Acyclic Edge Coloring of Chordal Graphs with Bounded Degree

Yulai Ma, Yongtang Shi
Center for Combinatorics and LPMC
Nankai University, Tianjin 300071, China

Weifan Wang *
Department of Mathematics
Zhejiang Normal University, Jinhua 321004, China

November 10, 2018

Abstract

An acyclic edge coloring of a graph G is a proper edge coloring such that no bichromatic cycles are produced. It was conjectured that every simple graph G with maximum degree Δ is acyclically edge-$(\Delta+2)$-colorable. In this paper, we confirm the conjecture for chordal graphs G with $\Delta \leq 6$.

Keywords: Acyclic edge coloring; Chordal graphs; Simplicial vertices; Maximum degree

1 Introduction

Only simple graphs are considered in this paper. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *proper edge-k-coloring* of a graph G is a mapping $c: E(G) \to \{1, 2, \ldots, k\}$ such that any two adjacent edges receive distinct colors. The graph G is *edge-k-colorable* if it has a proper edge-k-coloring, and the *chromatic index* of G is the minimum k such that G is edge-k-colorable, denoted by $\chi'(G)$.

A proper edge-k-coloring c of G is *acyclic* if there are no bichromatic cycles in G, i.e., the union of any two color classes induces a subgraph of G that is a forest. The *acyclic*

*Corresponding author. Email: wwf@zjnu.cn.

chromatic index of G, denoted by $a'(G)$, is the smallest integer k such that G is acyclically edge-k-colorable.

Let $\Delta(G)$ (Δ for short) denote the maximum degree of a graph G. Fiamčik [9] and later Alon, Sudakov and Zaks [3] independently conjectured the following.

Conjecture 1.1 *For any graph G, $a'(G) \leq \Delta + 2$.*

In 1991, Alon, McDiarmid and Reed [2] proved that $a'(G) \leq 64\Delta$ for any graph G. Molloy and Reed [11] improved the bound to 16Δ in 1998, and this was improved to $a'(G) \leq \lceil 9.62(\Delta - 1) \rceil$ in [12], and $a'(G) \leq 4\Delta - 4$ in [8], and $a'(G) \leq \lceil 3.74(\Delta - 1) \rceil + 1$ in [10]. Some special classes of graphs for this conjecture are also investigated, such as subcubic graphs [4, 14], graphs with $\Delta = 4$ [5, 13, 16], 2-degenerate graphs [6], planar graphs [7, 17, 15].

A *chord* of a cycle is an edge not in the cycle whose endpoints are in the cycle. A *hole* in a graph is an induced subgraph which is a cycle of length at least 4. A graph is a *chordal graph* if it has no hole. A vertex is *simplicial* if its neighborhood induces a clique. A vertex v is *almost-simplicial* in G if v has exactly one simplicial neighbor u, such that v is simplicial in $G - u$. Two simplicial vertices u and v are *simplicial twins*, if they are adjacent.

The purpose of this paper is to investigate the acyclic edge coloring of chordal graphs with smaller maximum degree. Main results are as follows:

Theorem 1.2 *If G is a chordal graph with $\Delta = 5$, then $a'(G) \leq 7$.*

Theorem 1.3 *If G is a chordal graph with $\Delta = 6$, then $a'(G) \leq 8$.*

Combining the known results stated as above, we know that Conjecture 1.1 holds for chordal graphs with $\Delta \leq 6$.

2 Lemmas

Let $[k] = \{1, 2, \ldots, k\}$. Assume that c is a partial acyclic edge-k-coloring of a chordal graph G using the color set $C = [k]$. For a vertex x in G, let $N_G(x)$ denote the neighborhood of x in G and $N_G[x] = N_G(x) \cup \{x\}$; let $d_G(x)$ denote the degree of a vertex x in G and $C_G(x)$ denote the set of colors assigned to edges incident to x under c. When there is no scope for ambiguity, we replace $N_G(x)$, $N_G[x]$, $d_G(x)$ and $C_G(x)$ with $N(x)$, $N[x]$, $d(x)$ and $C(x)$, respectively. Moreover, we use K_n to denote a complete graph with n vertices.

If the edges of a cycle $ux \ldots vu$ are alternatively colored with colors i and j, then we call the cycle an $(i, j)_{(u,v)}$-cycle. Similarly, if the edges of a path $ux \ldots v$ are alternatively colored with colors i and j, then we call the path an $(i, j)_{(u,v)}$-path. We use $(e_1, e_2, \ldots, e_m)_c =$

$(a_1, a_2, \ldots, a_m)$ to denote that $c(e_i) = a_i$ for $i \in [m]$. Let $(e_1, e_2, \ldots, e_n) \to (b_1, b_2, \ldots, b_n)$ denote that e_i is colored or recolored with the color b_i for $i \in [n]$. In particular, when $n = 1$, we write simply $e_1 \to b_1$.

Let W_0 denote the vertex set consisting of all simplicial vertices in G. Since G is a chordal graph, we have $W_0 \neq \emptyset$. Let $G_1 = G - W_0$ and W_1 denote the vertex set consisting of all simplicial vertices in G_1. Furthermore, it is easy to know that G_1 is a chordal graph. Let $\Delta_0 \in \{5, 6\}$ such that $\Delta(G) \leq \Delta_0$.

It should be explained that, in the following figures, all neighbors of black points have been shown in the figures, whereas others may be not. Now we present some lemmas, which will be useful in the following.

Lemma 2.1 ([1]) *If G is a non-trivial chordal graph, then there exists a pair of vertices u and v satisfying at least one of the following:*

(i) u and v are both simplicial vertices but not adjacent, and $N(u) \cap N(v) \neq \emptyset$;

(ii) u and v are simplicial twins;

(iii) u is simplicial, v is an almost-simplicial neighbor of u, and the degree of v in G is at least 2.

The following lemma is frequently used in studying the acyclic edge coloring. For completeness, we give its proof here.

Lemma 2.2 *Suppose that a graph G has an acyclic edge-$(\Delta + 2)$-coloring c. Let $P = uv_1v_2 \ldots v_kv_{k+1}$ be a maximal $(a, b)_{(u,v_{k+1})}$-path in G with $c(uv_1) = a$ and $b \notin C(u)$. If $w \notin V(P)$, then there does not exist an $(a, b)_{(u,w)}$-path in G under c.*

Proof. Suppose that there exists an $(a, b)_{(u,w)}$-path P' in G under c and assume $P' = uw_1 \ldots w_mw_{m+1}$ with $m \geq 1$ and $w = w_{m+1}$. Since $w \notin V(P)$ and $b \notin C(u)$, let v_i be the first vertex such that $v_{i+1} \neq w_{i+1}$. Then $c(v_iv_{i+1}) = c(v_iw_{i+1})$, a contradiction. □

Lemma 2.3 *Let G be a chordal graph and v a simplicial vertex of G. Then the graph $G - uv$ is a chordal graph, where $u \in N(v)$.*

Proof. If $G - uv$ is not a chordal graph, then $G - uv$ contains a cycle C without chords. Note that $u, v \in V(C)$. We claim that $|C| \geq 5$, since otherwise, let $C = uv_ivv_j$ and then $v_iv_j \notin E(G)$, a contradiction. If $|C| \geq 5$, then we can find a cycle C' in G such that $E(C') \subseteq E(C) \cup \{uv\}$ and $|C'| \geq 4$, which also has no chords, a contradiction. □

Lemma 2.4 *Let G be a chordal graph and x a cut-vertex. Denote $G'_1, G'_2, \ldots, G'_k$ as the components of $G-x$ and H_i as the subgraph induced by $V(G'_i)\cup\{x\}$, respectively. If $a'(H_i) \leq \Delta+2$ for each $i \in [k]$, then $a'(G) \leq \Delta+2$.*

Proof. Let c_i be an acyclic edge coloring of H_i using color set $C^{(i)} = [\Delta+2]$. For each H_i, by permuting colors in $C^{(i)}$, we can obtain a new acyclic edge-$(\Delta+2)$-coloring c'_i using color set $C^{(i')} = [\Delta+2]$, such that $C^{(1')}(x) = [d_{H_1}(x)]$ and $C^{(i')}(x) = [\sum_{j=1}^{i} d_{H_j}(x)] \setminus [\sum_{j=1}^{i-1} d_{H_j}(x)]$, where $i \in [k] \setminus \{1\}$. Then we obtain an acyclic edge-$(\Delta+2)$-coloring of G. □

Lemma 2.5 *Let G be a chordal graph and C a cycle of G. If $uv \in E(C)$, then $N(u) \cap N(v) \cap V(C) \neq \emptyset$.*

Proof. Let k be the length of C. By induction on k. If $|C| \leq 4$, then the lemma holds obviously. Suppose that $|C| \geq 5$. And assume that $C = u_1u_2\ldots u_ku_1$ and $u = u_1, v = u_k$. By the definition of a chordal graph, let u_iu_j be a chord of C with $1 \leq i < j \leq k$. Then we obtain a cycle $C_1 = u_1 \ldots u_iu_j \ldots u_ku_1$. By the induction hypothesis, we know $N(u) \cap N(v) \cap V(C_1) \neq \emptyset$. The proof is then complete. □

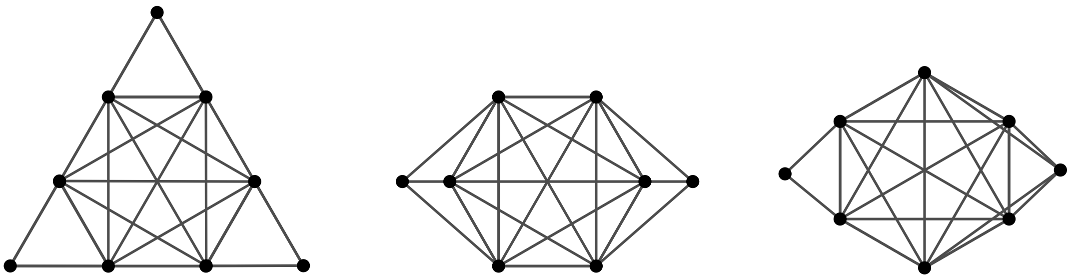
Lemma 2.6 *Let G be a 2-connected chordal graph. If G contains a copy of K_{Δ_0}, then $a'(G) \leq \Delta_0 + 2$. Furthermore, $a'(K_7) \leq 7$.*

Proof. First suppose that $G = K_7$. We complete this case by proving that K_7 is acyclic edge-7-colorable. It is easy to see that K_8 is edge-7-colorable. Assume that $V(K_8) = \{v_0, v_1, \ldots, v_7\}$. Place the vertices $v_1, v_2, \ldots, v_7$ cyclically about a regular 7-gon and place v_0 in the center of the 7-gon. Join every two vertices of K_8 by a straight line segment. For each i with $1 \leq i \leq 7$, the edge v_0v_i and all edges perpendicular to v_0v_i form a 1-factor F_i of K_8 and so $\mathscr{F} = \{F_1, F_2, \ldots, F_7\}$ is a 1-factorization of K_8. Assign each edge of F_i the color i for $1 \leq i \leq 7$. Then we observe that the subgraph induced by any two colors classes in K_8 is C_8. By deleting v_0 from K_8, we delete two edges of the C_8 induced by any two colors classes in K_8. And it provides an acyclic edge-7-coloring of K_7 with vertex set $\{v_1, v_2, \ldots, v_7\}$.

If we split one vertex in K_7 to three vertices with degree 2, then we obtain a graph A_1 as shown in Figure 1. And by constructing a bijection between $E(G)$ and $E(A_1)$, it is easy to obtain $a'(A_1) \leq 7$ since $a'(K_7) \leq 7$. Similarly, we get a graph A_2 and A_3 as shown in Figure 1. Obviously, $a'(A_2) \leq 7$ and $a'(A_3) \leq 7$.

Now suppose that G contains a copy of K_{Δ_0} but $G \neq K_7$, denoted H. Let $d(H, u) = \min\{d_G(v, u)|v \in V(H)\}$ and $S = \{u|d(H, u) = 1 \text{ and } u \in V(G) \setminus V(H)\}$. If $|S| = 1$, since G is 2-connected, then G is a subgraph of K_7. Suppose that $|S| \geq 2$. By Lemma 2.5, for any two vertices $s_1, s_2 \in S$, since $\Delta \leq \Delta_0$, then $s_1s_2 \notin E(G)$ and there does not exist a

(s_1, s_2)-path passing any vertex in $V(G) \setminus (V(H) \cup S)$. This means $V(G) = V(H) \cup S$. If $|S| \geq 4$ or $|S| \geq 3$ with $\Delta_0 = 5$, then there exists a vertex $s \in S$ which is adjacent to exactly one vertex y in $V(H)$. And obviously, y is a 1-vertex or a cut-vertex, a contradiction. If $|S| = 3$ with $\Delta_0 = 6$, then $G = A_1$. Suppose that $|S| = 2$. Then G is a subgraph of K_7 or A_2 or A_3, and so we are done. □



A_1 A_2 A_3

Figure 1: The configurations used in the proof of Lemma 2.6.

Lemma 2.7 *Let G be a chordal graph and v a simplicial vertex in G. Suppose that $d(v) = 3$ and $u \in N_G(v)$ with $d(u) \leq 5$. If $a'(G - uv) \leq \Delta_0 + 2$, then $a'(G) \leq \Delta_0 + 2$.*

Proof. Let c be an acyclic edge-$(\Delta_0 + 2)$-coloring of $G - uv$ with color set $C = [\Delta_0 + 2]$. Assume that $\{v, w, x\} \subseteq N_G(u)$, $N_G(v) = \{u, w, x\}$ and $(vw, vx)_c = (1, 2)$. Let $S = C \setminus (C(u) \cup C(v))$.

If $|C(u) \cap C(v)| = 0$, then let $uv \to a$, where $a \in S$.

Suppose that $|C(u) \cap C(v)| = 1$. By symmetry, assume that $C(u) \cap C(v) = \{1\}$ and $C(u) \subseteq \{1, 3, 4, 5\}$. For any $j \in S$, G contains a $(1, j)_{(u,v)}$-path, otherwise let $uv \to j$. If $c(ux) = 1$, then assume $c(uw) = 5$. Note that $S \subseteq C(x) \cap C(w)$. If there exists some $i \in \{3, 4\} \setminus (C(x) \cup C(w))$, then let $(vx, uv) \to (i, 2)$. WLOG, assume $3 \in C(x)$ and $4 \in C(w)$. We obtain $C(w) = \{1, 4, 5\} \cup S$ and $C(x) = \{1, 2, 3\} \cup S$. For any $i \in S$, G contains a $(3, i)_{(u,w)}$-path, otherwise let $(vw, uv) \to (3, i)$; G contains a $(4, i)_{(u,x)}$-path, otherwise let $(vw, vx, uv) \to (2, 4, i)$. Hence assume $\{y, z\} \subseteq N_G(u)$ and $(uy, uz)_c = (3, 4)$. If $1 \notin C(z)$, then let $(vx, uv) \to (4, 2)$. If $5 \notin C(y)$, then let $(vw, vx, uv) \to (3, 5, 2)$. Hence $(\{1, 4\} \cup S) \subseteq C(z)$ and $(\{3, 5\} \cup S) \subseteq C(y)$. If $2 \notin C(z) \cup C(y)$, then let $(ux, vw, vx, uv) \to (2, 2, 4, 1)$; if $2 \in C(z)$, then let $(uz, uw, vx, uv) \to (5, 2, 5, 4)$; if $2 \in C(y) \setminus C(z)$, then let $(uy, ux, uw, uv) \to (1, 5, 2, 3)$, we are done. Hence we suppose $y \in N_G(u)$ and $c(uy) = 1$. Then assume $(ux, uw)_c = (3, 5)$. If $3 \notin C(w)$, then let $vw \to 3$, we are done by the proof of the case of $c(ux) = 1$. Suppose $3 \in C(w)$. Then we obtain $C(w) = \{1, 3, 5\} \cup S$. If

$i \notin C(x)$ for some $i \in S$, then let $(vx, uv) \to (i, 2)$ by Lemma 2.2. Hence $S \subseteq C(x) \cap C(w)$. If $4 \notin C(x)$, then let $(vx, uv) \to (4, 2)$; if $4 \in C(x)$, then let $(vw, vx, uv) \to (4, 1, 2)$.

Now suppose that $|C(u) \cap C(v)| = 2$ and $C(u) \subseteq \{1, 2, 3, 4\}$. For any $j \in S$, G contains an $(i, j)_{(u,v)}$-path for some $i \in \{1, 2\}$, otherwise let $uv \to j$. If $i \notin C(w)$ for some $i \in S$, then let $vw \to i$ by Lemma 2.2, the proof is reduced to the case of $|C(u) \cap C(v)| = 1$. Hence $S \subseteq C(x) \cap C(w)$ by symmetry. By symmetry, if $c(ux) = 1$ and $c(uy) = 2$, then assume $(uw, wx)_c = (3, 5)$ and let $(vw, uv) \to (4, 5)$; if $c(uz) = 1$ and $c(uy) = 2$, then assume $(uw, ux, wx)_c = (3, 4, 5)$ and let $uv \to 5$. □

Lemma 2.8 *Let G be a 2-connected chordal graph and $u, v \in W_0$ with $d(u) \leq 4$ and $d(v) \leq 4$. If $uv \in E(G)$ and $a'(G - uv) \leq \Delta_0 + 2$, then $a'(G) \leq \Delta_0 + 2$. Furthermore, if $w \in W_1$, then $N_G(z) \subseteq N_{G_1}[w]$ for every $z \in N_G(w) \cap W_0$ with $d_G(z) \leq 4$.*

Proof. Let c be an acyclic edge-$(\Delta_0 + 2)$-coloring of $G - uv$.

If $d(u) = d(v) = 2$, then let $y \in N_G(u) \cap N_G(v)$ and we see that y is a cut-vertex, a contradiction. If $d(u) = d(v) = 3$, we are done by Lemma 2.7. Now suppose $d(u) = d(v) = 4$, and let $N_G(u) \cap N_G(v) = \{u_1, u_2, u_3\}$. If $\Delta_0 = 5$, then G contains a copy of K_{Δ_0}, we are done by Lemma 2.6. Suppose $\Delta_0 = 6$. If $|C(u) \cap C(v)| = 0$, then let $uv \to a$, where $a \in C \setminus (C(u) \cup C(v))$. Thus, $1 \leq |C(u) \cap C(v)| \leq 3$. Let $S \subseteq C \setminus (C(u) \cup C(v))$ be a color set such that G contains a $(1, j)_{(u,v)}$-path for each $j \in S$ but G contains no $(1, j')_{(u,v)}$-path for any $j' \in C \setminus (C(u) \cup C(v) \cup S)$.

Case 1 $|C(u) \cap C(v)| = 3$, say $C(u) \cap C(v) = \{1, 2, 3\}$.

By symmetry, assume $(uu_1, uu_2, uu_3)_c = \{1, 2, 3\}$ and $(vu_1, vu_2, vu_3)_c = \{2, 3, 1\}$. For any $j \in \{4, 5, 6, 7, 8\}$, G contains an $(i, j)_{(u,v)}$-path for some $i \in \{1, 2, 3\}$, otherwise let $uv \to j$.

If $|S| = 4$, then let $uv \to a$, where $a \in C \setminus (\{1, 2, 3\} \cup S)$, we are done. If $|S| = 3$, then WLOG, assume $S = \{4, 5, 6\}$ and G contains a $(2, 7)_{(u,v)}$-path and a $(3, 8)_{(u,v)}$-path as $\Delta \leq 6$. We obtain $C(u_1) = \{1, 2, 4, 5, 6, 7\}$ and $\{2, 3, 7, 8\} \subseteq C(u_2)$. Then let $(uu_1, uv) \to (8, a)$, where $a \in \{4, 5, 6\} \setminus C(u_2)$, we are done. If $|S| = 2$, then assume $S = \{4, 5\}$. By symmetry and $\Delta \leq 6$, we assume G contains a $(3, 8)_{(u,v)}$-path and a $(2, j)_{(u,v)}$-path for every $j \in \{6, 7\}$. Then, we obtain $C(u_1) = \{1, 2, 4, 5, 6, 7\}$ and $\{2, 3, 6, 7, 8\} \subseteq C(u_2)$. Then let $(uu_1, uv) \to (8, a)$, where $a \in \{4, 5\} \setminus C(u_2)$, we are done.

Case 2 $|C(u) \cap C(v)| = 2$, say $C(u) \cap C(v) = \{1, 2\}$.

By symmetry, assume $(uu_1, uu_2, uu_3)_c = \{1, 2, 3\}$ and $(vu_1, vu_2, vu_3)_c = \{2, 4, 1\}$. For any $j \in \{5, 6, 7, 8\}$, G contains an $(i, j)_{(u,v)}$-path for some $i \in \{1, 2\}$, otherwise let $uv \to j$. Thus, we know $C(u_1) = \{1, 2, 5, 6, 7, 8\}$. If $4 \notin C(u_3)$, then let $uu_3 \to 4$ and the proof is reduced to **Case 1**. Suppose $4 \in C(u_3)$. Let a be a color in $\{5, 6, 7, 8\} \setminus C(u_3)$. We know there is a $(2, a)_{(u,v)}$-path. Let $(uu_3, uv) \to (a, 3)$, then we are done by Lemma 2.2.

Case 3 $|C(u) \cap C(v)| = 1$, say $C(u) \cap C(v) = \{1\}$.

By symmetry, assume $(uu_1, uu_2, uu_3)_c = \{1, 2, 3\}$ and $(vu_1, vu_2, vu_3)_c = \{5, 4, 1\}$. For any $j \in \{6, 7, 8\}$, G contains a $(1, j)_{(u,v)}$-path, otherwise $uv \to j$.

If $4 \notin C(u_3)$, then $4 \in C(u_1)$ and $1 \in C(u_2)$, otherwise let $uu_3 \to 4$; $2 \in C(u_3)$, otherwise let $vu_1 \to 2$. The proof is reduced to **Case 2**. Hence, $C(u_1) = \{1, 4, 5, 6, 7, 8\}$ and $C(u_3) = \{1, 2, 3, 6, 7, 8\}$. Since $c(u_1u_2) \neq 4$ and $c(u_2u_3) \neq 2$, we know $\{c(u_1u_2), c(u_2u_3)\} \subseteq \{6, 7, 8\}$. WLOG, assume $(c(u_1u_2), c(u_2u_3))_c = \{7, 8\}$. If $3 \notin C(u_2)$, then let $vu_2 \to 3$; if $3 \in C(u_2)$, then let $uu_2 \to 5$. Then the proof is reduced to **Case 2**.

Suppose $4 \in C(u_3)$ and $2 \in C(u_1)$ by symmetry. If $3 \notin C(u_2)$, then let $vu_2 \to 3$; if $5 \notin C(u_2)$, then let $uu_2 \to 5$. Then the proof is reduced to **Case 2**. Hence suppose $3 \in C(u_2)$ and $5 \in C(u_2)$. Since $c(u_1u_2) \neq 2$ and $c(u_2u_3) \neq 4$, we know $\{c(u_1u_2), c(u_2u_3)\} \subseteq \{6, 7, 8\}$. Assume $(c(u_1u_3), c(u_1u_2), c(u_2u_3))_c = \{6, 7, 8\}$, then $C(u_2) = \{2, 3, 4, 5, 7, 8\}$. Let $(vu_2, uv) \to (6, 4)$, we are done. □

3 Proof of Theorem 1.2

The proof is proceeded by induction on $|E(G)|$. If $|E(G)| = 0$, then G is trivial and the theorem holds naturally. If $|V(G)| \leq 5$, then $\Delta(G) \leq 4$ and we have $a'(G) \leq 6$ by previously known results. So $|E(G)| \geq 1$ and $|V(G)| \geq 6$. If G is disconnected, then denote $G_1, \ldots, G_k$ as the components of G. And we know $a'(G) \leq \max\{a'(G_i)|i \in [k]\}$. Hence combining Lemma 2.4, WLOG, assume that G is 2-connected. For each $x \in W_0$ and $y \in N_G(x)$, $G - xy$ is a chordal graph by Lemma 2.3. Then by the induction hypothesis, $G - xy$ has an acyclic edge-7-coloring c using the color set $C = [7]$.

Let u be a simiplicial vertex in G, we have $d_G(u) = 2$ by Lemmas 2.6 and 2.7. Let $N(u) = \{v, u_1\}$ and c be an acyclic edge-7-coloring of $G - uv$. By Lemma 2.5, u_1 and v have a common neighbor different from u, denoted w. If $C(u) \cap C(v) = \emptyset$, then let $uv \to a$, where $a \in C \setminus (C(u) \cup C(v))$. Suppose $|C(u) \cap C(v)| = 1$ and WLOG, let $C(u) \cap C(v) = \{1\}$ and $C(v) \subseteq S = \{1, 2, 3, 4\}$.

Case 1 $c(vw) \neq 1$, say $(vu_1, vw, vv_1)_c = (3, 4, 1)$.

We know that G contains a $(1, i)_{(u,v)}$-path for each $i \in \{5, 6, 7\}$, otherwise let $uv \to i$. Hence $\{5, 6, 7\} \subseteq C(u_1)$. By symmetry, assume $c(u_1w) = 5$. For any $i \in \{5, 6, 7\}$, G contains a $(4, i)_{(u_1,v)}$-path, otherwise let $(uu_1, uv) \to (4, i)$. Hence $C(w) = \{1, 4, 5, 6, 7\}$. Let $(uu_1, uv) \to (2, 5)$.

Case 2 $c(vw) = 1$, say $c(vu_1) = 4$.

We obtain G contains a $(1, i)_{(u,v)}$-path for every $i \in \{5, 6, 7\}$, otherwise let $uv \to i$. Hence $\{5, 6, 7\} \subseteq C(u_1)$. Let $uu_1 \to 2$, the proof is reduced to **case 1** or the case of $C(u) \cap C(v) = \emptyset$.

4 Proof of Theorem 1.3

The proof is proceeded by induction on $|E(G)|$. With the similar proof of Theorem 1.2, we assume that $|E(G)| \geq 1$, $|V(G)| \geq 6$ and G is 2-connected. For each $x' \in W_0$ and $y' \in N_G(x')$, $G - x'y'$ is a chordal graph by Lemma 2.3. By the induction hypothesis, $G - x'y'$ has an acyclic edge-8-coloring c using the color set $C = [8]$. Then we have $2 \leq d_G(x') \leq 4$ by Lemma 2.6. And if y' is an almost-simplicial neighbor of x', then $2 \leq d_G(y') \leq 5$ by Lemma 2.6.

By Lemma 2.1, we can find a pair of vertices u_0 and v_0 satisfying at least one of (i), (ii) and (iii). WLOG, let u_0 be a simplicial vertex and call v_0 a partner of u_0. Then, let W_2 denote the vertex set consisting of all vertices which can be a candidate for u_0.

Case 1 There exists a vertex u with $d_G(u) = 2$ and $u \in W_2$.

If there is a partner v' of u such that $v' \in N(u)$, then let $v = v'$. Otherwise, for some partner v' of u, let v be a vertex in $N(v') \cap N(u)$. Let $N(u) = \{v, u_1\}$ and c be an acyclic edge-8-coloring of $G - uv$. By Lemmas 2.4 and 2.5, u_1 and v have a common neighbor different from u, denoted w. And we denote $\{u, u_1, w, v_1, \ldots, v_l\}$, $\{u, v, w, u_{11}, \ldots, u_{1k}\}$ and $\{u_1, v, w_1, \ldots, w_m\}$ as neighborhood of v, u_1 and w, respectively.

If $C(u) \cap C(v) = \emptyset$, then let $uv \to a$, where $a \in C \backslash (C(u) \cup C(v))$. Suppose $|C(u) \cap C(v)| = 1$. WLOG, assume $C(u) \cap C(v) = \{1\}$ and $C(v) \subseteq S = \{1, 2, 3, 4, 5\}$.

Subcase 1.1 $c(vw) \neq 1$, say $(vu_1, vw, vv_1)_c = (4, 5, 1)$.

Subcase 1.1.1 $c(u_1w) = 3$ or $c(u_1w) = 2$.

By symmetry, assume $c(u_1w) = 3$. It is obvious that $\{6, 7, 8\} \subseteq C(u_1)$, otherwise let $uv \to a$, where $a \in \{6, 7, 8\} \setminus C(u_1)$. If G contains no $(5, i)_{(u_1,v)}$-path for some $i \in \{6, 7, 8\}$, then let $(uu_1, uv) \to (5, i)$. Similarly, we know G contains a $(2, i)_{(u_1,v)}$-path for each $i \in \{6, 7, 8\}$. Assume $(ww_1, ww_2, ww_3)_c = (6, 7, 8)$, then $5 \in C(w_1) \cap C(w_2) \cap C(w_3)$.

- The graph G contains no $(4, j)_{(w,u_1)}$-path for some $j \in \{1, 2\}$.

If $4 \in C(w)$, then let $(u_1w, uu_1, uv) \to (j, 3, i)$; if $4 \notin C(w)$, then let $(u_1w, uu_1, uv) \to (a, 3, i)$, where $a \in \{1, 2\} \setminus C(w)$. Then we obtain that G contains a $(3, i)_{(u_1,v)}$-path for each $i \in \{6, 7, 8\}$, otherwise we are done by Lemma 2.2. Thus, $\{6, 7, 8\} \subseteq C(v_1) \cap C(v_2) \cap C(v_3)$ and $\{1, 2, 3, 5\} \subseteq C(u_{11}) \cap C(u_{12}) \cap C(u_{13})$. Since $d(w) \geq 5$ and $d(v) = 6$, then $v' \notin (N[u] \cup \{w\})$ and $v' \in \{v_1, v_2, v_3\} \cap W_0$. And we know $w \in N(v')$ or $u_1 \in N(v')$ as $d(v) = 6$. Then $|C(v')| \geq 5$ and $d(v') \geq 5$, a contradiction.

- The graph G contains a $(4, j)_{(w,u_1)}$-path for each $j \in \{1, 2\}$.

If $3 \notin C(v)$, then let $uv \to 3$. Suppose $3 \in C(v)$ and assume $(vv_2, vv_3)_c = (2, 3)$. If $a \in \{6, 7, 8\} \setminus C(v_3)$, then let $(vv_3, vu_1, u_1w, uv) \to (a, 3, 2, 4)$ by Lemma 2.2. Hence $\{6, 7, 8\} \subseteq C(v_3)$. Then $\{1, 2, 5\} \subseteq C(u_{11}) \cap C(u_{12}) \cap C(u_{13})$ and $\{4, 6, 7, 8\} \subseteq C(v_1) \cap C(v_2)$.

Since $d(w) \geq 5$ and $d(v) = 6$, we obtain $v' \notin (N[u] \cup \{w\})$ and $v' = v_3$ as $d(v_1) \geq 5$ and $d(v_2) \geq 5$. Then we claim that $C(v') = \{3, 6, 7, 8\}$. Hence $\{u_1, w\} \cap N(v') = \emptyset$. Then we have $d(v) \geq 7$, a contradiction.

Subcase 1.1.2 $c(u_1w) \in \{6, 7, 8\}$, say $c(u_1w) = 6$ by symmetry.

We know G contains a $(1, i)_{(u,v)}$-path for each $i \in \{6, 7, 8\}$, otherwise let $uv \to i$.

• $5 \in C(u_1)$.

If G contains no $(i, j)_{(u_1,v)}$-path for some $i \in \{2, 3\}$ and some $j \in \{6, 7, 8\}$, then let $(uu_1, uv) \to (i, j)$. WLOG, assume $(u_1u_{11}, u_1u_{12}, u_1u_{13})_c = (5, 7, 8)$, $(vv_1, vv_2, vv_3)_c = (1, 2, 3)$ and $(ww_1, ww_2, ww_3)_c = (1, 2, 3)$. Then $6 \in C(w_1) \cap C(w_2) \cap C(w_3)$. And we have $\{1, 2\} \subseteq C(u_{12}) \cap C(u_{13})$ and $\{6, 7, 8\} \subseteq C(v_1) \cap C(v_2) \cap C(v_3)$. Since $d(w) \geq 5$ and $d(v) = 6$, we obtain $v' \notin (N[u] \cup \{w\})$ and $v' \in \{v_1, v_2, v_3\} \cap W_0$. Then we know $w \in N(v')$ or $u_1 \in N(v')$ as $d(v) = 6$. If $4 \in C(w)$, then $|C(v')| \geq 5$ and $d(v') \geq 5$, a contradiction. Suppose $4 \notin C(w)$. By Lemma 2.2, we know G contains an $(i, 5)_{(u_1,v)}$-path for each $i \in \{1, 2, 3\}$, otherwise let $(uu_1, vw, uv) \to (i, a, 5)$, where $a \in \{7, 8\} \setminus C(w)$. Then $5 \in C(v_1) \cap C(v_2) \cap C(v_3)$ and $|C(v')| \geq 5$. So we obtain $d(v') \geq 5$, a contradiction.

• $5 \notin C(u_1)$.

Since $\Delta \leq 6$, there is a color $a \in \{2, 3\} \setminus C(u_1)$. WLOG, assume $a = 3$. we know G contains an $(i, j)_{(u_1,v)}$-path for each $i \in \{3, 5\}$ and each $j \in \{6, 7, 8\}$, otherwise let $(uu_1, uv) \to (i, j)$. If $2 \notin C(v)$, then let $(uu_1, uv) \to (5, 2)$. Suppose $2 \in C(v)$. Assume $(u_1u_{11}, u_1u_{12}, vv_1, vv_2,\ vv_3)_c = (7, 8, 1, 2, 3)$ and $(ww_1, ww_2, ww_3, ww_4)_c = (1, 3, 7, 8)$. Then $6 \in C(w_1) \cap C(w_2)$, $5 \in C(w_3) \cap C(w_4)$, $C(w) = \{1, 3, 5, 6, 7, 8\}$, $\{1, 3, 5\} \subseteq C(u_{11}) \cap C(u_{12})$ and $\{6, 7, 8\} \subseteq C(v_1) \cap C(v_3)$. And we have $6 \in C(v_2)$, otherwise let $(vv_2, uu_1, uv) \to (6, 5, 2)$. Note that for $\{i, j\} = \{7, 8\}$, if $i \in C(v_2)$ but $j \notin C(v_2)$, then $4 \in C(v_2)$, otherwise let $(vv_2, uu_1, uv) \to (j, 5, 2)$ by Lemma 2.2.

Suppose $7 \in C(v_2)$ or $8 \in C(v_2)$. Since $d(w) \geq 5$ and $d(v) = 6$, then $v' \notin (N[u] \cup \{w\})$ and $v' \in \{v_1, v_2, v_3\} \cap W_0$. And we know $w \in N(v')$ or $u_1 \in N(v')$ as $d(v) = 6$. Then $|C(v')| \geq 5$ and $d(v') \geq 5$, a contradiction.

Now suppose $7 \notin C(v_2)$ and $8 \notin C(v_2)$. We know G contains a $(4, j)_{(v,v_2)}$-path for each $j \in \{7, 8\}$, otherwise let $(uu_1, uv, vv_2) \to (5, 2, j)$. And G contains a $(6, j)_{(v_2,u_1)}$-path for each $j \in \{7, 8\}$, otherwise let $(u_1v, uu_1, u_1w, uv, vv_2) \to (6, 5, 4, 2, j)$. Then we obtain $\{1, 3, 4, 5, 6\} \subseteq C(u_{11}) \cap C(u_{12})$. Since $4 \notin C(w)$, the $(4, 7)_{(v,v_2)}$-path will not pass the vertex w, then there exists a vertex $s \in (N(u_1) \cap N(v)) \setminus \{u, w\}$ such that $4 \in C(s)$ by the Lemma 2.5. Since $|C(x) \cup C(y)| \geq 7$ for each $x \in \{u_{11}, u_{12}\}$ and each $y \in \{v_1, v_2, v_3\}$, we have $s = u_{13} = v_i$, where $i = 1$ or $i = 3$, such that $d(u_{13}) = 6$ and $c(u_1u_{13}) = 2$. Thus $C(s) = \{i, 2, 4, 6, 7, 8\}$, where $i = 1$ or $i = 3$. Let $u_1u_{13} \to 5$ by the Lemma 2.2, the proof is reduced to the case of $5 \in C(u_1)$.

Subcase 1.2 $c(vw) = 1$, say $c(vu_1) = 4$.

If $d(v) = 3$, then $|(C(u_1) \cup C(v))| \leq 7$, let $uv \to a$, where $a \in C \setminus (C(u_1) \cup C(v))$. Suppose $d(v) \geq 4$. Let $uu_1 \to a$, where $a \in C \setminus C(u_1)$. If $a \notin C(v)$, then let $uv \to b$, where $b \in C \setminus (\{a\} \cup C(v))$; if $a \in C(v)$, the proof is reduced to **Subcase 1.1**. □

For an edge $uv \in E(G)$, if $\{u, v\}$ is a separating set of G and $Q_1, Q_2, \ldots, Q_k$ are the components of $G - \{u, v\}$, then let $H_i = G[V(Q_i) \cup \{u, v\}]$ and $j = d_G(u) - d_{H_1}(u)$.

Claim 1 If $a'(H_1) + j \leq \Delta + 2$ and $j \leq 2$, then $a'(G) \leq \Delta + 2$.

Proof. Let $T = G \backslash V(Q_1)$. By the induction hypothesis, the subgraph T has an acyclic edge-(Δ+2)-coloring c'. And H_1 has an acyclic edge coloring c with colors of $\{c'(uv)\} \cup (C' \backslash C'_T(u))$. Permuting the colors in $\{c'(uv)\} \cup (C' \setminus C'_T(u))$ in H_1 such that $C_{H_1}(v) \cap C'_T(v) = c'(uv)$, then we obtain an acyclic edge-$(\Delta + 2)$-coloring of G and $a'(G) \leq \Delta + 2$. □

For the graph classes $\mathscr{C}_1, \ldots, \mathscr{C}_8$ as shown in the Figure 2, let H_1 be the subgraph induced by the vertices that have been shown in the square marked with dotted lines.

Claim 2 If G is a subgraph of B_1 or B_2 or $G \in \cup_{i=1}^{8} \mathscr{C}_i$, then $a'(G) \leq 8$.

Proof. For graph B_1, let $F_1 = \{ae, bd, cg\}$, $F_2 = \{ab, ce\}$, $F_3 = \{ad, bc, eh\}$, $F_4 = \{be, cd\}$, $F_5 = \{ac, de, bf\}$, $F_6 = \{bg, dh, ef\}$ and $F_7 = \{af, ch, dg\}$. Assign each edge of F_i the color i for $1 \leq i \leq 7$, then we obtain $a'(B_1) \leq 7$. And it is easy to know $a'(B_2) \leq 8$ since B_1 is a subgraph of B_2 and $|E(B_2)| = |E(B_1)| + 1$.

If $G \in \cup_{i=1}^{4} \mathscr{C}_i \cup \mathscr{C}_8$, since H_1 is a subgraph of K_7 and $d_G(u) - d_{H_1}(u) = 1$, then $a'(G) \leq 8$ by $a'(K_7) \leq 7$ and **Claim 1**. If $G \in \mathscr{C}_6 \cup \mathscr{C}_7$, since H_1 is a subgraph of B_1 and $d_G(u) - d_{H_1}(u) = 1$, then $a'(G) \leq 8$ by $a'(B_1) \leq 7$ and **Claim 1**.

Suppose that $G \in \mathscr{C}_5$ and furthermore assume that $V(H_1) = \{u, v, a, b, c, d, e\}$ as shown in $\mathscr{C}_5$. Let $T = G \setminus (V(H_1) \setminus \{u, v\})$. By the induction hypothesis, the subgraph T has an acyclic edge-8-coloring c. WLOG, assume $C_T(u) \setminus c(uv) \subseteq \{1, 2\}$ and $c(uv) = 6$. Then let $F_1 = \{ad, bc\}$, $F_2 = \{bd, ce\}$, $F_3 = \{vb, ue\}$, $F_4 = \{ua, ve\}$, $F_5 = \{va, be\}$, $F_6 = \{ab\}$, $F_7 = \{ub, ae\}$ and $F_8 = \{ac, de\}$. Assign each edge of F_i the color i for $1 \leq i \leq 8$, then we obtain $a'(G) \leq 8$. □

Case 2 $3 \leq d_G(u) \leq 4$ for each $u \in W_2$.

It is obvious $V(G_1) \neq \emptyset$ since $a'(K_t) \leq t + 1$ for $t \leq 7$. By Lemma 2.1, Let x and y be a pair of vertices in G_1 satisfying at least one of (i), (ii) and (iii). WLOG, assume x is a simplicial vertex, then $2 \leq d_{G_1}(x) \leq 4$ by Lemma 2.6. Denote $S = N_{G_1}(y) \cap N_{G_1}(x)$ and $s_0 = \max\{d(s) | s \in S\}$. If $z \in N_G(x) \cap W_0$, then $3 \leq d_G(z) \leq 4$, otherwise the proof is reduced to **Case 1**. Let $n_3(w) = |\{z | z \in N_G(w) \cap W_0$ and $d_G(z) = 3\}|$ and

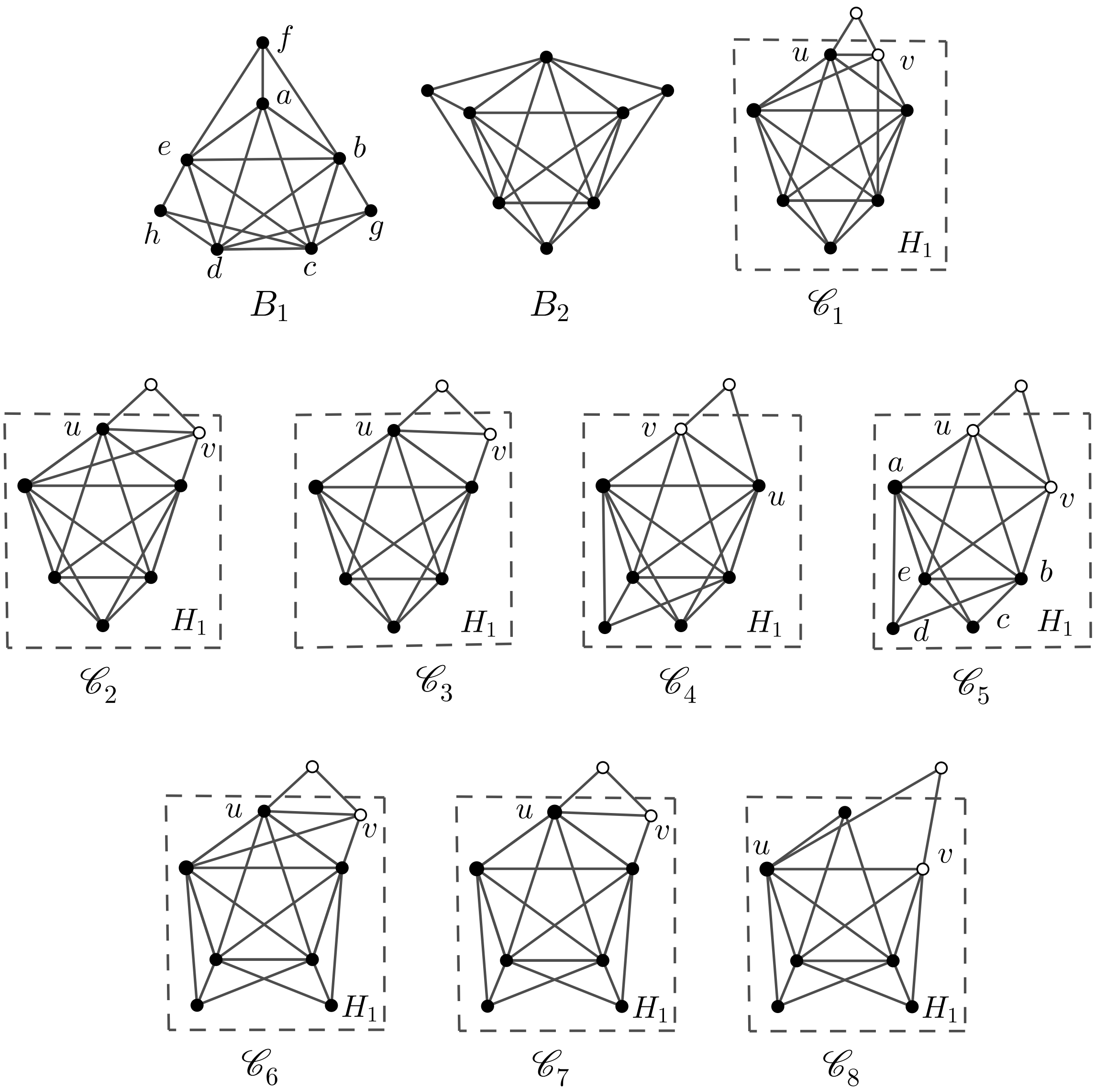


Figure 2: Configurations used in the proof of Claim 2.

$n_4(w) = |\{z|z \in N_G(w) \cap W_0 \text{ and } d_G(z) = 4\}|$ for any vertex w in G. If $d_{G_1}(x) = 2$, then $d(v) = 3$ for each $v \in N(x) \cap W_0$ by **Case 1**, Lemmas 2.6 and 2.8. If $d(x) \leq \Delta - 1$, then $a'(G) \leq \Delta(G) + 2$ by Lemma 2.7; if $d(x) = \Delta$, then G is a subgraph of K_7, we are done by Lemma 2.6. Hence suppose $3 \leq d_{G_1}(x) \leq 4$. Note that if $d_{G_1}(x) = 4$, then $n_3(x)+n_4(x) \geq 1$; if $d_{G_1}(x) = 3$ and $n_4(x) = 0$, then $n_3(x) = 3$; if $d_{G_1}(x) = 3$ and $n_4(x) = 1$, then $n_3(x) = 2$; otherwise, $d_{G_1}(x) = 3$ and $n_4(x) \geq 2$ by Lemma 2.7. Now we have to handle with three possibilities:

Subcase 2.1 The vertex y satisfies condition (i).

Since y is a simplicial vertex in G_1, we have $3 \leq d_{G_1}(y) \leq 4$ and $n_3(y) + n_4(y) \geq 1$.

Suppose $d_{G_1}(x) = 4$ and $d_{G_1}(y) = 3$. If $|S| \leq 2$ or $|S| = 3$ with $d(y) \geq 5$, since $n_3(y) + n_4(y) \geq 1$, then $d(s_0) \geq 7$ by Lemma 2.8, a contradiction. Hence $|S| = 3$ and $d(y) = 4$. By Lemmas 2.7 and 2.8, we are done. Hence suppose $d_{G_1}(x) = 4$ and $d_{G_1}(y) = 4$. If $|S| \leq 3$, since $n_3(y) + n_4(y) \geq 1$, then $d(s_0) \geq 7$, a contradiction. Hence $|S| = 4$. Since $\Delta = 6$, we have $n_3(y) = 0, n_4(y) = 1$ or $n_3(y) = 2, n_4(y) = 0$ by Lemmas 2.7 and 2.8. Then $N_G(x) \cap W_0 = \emptyset$, a contradiction.

Now suppose $d_{G_1}(x) = 3$ and $d_{G_1}(y) = 3$ by symmetry. We know $n_3(x) + n_4(x) \geq 2$ and $n_3(y) + n_4(y) \geq 2$. If $|S| = 1$, then $n_4(x) = 0$ and $n_3(x) = 3$ by Lemmas 2.7 and 2.8. And we know there is a cut-vertex $s_1 \in S$, a contradiction. Suppose $|S| \geq 2$. Since $n_3(x) + n_4(x) + n_3(y) + n_4(y) \geq 4$, then $d(s_0) \geq 7$ by Lemmas 2.7 and 2.8, a contradiction.

Subcase 2.2 The vertex y satisfies condition (iii).

It is obvious that $d_{G_1}(y) > d_{G_1}(x)$.

Suppose $d_{G_1}(x) = 4$. If $d_{G_1}(y) = 6$, then $N_G(x) \cap W_0 = \emptyset$, a contradiction. Hence $d_{G_1}(y) = 5$. If $n_4(x) + n_3(x) = 2$ with $n_4(x) \geq 1$, then $d(s_0) \geq 7$ by Lemma 2.8, a contradiction. If $n_4(x) = 1$ and $n_3(x) = 0$, then G is a subgraph of K_7 or $G \in \mathscr{C}_1$. Suppose $n_4(x) = 0$ and $n_3(x) = 2$, then $G = B_2$.

Now suppose $d_{G_1}(x) = 3$. If $d_{G_1}(y) \geq 5$, since $n_4(x)+n_3(x) \geq 2$, then $N_G(x) \cap W_0 = \emptyset$, a contradiction. Now suppose $d_{G_1}(y) = 4$. If $n_4(x)+n_3(x) = 3$ with $n_4(x) \geq 1$, then $d(s_0) \geq 7$ by Lemma 2.8, a contradiction. If $n_4(x) = 0$ and $n_3(x) = 3$, then $d_{G_1}(y) = 3$, a contradiction. Hence suppose $n_4(x) = 2$ and $n_3(x) = 0$, then G is a subgraph of K_7.

Subcase 2.3 The vertex y satisfies condition (ii).

Note that for each $z \in \{x, y\}$, if $n_4(z) = 0$, then $n_3(z) \geq 2$ by Lemma 2.7. Note that $n_3(y) + n_4(y) \geq 1$. Let $T = \{t|t \in W_0 \cap N_G(x) \cap N_G(y)\}$.

Subcase 2.3.1 $d_{G_1}(x) = 4$.

Suppose $|T| = 0$. If $n_4(x) + n_3(x) + n_4(y) + n_3(y) \geq 3$, then $d(s_0) \geq 7$ by Lemma 2.8, a contradiction. Hence $n_4(x) = 1$ and $n_4(y) = 1$, then G is a subgraph of K_7.

Suppose $|T| = 1$ and assume $t_1 \in T$. Note that $3 \leq d_G(t_1) \leq 4$. Suppose $d(t_1) = 4$. If $4 \geq n_4(x) + n_3(x) + n_4(y) + n_3(y) \geq 3$, then G is a subgraph of K_7 or B_2; if $n_4(x) + n_3(x) + n_4(y) + n_3(y) = 2$, then $G \in \mathscr{C}_2 \cup \mathscr{C}_3$ or G is a subgraph of K_7 or B_2 by Lemma 2.5. Now suppose $d(t_1) = 3$. We obtain $n_3(x) = n_3(y) = 2$ and G is a subgraph of B_2 by Lemmas 2.7 and 2.8.

Suppose $|T| = 2$ and assume $T = \{t_1, t_2\}$. If $d(t_1) = 4$ or $d(t_2) = 4$, then $G \in \mathscr{C}_4$ or G is a subgraph of K_7 or B_2 by Lemma 2.5 and **Case 1**. Suppose $d(t_1) = d(t_2) = 3$. We obtain $G \in \mathscr{C}_5 \cup \mathscr{C}_6 \cup \mathscr{C}_7$ or G is a subgraph of K_7 or B_1 by Lemma 2.5.

Subcase 2.3.2 $d_{G_1}(x) = 3$.

By Lemmas 2.7 and 2.8, we have $n_4(x) + n_3(x) \geq 2$ and $n_4(y) + n_3(y) \geq 2$. If $|T| = 0$, then $d(s_0) \geq 7$, a contradiction; if $|T| = 1$, then G is a subgraph of K_7; Suppose $|T| = 3$. If $n_4(x) \geq 2$, then G is a subgraph of K_7; if $n_4(x) = 1$, then G is a subgraph of K_7 by **Case 1**; if $n_4(x) = 0$, then $G \in \mathscr{C}_8$ or G is a subgraph of K_7 by **Case 1**. Now suppose $|T| = 2$. If $n_4(x) \geq 2$, then G is a subgraph of K_7. Hence suppose $n_4(x) + n_3(x) = 3$ by Lemmas 2.7. And we know G is a subgraph of K_7 by **Case 1**. □

Acknowledgments. The second author was partially supported by NSFC (No. 11771221 and 11811540390), Natural Science Foundation of Tianjin (No. 17JCQNJC00300) and the China-Slovenia bilateral project ″Some topics in modern graph theory″ (No. 12-6). The third author was partially supported by NSFC (No. 11771402).

References

[1] Abu-Khzam, F.N., Heggernes, P.: Enumerrating minimal dominating sets in chordal graphs. Inform. Process. Lett. **116**, 739-743 (2016)

[2] Alon, N., McDiarmid, C., Reed, B.: Acyclic coloring of graphs. Random Structures Algorithms **2**, 277-288 (1991)

[3] Alon, N., Sudakov, B., Zaks, A.: Acyclic edge colorings of graphs. J. Graph Theory **37**, 157-167 (2001)

[4] Basavaraju, M., Chandran, L.S.: Acyclic edge coloring of subcubic graphs. Discrete Math. **308**, 6650-6653 (2008)

[5] Basavaraju, M., Chandran, L.S.: Acyclic edge coloring of graphs with maximum degree 4. J. Graph Theory **61**, 192-209 (2009)

[6] Basavaraju, M., Chandran, L.S.: Acyclic edge coloring of 2-degeneeate graph. J. Graph Theory **69**, 1-27 (2012)

[7] Basavaraju, M., Chandran, L.S., Cohen, N., Havet, F., Müller, T.: Acyclic edge-coloring of planar graphs. SIAM J. Discrete Math. **25**, 463-478 (2011)

[8] Esperet, L., Parreau, A.: Acyclic edge-coloring using entropy compression. European J. Combin. **34**, 1019-1027 (2013)

[9] Fiamčik, J.: The acyclic chromatic class of a graph. Math. Slovaca **28**, 139-145 (1978) (in Russian)

[10] Giotis, L., Kirousis, L., Psaromiligkos, K.I., Thilikos, D.M.: Acyclic edge colouring through the Lovász Local Lemma. Theoret. Comput. Sci. **665**, 40-50 (2017)

[11] Molloy, M., Reed, B.: Further algorithmic aspects of Lovász local lemma, in Proceedings of the 30th Annual ACM Symposium on Theory of Computing, 524-529 (1998)

[12] Ndreca, S., Procacci, A., Scoppola, B.: Improved bounds on coloring of graphs. European J. Combin. **33**, 592-609 (2012)

[13] Shu, Q., Wang, Y., Ma, Y., Wang, W.: Acyclic edge coloring of 4-regular graphs without 3-cycles. Bull. Malays. Math. Sci. Soc. (2017), doi:10.1007/s40840-017-0484-x

[14] Skulrattanakulchai, S.: Acyclic colorings of subcubic graphs. Inform. Process. Lett. **92**, 161-167 (2004)

[15] Wang, T., Zhang, Y.: Further result on acyclic chromatic index of planar graphs. Discrete Appl. Math. **201**, 228-247 (2016)

[16] Wang, W., Ma, Y., Shu, Q., Wang, Y.: Acyclic edge coloring of 4-regular graphs (II). Bull. Malays. Math. Sci. Soc. (2017), doi: 10.1007/s40840-017-0592-7

[17] Wang, W., Shu, Q., Wang, Y.: A new upper bound on the acyclic chromatic indices of planar graphs. European J. Combin. **34**, 338-354 (2013)