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On skew Laplacian spectrum and energy of digraphs

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We consider the skew Laplacian matrix of a digraph $\overrightarrow{G}$ obtained by giving an arbitrary direction to the edges of a graph G having n vertices and m edges. With $\nu_1, \nu_2, \ldots, \nu_n$ to be the skew Laplacian eigenvalues of $\overrightarrow{G}$, the skew Laplacian energy $\mathrm{SLE}(\overrightarrow{G})$ of $\overrightarrow{G}$ is defined as $\mathrm{SLE}(\overrightarrow{G}) = \sum_{i=1}^{n} |\nu_i|$. In this paper, we analyze the effect of changing the orientation of an induced subdigraph on the skew Laplacian spectrum. We obtain bounds for the skew Laplacian energy $\mathrm{SLE}(\overrightarrow{G})$ in terms of various parameters associated with the digraph $\overrightarrow{G}$ and the underlying graph G and we characterize the extremal digraphs attaining these bounds. We also show these bounds improve some known bounds for some families of digraphs. Further, we show the existence of some families of skew Laplacian equienergetic digraphs.

Keywords: Digraph; skew Laplacian matrix; skew Laplacian spectrum; skew Laplacian energy.

AMS Subject Classification: 05C50, 05C20, 15B36

1. Introduction

Consider a simple graph G with n vertices and m edges and having the vertex set $V = \{v_1, v_2, \dots, v_n\}$. Let $\overrightarrow{G}$ be a digraph obtained by assigning arbitrarily a direction to each of the edges of G. The digraph $\overrightarrow{G}$ is called an orientation of G or oriented graph corresponding to G. Also, the graph G is called the underlying graph of $\overrightarrow{G}$. Let $d_i^+ = d^+(v_i)$, $d_i^- = d^-(v_i)$ and $d_i = d_i^+ + d_i^-$, $i = 1, 2, \dots, n$ be respectively the out-degree, in-degree and degree of the vertices of $\overrightarrow{G}$. The out-adjacency matrix of the digraph $\overrightarrow{G}$ is the $n \times n$ matrix $A^+ = A^+(\overrightarrow{G}) = (a_{ij})$, where $a_{ij} = 1$, if (v_i, v_j) is an arc and $a_{ij} = 0$, otherwise. The in-adjacency matrix of the digraph $\overrightarrow{G}$ is the $n \times n$ matrix $A^- = A^-(\overrightarrow{G}) = (a_{ij})$, where $a_{ij} = 1$, if (v_j, v_i) is an arc and $a_{ij} = 0$, otherwise. We note that $A^- = (A^+)^t$. The skew adjacency matrix of a digraph $\overrightarrow{G}$ is the $n \times n$ matrix $S = S(\overrightarrow{G}) = (s_{ij})$, where

$$s_{ij} = \begin{cases} 1, & \text{if there is an arc from } v_i \text{ to } v_j, \\ -1, & \text{if there is an arc from } v_j \text{ to } v_i, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $S(\overrightarrow{G})$ is a skew symmetric matrix, so all its eigenvalues are zero or purely imaginary. The energy of the matrix $S(\overrightarrow{G})$ is defined as $E_s(\overrightarrow{G}) = \sum_{i=1}^{n} |\xi_i|$, where $\xi_1, \xi_2, \dots, \xi_n$ are the eigenvalues of $S(\overrightarrow{G})$. This energy of a digraph $\overrightarrow{G}$ is called the skew energy [1]. For recent developments on the theory of skew spectrum and skew energy, we refer to the survey paper [11]. Let $D^+ = D^+(\overrightarrow{G}) = \mathrm{diag}(d_1^+, d_2^+, \dots, d_n^+), D^- = D^-(\overrightarrow{G}) = \mathrm{diag}(d_1^-, d_2^-, \dots, d_n^-)$ and $D(\overrightarrow{G}) = \mathrm{diag}(d_1, d_2, \dots, d_n)$ be respectively the diagonal matrix of vertex out-degrees, vertex in-degrees and vertex degrees of $\overrightarrow{G}$. Further, let A^+ and A^- be respectively the out-adjacency and in-adjacency matrix of a digraph $\overrightarrow{G}$. If $S(\overrightarrow{G})$ is the skew adjacency matrix of $\overrightarrow{G}$ and $A(G)$ is the adjacency matrix of the underlying graph G of the digraph $\overrightarrow{G}$, then it clear that $A(G) = A^+ + A^-$ and $S(\overrightarrow{G}) = A^+ - A^-$. Analogous to the definition of Laplacian matrix of a graph, Cai *et al.* [3] called the matrix $\widetilde{\mathrm{SL}}(\overrightarrow{G}) = \widetilde{D}(\overrightarrow{G}) - S(\overrightarrow{G})$, where $\widetilde{D}(\overrightarrow{G}) = D^+(\overrightarrow{G}) - D^-(\overrightarrow{G})$, as the *skew Laplacian matrix* of the digraph $\overrightarrow{G}$. Clearly, the matrix $\widetilde{\mathrm{SL}}(\overrightarrow{G})$ is not symmetric and so its eigenvalues need not be real. The characteristic polynomial

$$P_{\mathrm{sl}}(\overrightarrow{G}, x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n,$$

of the matrix $\widetilde{\mathrm{SL}}(\overrightarrow{G})$ is called the *skew Laplacian characteristic polynomial* of the digraph $\overrightarrow{G}$. The zeros of the polynomial $P_{\mathrm{sl}}(\overrightarrow{G}, x)$, that is, the eigenvalues of the matrix $\widetilde{\mathrm{SL}}(\overrightarrow{G})$ are the skew Laplacian eigenvalues of the digraph $\overrightarrow{G}$ and are denoted by $\nu_1, \nu_2, \dots, \nu_n$. The skew Laplacian spectrum of the digraph $\overrightarrow{G}$ is denoted by $\mathrm{Spect}_{\mathrm{sl}}(\overrightarrow{G})$. The sign of the even cycle $C_k = u_1 u_2 \dots u_k u_1$, denoted by $\mathrm{sgn}(C_k)$, is defined as $\mathrm{sgn}(C_k) = s_{12}s_{23}\dots s_{k-1k}s_{k1}$. An even oriented cycle C_k is called evenly-oriented (oddly-oriented) if its sign is positive (negative). If every even cycle in $\overrightarrow{G}$ is evenly-oriented, $\overrightarrow{G}$ is called evenly-oriented. An even oriented cycle C_{2k} is said to be uniformly oriented if $\mathrm{sgn}(C_{2k}) = (-1)^k$.

The following observations are immediate from the definition of $\widetilde{\mathrm{SL}}$.

Theorem 1.1 ([3]). (i) *If $\nu_1, \nu_2, \dots, \nu_n$ are the eigenvalues of $\widetilde{\mathrm{SL}}(\overrightarrow{G})$, then $\sum_{i=1}^{n} \nu_i = 0$.*

(ii) *0 is an eigenvalue of $\widetilde{\mathrm{SL}}(\overrightarrow{G})$ with multiplicity at least p, where p is the number of components of $\overrightarrow{G}$ with all ones vector $(1, 1, \dots, 1)$ as the corresponding eigenvector.*

(iii) *If $P_{\mathrm{sl}}(\overrightarrow{G}, x) = x^n + \sum_{i=1}^{n} a_i x^{n-i}$ is the skew Laplacian characteristic polynomial of the digraph $\overrightarrow{G}$, then $a_1 = 0$, $a_2 = m + \sum_{i<j}(d_i^+ - d_i^-)(d_j^+ - d_j^-)$, $a_n = 0$.*

Evidently much research has been done on spectral theory of skew matrices of oriented graphs, see [11], but the research on the skew Laplacian spectrum of a digraph $\overrightarrow{G}$ has recently started and it will be of great interest to develop the theory in this direction. Although the skew Laplacian matrix of a digraph was so defined that it uses the structure of the digraph and at the same time enjoys the same characteristics as possessed by the Laplacian matrix of a graph, it seems the definition of $\widetilde{\mathrm{SL}}$ uses the structure of the digraph, but not all the properties of $L(G)$ are possessed by $\widetilde{\mathrm{SL}}$. It is well known that 0 is an eigenvalue of $L(G)$ with multiplicity equal to the number of components of G. In fact, the eigenvalue 0 in the spectrum of $L(G)$ decides the connectedness of the graph G. This need not be true for the matrix $\widetilde{\mathrm{SL}}$, as is clear from the following observation, the proof of which follows from [15, Theorem 2.1].

Theorem 1.2. *Let G be a bipartite graph and let $\overrightarrow{G}$ be the corresponding digraph of G. If $\overrightarrow{G}$ is an Eulerian digraph such that each even cycle of G is oriented uniformly in $\overrightarrow{G}$, then the multiplicity of 0 in the spectrum of $\widetilde{\mathrm{SL}}$ is same as the multiplicity of 0 in the spectrum of $A(G)$.*

As usual, we denote the complete graph on n vertices by K_n, the complete bipartite graph on $s+t$ vertices by $K_{s,t}$, the cycle on n vertices by C_n. For other undefined notations and terminology from graphs and spectral graph theory, the readers are referred to [2, 10, 13]. Let $K_{r,s}$ be the complete bipartite graph with both r and s even. Orient the edges of $K_{r,s}$ in such a way that in the resulting digraph $\overrightarrow{G}$ all the even cycles are oriented uniformly. Since 0 is an adjacency eigenvalue of $K_{r,s}$ of multiplicity $r+s-2$, from Theorem 2, it follows that 0 is the skew Laplacian eigenvalue of $\overrightarrow{G}$ of multiplicity $r+s-2$.

The rest of the paper is organized as follows. In Sec. 2, we see the effect of changing the orientation of an induced subdigraph on the skew Laplacian spectrum. In Sec. 3, we obtain bounds for the skew Laplacian energy $\mathrm{SLE}(\overrightarrow{G})$ in terms of various parameters associated with the digraph $\overrightarrow{G}$ and the underlying graph G. We show that these bounds improve some known bounds for certain families of digraphs. In Sec. 4, we show the existence of some families of skew Laplacian equienergetic digraphs.

2. Some Observations Regarding Skew Laplacian Spectrum

Let $\widetilde{\mathrm{SL}}$ be the skew Laplacian matrix of the digraph $\overrightarrow{G}$. If we reverse the direction of all the edges of $\overrightarrow{G}$, we obtain a new digraph $\overleftarrow{G}$, which we call the *converse digraph* of $\overrightarrow{G}$. Clearly $-\widetilde{\mathrm{SL}}$ is the skew Laplacian matrix of $\overleftarrow{G}$. Therefore, we have the following observation.

Theorem 2.1. *If $\overleftarrow{G}$ is the converse digraph of the digraph $\overrightarrow{G}$, then*

$$\mathrm{Spect}_{\mathrm{sl}}(\overleftarrow{G}) = -\mathrm{Spect}_{\mathrm{sl}}(\overrightarrow{G}).$$

Let $\overrightarrow{H}$ be an induced subdigraph of $\overrightarrow{G}$ corresponding to the induced subgraph H of G and let $\overrightarrow{H}^* = \overrightarrow{H} \cup (n - n(H))K_1$, that is, $\overrightarrow{H}$ together with $n - n(H)$ isolated vertices. Let $\overrightarrow{G} - E(\overrightarrow{H})$ be the subdigraph obtained by removing the arcs of $\overrightarrow{H}$ in $\overrightarrow{G}$ and $\overrightarrow{G} - \overrightarrow{H}$ be the subdigraph obtained by deleting the vertices together with the arcs of $\overrightarrow{H}$. Suppose both $\overrightarrow{H}$ and $\overrightarrow{G} - \overrightarrow{H}$ are Eulerian subdigraphs of $\overrightarrow{G}$. Let $\overrightarrow{G}_1$ be the digraph obtained by reversing the direction of all the arcs in $\overrightarrow{H}$ and keeping the other arcs unchanged. Let $\overrightarrow{G}_2$ be the digraph obtained by reversing the direction of all the arcs in $\overrightarrow{G} - E(\overrightarrow{H})$ and keeping the other arcs unchanged.

Again, let both $\overrightarrow{H}$ and $\overrightarrow{G} - \overrightarrow{H}$ be Eulerian subdigraphs of $\overrightarrow{G}$, and $\overrightarrow{G}_3$ be the digraph obtained by reversing the direction of the arcs having one end in $\overrightarrow{H}$ and other end in $\overrightarrow{G} - \overrightarrow{H}$. Let $\overrightarrow{G}_4$ be the digraph obtained from $\overrightarrow{G}$ by reversing the direction of the arcs in both $\overrightarrow{H}$ and $\overrightarrow{G} - \overrightarrow{H}$ and keeping the other arcs unchanged. Therefore, we have the following result.

Theorem 2.2. *Let $\overrightarrow{G}$ be an orientation of a graph G and let $\overrightarrow{H}$ be an induced subdigraph of $\overrightarrow{G}$ corresponding to the subgraph H of G. If the subdigraphs $\overrightarrow{H}$ and $\overrightarrow{G} - \overrightarrow{H}$ of $\overrightarrow{G}$ are Eulerian, then*

$$\text{(i)}\ \mathrm{Spect}_{\mathrm{sl}}(\overrightarrow{G}_1) = -\mathrm{Spect}_{\mathrm{sl}}(\overrightarrow{G}_2), \quad \text{(ii)}\ \mathrm{Spect}_{\mathrm{sl}}(\overrightarrow{G}_3) = -\mathrm{Spect}_{\mathrm{sl}}(\overrightarrow{G}_4),$$

where $\overrightarrow{G}_1$, $\overrightarrow{G}_2$, $\overrightarrow{G}_3$ and $\overrightarrow{G}_4$ are the digraphs defined above.

If $\overrightarrow{G}$ is itself an Eulerian digraph, the conclusion of Theorem 2.2 holds for all induced subdigraphs. A subset W of the vertex set $V(\overrightarrow{G})$ is said to be independent if the induced subdigraph $\langle W\rangle$ is an empty digraph. In other words, W is an independent subset of $V(\overrightarrow{G})$ if the vertices in W are mutually nonadjacent. By similar reasoning as above, we have the following observation.

Theorem 2.3. *Let $\overrightarrow{G}$ be an orientation of a graph G and let $\overrightarrow{H}$ be an induced subdigraph of $\overrightarrow{G}$ corresponding to the subgraph H of G. If the subdigraph $\overrightarrow{H}$ is Eulerian and the subdigraph $\overrightarrow{G}-\overrightarrow{H}$ is independent, then*

$$\text{(i)}\quad \mathrm{Spect}_{\mathrm{sl}}(\overrightarrow{G}_1) = -\mathrm{Spect}_{\mathrm{sl}}(\overrightarrow{G}_2),\quad \text{(ii)}\quad \mathrm{Spect}_{\mathrm{sl}}(\overrightarrow{G}_3) = -\mathrm{Spect}_{\mathrm{sl}}(\overrightarrow{G}_4),$$

where $\overrightarrow{G}_1$, $\overrightarrow{G}_2$, $\overrightarrow{G}_3$ and $\overrightarrow{G}_4$ are the digraphs defined as above.

3. Bounds for the Skew Laplacian Energy

The skew Laplacian energy of $\overrightarrow{G}$, denoted by $\mathrm{SLE}(\overrightarrow{G})$, is defined as

$$\mathrm{SLE}(\overrightarrow{G}) = \sum_{j=1}^{n} |\nu_j|, \tag{3.1}$$

where $\nu_1, \nu_2, \ldots, \nu_n$ are the skew Laplacian eigenvalues of $\overrightarrow{G}$. This concept was introduced in 2013 by Cai *et al.* [3]. The idea was to conceive a graph energy like quantity for a digraph, which instead of skew adjacency eigenvalues is defined in terms of skew Laplacian eigenvalues and that hopefully would preserve the main features of the original graph energy. The definition of $\mathrm{SLE}(\overrightarrow{G})$ was therefore so chosen that all the properties possessed by graph energy should be preserved. The skew Laplacian energy is an extension of skew energy of a digraph just as Laplacian energy (see [4, 5, 8, 12] and the references therein) is an extension of graph energy (see [6] and the references therein).

Now, we obtain the bounds for skew Laplacian energy $\mathrm{SLE}(\overrightarrow{G})$ and we will see that these bounds for $\mathrm{SLE}(\overrightarrow{G})$ are better than some of the previously known bounds. The following result gives a relation between skew energy and the skew Laplacian energy of an oriented graph G.

Theorem 3.1. *Let $\overrightarrow{G}$ be an orientation of G.*

(1) *If $\overrightarrow{G}$ is an Eulerian digraph, then* $\mathrm{SLE}(\overrightarrow{G}) = E_s(\overrightarrow{G})$.

(2) $\mathrm{SLE}(\overrightarrow{G}) \geq 2\sum_{\nu_i\in U_1'} |\nu_i| + 2|\sum_{\nu_i\in U_1'} \mathrm{Re}(\nu_i)|$, *where U_1 is the set of the eigenvalues of the form $\nu_i = a_i + ib_i$, with $b_i \neq 0$ and U_1' is the subset of U_1 containing either ν_i or $\bar{\nu}_i$ but not both. Equality occurs if and only if $\nu_i = s\nu_j$, for all $\nu_i, \nu_j \in U_2$, $s \in \mathbb{R}$.*

Proof. Let $\nu_1, \nu_2, \ldots, \nu_n$ be the eigenvalues of the matrix $\widetilde{\mathrm{SL}} = \tilde{D} - S(\overrightarrow{G})$. If the digraph $\overrightarrow{G}$ of G is Eulerian, then $\tilde{D} = 0$ and so $\widetilde{\mathrm{SL}} = -S(\overrightarrow{G})$. From this, the first part follows.

Let $U_1 = \{\nu_i : \nu = a_i + ib_i, b_i \neq 0\}$ and $U_2 = \{\nu_i : \nu = a_i + ib_i, b_i = 0\}$. Since the skew Laplacian characteristic polynomial $P_{\mathrm{sl}}(\overrightarrow{G}, x)$ has real coefficients, so if $\nu_i \in U_1$, then $\bar{\nu_i} \in U_1$, where $\bar{\nu_i}$ is the complex conjugate of ν_i. With out loss of generality, suppose that $U_1 = \{\nu_1, \nu_2, \ldots, \nu_k, \bar{\nu_1}, \bar{\nu_2}, \ldots, \bar{\nu_k}\}$. By Theorem 1.1, we have

$$\sum_{i=1}^{n} \nu_i = 0 \Rightarrow \sum_{\nu_i \in U_1} \nu_i + \sum_{\nu_i \in U_2} \nu_i = 0 \Rightarrow \sum_{\nu_i \in U_1} \mathrm{Re}(\nu_i) + \sum_{\nu_i \in U_2} \nu_i = 0$$

$$\Rightarrow \sum_{\nu_i \in U_2} \nu_i = -\sum_{\nu_i \in U_1} \mathrm{Re}(\nu_i) \Rightarrow \left| \sum_{\nu_i \in U_1} \mathrm{Re}(\nu_i) \right| = \left| \sum_{\nu_i \in U_2} \nu_i \right| \leq \sum_{\nu_i \in U_2} |\nu_i|.$$

From this and (1), we have

$$\mathrm{SLE}(\overrightarrow{G}) = \sum_{i=1}^{n} |\nu_i| = \sum_{\nu_i \in U_1} |\nu_i| + \sum_{\nu_i \in U_2} |\nu_i| \geq \sum_{\nu_i \in U_1} |\nu_i| + \left| \sum_{\nu_i \in U_1} \mathrm{Re}(\nu_i) \right|$$

$$= 2 \sum_{\nu_i \in U_1'} |\nu_i| + 2 \left| \sum_{\nu_i \in U_1'} \mathrm{Re}(\nu_i) \right|,$$

where U_1' is the subset of U_1 containing either ν_i or $\bar{\nu_i}$ but not both. Equality occurs if and only if $|\sum_{\nu_i \in U_2} \nu_i| = \sum_{\nu_i \in U_2} |\nu_i|$, that is, if and only if all $\nu_i \in U_2$ are collinear, that is, $\nu_i = s\nu_j$, for all $\nu_i, \nu_j \in U_2$, $s \in \mathbb{R}$. □

If $\overrightarrow{G}$ is an Eulerian digraph, then $\widetilde{\mathrm{SL}} = -S(\overrightarrow{G})$, a skew symmetric matrix. So all its eigenvalues are either zero or purely imaginary, that is, $\nu_i = 0$, for all $\nu_i \in U_2$. From this, it follows that for Eulerian digraphs equality occurs in Theorem 3.1. If all the skew Laplacian eigenvalues of a digraph $\overrightarrow{G}$ are real, then $U_1 = \phi$ and so we have the following observation, the proof of which follows from Theorem 3.1.

Corollary 3.1. *If all the skew Laplacian eigenvalues of a digraph $\overrightarrow{G}$ are real, then*

$$\mathrm{SLE}(\overrightarrow{G}) = 2 \sum_{\nu_i \in U_2'} \nu_i,$$

where U_2' is the set of positive eigenvalues in U_2.

For Eulerian digraphs $\overrightarrow{G}$, Theorem 3.1 implies that the skew Laplacian energy is same as the corresponding skew energy. So, all the theorems and problems that have been considered for the skew energy also hold for the skew Laplacian energy of Eulerian digraphs. One of the problems which is considered for the skew energy is that of determining the digraphs which attains the upper bound

$$E_s(\overrightarrow{G}) \leq n\sqrt{\Delta}, \tag{3.2}$$

where Δ is the maximum vertex degree. This problem is an active component of the present research and some families of digraphs have been characterized in this direction [11]. The following observation can be found in [10].

Lemma 3.1. *Let X be a square complex matrix of order n having singular values $\sigma_1(X) \geq \cdots \geq \sigma_n(X)$ and eigenvalues $\lambda_1(X), \lambda_2(X), \ldots, \lambda_n(X)$ with $|\lambda_1(X)| \geq \cdots \geq |\lambda_n(X)|$. Then, for $1 \leq k \leq n$, we have*

$$\sum_{i=1}^{k} |\lambda_i(X)|^p \leq \sum_{i=1}^{k} \sigma_i(X)^p,$$

for any positive real number p. Equality occurs if and only if X is a normal matrix.

A graph is said to be nilpotent if its adjacency matrix is nilpotent. It is well known that a graph is nilpotent if and only if it is a totally disconnected graph. We call a digraph $\overrightarrow{G}$ skew-nilpotent digraph if its skew Laplacian matrix $\widetilde{\mathrm{SL}}(\overrightarrow{G})$ is a nilpotent matrix. It is clear that the totally disconnected digraph $\overline{K}_n$ is a skew-nilpotent digraph. We note that there are digraphs having at least one edge which are skew-nilpotent digraphs. The simplest example is the digraph $\overrightarrow{G} \cong t\overrightarrow{K_2} \cup (n-2t)K_1$. The skew Laplacian matrix of the digraph $t\overrightarrow{K_2} \cup (n-2t)K_1$ is $\widetilde{\mathrm{SL}}(t\overrightarrow{K_2} \cup (n-2t)K_1) = \mathrm{diag}(X_1, X_1, \ldots, X_1, X_2 \ldots, X_t)$, where all X_i are zero matrices for $i \geq 2$ and $X_1 = \left(\begin{smallmatrix} 1 & -1 \\ 1 & -1 \end{smallmatrix}\right)$ is repeated t times. By direct calculation, it can be seen that the eigenvalues of X_1 are $0, 0$. Since the eigenvalues of $\widetilde{\mathrm{SL}}(t\overrightarrow{K_2} \cup (n-2t)K_1)$ are the union of the eigenvalues of $X_1, X_1, \ldots, X_1, X_2 \ldots, X_t$ and 0 is the only eigenvalue of each X_i, $i \geq 2$, it follows that the eigenvalues of $\widetilde{\mathrm{SL}}(t\overrightarrow{K_2} \cup (n-2t)K_1)$ are all zero and so $t\overrightarrow{K_2} \cup (n-2t)K_1$ is a skew-nilpotent digraph. It will of interest to characterize all skew-nilpotent digraphs and so we leave the following problem.

Problem 1. Characterize all skew-nilpotent digraphs with at least one edge.

Now, we obtain an upper bound for $\mathrm{SLE}(\overrightarrow{G})$ in terms of the skew Laplacian rank r_{sl} and the parameter $M_1 = m + \frac{1}{2}\sum_{i=1}^{n}(d_i^+ - d_i^-)^2$ associated to the digraph $\overrightarrow{G}$.

Theorem 3.2. *Let G be a connected graph with n vertices having m edges and let $\overrightarrow{G}$ be an orientation of G. Then*

$$\mathrm{SLE}(\overrightarrow{G}) \leq \sqrt{2M_1 r_{\mathrm{sl}}}, \tag{3.3}$$

where $M_1 = m + \frac{1}{2}\sum_{i=1}^{n}(d_i^+ - d_i^-)^2$ and r_{sl} is the rank of the matrix $\widetilde{\mathrm{SL}}$. Equality occurs if and only if $\overrightarrow{G}$ is an Eulerian digraph having skew Laplacian eigenvalues

$$0^{[n-r_{\mathrm{sl}}]}, \quad (ia)^{[\frac{r_{\mathrm{sl}}}{2}]}, \quad (-ia)^{[\frac{r_{\mathrm{sl}}}{2}]}, \quad a > 0.$$

Proof. Let $\widetilde{\mathrm{SL}} = (l_{ij})$ be the skew Laplacian matrix of $\overrightarrow{G}$ and let r_{sl} be the rank of the matrix $\widetilde{\mathrm{SL}}$. Let $\nu_1, \nu_2, \ldots, \nu_{n-1}, \nu_n = 0$ be the skew Laplacian eigenvalues of $\overrightarrow{G}$. Applying Cauchy–Schwarz's inequality to the vectors $(|\nu_1|, |\nu_2|, \ldots, |\nu_{r_{\mathrm{sl}}}|)$ and $(1, 1, \ldots, 1)$, we get

$$\mathrm{SLE}(\overrightarrow{G}) = \sum_{i=1}^{r_{\mathrm{sl}}} |\nu_i| \leq \sqrt{r_{\mathrm{sl}} \sum_{i=1}^{r_{\mathrm{sl}}} |\nu_i|^2},$$

with equality if and only if $|\nu_1| = |\nu_2| = \cdots = |\nu_{r_{\mathrm{sl}}}|$.

Taking $p = 2$, $k = r_{\mathrm{sl}}$, $\lambda_i = \nu_i$ and $X = \widetilde{\mathrm{SL}}$ in Lemma 3.1, we get

$$\sum_{i=1}^{r_{\mathrm{sl}}} |\nu_i|^2 \leq \sum_{i=1}^{r_{\mathrm{sl}}} \sigma_i(\widetilde{\mathrm{SL}})^2. \tag{3.4}$$

Since

$$\sum_{i=1}^{r_{\mathrm{sl}}} \sigma_i(\widetilde{\mathrm{SL}})^2 = \mathrm{tr}(\widetilde{\mathrm{SL}}^* \widetilde{\mathrm{SL}}) = \sum_{i,j=1}^{n} |l_{ij}|^2 = 2m + \sum_{i=1}^{n} \left(d_i^+ - d_i^-\right)^2,$$

from Eq. (4), it follows that

$$\mathrm{SLE}(\overrightarrow{G}) \leq \sqrt{r_{\mathrm{sl}}\left(2m + \sum_{i=1}^{n} \alpha_i^2\right)} = \sqrt{2r_{\mathrm{sl}} M_1}.$$

This completes the proof of the first part of the theorem. Equality occurs in (3) if and only if equality occurs in Lemma 3.1 and $|\nu_1| = |\nu_2| = \cdots = |\nu_{r_{\mathrm{sl}}}|$. Since equality occurs in Lemma 3.1 if and only if $\widetilde{\mathrm{SL}}$ is a normal matrix and as shown in [8], the matrix $\widetilde{\mathrm{SL}}$ is normal if and only if $\overrightarrow{G}$ is an Eulerian digraph, it follows that equality occurs in (3) if and only if $\overrightarrow{G}$ is an Eulerian digraph with $|\nu_1| = |\nu_2| = \cdots = |\nu_{r_{\mathrm{sl}}}|$. Since 0 is always an eigenvalue of $\widetilde{\mathrm{SL}}$, it follows that the skew Laplacian spectrum of the Eulerian digraph $\overrightarrow{G}$ is $\{\nu_1, \nu_2, \ldots, \nu_{r_{\mathrm{sl}}}, 0^{[n-r_{\mathrm{sl}}]}\}$, with $|\nu_1| = |\nu_2| = \cdots = |\nu_{r_{\mathrm{sl}}}| = a$. The following cases arise.

Case 1. If $a = 0$, all the skew Laplacian eigenvalues of $\overrightarrow{G}$ are 0, so $\overrightarrow{G}$ is either an empty digraph or $\overrightarrow{G}$ is a skew-nilpotent digraph with at least one edge. If $\overrightarrow{G}$ is an empty, then equality cannot occur as G is connected. We will show there is no Eulerian skew-nilpotent digraph. For if $\overrightarrow{G}$ is a skew-nilpotent Eulerian digraph, then the skew matrix of $\overrightarrow{G}$ is a nilpotent matrix. Since the skew matrix of $\overrightarrow{G}$ is a normal matrix and it is well known [9] that a normal matrix is nilpotent if and only if it is a zero matrix. It follows that there is no Eulerian skew-nilpotent digraph $\overrightarrow{G}$ with at least one edge.

Case 2. If $a \neq 0$ and $\overrightarrow{G}$ is Eulerian, then using the fact that $\widetilde{\mathrm{SL}} = -S(\overrightarrow{G})$ is a real skew-symmetric matrix, it follows that the nonzero skew Laplacian eigenvalues of $\overrightarrow{G}$ are purely imaginary and so they occur in conjugate pairs. Thus, it follows that the skew Laplacian eigenvalues of the digraph $\overrightarrow{G}$ should be of the form $0^{[n-r_{\rm sl}]}, (ia)^{[\frac{r_{\rm sl}}{2}]}, (-ia)^{[\frac{r_{\rm sl}}{2}]}$, $a > 0$. This completes the proof. □

Since 0 is always a skew Laplacian eigenvalue of $\overrightarrow{G}$, it follows that $r_{\rm sl} \leq n - 1$ and so we make the following observation which is immediate from Theorem 3.2.

Corollary 3.2. *Let G be a connected graph with n vertices and m edges and let $\overrightarrow{G}$ be an orientation of G. Then*

$$\mathrm{SLE}(\overrightarrow{G}) \leq \sqrt{2M_1(n-1)},$$

where $M_1 = m + \frac{1}{2}\sum_{i=1}^{n}(d_i^+ - d_i^-)^2$. Equality occurs if and only if $\overrightarrow{G}$ is an Eulerian digraph having skew Laplacian eigenvalues $0, (ai)^{[\frac{n-1}{2}]}, (-ai)^{[\frac{n-1}{2}]}(a > 0)$.

This upper bound for skew Laplacian energy $\mathrm{SLE}(\overrightarrow{G})$ of a digraph $\overrightarrow{G}$ was obtained in [3]. Since for Eulerian digraphs $M_1 = m$ and $r_{\rm sl} = r_s$, where r_s is the rank of $S(\overrightarrow{G})$, we have the following observation from Theorem 3.2.

Corollary 3.3. *Let G be a connected graph with n vertices and m edges and let $\overrightarrow{G}$ be an orientation of G. If $\overrightarrow{G}$ is Eulerian, then*

$$\mathrm{SLE}(\overrightarrow{G}) \leq \sqrt{2mr_s}, \tag{3.5}$$

with equality if and only if the skew Laplacian eigenvalues of $\overrightarrow{G}$ are $0^{[n-r_s]}$, $(ai)^{[\frac{r_s}{2}]}$, $(-ai)^{[\frac{r_s}{2}]}$, $(a > 0)$.

In [15], it is shown that for a bipartite graph G, there is always a digraph $\overrightarrow{G}$ having skew spectrum i times the adjacency spectrum of G. So, for such orientations of bipartite graphs, skew rank r_s is same as the corresponding adjacency rank. From this, it follows that if the edges of a bipartite graph G are so directed that the resulting orientation $\overrightarrow{G}$ is Eulerian with all the even cycles oriented uniformly, we can always find a digraph whose skew Laplacian rank is less than $n - 1$. The simplest example is the complete bipartite graph $K_{s,t}$, the adjacency rank of this graph is 2. For all such digraphs the upper bound given by (2) is always better than the upper bound given by Corollary 3.2. Further, since $\sqrt{2mr_s} \leq \sqrt{2m(n-1)} \leq \sqrt{n(n-1)\Delta} < n\sqrt{\Delta}$, as $2m \leq n\Delta$, it follows that Eulerian digraphs never attain the upper bound (2) and so they are not the maximum skew energy digraphs.

The following gives another upper bound for $\mathrm{SLE}(\overrightarrow{G})$, in terms of the skew Laplacian rank $r_{\rm sl}$ and the parameter $M_1 = m + \frac{1}{2}\sum_{i=1}^{n}(d_i^+ - d_i^-)^2$ associated to the digraph $\overrightarrow{G}$.

Theorem 3.3. *Let G be a connected graph G with n vertices having m edges and let $\overrightarrow{G}$ be the digraph of G. Then*

$$\mathrm{SLE}(\overrightarrow{G}) \le \sqrt{\frac{2M_1}{r_{\mathrm{sl}}}} + \sqrt{(r_{\mathrm{sl}}-1)\left(2M_1 - \frac{2M_1}{r_{\mathrm{sl}}}\right)}. \tag{3.6}$$

Equality occurs if and only if $\overrightarrow{G}$ is an Eulerian digraph having skew Laplacian eigenvalues

$$0^{[n-r_{\mathrm{sl}}]}, \quad (ia)^{[\frac{r_{\mathrm{sl}}}{2}]}, \quad (-ia)^{[\frac{r_{\mathrm{sl}}}{2}]}, \quad a = \sqrt{\frac{2M_1}{r_{\mathrm{sl}}}}.$$

Proof. Let $\widetilde{\mathrm{SL}} = (l_{ij})$ be the skew Laplacian matrix of $\overrightarrow{G}$ and let r_{sl} be the rank of the matrix $\widetilde{\mathrm{SL}}$. Let $\nu_1, \nu_2, \ldots, \nu_{n-1}, \nu_n = 0$ be the skew Laplacian eigenvalues of $\overrightarrow{G}$ with $\rho_{\mathrm{sl}} = |\nu_1| \ge |\nu_2| \ge \cdots \ge |\nu_{n-1}| \ge 0$. Applying Cauchy–Schwarz's inequality to the vectors $(|\nu_2|, |\nu_3|, \ldots, |\nu_{r_{\mathrm{sl}}}|)$ and $(1, 1, \ldots, 1)$, we get

$$\begin{aligned}\mathrm{SLE}(\overrightarrow{G}) - |\nu_1| = \sum_{i=2}^{r_{\mathrm{sl}}} |\nu_i| &\le \sqrt{(r_{\mathrm{sl}}-1)\sum_{i=2}^{r_{\mathrm{sl}}} |\nu_i|^2} \\ &\le \sqrt{(r_{\mathrm{sl}}-1)\left(\sum_{i=1}^{r_{\mathrm{sl}}} |\nu_i|^2 - |\nu_1|^2\right)},\end{aligned}$$

with equality if and only if $|\nu_2| = |\nu_3| = \cdots = |\nu_{r_{\mathrm{sl}}}|$.

Taking $p = 2$, $k = r_{\mathrm{sl}}$, $\lambda_i = \nu_i$ and $X = \widetilde{\mathrm{SL}}$ in Lemma 3.1, we get

$$\sum_{i=1}^{r_{\mathrm{sl}}} |\nu_i|^2 \le \sum_{i=1}^{r_{\mathrm{sl}}} \sigma_i(\widetilde{\mathrm{SL}})^2. \tag{3.7}$$

Since

$$\sum_{i=1}^{r_{\mathrm{sl}}} \sigma_i(\widetilde{\mathrm{SL}})^2 = \mathrm{tr}(\widetilde{\mathrm{SL}}^* \widetilde{\mathrm{SL}}) = \sum_{i,j=1}^{n} |l_{ij}|^2 = 2m + \sum_{i=1}^{n} \alpha_i^2,$$

from Eq. (7), it follows that

$$\mathrm{SLE}(\overrightarrow{G}) \le \rho_{\mathrm{sl}} + \sqrt{(r_{\mathrm{sl}}-1)(2M_1 - \rho_{\mathrm{sl}}^2)}.$$

Consider the function

$$f(x) = x + \sqrt{(r_{\mathrm{sl}}-1)(2M_1 - x^2)}.$$

It is easy to see that $f(x)$ is a decreasing function for $\sqrt{\frac{2M_1}{r_{\mathrm{sl}}}} \le x \le \sqrt{2M_1}$. So, we have $f(x) \le f(\sqrt{\frac{2M_1}{r_{\mathrm{sl}}}})$, that is,

$$f(x) \le \sqrt{\frac{2M_1}{r_{\mathrm{sl}}}} + \sqrt{(r_{\mathrm{sl}}-1)\left(2M_1 - \frac{2M_1}{r_{\mathrm{sl}}}\right)},$$

which gives

$$\mathrm{SLE}(\overrightarrow{G}) \leq \sqrt{\frac{2M_1}{r_{\rm sl}}} + \sqrt{(r_{\rm sl}-1)\left(2M_1 - \frac{2M_1}{r_{\rm sl}}\right)},$$

completing the proof of the first part of the theorem. Equality case is similar to that of Theorem 3.2. □

From the proof of Theorem 3.3, if we know a lower bound for $\rho_{\rm sl}$, the upper bound given in this theorem can be improved. Thus, it will be attractive to find the possible lower bounds for the skew Laplacian spectral radius $\rho_{\rm sl}$ which relates it to the structure of the digraph.

4. Skew Laplacian Equienergetic Digraphs

Two digraphs of same order are said to be *skew equienergetic digraphs*, if they are noncospectral with respect to their skew spectrum and have the same skew energy [14]. Like wise, two digraphs of same order are said to be skew Laplacian equienergetic digraphs, if they are noncospectral with respect to their skew Laplacian spectrum and have the same skew Laplacian energy.

Ramane *et al.* [14] have shown the existence of nonskew cospectral Eulerian digraphs of order n having same skew energy for all $n \geq 6$. Since for Eulerian digraphs skew energy and skew Laplacian energy are same. So the families of digraphs obtained in [14] are also skew Laplacian equienergetic. It will be interesting to obtain non-Eulerian digraphs which are noncospectral with respect skew Laplacian spectrum and have the same skew Laplacian energy. Our aim is be to show the existence of families of non-Eulerian skew Laplacian equienergetic digraphs.

The following observations are immediate from Theorems 2.1–2.3.

Theorem 4.1. *Let $\overrightarrow{G}$ be an orientation of a connected graph G. Then*

(1) *$\overrightarrow{G}$ and $\overrightarrow{G}$ are nonisomorphic, nonskew Laplacian cospectral digraphs with* $\mathrm{SLE}(\overrightarrow{G}) = \mathrm{SLE}(\overleftarrow{G})$.

(2) *$\overrightarrow{G}_1$ and $\overrightarrow{G}_2$ are nonisomorphic, nonskew Laplacian cospectral digraphs with* $\mathrm{SLE}(\overrightarrow{G}_1) = \mathrm{SLE}(\overrightarrow{G}_2)$.

(3) *$\overrightarrow{G}_3$ and $\overrightarrow{G}_4$ are nonisomorphic, nonskew Laplacian cospectral digraphs with* $\mathrm{SLE}(\overrightarrow{G}_3) = \mathrm{SLE}(\overrightarrow{G}_4)$,

where $\overrightarrow{G}_1, \overrightarrow{G}_2, \overrightarrow{G}_3, \overrightarrow{G}_4$ are the digraphs as defined in Theorems 2.2 *and* 2.3.

The first part of Theorem 4.1, tell us that the digraph and its converse have the same skew Laplacian energy. Similarly, other parts reveal that by changing the orientation of the arcs between some suitable induced subdigraphs of $\overrightarrow{G}$ does not effect the skew Laplacian energy. Let $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ be orientations of G_1 and G_2

respectively and let $\overrightarrow{G} = \overrightarrow{G_1} \to \overrightarrow{G_2}$, be the digraph obtained by taking the union of the digraphs $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ and joining each vertex v in $\overrightarrow{G_1}$ with every vertex u in $\overrightarrow{G_2}$ by an arc directed from v to u. It is clear that the underlying graph of $\overrightarrow{G}$ is the join of G_1 and G_2. The following theorem obtained in [7] gives the skew characteristic polynomial of the digraph $\overrightarrow{G} = \overrightarrow{G_1} \to \overrightarrow{G_2}$ in terms of the skew characteristic polynomial of the digraphs $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$.

Theorem 4.2. *If $\overrightarrow{G} = \overrightarrow{G_1} \to \overrightarrow{G_2}$, then*

$$P_{\mathrm{sl}}(\overrightarrow{G}, x) = \frac{x(x - n_2 + n_1)}{(x + n_1)(x - n_2)} P_{\mathrm{sl}}(\overrightarrow{G_1}, x - n_2) P_{\mathrm{sl}}(\overrightarrow{G_2}, x + n_1),$$

where n_1 and n_2 are respectively, the orders of digraphs $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$.

If $\nu_i, 0$, for $i = 1, 2, \ldots, n_1 - 1$, are the skew Laplacian eigenvalue of G_1, and $\xi_i, 0$, for $i = 1, 2, \ldots, n_2 - 1$ are the skew Laplacian eigenvalue of G_2, then from Theorem 4.2, it is clear that the skew Laplacian eigenvalues of $\overrightarrow{G_1} \to \overrightarrow{G_2}$ are

$$\nu_i + n_2, \quad \xi_k - n_1, \quad n_2 - n_1, \quad 0, \quad i = 1, 2, \ldots, n_1 - 1, \quad k = 1, 2, \ldots, n_2 - 1.$$

Therefore, the skew Laplacian energy of the digraph $\overrightarrow{G_1} \to \overrightarrow{G_2}$ is given by

$$\mathrm{SLE}(\overrightarrow{G_1} \to \overrightarrow{G_2}) = |n_2 - n_1| + \sum_{i=1}^{n_1 - 1} |\nu_i + n_2| + \sum_{i=1}^{n_2 - 1} |\xi_i - n_1|.$$

Suppose that all ν_i, ξ_k are real with $|\nu_i| \le n_2$, $|\xi_k| \le n_1$, for all $i = 1, 2, \ldots, n_1 - 1$; $k = 1, 2, \ldots, n_2 - 1$ and $n_1 \le n_2$. Then,

$$|\nu_i + n_2| = \begin{cases} n_2 + |\nu_i|, & \text{if } \nu_i \ge 0, \\ n_2 - |\nu_i|, & \text{if } \nu_i < 0, \end{cases} \qquad |\xi_i - n_1| = \begin{cases} n_1 - |\xi_i|, & \text{if } \xi_i \ge 0, \\ n_1 + |\xi_i|, & \text{if } \xi_i < 0, \end{cases}$$

and so

$$\begin{aligned} \mathrm{SLE}(\overrightarrow{G_1} \to \overrightarrow{G_2}) &= |n_2 - n_1| + \sum_{i=1}^{n_1 - 1} |\nu_i + n_2| + \sum_{i=1}^{n_2 - 1} |\xi_i - n_1| \\ &= n_2 - n_1 + \sum_{\nu_i \ge 0} (n_2 + |\nu_i|) + \sum_{\nu_i < 0} (n_2 - |\nu_i|) \\ &\quad + \sum_{\xi_i \ge 0} (n_1 - |\xi_i|) + \sum_{\xi_i < 0} (n_1 + |\xi_i|) \\ &= 2n_1(n_2 - 1) + \sum_{\nu_i, \xi_i \ge 0} (|\nu_i| - |\xi_i|) + \sum_{\nu_i, \xi_i < 0} (|\xi_i| - |\nu_i|). \end{aligned}$$

From this, we arrive at the following observation.

Theorem 4.3. *Let G_1 be a graph with n_1 vertices and m_1 edges and let $\overrightarrow{G_1}$ be an orientation of G_1. If all the skew Laplacian eigenvalues $\nu_1, \nu_1 \ldots, \nu_{n_1 - 1}, 0$ of G_1 are* *real with* $|\nu_i| \leq n_1$, *for all* i, *then*

$$\mathrm{SLE}(\overrightarrow{G_1} \to \overrightarrow{G_1}) = 2n_1(n_1 - 1).$$

If all the skew Laplacian eigenvalues of G_1 *are purely imaginary or zero with* $|\nu_i| \leq n_1$, *for all* i, *then*

$$\mathrm{SLE}(\overrightarrow{G_1} \to \overrightarrow{G_1}) = 2\sum_{i=1}^{n_1-1} |\nu_i + n_1|.$$

Proof. Proof follows from the above discussion. □

Theorem 4.3 implies that the skew Laplacian energy $\mathrm{SLE}(\overrightarrow{G_1} \to \overrightarrow{G_1})$ of the digraph $\overrightarrow{G_1} \to \overrightarrow{G_1}$ is a function of n_1, the number of vertices of $\overrightarrow{G_1}$, provided all the skew Laplacian eigenvalues $\nu_1, \nu_1 \ldots, \nu_{n_1-1}, 0$ of G_1 are real with $|\nu_i| \leq n_1$, for all i. Therefore, we have the following observation.

Corollary 4.1. *Let* $\overrightarrow{G_1}$ *and* $\overrightarrow{H_1}$ *be two digraphs each having order* n_1. *Let* $\nu_i, 0$, *for* $i = 1, 2, \ldots, n_1 - 1$ *be the skew Laplacian eigenvalues of* G_1 *and* $\xi_i, 0$, *for* $i = 1, 2, \ldots, n_1 - 1$, *be the skew Laplacian eigenvalues of* H_1. *If for* $i = 1, 2, \ldots, n_1 - 1$ *each of* ν_i, ξ_i *are real with* $|\nu_i|, |\xi_i| \leq n_1$, *then*

$$\mathrm{SLE}(\overrightarrow{G_1} \to \overrightarrow{G_1}) = \mathrm{SLE}(\overrightarrow{H_1} \to \overrightarrow{H_1}).$$

If the digraphs $\overrightarrow{G_1}$ and $\overrightarrow{H_1}$ in Corollary 4.1 are nonskew Laplacian cospectral, we obtain an infinite families of skew Laplacian equienergetic digraphs.

Let $K_{r,s}$ be a complete bipartite graph of order $n_1 = r + s$ having partite sets $V_1 = \{x_1, x_2, \ldots, x_r\}$ and $V_2 = \{y_1, y_2, \ldots, y_s\}$. Consider different orientations of $K_{r,s}$. Let $\overrightarrow{H}_1$ be the orientation when all the edges are directed from V_1 to V_2 and $\overrightarrow{H}_2$ be the orientation when all the edges are directed from V_2 to V_1. The following lemma [7] is about the skew Laplacian eigenvalues of $\overrightarrow{H}_1$ and $\overrightarrow{H}_2$.

Lemma 4.1. *The skew Laplacian spectrum of* $\overrightarrow{H}_1$ *is* $\{s-r, 0, s^{[r-1]}, (-r)^{[s-1]}\}$ *and the skew Laplacian spectrum of* $\overrightarrow{H}_2$ *is* $\{-(s-r), 0, (-s)^{[r-1]}, r^{[s-1]}\}$.

Any orientation of a complete graph K_n on n vertices is said to be a tournament. If $v_i \to v_j$ is an arc in a tournament, the vertex v_i is said to dominate the vertex v_j. A tournament is said to be transitive if u dominates v and v dominates w implies u dominates w, for all the vertices u, v, w of the tournament. We denote a transitive tournament of order n_1 by T_{n_1}. The following theorem gives the skew Laplacian spectrum of a transitive tournament and can be found in [7].

Lemma 4.2. *For a transitive tournament of order* n, *the skew Laplacian spectrum is* $\{\pm(n-2j) : j = 1, 2, 3, \ldots \lfloor \frac{n_1}{2} \rfloor\}$, *when* n *is even and equal to* $\{0, \pm(n-2j) : j = 1, 2, \ldots, \lfloor \frac{n_1}{2} \rfloor$, *when* n *is odd.*

The following result gives a family of non-Eulerian skew Laplacian equienergetic digraphs of order $n \equiv 0(\text{mod } 2)$.

Theorem 4.4. *The digraphs* $H = \overrightarrow{H_1} \rightarrow \overrightarrow{H_1}$ *and* $G = T_{n_1} \rightarrow T_{n_1}$ *are non-Eulerian skew Laplacian equienergetic digraphs of order* $n \equiv 0(\text{mod } 2)$, *where* H_1 *and* T_{n_1} *are the digraphs defined in Lemmas* 4.1 *and* 4.2.

Proof. By Lemma 4.1, the skew Laplacian eigenvalues of $\overrightarrow{H_1}$ are $s - r, 0, s^{[r-1]}, (-r)^{[s-1]}$, $r + s = n_1$. Clearly, all the eigenvalues of $\overrightarrow{H_1}$ are real with their moduli less or equal to $n_1 = r + s$. Therefore, by Corollary 4.1, we have $\text{SLE}(\overrightarrow{H_1} \rightarrow \overrightarrow{H_1}) = 2n_1(n_1 - 1)$. Also, for n_1 even, by Lemma 4.2, it follows that the skew Laplacian eigenvalues of T_{n_1} are $\pm(n_1 - 2j) : j = 1, 2, 3, \dots \lfloor \frac{n_1}{2} \rfloor$. Clearly, all the eigenvalues of T_{n_1} are real with their moduli less or equal to n_1. Therefore, by Corollary 4.1, we have $\text{SLE}(T_{n_1} \rightarrow T_{n_1}) = 2n_1(n_1 - 1)$. It is also clear that the $\overrightarrow{H_1} \rightarrow \overrightarrow{H_1}$ and $T_{n_1} \rightarrow T_{n_1}$ are non-Eulerian nonskew Laplacian cospectral digraphs. The case when n_1 is odd can be similarly done. □

Let $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{n_1}$ be a Hamiltonian path and for $i = 1, 2, \dots, n_1 - 1$, let $e = v_i v_{i+1}$ be an arc in a transitive tournament T_{n_1}. Let $T_{n_1} - e$ be the digraph obtained by removing the arc $e = v_i v_{i+1}$ from T_{n_1}. In [7], it is shown that the skew Laplacian spectrum of $T_{n_1} - e$ is same as the skew Laplacian spectrum of T_{n_1}. Therefore, we have the following observations.

Theorem 4.5. (1) *The digraphs* $H = \overrightarrow{H_1} \rightarrow \overrightarrow{H_1}$ *and* $G = T_{n_1} - e \rightarrow T_{n_1}$ *are non-Eulerian skew Laplacian equienergetic digraphs of order* $n \equiv 0(\text{mod } 2)$, *where* H_1 *and* T_{n_1} *are the digraphs defined in Lemmas* 4.1, 4.2 *and* $e = v_i v_{i+1}$ *is an arc in a Hamiltonian path of* T_{n_1}.

(2) *The digraphs* $H = \overrightarrow{H_1} \rightarrow \overrightarrow{H_1}$ *and* $G = T_{n_1} \rightarrow T_{n_1} - e$ *are non-Eulerian skew Laplacian equienergetic digraphs of order* $n \equiv 0(\text{mod } 2)$, *where* H_1 *and* T_{n_1} *are the digraphs defined in Lemmas* 4.1, 4.2 *and* $e = v_i v_{i+1}$ *is an arc in a Hamiltonian path of* T_{n_1}.

(3) *The digraphs* $H = \overrightarrow{H_1} \rightarrow \overrightarrow{H_1}$ *and* $G = T_{n_1} - e \rightarrow T_{n_1} - e$ *are non-Eulerian skew Laplacian equienergetic digraphs of order* $n \equiv 0(\text{mod } 2)$, *where* H_1 *and* T_{n_1} *are the digraphs defined in Lemmas* 4.1, 4.2 *and* $e = v_i v_{i+1}$ *is an arc in a Hamiltonian path of* T_{n_1}.

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