3 or v_7, then $x \in T_1$ or $x \in T_2$. So x is not adjacent to v_3 or v_7. Since x is adjacent to only one vertex in $\{v_4, v_5, v_6\}$, $G[\{v_3, v_4, v_5, v_6, v_7, x\}]$ contains an induced P_5, a contradiction. Now suppose that x has four neighbors on C. Then x must have two consecutive neighbors, say v_1, v_2 by symmetry. If x does not have three consecutive neighbors, then x is not adjacent to v_3 or v_7. Then $\{v_7, v_1, v_2, v_3, x\}$ induces a C_5. Since G is W_5-free, x is not adjacent to v_5, and so is adjacent to v_4 and v_6. But then $\{v_1, v_4, v_6, x\}$ induces a K_4. Thus, x is adjacent to v_3 or v_7, say v_3 by symmetry. If x is adjacent to v_7 or v_4, then $x \in F_5$ or $x \in F_6$. Otherwise x is adjacent to exactly one of v_5 or v_6. But then $\{v_4, v_5, v_6, v_7, x\}$ induces a P_5.

Now let $z \in V \setminus V(C)$ that has no neighbor in C. Since G is connected and P_5-free, z has a neighbor in T or F. If z is adjacent to $t_1 \in T_1$, then z, t_1, v_2, v_5, v_3 is an induced P_5. If t_1 is adjacent to $f_1 \in F_1$, then z, f_1, v_3, v_7, v_2 is an induced P_5. So there is no such vertex. This proves the claim. □

Note that since G is K_4-free, F_i and T_i are independent sets for each $1 \leq i \leq 7$. We now investigate the adjacency among T_i and F_i for $1 \leq i \leq 7$.

Claim 2. *For each i, T_i is anticomplete to T_{i+1}, and is complete to T_{i+3}.*

Proof. By symmetry, it suffices to prove the claim for $i = 1$. Let $t_1 \in T_1$. If t_1 is adjacent to $t_2 \in T_2$, then t_2, t_1, v_7, v_4, v_6 is an induced P_5. If t_1 is not adjacent to $t_3 \in T_3$, then $\{v_2, t_3, v_3, t_1, v_1, \}$ If t_1 is not adjacent to $t_4 \in T_4$, then t_1, v_2, v_6, v_3, t_4 is an induced P_5. □

Claim 3. *For each i, F_i is complete to $T_{i-1} \cup T_i \cup T_{i+1}$, and anticomplete to T_{i+3}.*

Proof. Let $f \in F_1$. Note that $C' = V(C) \setminus \{v_1\} \cup \{f\}$ induces a 7-antihole, where f plays the role of v_1. If $t_1 \in T_1$ is not adjacent to f, then it is adjacent to two nonconsecutive vertices on C', contradicting Claim 1. If $t \in T_2 \cup T_7$ is not adjacent to f, then t is adjacent to exactly two consecutive vertices on C' , contradicting Claim 1. If $t \in T_4$ is adjacent to f, the neighbors of t on C' are not consecutive, contradicting Claim 1. This proves the claim. □

Claim 4. *For each i, F_i is anticomplete to F_{i+1}, and complete to F_{i+3}.*

Proof. It suffices to prove for $i = 1$. Let $f \in F_1$. If f is adjacent to $f' \in F_2$, then $\{f, f', v_4, v_6\}$ induces a K_4. If f is not adjacent to $f' \in F_4$, then the neighbors of f' on $C' = V(C) \setminus \{v_1\} \cup \{f\}$ are not consecutive, contradicting Claim 1. This proves the claim. □

Claim 5. *For each $t \in T_i$, $N(t) \subseteq N(v_{i-3}) \cup N(v_{i+3})$.*

Proof. We prove for $i = 1$. Let x be a common nonneighbor of v_4 and v_5. If $x = v_4$ or v_5, then $x \notin N(t)$ by the definition of t. If $x \in T$, then $x \in T_7 \cup T_1 \cup T_2$, and $x \notin N(t)$ by Claim 2. If $x \in F$, then $x \in F_4 \cup F_5$, and $x \notin N(t)$ by Claim 3. This proves the claim. □

Let L be the 4-list assignment of G such that

- $L(v_1) = \{1\}$, $L(v_2) = L(v_3) = \{2\}$, $L(v_4) = L(v_5) = \{3\}$, and $L(v_6) = L(v_7) = \{4\}$,
- and $L(v) = [k]$ for every $v \in V \setminus V(C)$.

Claim 6. *G is L-colorable if and only if $G - (T_6 \cup T_1 \cup T_3)$ is L-colorable.*

Proof. Suppose that $G - (T_6 \cup T_1 \cup T_3)$ has an L-coloring ϕ. We color every vertex in T_1 with color 3, color every vertex in T_3 with color 4, and color eevery vertex in T_6 with color 2. This extended coloring is an L-coloring of G by Claim 5. □

We now prove that G is L-colorable, which implies that G is 4-colorable. By Claim 6, it suffices to show that $G - (T_6 \cup T_1 \cup T_3)$ is L-colorable. We shall do this in a number of steps.

The first step: propagate from C. We propagate from $v_1, \ldots, v_7$.

- The list of every vertex in F_1, F_3, F_4, F_5, F_6 is $\{1\}, \{2\}, \{3\}, \{3\}, \{4\}$ respectively in this order. Every vertex in F_2 has list $\{1, 2\}$ and every vertex in F_7 has list $\{1, 4\}$.
- Every vertex in T_2 has list $\{3, 4\}$, every vertex in T_4 has list $\{1, 4\}$, every vertex in T_5 has list $\{1, 2\}$, and every vertex in T_7 has list $\{2, 3\}$.

Let G' be the subgraph of G with list assignment L' described in Figure 2. Note that G' is not an induced subgraph of G. It follows from Claims 2-4 that G is L-colorable if and only if G' is L'-colorable. (Some vertex subsets such as F_1 and edges such as those between T_2 and F_2 are irrelevant in terms of coloring because of either disjoint lists or nonadjacency between vertices.)

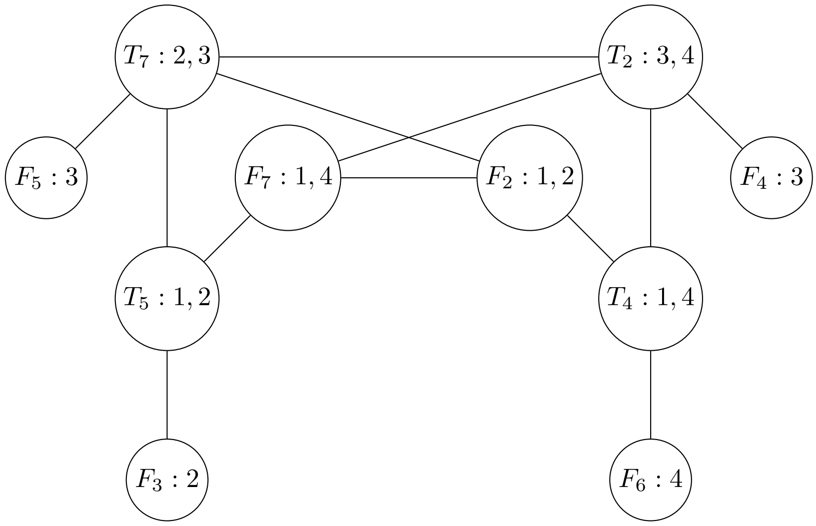

Figure 2: The instance (G', L'). A line between any two sets means that the edges between the two sets are arbitrary. No line means that edges are irrelevant in terms of coloring.

The second step: propagate exhaustively from F_3, F_4, F_5, F_6. We propagate the coloring from all vertices in $F_3 \cup F_4 \cup F_5 \cup F_6$ exhaustively.

Let $T'_7 = N(F_5) \cap T_7$, $T'_5 = N(T'_7 \cup F_3) \cap T_5$, and $F'_7 = N(T'_5) \cap F_7$. Since every vertex in F_5 has list $\{3\}$, every vertex in T'_7 is must be colored with 2 in any L'-coloring. Similarly, every vertex in T'_5 and in F'_7 must be colored with 1 and 4, respectively. Symmetrically, let $T'_2 = N(F_4) \cap T_2$, $T'_4 = N(T'_2 \cup F_6) \cap T_4$, and $F'_2 = N(T'_4) \cap F_2$. Then every vertex in T'_2 must be colored with 4, every vertex in T'_4 must be colored with 1, and every vertex in F'_2 must be colored with 2. Let L'' denote the resulting list assignment. For every set $S \in \{T_2, T_4, T_5, T_7, F_2, F_7\}$, let $S'' = S \setminus S'$. Let $G'' = G[T''_2 \cup T''_4 \cup T''_5 \cup T''_7 \cup F''_2 \cup F''_7]$.

Claim 7. *For each i, every vertex in T_i is anticomplete to either F_{i-2} or F_{i+2}*

Proof. If $t_1 \in T_1$ has a neighbor $f_3 \in F_3$ and a neighbor $f_6 \in F_6$, then $\{t_1, f_3, f_6, v_1\}$ induces a K_4. $\square$

Claim 8. *For each i, every vertex in F_i is anticomplete to either T_{i-2} or T_{i+2}.*

Proof. Suppose that $f_1 \in F_1$ has a neighbor $t_3 \in T_3$ and a neighbor $t_6 \in T_6$. Then $Q = v_1, v_4, t_3, t_6, v_5$ is a 5-hole with f_1 having four neighbors on Q. This contradicts that G is F-free. $\square$

By Claim 7, T'_7 and T'_2 are anticomplete to F_2 and F_7, respectively. By Claim 8, F'_7 and F'_2 are anticomplete to T_2 and T_7, respectively. Therefore, G' is L'-colorable if and only if G'' is L''-colorable.

The final step: color G''. We finish the proof by giving an L''-coloring of G''.

- Color every vertex in F''_7 with color 4 and every vertex in F''_2 with color 1.

- Assign color 4 to those vertices in T_4'' that are neighbors of F_2'', and assign color 1 to the remaining vertices in T_4''.
- Assign color 3 to those vertices in T_2'' that are neighbors of F_7'' or neighbors of vertices in T_4'' with color 4, and assign color 4 to the remaining vertices in T_2''.
- Assign color 2 to those vertices in T_7'' that are neighbors of T_2'' with color 3, and assign color 3 to the remaining vertices in T_7''.
- Assign color 1 to those vertices in T_5'' that are neighbors of T_7'' with color 2, and assign color 2 to the remaining vertices in T_5''.

It is routine to verify that the assignment is an L''-coloring of G''. This completes the proof. □

We are now ready to prove Theorem 7.

Proof of Theorem 7. Let G be a 5-vertex-critical (P_5, K_4)-free graph. If G contains an induced W_5 or F, then G is either G_1 or G_2 by Lemma 3. So we can assume that G is (W_5, F)-free as well. By Lemma 5, G must be 7-antihole-free, and so is G_1 by Lemma 4. □

3.2 $P_1 + P_3$-Free Graphs

Theorem 10. *For every fixed integer $k \geq 1$, there are finitely many k-vertex-critical $P_1 + P_3$-free graphs.*

Proof. Let G be a k-vertex-critical $P_1 + P_3$-free graph. If G contains a K_k, then G is isomorphic to K_k. So we assume in the following that G is K_k-free. Let $K = \{v_1, \ldots, v_t\}$ be a maximal clique, where $1 \leq t < k$. Since K is maximal, every vertex in $V \setminus K$ is not adjacent to at least one vertex in K. We partition $V \setminus K$ into the following subsets.

- F_1 is the set of nonneighbors of v_1.
- For $2 \leq i \leq t$, F_i is the set of nonneighbors of v_i that are not in $F_1 \cup \cdots \cup F_{i-1}$.

By the definition, v_i is complete to F_j if $i < j$. Since G is $P_1 + P_3$-free, each F_i is P_3-free, and so is a disjoint union of cliques.

Claim 9. *If F_i has at least two components, then every neighbor of v_i is either complete or anticomplete to F_i.*

Proof. Let v be a neighbor of v_i. Suppose that v has a neighbor f in F_i. Let K be the component of F_i containing f. If v is not adjacent to some vertex $f' \in F_1 \setminus K$, then $\{f', f, v, v_i\}$ induces a $P_1 + P_3$, a contradiction. So v is complete to $F_i \setminus K$. Since F_i has at least two components, v has a neighbor in a component other than K. It follows from the same argument that v is complete to K. This completes the proof. □

Claim 10. *For every nonneighbor v of v_i and every component K of F_i, v is either complete or anticomplete to K.*

Proof. If v is mixed on an edge xy in K, then $\{v, v_i, x, y\}$ induces a $P_1 + P_3$, a contradiction. □

By Claim 9 and Claim 10, if F_i has at least two components, every component of F_i is a homogeneous set of G. Moreover, since v_i is complete to F_j for $i < j$, no vertex in $\{v_j\} \cup F_j$ with $j > i$ is mixed on two components of F_i. We next show that each F_i has bounded size.

Claim 11. *$|F_1| \leq k$.*

Proof. We show that F_1 is connected. Suppose not. Let K and K' be two component of F_1 with $|K| \leq |K'|$. Then $N(K) = N(K')$. By Lemma 2, G is not k-vertex-critical. This is a contradiction. Therefore, F_1 is a clique and so has at most k vertices. □

Claim 12. *For each $1 \leq i \leq t$, F_i has bounded size.*

Proof. We prove this by induction on i. By Claim 11, the statement is true for $i = 1$. Now assume that $i \geq 2$ and F_j has bounded size for each $1 \leq j < i$. If F_i is connected, then $|F_i| \leq k$ and we are done. So we assume that F_i has at least two components. We will show that the number of components in F_i is bounded and this will complete the proof. For this purpose, we construct a graph X as follows.

- $V(X)$ is the set of all components of F_i.
- Two components K and K' of F_i are connected by an edge in X if and only if $N(K) \subseteq N(K')$ or $N(K') \subseteq N(K)$.

Note that X is a comparability graph. Next we show that $\omega(X) \leq k$. Suppose that $K_1, \ldots, K_t$ is a maximum clique in X with $t > k$. We may assume that $N(K_1) \subseteq N(K_2) \subseteq \cdots \subseteq N(K_t)$. It follows from Lemma 2 that $|K_i| > |K_j|$ for $i < j$, i.e., $|K_1| > |K_2| > \cdots > |K_t| \geq 1$. So $|K_1| \geq k$. This is a contradiction, since G is K_k-free. This proves that $\omega(X) \leq k$. Since X is perfect by Theorem 5, $V(X)$ can be partitioned into at most k independent sets $S_1, \ldots, S_k$. We show that each S_p has bounded size. Let K and K' be two components in S_p. Then there are vertices x and x' such that $x \in N(K) \setminus N(K')$ and $x' \in N(K') \setminus N(K)$. Note that $x, x' \in T_i = \bigcup_{1 \leq j < i} F_j \cup \{v_j\}$. If $|S_p| > 2|T_i|^2$, by the pigeonhole principle, there are two pairs $\{K, K'\}$ and $\{L, L'\}$ of components that correspond to the same pair $\{x, x'\}$ in T_i. Then $\{K, x, L, K'\}$ induces a $P_1 + P_3$. This shows that each S_p has size at most $2|T_i|^2$, which is a constant by the inductive hypothesis. Therefore, X has constant number of vertices, i.e., F_i has constant number of components. This completes the proof. □

By Claim 11 and Claim 12, each $|F_i| \leq M$ for some constant M (depending only on k). Therefore, G has bounded size. □

3.3 $P_2 + 2P_1$-Free Graphs

Theorem 11. *For every fixed integer $k \geq 1$, there are finitely many k-vertex-critical $(P_5, P_2 + 2P_1)$-free graphs.*

Proof. Let G be a k-vertex-critical $(P_5, P_2 + 2P_1)$-free graph. If G contains a K_k, then G is isomorphic to K_k. So we assume in the following that G is K_k-free. Let $K = \{v_1, \ldots, v_t\}$ be a maximal clique, where $1 \leq t < k$. Since K is maximal, every vertex in $V \setminus K$ is not adjacent to at least one vertex in K. We partition $V \setminus K$ into the following subsets.

- F_1 is the set of nonneighbors of v_1.
- For $2 \leq i \leq t$, F_i is the set of nonneighbors of v_i that are not in $F_1 \cup \cdots \cup F_{i-1}$.

By the definition, v_i is complete to F_j if $i < j$. Since G is $2P_1 + P_2$-free, each F_i is $P_1 + P_2$-free, and so is a complete multipartite graph. Since G has no K_k, each F_i has at most k parts.

Claim 13. *Let S be a part of F_i and T be a part in F_j with $i < j$. Then $G[S \cup T]$ is a $2P_2$-free graph.*

Proof. Suppose not. Let s_1t_1 and s_2t_2 be an induced $2P_2$, where $s_i \in S$ and $t_i \in T$ for $i = 1, 2$. Then since v_i is not adjacent to s_1, s_2 and is adjacent to t_1, t_2, it follows that s_1, t_1, v_i, t_2, s_2 induces a P_5, a contradiction. □

Claim 14. *Let S be a part of F_i and T be a part in F_j with $i < j$. Every vertex in T is adjacent to all but at most one vertex in S.*

Proof. Suppose that $t \in T$ is not adjacent to two vertices s, s' in S. Then $\{v_i, s, s', t\}$ induces a $2P_1 + P_2$, a contradiction. □

Next we show that each part in F_i has bounded size.

Claim 15. *F_t is an independent set of bounded size.*

Proof. If F_t has at least two parts, then any two vertices from two different parts and $K \setminus \{v_t\}$ form a clique of size $|K| + 1$, contradicting the choice of K. So F_i is an independent set. By Claim 14, each vertex in F_t is adjacent to all but at most one vertex in any part of F_i with $1 \leq i \leq t-1$. For each part S in $F_1 \cup \cdots \cup F_{t-1}$, we introduce a binary variable $X_S \in \{0, 1\}$. If $X_S = 0$, it indicates that a vertex in F_t is complete to S while $X_S = 1$ indicates that a vertex in F_t is adjacent to all vertices in S except one vertex. A *type* is a binary vector $(X_S)_{S \text{ is a part of } F_i \text{ with } i<t}$. Since the number of parts in each F_i is at most k, there are at most $2^{kt} \leq 2^{k^2}$ types. If $|F_t| > 2^{k^2}$, then there are two vertices $x, y \in F_t$ that have the same type. Let us fix a part $S \in F_1 \cup \cdots \cup F_{t-1}$. If $X_S = 0$, then both x and y are complete to S. If $X_S = 1$, then each of x and y has a unique nonneighbor x' and y' in S. If $x' \neq y'$, then $\{x, x', y, y'\}$ induces a $2P_2$, which contradicts Claim 13. So $x' = y'$ and thus x and y have the same neighbors in S. Since x and y have the same type, $N(x) = N(y)$, contradicting Lemma 2. □

Claim 16. *For each $1 \leq i \leq t$, F_i has bounded size.*

Proof. The statement is true for $i = t$ by Claim 15. Now suppose that $i < t$ and F_j has bounded size for each $i < j \leq t$. For each part S in $F_1 \cup \cdots \cup F_{i-1}$, we introduce a binary variable $X_S \in \{0, 1\}$. Moreover, for each vertex u in $\{v_j\} \cup F_j$ for $j > i$, we introduce a binary variable $X_{\{u\}} \in \{0, 1\}$. The meaning of $X_{\{u\}}$ is to indicate whether a vertex in F_i is a neighbor or a nonneighbor of u. A *type* is a binary vector

$$(X_S)_{S \text{ is a part of } F_\ell \text{ with } \ell<i \text{ or } S=\{u\} \text{ for some vertex } u\in\{v_j\}\cup F_j \text{ with } j>i}.$$

By the inductive hypothesis, each F_j with $j > i$ has bounded size. Therefore, there is a constant M depending only on k such that the number of types is at most M. If a part T in F_i has size larger than M, there are two vertices $x, y \in T$ having the same type. Using the exact same argument in Claim 15, it follows that $N(x) = N(y)$. This contradicts Lemma 2. Therefore, each part in F_i has bounded size and so does F_i. □

By Claim 15 and Claim 16, G has bounded size. $\square$

3.4 Diamond-Free Graphs

Theorem 12. *For every fixed integer $k \geq 1$, there are finitely many k-vertex-critical (P_5, diamond)-free graphs.*

Proof. Let G be a k-vertex-critical (P_5, diamond)-free graph. We show that $|G| \leq \max\{k, 57\}$. If G contains a K_k, then G is isomorphic to K_k and thus $|G| \leq \max\{k, 57\}$. So assume that G is K_k-free. Since G is imperfect, G contains an induced C_5 by Theorem 6. Let $C = v_1, v_2, v_3, v_4, v_5$ be an induced C_5. For each $1 \leq i \leq 5$, we define

$$\begin{aligned} Z &= \{v \in V \setminus C : N_C(v) = \emptyset\}, \\ R_i &= \{v \in V \setminus C : N_C(v) = \{v_{i-1}, v_{i+1}\}\}, \\ Y_i &= \{v \in V \setminus C : N_C(v) = \{v_{i-2}, v_i, v_{i+2}\}\}. \end{aligned}$$

Let $R = \cup_{1 \leq i \leq 5} R_i$ and $Y = \cup_{1 \leq i \leq 5} Y_i$.

Claim 17. $V(G) = V(C) \cup Z \cup R \cup Y$.

Proof. Let $v \in V(G) \setminus V(C)$. If v has three consecutive neighbors v_i, v_{i+1}, v_{i+2} on C, then $\{v, v_i, v_{i+1}, v_{i+2}\}$ induces a diamond. So if v has at least three neighbors on C, $v \in Y$. If v has no neighbors on C, then $v \in Z$. Now assume that $1 \leq |N(v) \cap C| \leq 2$. If $N(v) \subseteq \{v_{i-2}, v_{i+2}\}$ for some i, say v is adjacent to v_{i+2}, then $C \setminus \{v_{i-2}\} \cup \{v\}$ induces a P_5. So $v \in C$. This completes the proof. $\square$

We first bound Y.

Claim 18. *Each R_i and Y_i is an independent set.*

Proof. Suppose that R_i contains two adjacent vertices x and y, then $\{x, y, v_{i-1}, v_{i+1}\}$ induces a diamond. The proof for Y_i is the same. $\square$

Claim 19. *For each $1 \leq i \leq 5$, $|Y_i| \leq 1$.*

Proof. If Y_i contains two nonadjacent vertices x, y, then $\{x, y, v_{i-2}, v_{i+2}\}$ induces a diamond. So Y_i is a clique. By Claim 18, $|Y_i| \leq 1$. $\square$

Next we bound Z.

Claim 20. *Z is anticomplete to R.*

Proof. Let $z \in Z$. If z has a neighbor $r \in R_1$, then z, r, v_2, v_3, v_4 induces a P_5. $\square$

Claim 21. *Each vertex in Y is either complete or anticomplete to a component of Z.*

Proof. Let $y \in Y_1$. If y is mixed on an edge wz in Z with $yw \notin E$ and $yz \in E$, then w, z, y, v_4, v_5 induces a P_5. $\square$

Claim 22. $|Z| \leq 32$.

Proof. We first show that Z is an independent set. Let Q be any component of Z. Then $N(Q) \subseteq Y$ by Claim 20. By Lemma 1, $N(Q)$ contains two nonadjacent vertices $y, y' \in Y$. By Claim 21, $\{y, y'\}$ is complete to Q. Since G is diamond-free, Q is a singleton. This proves that Z is an independent set. If $|Z| > 32$, then there are two vertices in Z that have the same neighborhood by Claim 19. This contradicts Lemma 2. $\square$

Finally, we bound R.

Claim 23. *R_i and R_{i+1} are complete to each other.*

Proof. Let $r_3 \in R_3$ and $r_4 \in R_4$. If $r_3r_4 \notin E$, then r_4, v_5, v_1, v_2, r_3 induces a P_5. $\square$

Claim 24. *$G[R_i \cup R_{i+2}]$ contains at most one edge.*

Proof. By symmetry, we prove for $i = 1$. Let $r \in R_1$. If r has two neighbors in R_3, then these two vertices together with v_2, r induce a diamond by Claim 18. So every vertex in R_1 has at most one neighbor in R_3. Similarly, every vertex in R_3 has at most one neighbor in R_1. If $G[R_1 \cup R_3]$ contains two edges xy and $x'y'$ with $x, x' \in R_1$ and $y, y' \in R_3$, then y, x, v_5, x', y' induce a P_5. $\square$

Claim 25. *R_i is complete to Y_i and is anticomplete to Y_j for $j \neq i$.*

Proof. Let $r \in R_1$. If r is not adjacent to $y \in Y_1$, y, v_3, v_2, r, v_5 induces a P_5. If r is adjacent to $y \in Y_2$, then $\{r, y, v_2, v_5\}$ induces a diamond. If r is adjacent to $y \in Y_3$, then $\{r, y, v_1, v_5\}$ induces a diamond. This completes the proof. $\square$

Claim 26. *For each $1 \leq i \leq 5$, $|R_i| \leq 3$.*

Proof. Suppose that $|R_1| \geq 4$. Then by Claims 23-25, there are two vertices in R_1 that have the same neighborhood in G. This contradicts Lemma 2. $\square$

By Claim 17, it follows that $|G| = |V(C)| + |Y| + |R| + |Z| \leq 5+5+15+32 = 57$. This proves the theorem. $\square$

4 A Complete Classification

In this section, we prove our main result Theorem 4.

Proof of Theorem 4. An infinite family of 5-vertex-critical $2P_2$-free graphs is constructed in [22]. It can be easily checked that these graphs are $P_1 + K_3$-free. Since $2P_2$ and $P_1 + K_3$ do not contain any universal vertices, for every fixed $k \geq 6$ one can obtain an infinite family of k-vertex-critical $2P_2$-free graphs and $(P_5, P_1 + K_3)$-free graphs by adding $k - 5$ universal vertices to the 5-vertex-critical family in [22].

Now assume that H is not $2P_2$ or $P_1 + K_3$. Let G be a k-vertex-critical (P_5, H)-free graph. We may assume that G is K_k-free for otherwise G is K_k. If $H = 4P_1$, then Ramsey's theorem [31] shows that $|G| \leq R(4, k) - 1$. If $H = K_4$, then there are no k-vertex-critical (P_5, K_4)-free graphs for any $k \geq 6$ [15]. Moreover, there are only two 5-vertex-critical (P_5, K_4)-free graphs by Theorem 7. If H is a diamond or $P_2 + 2P_1$, then the finiteness follows from Theorem 12 and Theorem 11, respectively. If $H = C_4$, then the finiteness follows from [20]. If $H = P_4$, then G is perfect and so $(k-1)$-colorable, a contradiction. If H is a claw, then the finiteness follows from [26]. If H is $P_1 + P_3$, then the finiteness follows from Theorem 10. If H is a paw, then G is either triangle-free or a complete multipartite graph by a result of Olariu [29]. In either case, G is $(k-1)$-colorable, a contradiction. $\square$

In view of Theorem 4, it is natural to ask the following question, which we leave as a possible future direction.

Problem. Which five-vertex graphs H could lead to finitely many k-vertex-critical (P_5, H)-free graphs?

As mentioned in the introduction, it was shown in [10] that $H = \overline{P_5}$ is one such graph.

5 Appendix

The source code of Algorithm 1 and Algorithm 2 which we used in the proofs of Lemma 3 and Lemma 4 can be downloaded from [18]. We refer to [19] for more details on how we verified the correctness of our implementation. We executed the program on an Intel i9-9900 CPU at 3.10GHz and in each case the program terminated in a few seconds.

Below we give the adjacency list of the two 5-vertex-critical (P_5, K_4)-free graphs G_1 and G_2 from Theorem 7. They can also be obtained from the database of interesting graphs at the *House of Graphs* [3] by searching for the keywords "5-critical P5K4-free"[1].

- Graph G_1: {0: 1 2 10 12; 1: 0 8 10 12; 2: 0 9 10 12; 3: 4 5 11 12; 4: 3 6 11 12; 5: 3 7 11 12; 6: 4 7 11 12; 7: 5 6 11 12; 8: 1 9 10 12; 9: 2 8 10 12; 10: 0 1 2 8 9 11; 11: 3 4 5 6 7 10; 12: 0 1 2 3 4 5 6 7 8 9}
- Graph G_2: {0: 1 2 12 13; 1: 0 3 12 13; 2: 0 4 12 13; 3: 1 4 12 13; 4: 2 3 12 13; 5: 6 7 9 11 12; 6: 5 8 10 11 13; 7: 5 8 9 11 13; 8: 6 7 10 11 12; 9: 5 7 10 12 13; 10: 6 8 9 12 13; 11: 5 6 7 8 12 13; 12: 0 1 2 3 4 5 8 9 10 11; 13: 0 1 2 3 4 6 7 9 10 11}

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