Further results on the rainbow vertex-disconnection of graphs*

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Abstract

Let G be a nontrivial connected and vertex-colored graph. A subset X of the vertex set of G is called rainbow if any two vertices in X have distinct colors. The graph G is called *rainbow vertex-disconnected* if for any two vertices x and y of G, there exists a vertex subset S such that when x and y are nonadjacent, S is rainbow and x and y belong to different components of $G-S$; whereas when x and y are adjacent, $S+x$ or $S+y$ is rainbow and x and y belong to different components of $(G-xy)-S$. Such a vertex subset S is called an *x-y rainbow vertex-cut* of G. For a connected graph G, the *rainbow vertex-disconnection number* of G, denoted by $rvd(G)$, is the minimum number of colors that are needed to make G rainbow vertex-disconnected.

In this paper, we obtain bounds of the rainbow vertex-disconnection number of a graph in terms of the minimum degree and maximum degree of the graph. We give a tighter upper bound for the maximum size of a graph G with $rvd(G)=k$ for $k\geq \frac{n}{2}$. We then characterize the graphs of order n with rainbow vertex-disconnection number $n-1$ and obtain the maximum size of a graph G with $rvd(G)=n-1$. Moreover, we get a sharp threshold function for the property $rvd(G(n,p))=n$ and prove that almost all graphs G have $rvd(G)=rvd(\overline{G})=n$. Finally, we obtain some Nordhaus-Gaddum-type results: $n-5\leq rvd(G)+rvd(\overline{G})\leq 2n$ and $n-1\leq rvd(G)\cdot rvd(\overline{G})\leq n^2$ for the rainbow vertex-disconnection numbers of nontrivial connected graphs G and $\overline{G}$ with order $n\geq 24$.

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. Let $G = (V(G), E(G))$ be a nontrivial connected graph with vertex set $V(G)$ and edge set $E(G)$. The *order* of G is denoted by $n = |V(G)|$ and the *size* of G is denoted by $|E(G)|$. For a vertex $v \in V$, the *open neighborhood* and *closed neighborhood* of v in G are the set $N_G(v) = \{u \in V(G)|uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. The *degree* of v in G is $d_G(v) = |N_G(v)|$. The minimum and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Let P_n denote a path with order n. Let V_1, V_2 be two disjoint vertex subsets of G. We denote the set of edges between V_1 and V_2 in G by $E(V_1, V_2)$. We follow [8, 9] for graph theoretical notation and terminology not defined here.

In [11], Chartrand et al. firstly studied the rainbow edge-cut by introducing the concept of rainbow disconnection of graphs. Let G be a nontrivial connected and edge-colored graph. An *edge-cut* of G is a set R of edges of G such that $G - R$ is disconnected. If any two edges in R have different colors, then R is a *rainbow cut*. A rainbow cut R is called a *u-v rainbow cut* if the vertices u and v belong to different components of $G - R$. An edge-coloring of G is a rainbow disconnection coloring if for every two distinct vertices u and v of G, there exists a u-v rainbow cut in G, separating them. The *rainbow disconnection number* $rd(G)$ of G is the minimum number of colors required by a rainbow disconnection coloring of G.

For vertex-colorings of graphs, the authors in [4] introduced the concept of rainbow vertex-disconnection number. They gave some applications of the rainbow vertex-disconnection numbers of graphs. For more results on rainbow and other colored disconnections of graphs, we refer the readers to [2, 3, 4, 5, 6, 12, 16].

For a connected and vertex-colored graph G, let x and y be two vertices of G. If x and y are nonadjacent, then an *x-y vertex-cut* is a subset S of $V(G)$ such that x and y belong to different components of $G - S$. If x and y are adjacent, then an *x-y vertex-cut* is a subset S of $V(G)$ such that x and y belong to different components of $(G - xy) - S$. A vertex subset S of G is *rainbow* if no two vertices of S have the same color. An *x-y rainbow vertex-cut* is an x-y vertex-cut S such that if x and y are nonadjacent, then S is rainbow; if x and y are adjacent, then $S + x$ or $S + y$ is rainbow.

A vertex-colored connected graph G is called *rainbow vertex-disconnected* if for any two vertices x and y of G, there exists an x-y rainbow vertex-cut. In this case, the vertex-coloring c is called a *rainbow vertex-disconnection coloring* of G. For a connected graph G, the *rainbow vertex-disconnection number* of G, denoted by $rvd(G)$, is the minimum number of colors that are needed in order to make G rainbow vertex-disconnected. A rainbow vertex-disconnection coloring with $rvd(G)$ colors is called an *rvd-coloring* of G.

An *injective coloring* of a graph G is a vertex-coloring of G such that the colors of any two vertices with a common neighbor are different. The *injective chromatic number* $\chi_i(G)$ of a graph G is the minimum number of colors such that G has an injective coloring using this number of colors. The injective coloring was first introduced in [15] by Hahn et al. in 2002 and originated from complexity theory [18].

In this paper, we study the relationships among the graph parameters: rainbow vertex-disconnection number, injective chromatic number, minimum degree and maximum degree. We obtain the following result in Section 2:

$$\delta(G) \leq rvd(G) \leq \chi_i(G) \leq \Delta(G)(\Delta(G) - 1) + 1.$$

In Section 3 we give a tighter upper bound for the maximum size of a graph G with $rvd(G) = k$ for $k \geq \frac{n}{2}$. In Section 4 we characterize the graphs with rainbow vertex-disconnection number $n - 1$ and obtain the maximum size of graphs G with $rvd(G) = n - 1$. In Section 5 we consider the sharp threshold function of random graphs $G(n, p)$ with $rvd(G(n, p)) = n$ and obtain that almost all graphs G have $rvd(G) = rvd(\overline{G}) = n$. In Section 6 we get some Nordhaus-Gaddum-type results for the rainbow vertex-disconnection number, and leave a conjecture for further study.

2 Preliminaries

In this section, we first introduce some known results from [4]. Then we obtain some bounds for the rainbow vertex-disconnection number of a graph.

Lemma 2.1 *[4] Let G be a nontrivial connected graph, and let u and v be two vertices of G having at least two common neighbors. Then u and v receive different colors in any rvd-coloring of G.*

Lemma 2.2 *[4] Let G be a nontrivial connected graph of order n. Then $rvd(G) = n$ if and only if any two vertices of G have at least two common neighbors.*

Theorem 2.3 *Let G be a connected graph of order n with minimum degree δ. If $\delta \geq \frac{n+2}{2}$, then $rvd(G) = n$.*

Proof. Since $\delta \geq \frac{n+2}{2}$, there exist at least $\frac{n+2}{2} \times 2 - n = 2$ common neighbors for any two vertices of G. By Lemma 2.2, we have $rvd(G) = n$. □

Let x and y be two vertices of a graph G. The *local connectivity* $\kappa_G(x, y)$ of two nonadjacent vertices x and y is the minimum number of vertices required to separate x from y. If x and y are adjacent vertices, the local connectivity $\kappa_G(x, y)$ of x and y is defined as $\kappa_{G-xy}(x, y) + 1$. The *connectivity* $\kappa(G)$ of G is the minimum number of vertices of G whose removal results in a disconnected graph or a trivial graph. The *upper connectivity* $\kappa^+(G)$ of G is the upper bound of the function $\kappa_G(x, y)$ on G.

Lemma 2.4 *[4] Let G be a nontrivial connected graph of order n. Then $\kappa(G) \leq \kappa^+(G) \leq rvd(G) \leq n$.*

Lemma 2.5 *[17] Let K be a complete subgraph of G with $E(G-K) \neq \emptyset$. Then there exists an edge $a_1a_2 \in E(G-K)$ such that $k(a_1, a_2) = min\{d(a_1), d(a_2)\}$.*

Lemma 2.6 *[15] Let G be a graph with maximum degree Δ. Then, $\chi_i(G) \leq \Delta(\Delta - 1) + 1$.*

Theorem 2.7 *Let G be a nontrivial connected graph with maximum degree Δ. Then $\delta(G) \leq \kappa^+(G) \leq rvd(G) \leq \chi_i(G) \leq \Delta(\Delta - 1) + 1$.*

Proof. By Lemmas 2.4 and 2.5, we have $rvd(G) \geq \kappa^+(G) \geq \delta(G)$. Let c be an injective coloring of G. Let u and v be any two vertices of G. Since the colors of any two vertices with a common neighbor are different under c, $N_G(u)$ is rainbow. If u and v are adjacent, then $N_G(u) \setminus \{v\}$ is a u-v rainbow vertex-cut. If u and v are not adjacent, then $N_G(u)$ is a u-v rainbow vertex-cut. Thus, c is a rainbow vertex-disconnection coloring of G. By Lemma 2.6, we have $rvd(G) \leq \chi_i(G) \leq \Delta(\Delta-1)+1$. □

3 Bounds on the maximum size

In this section, we give a tighter upper bound for the maximum size of a graph G with $rvd(G) = k$ for $k \geq \frac{n}{2}$, which is better for large k than that in the following lemma reported in [4].

Lemma 3.1 *[4] For $k \geq 4$, let G be a graph of order n with $rvd(G) = k$. Then,* $\frac{1}{2}k(n-1) - \binom{k}{2} \leq |E(G)|_{\max} \leq k(n-1) - \binom{k}{2}$.

We need a lemma first.

Lemma 3.2 *Let G be a nontrivial connected graph with $rvd(G) = k$. Let $V_1, V_2, V_3, \cdots, V_k$ be the set of color classes of an rvd-coloring of G. Then for $i \in [k]$ and $|V_i| \geq 2$, we have*

$$\sum_{v \in V_i} d_G(v) \leq n + \binom{|V_i|}{2}.$$

Let $S = \{v_i | v_i \in V_i \text{ and } |V_i| = 1\}$. We have

$$\sum_{v \in S} d_G(v) \leq (\frac{n+k}{2} - 1)|S|.$$

Proof. Without loss of generality, we assume that $|V_1| \leq |V_2| \leq \cdots \leq |V_k|$ and $s = |S|$. Then $S = \{v_1, v_2, \cdots, v_s\}$. For vertices v_1 and v_2, since V_j $(j = s+1, s+2, \cdots, k)$ is monochromatic, the vertices v_1 and v_2 have at most one common neighbor in V_j. Otherwise, assume that $u_1, u_2 \in V_j$ are the common neighbors of v_1 and v_2. Then we have that v_1, v_2 are two common neighbors of u_1 and u_2. So u_1, u_2 have different colors, a contradiction. So, we obtain $|E(v_1, V_j)| + |E(v_2, V_j)| \leq |V_j| + 1$. Then we have

$$\begin{aligned}
d_G(v_1) + d_G(v_2) &= |E(v_1, S - v_1)| + \sum_{j \in \{s+1, \cdots, k\}} |E(v_1, V_j)| \\
&\quad + |E(v_2, S - v_2)| + \sum_{j \in \{s+1, \cdots, k\}} |E(v_2, V_j)| \\
&\leq 2(s-1) + \sum_{j \in \{s+1, \cdots, k\}} (|V_j| + 1) \\
&= 2(s-1) + n - s + k - s \\
&= n + k - 2.
\end{aligned}$$

Since the above inequality holds for any two vertices in S, we can derive that $\sum_{i \in [s]} d_G(v_i) \leq \frac{(n+k-2)s}{2}$.

Now consider the degrees of vertices in V_j. Let $\widetilde{d}(v) = |E(v, V_j)|$, where $v \in V(G) - V_j$. Let $T = \{v | \widetilde{d}(v) \geq 2\}$. Since V_j is monochromatic, there are $\binom{|V_j|}{2}$ pairs of vertices in V_j which have at most one common neighbor by Lemma 2.1. Assume that $|E(V_j)| \leq \frac{|V_i|}{2}$. For $v \in T$, when $\widetilde{d}(v)$ increases one, this will increase at least one pair of vertices in V_j which has one common neighbor v. Then we have

$$\begin{aligned}|E(V_j, V(G) - V_j)| &= |V(G) - V_j - T| + \sum_{v \in T} \widetilde{d}(v) \\ &= n - |V_j| + \sum_{v \in T} (\widetilde{d}(v) - 1) \\ &\leq n - |V_j| + \binom{|V_j|}{2}.\end{aligned}$$

Thus, we obtain

$$\sum_{v \in V_j} d_G(v) = 2|E(V_j)| + |E(V_j, V(G) - V_j)| \leq n + \binom{|V_j|}{2}.$$

If $|E(V_j)| > \frac{|V_j|}{2}$, assume that there are p connected components $T_1, T_2, \cdots, T_p$ in V_j, which are trees. Each T_i ($i \in [p]$) has at least $|T_i| - 2$ pairs of vertices which have a common neighbor in V_j. Since

$$\sum_{i \in [p]} (|T_i| - 2) = \sum_{i \in [p]} |T_i| - 2p = |V_j| - 2(|V_j| - |E_j|) = 2|E_j| - |V_j|,$$

we have at least $2|E_j| - |V_j|$ pairs of vertices of V_j which have no common neighbor in $V(G) - V_j$. So, we have

$$\begin{aligned}\sum_{v \in V_j} d_G(v) &= 2|E(V_j)| + |E(V_j, V(G) - V_j)| \\ &\leq 2|E(V_j)| + n - |V_j| + \binom{|V_j|}{2} - (2|E_j| - |V_j|) \\ &= n + \binom{|V_j|}{2}.\end{aligned}$$

□

Theorem 3.3 *Let G be a nontrivial connected graph with $rvd(G) = k$ for $k \geq \frac{n}{2}$. Then $|E(G)|_{\max} \leq \frac{(n+k-2)(2k-n)}{4} + \frac{(n-k)(n+1)}{2}$.*

Proof. Let $V_1, V_2, V_3, \cdots, V_k$ be the set of color classes of an rvd-coloring of G. Assume that $S = \{v_i | v_i \in V_i \text{ and } |V_i| = 1\}$ and $s = |S|$. For any two V_{j_1} and V_{j_2} with $|V_{j_1}| \geq |V_{j_2}| \geq 3$, we move one vertex u from V_{j_2} to V_{j_1}. Then we have

$$\begin{aligned}&\sum_{v \in V_{j_1} \cup \{u\}} d_G(v) + \sum_{v \in V_{j_2} \setminus \{u\}} d_G(v) \\ &\leq n + \binom{|V_{j_1}| + 1}{2} + n + \binom{|V_{j_2}| - 1}{2} \\ &= n + \binom{|V_{j_1}|}{2} + n + \binom{|V_{j_2}|}{2} + |V_{j_1}| - (|V_{j_2}| - 1).\end{aligned}$$

We find the bound is larger after moving. So, there will be $k-s-1$ color classes with order 2 and one color class with order $n-s-2(k-s-1)=n-2k+s+2$. Now we define the upper bound function $f(s)$ as follows:

$$f(s)=\frac{(n+k-2)s}{2}+(k-s-1)(n+1)+n+\binom{n-2k+s+2}{2}.$$

Since $k\geq s\geq 2k-n$ and the axis of symmetry of function $f(s)$ is $s=\frac{3k-n+1}{2}$, we get the maximum value of $f(s)$ at $s=2k-n$. Since $f(2k-n)=\frac{n+k-2}{2}(2k-n)+(n-k)(n+1)$, by Lemma 3.2, we obtain $|E(G)|_{\max}\leq\frac{1}{2}\sum_{v\in V(G)}d_G(v)\leq\frac{1}{2}f(2k-n)=\frac{(n+k-2)(2k-n)}{4}+\frac{(n-k)(n+1)}{2}$. This upper bound is tighter than the upper bound $k(n-1)-\binom{k}{2}$ in Lemma 3.1 for $k\geq\frac{n}{2}$. $\square$

4 Graphs with rainbow vertex-disconnection number $n-1$

Let x and y be two vertices of a graph G. We denote the set of common neighbors of x and y by $M_G(x,y)$. Let $m_G(x,y)=|M_G(x,y)|$. Let $S_G(x,y)$ be an x-y rainbow vertex-cut in G. Let $D_G(x,y)$ be the rainbow vertex set such that if x,y are adjacent, then $S_G(x,y)+x\subseteq D_G(x,y)$ or $S_G(x,y)+y\subseteq D_G(x,y)$ and $D_G(x,y)$ is rainbow; if x,y are nonadjacent, then $S_G(x,y)\subseteq D_G(x,y)$ and $D_G(x,y)$ is rainbow. In order to prove that there exists an x-y rainbow vertex-cut in G, we only need to find $D_G(x,y)$.

Theorem 4.1 *Let G be a nontrivial connected graph of order n. Then $rvd(G)=n-1$ if and only if G satisfies the following three conditions:*

1. *There exists at least one pair (x,y) of vertices with $m_G(x,y)\leq 1$.*

2. *For any two pairs (x,y) and (p,q) of vertices with $m_G(x,y)\leq 1$ and $m_G(p,q)\leq 1$, Fig. 1.(1) or (2) is a subgraph of G containing the vertex set $\{x,y,p,q\}$.*

3. *For any three pairs (x,y), (x,z), (y,z) of vertices with $m_G(x,y)\leq 1$, $m_G(x,z)\leq 1$ and $m_G(y,z)\leq 1$, Fig. 1.(3) or (4) is a subgraph of G containing the vertex set $\{x,y,z\}$.*

Proof. Let $rvd(G)=n-1$. Assume, to the contrary, that the graph G does not satisfy at least one of the conditions. Then there are three cases to discuss.

Case 1. Each pair of vertices have at least two common neighbors.

By Lemma 2.2, we have $rvd(G)=n$, a contradiction.

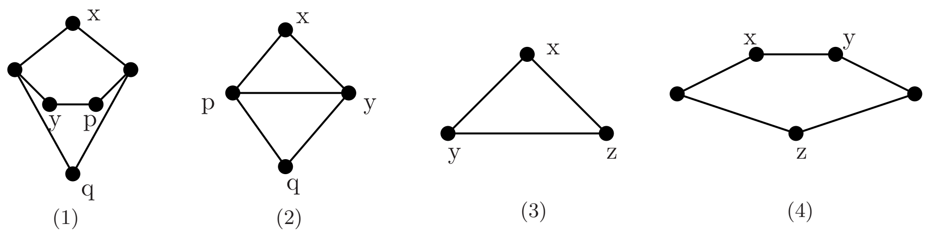

Figure 1: The graphs of condition 2 and 3.

Case 2. There exist two pairs (x, y) and (p, q) of vertices with $m_G(x, y) \leq 1$ and $m_G(p, q) \leq 1$ which do not satisfy Condition 2.

Define a vertex-coloring c of G with $n-2$ colors such that $c(x) = c(y) = 1$, $c(p) = c(q) = 2$ and the remaining vertices have different colors from $3, 4, \cdots, n-2$. Since $rvd(G) = n-1$, we have that c is not a rainbow vertex-disconnection coloring of G. Then there exist two vertices u, v which have no u-v rainbow vertex-cut. Next, we claim that such vertices u, v do not exist.

Let P_1 be the u-v path of length two through a vertex with color 1. Let P_2 be the u-v path of length two through another vertex with color 2. Let P_3 be the u-v path of length three through two vertices with color 1 and color 2. Since $m_G(x, y) \leq 1$ and $m_G(p, q) \leq 1$, there is at most one path P_1, at most one path P_2 and at most two internally disjoint paths P_3.

Consider that u and v are not adjacent. If $u \in \{x, y, p, q\}$ or $v \in \{x, y, p, q\}$, without loss of generality, assuming $u = x$, then $N_G(u)$ or $N_G(v)$ is a u-v rainbow vertex-cut. So $u, v \notin \{x, y, p, q\}$. There are several cases to deal with. Because of the symmetry of P_1 and P_2, some cases can be omitted. If there are no P_1, P_2 and P_3, then $D_G(u, v) = V(G) \setminus \{u, v, y, q\}$. If there is one P_1 but no P_2, P_3, assuming $P_1 = uxv$, then $D_G(u, v) = V(G) \setminus \{u, v, y, p\}$. If there is one P_3 but no P_1, P_2, assuming $P_3 = uxpv$, then $D_G(u, v) = V(G) \setminus \{u, v, y, p\}$. If there are P_1, P_2 but no P_3, assuming $P_1 = uxv$ and $P_2 = upv$, then $D_G(u, v) = V(G) \setminus \{u, v, y, q\}$. If there are P_1, P_3 but no P_2, then assume $P_1 = uxv$. When there exists one path P_3 which is internally disjoint with P_1, assuming $P_3 = uypv$, we have $D_G(u, v) = V(G) \setminus \{u, v, y, q\}$. When all the paths P_3 pass the vertex x, since $m_G(p, q) \leq 1$, we only have one path P_3. Then $D_G(u, v) = V(G) \setminus \{u, v, y, p\}$. If there are P_1, P_2 and P_3, then assume $P_1 = uxv$ and $P_2 = upv$. When there exists one path P_3 which is internally disjoint with P_1 and P_2, we have that Fig. 1.(1) is a subgraph of G containing $\{x, y, p, q\}$, a contradiction. When each path P_3 has a common vertex (not u, v) with P_1 or P_2, we have $D_G(u, v) = V(G) \setminus \{u, v, y, q\}$.

So, u and v are adjacent. When $u, v \notin \{x, y, p, q\}$, similar to the situation where u and v are nonadjacent, there exists a u-v rainbow vertex-cut. When $u \in \{x, y, p, q\}$ and $v \notin \{x, y, p, q\}$, we have $N(u) \setminus \{v\}$ or $N(v) \setminus \{u\}$ is a u-v rainbow vertex-cut. So, $u, v \in \{x, y, p, q\}$. If the colors of u, v are the same, then $N(u) \setminus \{v\}$ or $N(v) \setminus \{u\}$ is a u-v rainbow vertex-cut. So, the colors of u and v are different. Without loss of generality, we have $u = x$, $v = p$. If there is no $P_1 = uyv$, then $D_G(u, v) = V(G) \setminus \{y, v\}$. If there is no $P_2 = uqv$, then $D_G(u, v) = V(G) \setminus \{u, q\}$. So, there exist two paths uyv and uqv. Thus, Fig. 1.(2) is a subgraph of G containing $\{x, y, p, q\}$, which is a contradiction.

Case 3. There exist three pairs (x, y), (x, z), (y, z) of vertices with $m_G(x, y) \leq 1$, $m_G(y, z) \leq 1$ and $m_G(z, x) \leq 1$ which do not satisfy Condition 3.

Define a vertex-coloring c of G with $n-2$ colors such that $c(x) = c(y) = c(z) = 1$, and the remaining vertices have different colors from $2, 3, \cdots, n-2$. Since $rvd(G) = n-1$, we have that c is not a rainbow vertex-disconnection coloring of G. Then there exist two vertices u and v which do not have a u-v rainbow vertex-cut. Next, we claim that such vertices u, v do not exist.

Let Q_1 be the u-v path of length two through a vertex with color 1. Let Q_2 be the u-v path of length three through two vertices with color 1. Since $m_G(x, y) \leq 1$, $m_G(y, z) \leq 1$ and $m_G(z, x) \leq 1$, there is at most one path Q_1 and at most one path Q_2.

Assume $u, v \notin \{x, y, z\}$. Then there exist two internally disjoint paths Q_1 and Q_2. (Otherwise, if there are no paths Q_1 and Q_2, then $D_G(u, v) = V(G) \setminus \{y, z, u\}$; if there is a path Q_1 but no path Q_2, assuming $Q_1 = uxv$, then $D_G(u, v) = V(G) \setminus \{y, z, u\}$; if there is a path Q_2 but no path Q_1, assuming $Q_2 = uxyv$, then $D_G(u, v) = V(G) \setminus \{y, z, u\}$; if there exist Q_1 and Q_2, but Q_1 and Q_2 having a common vertex (not u,v), say x, then $D_G(u, v) = V(G) \setminus \{y, z, u\}$.) So, Fig. 1.(4) is a subgraph of G containing $\{x, y, z\}$, a contradiction.

Assume $u \in \{x, y, z\}$. Without loss of generality, let $u = x$. Suppose $v \notin \{y, z\}$. If there exists Q_1, assuming $Q_1 = uyv$, then $D_G(u, v) = V(G) \setminus \{z, u\}$; if there is no Q_1, then $D_G(u, v) = V(G) \setminus \{z, u\}$. So, we have $v \in \{y, z\}$. Assume $v = y$. When vertices u and v are not adjacent, then $V(G) \setminus \{u, v\}$ is a u-v rainbow vertex-cut. So, vertices u and v are adjacent. If there is no path Q_1, then $D_G(u, v) = V(G) \setminus \{z, u\}$. If there is a path Q_1, then $Q_1 = uzv$. So, Fig. 1.(3) is a subgraph of G containing $\{x, y, z\}$, a contradiction.

Now we are ready to show that a graph G satisfying the three conditions has $rvd(G) = n-1$.

Let c be any rvd-coloring of G. For the sake of contradiction, assume $rvd(G) \leq n-2$. If there are at least two colors which are repeated, then there exist four vertices v_1, v_2, v_3, v_4 with $c(v_1) = c(v_2)$ and $c(v_3) = c(v_4)$. By Lemma 2.1, we have $m_G(v_1, v_2) \leq 1$ and $m_G(v_3, v_4) \leq 1$. Then Fig. 1.(1) or (2) is a subgraph of G containing $\{v_1, v_2, v_3, v_4\}$. So, there are at least three colors for vertices v_1, v_2, v_3 and v_4, a contradiction. If there is only one color which is repeated, then there exist at least three vertices v_1, v_2, v_3 with $c(v_1) = c(v_2) = c(v_3)$. Similarly, we have $m_G(v_1, v_2) \leq 1$, $m_G(v_1, v_3) \leq 1$ and $m_G(v_2, v_3) \leq 1$ by Lemma 2.1. Then Fig. 1.(3) or (4) is a subgraph of G containing $\{v_1, v_2, v_3\}$. So, there are at least two colors for vertices v_1, v_2 and v_3, a contradiction. Thus, $rvd(G) \geq n-1$. By Lemma 2.2, we have $rvd(G) \leq n-1$. $\square$

Theorem 4.2 *Let G be a nontrivial connected graph of order n with $rvd(G) = n-1$. Then*

$$|E(G)|_{max} = \begin{cases} 1, & n = 2, \\ \frac{1}{2}n(n-1) - n + 3, & n \geq 3. \end{cases}$$

Proof. When $n = 2$, the graph is K_2. Consider $n \geq 3$. Since $rvd(G) = n-1$, there exists at least one pair (x, y) of vertices with $m_G(x, y) \leq 1$ by Theorem 4.1. If x and y are adjacent, then $d_G(x) + d_G(y) \leq n+1$ and there are at least $2(n-1) - d_G(x) - d_G(y) \geq n-3$ edges which are not in G. If x and y are not adjacent, then $d_G(x) + d_G(y) \leq n-1$ and there are at least $2(n-1) - d_G(x) - d_G(y) - 1 \geq n-2$ edges which are not in G. Thus, $|E(G)| \leq \frac{1}{2}n(n-1) - n + 3$. Let H be a graph with $V(H) = \{v_1, v_2, \cdots, v_n\}$, which is obtained from K_n by deleting edges $v_n v_i$ $(i = [n-3])$. We have $rvd(H) = n-1$ and $E(H) = \frac{1}{2}n(n-1) - n + 3$. $\square$

Remark: This improves the result of Theorem 3.3 for the case $k = n-1$, where only bounds were given.

5 Results for random graphs

Let $G = G(n, p)$ be the random graphs on n vertices and edge probability p. In the study of properties of random graphs, many researchers observed that there are sharp *threshold functions* for various natural graph properties. For a graph property A and for a function $p = p(n)$, we say that $G(n, p)$ satisfies A *almost surely* if the probability that $G(n, p(n))$ satisfies A tends to 1 as n tends to infinity. We say that a function $f(n)$ is a *sharp threshold function* for the property A if there are two positive constants c and C such that $G(n, cf(n))$ almost surely does not satisfy A and $G(n,p)$ satisfies A almost surely for all $p \geq Cf(n)$. It is well-known that all monotone graph properties have a sharp threshold function, see [7] and [14]. In [10], the authors obtained the sharp threshold function for the property $rc(G(n,p)) \leq 2$ by proving the property that any two vertices of $G(n,p)$ have at least $2\log n$ common neighbors. By Lemmas 2.2 and 5.1 we can obtain Theorem 5.2 immediately.

Lemma 5.1 *[10] $p = \sqrt{\log n/n}$ is a sharp threshold function for the property $rc(G(n,p)) \leq 2$.*

Theorem 5.2 *$p = \sqrt{\log n/n}$ is a sharp threshold function for the property $rvd(G(n,p)) = n$.*

Lemma 5.3 *[1](Chernoff Bound) If X is a binomial random variable with expectation μ, and $0 < \delta < 1$, then*

$$Pr[X < (1-\delta)\mu] \leq exp(-\frac{\delta^2\mu}{2})$$

and if $\delta > 0$,

$$Pr[X > (1+\delta)\mu] \leq exp(-\frac{\delta^2\mu}{2+\delta}).$$

Theorem 5.4 *Almost all graphs G have $rvd(G) = rvd(\overline{G}) = n$.*

Proof. Consider the random graphs $G(n, \frac{1}{2})$. By Lemma 2.2, it suffices to show that almost surely any two vertices of $G(n, \frac{1}{2})$ have at least 2 common neighbors and two common nonadjacent vertices. Let x and y be two vertices of $G(n, \frac{1}{2})$. By union bound, it suffices to show that x, y do not have two common neighbors or two common nonadjacent vertices with probability $o(\frac{1}{n^2})$. Let X_A be the number of common neighbors of x and y. Let X_B be the number of vertices which are not adjacent to x, y. Let A_i be the event that vertex i is the common neighbor of x and y. Let B_i be the event that $i \notin N_G(x) \cup N_G(y)$. Then

$$Pr(A_i) = Pr(B_i) = \frac{1}{4}, \quad E(X_A) = E(X_B) = \sum_i Pr(A_i) = \frac{n-2}{4}.$$

By Lemma 5.3,

$$Pr(X_A < \frac{2n-4}{9}) = Pr(X_B < \frac{2n-4}{9}) \leq exp(-\frac{n-2}{648}),$$

where $\delta_A = \delta_B = \frac{1}{9}$. Then, when n is sufficiently large, we have

$$Pr(X_A < 2 \text{ or } X_B < 2) \leq 2exp(-\frac{n-2}{648}) = o(\frac{1}{n^2}).$$

So, almost all graphs G have $rvd(G) = rvd(\overline{G}) = n$. □

6 Nordhaus-Gaddum-type results

In this section, we study the Nordhaus-Gaddum-type problem for the rainbow vertex-disconnection number of graphs. We assume that both a graph G and its complement $\overline{G}$ are connected of order n. So, we have $n \geq 4$.

Lemma 6.1 *[4] If G is a nontrivial connected graph and H is a connected subgraph of G, then $rvd(H) \leq rvd(G)$.*

Lemma 6.2 *[4] For an integer $n \geq 2$,*

$$rvd(K_n) = \begin{cases} n-1, & if\ n = 2, 3, \\ n, & if\ n \geq 4. \end{cases}$$

Lemma 6.3 *$rvd(K_n - e) = n$ for $n \geq 5$ and $rvd(K_n - 2e) = n$ for $n \geq 6$.*

Proof. If $n = 5$, then $|E(K_5 - e)| = 9$. By Theorem 4.2, we have $rvd(K_5 - e) = 5$. For $n \geq 6$, since $E(K_n - 2e) = \frac{1}{2}n(n-1) - 2$, we have $rvd(K_n - 2e) = n$ by Theorem 4.2. By Lemma 6.1, we have $rvd(K_n - e) \geq rvd(K_n - 2e) = n$. □

Lemma 6.4 *[4] Let G be a nontrivial connected graph. Then $rvd(G) = 1$ if and only if G is a tree.*

Lemma 6.5 *If G is a connected graph of order n with $5 \leq n \leq 7$ and $rvd(G) = 1$, then $rvd(\overline{G}) \geq n - 2$ and the lower bound is sharp.*

Proof.

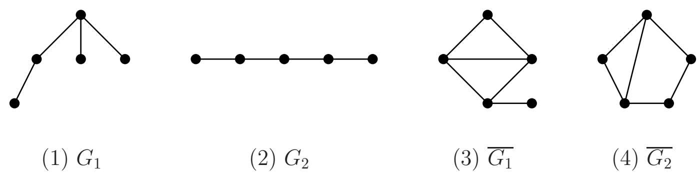

Figure 2: The trees and their complement graphs with order $n = 5$.

By Lemma 6.4, G is a tree. For $n = 5$, the graph G is G_1 or G_2 as shown in Fig. 2. Since $rvd(\overline{G_1}) = rvd(\overline{G_2}) = 3$, we have $rvd(\overline{G}) = n - 2$.

For $n = 6$, the graph G is one of G_1 through G_5 as shown in Fig. 3.(1) through (5). For G_1 through G_3, the four vertices in dashed line cycle form a K_4 in $\overline{G}$. By Lemmas 6.2 and 6.1, we have $rvd(\overline{G}) \geq rvd(K_4) \geq 4$. If G is the graph G_4, then

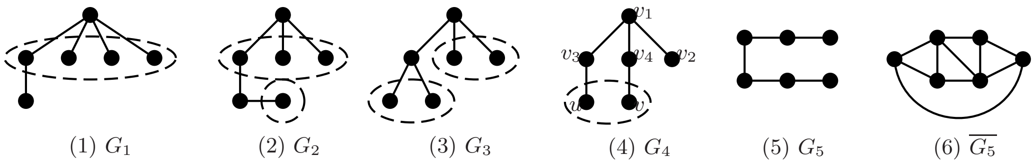

Figure 3: The trees with order $n = 6$ and the complement graph of P_6.

there are four internally disjoint paths between u and v in $\overline{G}$, which are paths uv, uv_1v, uv_2v and uv_4v_3v. So, by Lemma 2.4 we have $rvd(\overline{G}) \geq 4$. If G is the graph G_5, then $\overline{G}$ is the graph as shown in Fig. 3.(6). We have $rvd(\overline{G_5}) = rvd(\overline{P_6}) = 4$. So, $rvd(\overline{G}) \geq n - 2$.

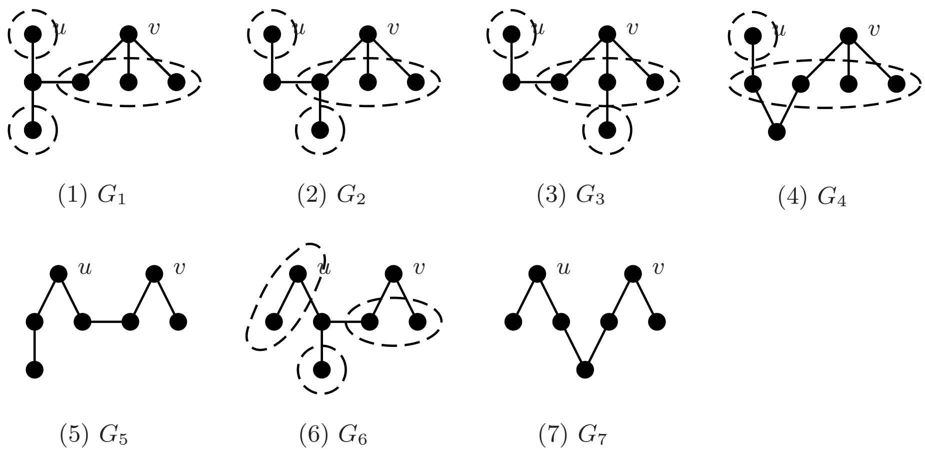

Figure 4: The trees with order $n = 7$ and $|E(H)| = 2$.

For $n = 7$, if any two vertices of the graph $\overline{G}$ have at least two common neighbors, then $rvd(\overline{G}) = n$ by Lemma 2.2. So, assume that there exist two vertices x, y in $\overline{G}$ with $m_{\overline{G}}(x, y) \leq 1$. Then there is at most one vertex which is not adjacent to x and y in G. Let $H = G - x - y$. If $|E(H)| \geq 3$, assuming that $\{e_1, e_2, e_3\} \subseteq E(H)$, then there are at least two edges with the common vertex, say $e_1 = v_1v_2$ and $e_2 = v_2v_3$. Since G is a tree, we have one vertex $t \in \{v_1, v_2, v_3\}$ but $t \notin N_G(x) \cup N_G(y)$ and there is a path $P_1 = xe_1y$ or xe_2y or xe_1e_2y in G. So, the endpoints of e_3 are adjacent to x and y, respectively. There is a path $P_2 = xe_3y$. Then according to P_1 and P_2, there is a cycle in graph G, a contradiction. So, $|E(H)| \leq 2$. When $|E(H)| \leq 1$, we have $\overline{H} = K_5$ or $K_5 - e$. By Lemmas 6.1, 6.2 and 6.3, we have $rvd(\overline{G}) \geq rvd(\overline{H}) = 5$. When $|E(H)| = 2$, G is as shown in Fig. 4. If G is one of G_1 through G_4 or G_6, then the vertices in dashed line cycle form a K_5 or $K_5 - e$ in $\overline{G}$. So, $rvd(G) \geq 5$ by Lemmas 6.2 and 6.3. If G is G_5 or G_7, i.e. P_7, then we have $rvd(\overline{G}) = rvd(\overline{P_7}) = 5$. $\square$

Lemma 6.6 *If G is a connected graph of order $n \geq 8$ and $rvd(G) = 1$, then $rvd(\overline{G}) \geq n-1$ and the lower bound is sharp.*

Proof. By Lemma 6.4, G is a tree. Assume $n \geq 8$. If any two vertices of the graph $\overline{G}$ have at least two common neighbors, then $rvd(\overline{G}) = n$ by Theorem 2.2. So, we consider that there exist two vertices x and y in $\overline{G}$ with $m_{\overline{G}}(x, y) \leq 1$. Then there is at most one vertex which is not adjacent to x and y in G. Without loss of generality, let $N_G(y) \geq N_G(x)$. Then $|N_G(y) \setminus \{x\}| \geq 3$. Let $H = G - x - y$.

If x and y have no common nonadjacent vertex in G, then $V(H) \subseteq N_G(x) \cup N_G(y)$. Since there exists at most one edge in graph H, we have $rvd(\overline{H}) = n-2$ by Lemmas 6.2 and 6.3. Let $v \in V(H)$. If $M_G(x, y) = \{s\}$, then $N_G(y) \geq 4$. We have $N_G(y) \setminus \{s, v\} \subseteq M_{\overline{G}}(x, v)$. So, $m_{\overline{G}}(x, v) \geq 2$. By Lemma 2.1, we obtain $rvd(\overline{G}) \geq n-1$. Consider $M_G(x, y) = \emptyset$. If $xy \in E(G)$, then $N_G(y) \setminus \{x, v\} \subseteq M_{\overline{G}}(x, v)$. So, $m_{\overline{G}}(x, v) \geq 2$ and $rvd(\overline{G}) \geq n-1$. If $xy \notin E(G)$, then there is an edge $v_1v_2 \in E(G)$ with $v_1 \in N_G(x)$ and $v_2 \in N_G(y)$. Then $N_G(y) \setminus \{v_2\} \subseteq M_{\overline{G}}(x, v)$ for $v \in N_G(x)$ and $N_G(y) \setminus \{v\} \subseteq M_{\overline{G}}(x, v)$ for $v \in V(H) \setminus N_G(x)$. So, $m_{\overline{G}}(x, v) \geq 2$ for $v \in V(H)$ and $rvd(\overline{G}) \geq n-1$.

Now we consider that there exists a vertex t which is not adjacent to x and y in G. If $xy \in E(G)$, then the vertex t is adjacent to a vertex t' of $N_G(x) \setminus \{y\}$ or $N_G(y) \setminus \{x\}$ in G. So, $rvd(\overline{H}) = rvd(K_{n-2} - e) = n-2$ by Lemma 6.3. If $t' \in N_G(x) \setminus \{y\}$ in G, then $N_G(y) \setminus \{x, v\} \subseteq M_{\overline{G}}(x, v)$; if $t' \in N_G(y) \setminus \{x\}$ in G, then $N_G(y) \setminus \{x, t'\} \subseteq M_{\overline{G}}(x, v)$ for $v = t$ and $N_G(y) \setminus \{x, v\} \subseteq M_{\overline{G}}(x, v)$ for $v \in V(H) \setminus \{t\}$. We have $m_{\overline{G}}(x, v) \geq 2$ for $v \in V(H)$. So $rvd(\overline{G}) \geq n-1$.

Assume $xy \notin E(G)$. If $M_G(x, y) = \{s'\}$, then t is adjacent to a vertex h_1 from $V(H)$ in G. We have $rvd(\overline{H}) = rvd(K_{n-2} - e) = n-2$ by Lemma 6.3. If $h_1 \in N_G(x)$ or $N_G(y) \geq 4$ in G, then $N_G[y] \setminus \{s', h_1\} \subseteq M_{\overline{G}}(x, t)$, $N_G(y) \setminus \{s'\} \subseteq M_{\overline{G}}(x, v)$ for $v \in N_G(x)$ and $N_G(y) \cup \{t\} \setminus \{v, s'\} \subseteq M_{\overline{G}}(x, v)$ for $v \in N_G(y) \setminus \{s'\}$. So, by Lemma 2.1 we have that $V(\overline{G}) \setminus \{y\}$ is rainbow. If $h_1 \in N_G(y)$ and $N_G(y) = 3$ in G, then $N_G(x) = 3$. By symmetry, we know that $V(\overline{G}) \setminus \{x\}$ is rainbow. Thus, $rvd(\overline{G}) \geq n-1$. If $M_G(x, y) = \emptyset$, then $rvd(\overline{H}) = rvd(K_{n-2} - 2e) = n-2$ by Lemma 6.3. Let h_2 be the neighbor of t in $N_G(y)$ (if h_2 does not exist, then let $\{h_2\} = \emptyset$). Let v_xv_y be the edge of G with $v_x \in N_G(x)$ and $v_y \in N_G(y)$ (if v_xv_y does not exist, then let $\{v_x\} = \{v_y\} = \emptyset$). We have $N_G(y) \setminus \{h_2\} \subseteq M_{\overline{G}}(x, t)$, $N_G(y) \setminus \{v_y\} \subseteq M_{\overline{G}}(x, v_x)$ and $N_G(y) \setminus \{v\} \subseteq M_{\overline{G}}(x, v)$ for $v \in V(H) \setminus \{t, v_x\}$. So, $m_{\overline{G}}(x, v) \geq 2$ for $v \in V(H)$ and $rvd(\overline{G}) \geq n-1$.

For the sharpness of the lower bound, let G_0 be a tree obtained from $K_{1,n-2}$ by adding a new vertex which is adjacent to one of the leaves. Then we have $rvd(\overline{G_0}) =$

$n-1$. $\square$

A *block* of a graph is a maximal connected induced subgraph of G containing no cut vertices. An *end-block* is a block with exactly one cut vertex of G.

Lemma 6.7 *[4] Let G be a nontrivial connected graph. Then $rvd(G)=2$ if and only if each block of G is either K_2 or a cycle and at least one block of G is a cycle.*

Lemma 6.8 *If G is a connected graph of order n and $rvd(G)=2$, then $rvd(\overline{G}) \geq n-3$.*

Proof. If $G=C_n$, then $\delta(\overline{G})=n-3$. By Theorem 2.7, we have $rvd(\overline{G}) \geq n-3$. Assume that G is not C_n. By Lemma 6.7, there exist at least two end-blocks B_i and B_j which are a K_2 or a cycle. If B_i and B_j are both K_2, then we select one of the endpoints with degree one in G from B_i, B_j, respectively, say x and y. Then x, y are adjacent in $\overline{G}$ and $M_{\overline{G}}(x,y)=V(\overline{G}) \setminus \{N_G[x], N_G[y]\}$. So, $\kappa_{\overline{G}}(x,y) \geq n-3$. Thus, $rvd(\overline{G}) \geq n-3$ by Lemma 2.4. Assume that B_i is a cycle. If B_i is a triangle $v_1v_2v_3v_1$, where v_3 is a cut vertex of G, then $M_{\overline{G}}(v_1,v_2)=V(\overline{G}) \setminus \{v_1,v_2,v_3\}$. So, $rvd(\overline{G}) \geq n-3$. If B_i is a $C_4=v_1v_2v_3v_4v_1$, where v_4 is a cut vertex of G, then $M_{\overline{G}}(v_1,v_3)=V(\overline{G}) \setminus \{v_1,v_2,v_3,v_4\}$. Since v_1, v_3 are adjacent in $\overline{G}$, we have $rvd(\overline{G}) \geq \kappa_{\overline{G}}(v_1,v_3) \geq n-3$ by Lemma 2.4. If B_i is a cycle $C_t=v_1v_2\cdots v_{t-1}v_tv_1$ with order $t \geq 5$, where v_t is a cut vertex of G, then $M_{\overline{G}}(v_1,v_2)=V(\overline{G}) \setminus \{v_1,v_2,v_3,v_t\}$. Since there is a path $v_1v_3v_tv_2$ in $\overline{G}$, we have $rvd(\overline{G}) \geq \kappa_{\overline{G}}(v_1,v_2) \geq n-3$ by Lemma 2.4. $\square$

Theorem 6.9 *Let G and $\overline{G}$ be connected graphs of order n.*

1. If $n=4$, then $rvd(G) \cdot rvd(\overline{G})=1$.

2. If $5 \leq n \leq 7$, then $n-2 \leq rvd(G) \cdot rvd(\overline{G}) \leq n^2$ and the lower bound is sharp.

3. If $n \geq 8$, then $n-1 \leq rvd(G) \cdot rvd(\overline{G}) \leq n^2$. Furthermore, the lower bound is sharp and the upper bound is sharp for $n \geq 12$.

Proof. If $n=4$, then $G=\overline{G}=P_4$. By Lemma 6.4, $rvd(G) \cdot rvd(\overline{G})=1$. The upper bounds are obvious for $n \geq 5$. For the lower bounds, by Lemmas 6.5 and 6.8, we only need to consider $n \geq 8$. we assume that $\delta(G) \leq \delta(\overline{G})$. Then $\delta(G) \leq \frac{n-1}{2}$. When $rvd(G)=1$, we have $rvd(G) \cdot rvd(\overline{G}) \geq n-1$ by Lemma 6.6. When $rvd(G)=2$, we have $rvd(G) \cdot rvd(\overline{G}) \geq 2(n-3)=n-1+n-5 \geq n-1$ by Lemma 6.8. By symmetry, we consider $rvd(G) \geq 3$ and $rvd(\overline{G}) \geq 3$. So, for $rvd(G) \cdot rvd(\overline{G})$ we only need to consider $n \geq 11$. Let

$$f = \begin{cases} \lceil \frac{n-1}{3} \rceil, & if\ \delta(G) \leq 3, \\ \lceil \frac{n-1}{\delta(G)} \rceil, & if\ \delta(G) \geq 4. \end{cases}$$

If $rvd(\overline{G}) \geq f$, then for $\delta(G) \leq 3$, $rvd(G) \cdot rvd(\overline{G}) \geq 3f \geq n-1$; for $\delta(G) \geq 4$, $rvd(G) \cdot rvd(\overline{G}) \geq \delta(G) \cdot f \geq n-1$ by Theorem 2.7.

Suppose $rvd(\overline{G}) \leq f-1$. Let u be the vertex with $d_G(u) = \delta(G)$. Let $H = G - N_G[u]$ and v be any vertex of the graph H. Since u is adjacent to v in the graph $\overline{G}$, we have $m_{\overline{G}}(u,v) \leq f-2$. So, there are at most $f-2$ vertices which are not adjacent to u and v in G. Hence, we have

$$\begin{aligned} d_H(v) &\geq |H| - 1 - (f-2) \\ &= n - \delta(G) - f. \end{aligned}$$

If H is connected, then $rvd(G) \geq rvd(H) \geq \delta(H) \geq n - \delta(G) - f$ by Lemma 6.1 and Theorem 2.7. If H is not connected, then we denote a component with maximum rainbow vertex-disconnection number by H_1. Then $rvd(G) \geq rvd(H_1) \geq \delta(H_1) \geq n - \delta(G) - f$ by Lemma 6.1 and Theorem 2.7.

When $\delta(G) \leq 3$, $rvd(G) \cdot rvd(\overline{G}) \geq (n - \delta(G) - f) \cdot 3 \geq n - 1 + n - 10 \geq n - 1$. When $\delta(G) \geq 4$, since $\delta(G) \leq \frac{n-1}{2}$, we have $rvd(G) \cdot rvd(\overline{G}) \geq (n - \delta(G) - f) \cdot \delta(G) \geq n - 1 + \delta(G) \cdot (\delta(G) - 4) \geq n - 1$.

By Lemmas 6.5 and 6.6, the lower bound is sharp. For the upper bound, let G be a graph with order $n = 4k + t$ ($k \geq 3$ and $t = 0, 1, 2, 3$). The vertex set $V(G)$ can be partitioned into four cliques, $V_1 = \{v_1, v_2, \cdots, v_k\}$, $V_2 = \{v_{k+1}, v_{k+2}, \cdots, v_{2k}\}$, $V_3 = \{v_{2k+1}, v_{2k+2}, \cdots, v_{3k}\}$ and $V_4 = \{v_{3k+1}, v_{3k+2}, \cdots, v_{4k}, v_{4k+1}, \cdots, v_{4k+t}\}$. Each vertex set $\{v_i, v_{k+i}, v_{2k+i}, v_{3k+i}\}$ forms a clique, where $i \in [k-1]$. The vertex set $\{v_k, v_{2k}, v_{3k}, v_{4k}, v_{4k+1}, \cdots, v_{4k+t}\}$ also forms a clique. Then we have $rvd(G) = rvd(\overline{G}) = n$. □

Theorem 6.10 *Let G and $\overline{G}$ be connected graphs of order n. Then $n-7 \leq rvd(G) + rvd(\overline{G}) \leq 2n$.*

Proof. The upper bound is obvious. Now we consider the lower bound. For $n \geq 8$, when $rvd(G) = 1$ or 2, we have $rvd(G) + rvd(\overline{G}) \geq n - 1$ by Lemmas 6.6 and 6.8. Assume that $rvd(\overline{G}) \geq rvd(G) \geq 3$. If $rvd(G) \geq \lceil \frac{n-1}{2} \rceil$, then $rvd(G) + rvd(\overline{G}) \geq n-1$.

So, we consider $3 \leq rvd(G) \leq rvd(\overline{G}) \leq \lceil \frac{n-1}{2} \rceil - 1$ and $n \geq 13$. Let $rvd(G) = k$ and $\{V_1, V_2, \cdots, V_k\}$ be the set of color classes of an rvd-coloring of G. Since $\frac{n}{k} > 2$, there are three cases to consider.

Case 1. There exists a V_i with $|V_i| \geq 4$.

Let D_i be the subset of V_i with four vertices. For any two vertices of D_i, they have at most one common nonadjacent vertex in $\overline{G}$ by Lemma 2.1. Let $S = \{u|$ the vertex u is not adjacent to at least two vertices of D_i in $\overline{G}\}$. Let $T = V(\overline{G}) \setminus (D_i \cup S)$. Then $|S| \leq \binom{4}{2} = 6$ and $|N_{\overline{G}}(v) \cap D_i| \geq 3$ for $v \in T$. For any two vertices x, y of T, there are at least two common neighbors from D_i in $\overline{G}$. By Lemma 2.1, the vertex set T is rainbow in $\overline{G}$. Thus, $rvd(\overline{G}) \geq n - 10$ and $rvd(G) + rvd(\overline{G}) \geq n - 7$.

Case 2. There exist V_i, V_j with $|V_i| = |V_j| = 3$ and $|V_s| \leq 3$ for $s \in [k]$.

For any two vertices of V_i or V_j, they have at most one common nonadjacent vertex in $\overline{G}$ by Lemma 2.1. Let $S_1 = \{u|$ the vertex u is not adjacent to at least two vertices of V_i in $\overline{G}\}$ and $S_2 = \{u|$ the vertex u is not adjacent to at least two vertices of V_j in $\overline{G}\}$. Let $T = V(\overline{G}) \setminus (V_i \cup V_j \cup S_1 \cup S_2)$. Then $|S_1 \cup S_2| \leq 6$. We have $|N_{\overline{G}}(v) \cap V_i| \geq 2$ and $|N_{\overline{G}}(v) \cap V_j| \geq 2$ for $v \in T$. For any two vertices x, y of T, x and y have at least one common neighbor from V_i and another from V_j in $\overline{G}$. By Lemma 2.1, the vertex set T is rainbow in $\overline{G}$. Thus, $rvd(\overline{G}) \geq n - 12$ and $rvd(G) + rvd(\overline{G}) \geq \lceil \frac{n}{3} \rceil + n - 12 \geq n - 7$.

Case 3. There is only one V_i with $|V_i| = 3$ and $|V_s| \leq 2$ for $s \in [k] \setminus \{i\}$.

We have $rvd(\overline{G}) \geq rvd(G) \geq \frac{n-3}{2} + 1 = \frac{n-1}{2}$. So, $rvd(G) + rvd(\overline{G}) \geq n - 1$. $\square$

Theorem 6.11 *Let G and $\overline{G}$ be connected graphs of order $n \geq 24$. Then $n - 5 \leq rvd(G) + rvd(\overline{G}) \leq 2n$ and the upper bound is sharp.*

Proof. Let $rvd(G) = k$ and $\{V_1, V_2, \cdots, V_k\}$ be the set of color classes of an rvd-coloring of G. Then for any triple $\{v_1, v_2, v_3\} \subseteq V_i$, where $i \in [k]$, let $S = V(\overline{G}) \setminus \{v_1, v_2, v_3\}$. Since $m_G(v_1, v_3) \leq 1$, we have v_3 is adjacent to at least $|S - N_{\overline{G}}(v_1)| - 1$ vertices of the vertex set $S - N_{\overline{G}}(v_1)$ in $\overline{G}$. Since $m_G(v_2, v_3) \leq 1$, we have that v_3 is adjacent to at least $|S - N_{\overline{G}}(v_2)| - 1$ vertices of the vertex set $S - N_{\overline{G}}(v_2)$ in $\overline{G}$. Since $m_G(v_1, v_2) \leq 1$, we obtain $|(S - N_{\overline{G}}(v_1)) \cap (S - N_{\overline{G}}(v_2))| \leq 1$. If $d_{\overline{G}}(v_1) < \frac{n+2}{2}$ and

$d_{\overline{G}}(v_2) < \frac{n+2}{2}$, then

$$\begin{aligned} d_{\overline{G}}(v_3) &\geq |S - N_{\overline{G}}(v_1)| + |S - N_{\overline{G}}(v_2)| - 3 \\ &= 2n - 9 - d_{\overline{G}}(v_1) - d_{\overline{G}}(v_2) \\ &> n - 11 \\ &\geq \frac{n+2}{2}. \end{aligned}$$

So, for $\overline{G}$ there is at least one vertex with degree more than $\frac{n+2}{2}$ in $\{v_1, v_2, v_3\}$. Let T be the set of vertices with degrees larger than $\frac{n+2}{2}$ in $\overline{G}$. Then we have $|T| \geq \sum_{i\in[k]}(|V_i| - 2) = n - 2k$. For any two vertices x and y in T, $d_{\overline{G}}(x) + d_{\overline{G}}(y) \geq n + 2$. So, we have that T is rainbow by Lemma 2.1. Thus, $rvd(\overline{G}) \geq n - 2k$. When $k \leq 5$, we have $rvd(G) + rvd(\overline{G}) \geq n - k \geq n - 5$. Now consider $k \geq 6$. Since $n \geq 24$, we have $rvd(G) + rvd(\overline{G}) \geq n - 4$ for the Case 2 and Case 3 of Theorem 6.10. For the Case 1 of Theorem 6.10, we have $rvd(G) + rvd(\overline{G}) \geq 6 + n - 10 \geq n - 4$.

The upper bound is sharp, which can be achieved by graphs with order $n = 4k + t$ ($k \geq 6$ and $t = 0, 1, 2, 3$), described in Theorem 6.9. □

In fact, we think that the lower bound of $rvd(G) + rvd(\overline{G})$ could be improved further. When $rvd(G) = 1$, we have $rvd(G) + rvd(\overline{G}) \geq n$ for $n \geq 8$ by Lemma 6.6. So, we pose the following conjecture for further study.

Conjecture 6.12 *Let G and $\overline{G}$ be nontrivial connected graphs of order $n \geq 8$. Then $rvd(G) + rvd(\overline{G}) \geq n$.*

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