On proper (strong) rainbow connection of graphs*

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Abstract

A path in an edge-colored graph G is called a *rainbow path* if no two edges on the path have the same color. The graph G is called *rainbow connected* if between every pair of distinct vertices of G, there is a rainbow path. Recently, Johnson et al. considered this concept with the additional requirement that the coloring of G be proper. The *proper rainbow connection number* of G, denoted by $prc(G)$, is the minimum number of colors needed to properly color the edges of G so that G is rainbow connected. Similarly, the *proper strong rainbow connection number* of G, denoted by $psrc(G)$, is the minimum number of colors needed to properly color the edges of G such that for any two distinct vertices of G, there is a rainbow geodesic (shortest path) connecting them. In this paper, we characterize those graphs with proper rainbow connection numbers equal to the size or within 1 of the size. Moreover, we completely solve a question proposed by Johnson et al. by proving that if $G = K_{p_1}\square\cdots\square K_{p_n}$, where $n \geq 1$, and $p_1, ..., p_n > 1$ are integers, then $prc(G) = psrc(G) = \chi'(G)$, where $\chi'(G)$ denotes the chromatic index of G. Finally, we investigate some sufficient conditions for a graph G to satisfy $prc(G) = rc(G)$, and make some slightly positive progress by using a relation between $rc(G)$ and the girth of the graph.

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1 Introduction

All graphs considered in this paper are simple, finite, and undirected. We follow the terminology and notation of Bondy and Murty in [1] for those not defined here. For a connected graph G, we use $V(G), E(G), v(G), e(G), \Delta(G)$ and $diam(G)$ to denote the vertex set, edge set, order, size, maximum degree and diameter of G, respectively. Suppose that $X \subset V(G)$, we use $G[X]$ to denote the subgraph of G *induced by* X, that is, the subgraph of G whose vertex set is X and whose edge set is the set of all those edges of G that have both ends in X. An edge xy is called a *leaf* if one of its end vertices, say x, has degree one, and x is called a *pendent vertex*. Let K_n and C_n denote a complete graph and a cycle on n vertices, respectively.

Let G be a nontrivial connected graph with an *edge-coloring* $c : E(G) \to \{0, 1, \ldots, t\}$, $t \in \mathbb{N}$, where adjacent edges may be colored with the same color. If adjacent edges of G receive different colors by c, then c is a *proper coloring*. The minimum number of colors needed in a proper coloring of G is the *chromatic index* of G and denoted by $\chi'(G)$. All colorings of graphs in this work are assumed to be colorings of the edges unless explicitly stated otherwise.

A path in an edge-colored graph G is called a *rainbow path* if no two edges on the path have the same color. The graph G is called *rainbow connected* if for any two distinct vertices of G, there is a rainbow path connecting them. In this case, the coloring c is called a *rainbow connection coloring* (*RC-coloring* for short) of G. For a connected graph G, the *rainbow connection number* of G, denoted by $rc(G)$, is defined as the minimum number of colors that are needed to make G rainbow connected. Similarly, the graph G is called *strong rainbow connected* if between every pair of distinct vertices of G, there is a rainbow geodesic (shortest path) connecting them. In this case, the coloring c is called a *strong rainbow connection coloring* (*SRC-coloring* for short) of G. For a connected graph G, the *strong rainbow connection number* of G, denoted by $src(G)$, is defined as the minimum number of colors that are required to make G strong rainbow connected. Obviously, $rc(G) \leq src(G)$ for all connected graphs G. Moreover, $rc(G) = src(G) = 1$ if and only if G is a complete graph. These concepts were first introduced by Chartrand et al. in [2] and have been well-studied since then. For further details, we refer the reader to a survey [4] (with an updated version available at [5]) and a book [6].

Recently, Johnson et al. [3] considered rainbow connection colorings with the additional requirement that the coloring be proper. The *proper rainbow connection number* of a connected graph G, denoted by $prc(G)$, is the minimum number of colors needed to properly color the edges of G to make G rainbow connected. This coloring c is called a *proper rainbow connection coloring* (*PRC-coloring* for short) of G. This concept was defined in [3] along with a "strong" version, the *proper strong rainbow connection number*, requiring that the rainbow paths be geodesics, denoted by $psrc(G)$ (the coloring involved is written as *PSRC-coloring* for short). Some preliminary observations were made.

Proposition 1 ([3]). *Let G be a connected graph. Then we have*

$$diam(G) \leq rc(G) \leq src(G) \leq psrc(G) \leq e(G), \tag{1}$$

$$rc(G) \leq prc(G), \tag{2}$$

and

$$\chi'(G) \leq prc(G) \leq psrc(G). \tag{3}$$

Theorem 1 ([3]). $prc(K_n) = psrc(K_n) = \chi'(K_n) = \begin{cases} n-1 = \Delta(K_n) & \text{if } n \text{ is even;} \\ n = \Delta(K_n)+1 & \text{if } n \text{ is odd.} \end{cases}$

It is easy to see that if G is a tree, then $prc(G) = psrc(G) = e(G)$. The opposite direction does not hold since $prc(K_3) = psrc(K_3) = 3 = e(K_3)$, which brings us to the first question.

Question 1 ([3]). *Can we characterize the connected graphs G such that $prc(G) = e(G)$?*

In Section 2, we characterize all the graphs G with $prc(G) = e(G)$. Additionally, we characterize all the graphs with $prc(G) = e(G) - 1$.

The *Cartesian product* of simple graphs G and H is the graph $G\Box H$ whose vertex set is $V(G) \times V(H)$ and whose edge set is the set of all pairs $(u_1, v_1)(u_2, v_2)$ such that either $u_1u_2 \in E(G)$ and $v_1 = v_2$, or $v_1v_2 \in E(H)$ and $u_1 = u_2$. The authors in [3] obtained an easy result about Cartesian products.

Proposition 2 ([3]). *Suppose that $n, p_1, \cdots, p_n > 1$ are integers, and $G = K_{p_1}\Box\cdots\Box K_{p_n}$. Then $rc(G) = src(G) = n$ and*

$$\sum_{i=1}^{n}(p_i - 1) \leq prc(G) \leq psrc(G) \leq \sum_{i=1}^{n}\chi'(K_{p_i}).$$

These inequalities are all equal if all the p_is are even since $\chi'(K_{p_i}) = p_i - 1$ (in fact, in this case $prc(G) = psrc(G) = \chi'(G)$). The authors in [3] asked the following question.

Question 2 ([3])**.** *What happens when some of the p_is are odd?*

In Section 3, we prove that if $G = K_{p_1}\square\cdots\square K_{p_n}$ where $n \geq 1$ and $p_1, \cdots, p_n > 1$ are positive integers, then $prc(G) = psrc(G) = \chi'(G)$.

In the final section, we investigate some sufficient conditions for a graph G to satisfy $prc(G) = rc(G)$, and make some slightly positive progress by using a relation between $rc(G)$ and the girth of the graph.

2 Graphs with large proper rainbow connection numbers

Let c be an edge-coloring of a graph G. We use $c(e)$ to denote the color of an edge e. For a subgraph H of G, let $c(H)$ be the set of colors of the edges of H. First list some useful results.

Proposition 3 ([2])**.** *Let G be a nontrivial connected graph. Then*

(i) $rc(G) = src(G) = e(G)$ if and only if G is a tree,

(ii) $rc(G) = 2$ if and only if $src(G) = 2$.

Proposition 4 ([2])**.** *For each integer $n \geq 4$, $rc(C_n) = src(C_n) = \lceil \frac{n}{2} \rceil$.*

Obviously, the following holds.

Corollary 1. *For each integer $n \geq 4$, $prc(C_n) = psrc(C_n) = \lceil \frac{n}{2} \rceil$.*

Now, we are ready to characterize all connected graphs G with $prc(G) = e(G)$.

Theorem 2. *Let G be a connected graph. Then $prc(G) = e(G)$ if and only if G is a tree or K_3.*

Proof. By Proposition 1, Proposition 3 and Theorem 1, it is easy to see that $prc(G) = e(G)$ if G is a tree or K_3.

For the opposite direction, it suffices to prove that if $prc(G) = e(G)$, then G is a tree or K_3. Suppose that G is neither a tree nor K_3. Let ℓ be the circumference of G and let C denote a cycle of order ℓ in G. If $\ell \geq 4$, then construct a coloring of G by coloring C with $\lceil \frac{\ell}{2} \rceil$ colors (by Corollary 1), and assigning distinct colors to the remaining edges of G. It can be checked that this is a PRC-coloring of G with $\lceil \frac{\ell}{2} \rceil + e(G) - \ell < e(G)$ colors. If $\ell = 3$, set $C = u_1u_2u_3u_1$. Let G_1, G_2, G_3 denote the components of $G - E(C)$, where $u_i \in V(G_i), i = 1, 2, 3$. Since $G \neq K_3$, there exists a nontrivial component, say G_1.

Give distinct colors to the edges u_1u_2, u_1u_3, and the edges in $G-E(C)$. Assign one color used in $E(G_1)$ to the remaining edge u_2u_3. It is easy to check that G is proper rainbow connected with $e(G)-1$ colors.

The proof is thus complete. $\square$

We are also able to classify those graphs whose proper rainbow connection numbers are close to the maximum possible value. Let $\mathscr{H}'$ and $\mathscr{H}''$ be the two graph classes as shown in Figure 1, where the order of $H' \in \mathscr{H}'$ is at least 4 and the order of $H'' \in \mathscr{H}''$ is at least 5, respectively. The dashed edge therefore represents a path of length at least 1.

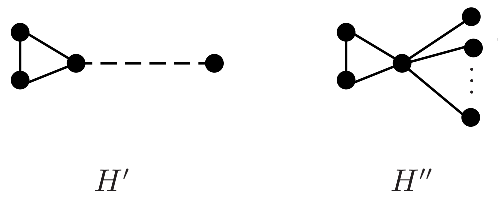

Figure 1: The graphs $H' \in \mathscr{H}'$ and $H'' \in \mathscr{H}''$, respectively.

Theorem 3. *Let G be a connected graph. Then $prc(G) = e(G)-1$ if and only if $G \in \mathscr{H}'$ or $G \in \mathscr{H}''$.*

Proof. First suppose $G \in \mathscr{H}'$ or $G \in \mathscr{H}''$. Then $prc(G) \leq e(G)-1$ by Theorem 2. Let C_3 be the triangle of G and T the nontrivial component of $G-E(C_3)$. In any PRC-coloring of G, there must be at least three colors used in the triangle as well as $e(T)$ colors used in T different from two of the colors used in the triangle. Hence, $prc(G) \geq e(T)+2 = e(G)-1$. Thus, we have $prc(G) = e(G)-1$.

Next, we need to verify the converse. Let G be a connected graph with $prc(G) = e(G)-1$. By Theorem 2, there is a cycle in G. Recall that $prc(K_3) = 3$ and $prc(C_n) = \lceil \frac{n}{2} \rceil < n-1$ for $n \geq 4$. Let ℓ denote the circumference of G and let C denote a cycle of order ℓ in G. If $\ell \geq 4$, then consider a coloring of G by using $\lceil \frac{\ell}{2} \rceil$ colors on the edges of C and distinct colors on the remaining edges of G. It can be checked that this is a PRC-coloring of G with $\lceil \frac{\ell}{2} \rceil + e(G) - \ell < e(G)-1$ colors. Thus, $\ell = 3$ and set $C = u_1u_2u_3u_1$. Let G_1, G_2, G_3 denote the components of $G-E(C)$, where $u_i \in V(G_i), i = 1, 2, 3$. If there exist at least two nontrivial components among G_1, G_2 and G_3, say G_1 and G_2, we construct a coloring of G using at most $e(G)-2$ colors. Give distinct colors to the edges in $G-E(C)$ and the edge u_1u_2. Assign one color used in $E(G_1)$ to the edge u_2u_3 and one color used in $E(G_2)$ to the edge u_1u_3. It can be verified that G is proper rainbow connected with $e(G)-2$ colors. Together with $prc(K_3) = e(K_3)$, we conclude that there exists exactly one nontrivial component among G_1, G_2 and G_3, say G_1. Suppose G_1 is not a tree, which means that G_1 contains a cycle K_3, since $\ell = 3$. If $G_1 = K_3$, that is, $G \cong G_0$, then the edge-coloring of G_0 as shown in Figure 2 makes G proper rainbow connected, meaning that $prc(G) \leq e(G) - 2$ in this case. Otherwise, by Theorem 2 we first give a PRC-coloring of G_1 with at most $e(G_1) - 1$ colors. Next, give the edges incident with u_1 distinct colors and the remaining edge u_2u_3 a color used in $E(G_1)$. Hence, we obtain a PRC-coloring of G with at most $e(G) - 2$ colors. This means that G_1 must be a tree.

It is easy to verify when $|G_1| \leq 3$, so we just need to consider two cases under the assumption that $|G_1| \geq 4$. We first consider the case that G_1 is a path. If u_1 is a pendent vertex of G_1, then $G \in \mathscr{H}'$ and satisfies $prc(G) = e(G) - 1$. Otherwise, let vv' and ww' be the two leaves of G_1, where v and w are two pendent vertices of G_1. Without loss of generality, suppose that $d(u_1, w) \geq d(u_1, v)$. Then $d(u_1, w) \geq 2$ since $|G_1| \geq 4$. No matter whether u_1 is just the vertex v', we give a coloring of G as follows: we first color $e(G_1)$ and u_1u_2 with $e(G) - 2$ different colors; then let $c(u_1u_3) = c(ww')$ and $c(u_2u_3) = c(vv')$. It is easy to check that this is a PRC-coloring of G with at most $e(G) - 2$ colors, a contradiction. Next we consider the case that G_1 is not a path. If G_1 is a star and u_1 is the center of G_1, then $G \in \mathscr{H}''$ and satisfies $prc(G) = e(G) - 1$. Otherwise, there exists a vertex u of degree at least 3 in G_1 and $d(u, u_1)$ is as large as possible. Let vv' and ww' be two leaves of G, where v and w are two pendent vertices of G whose distances from u are as small as possible. We first color $e(G_1)$ and u_2u_3 with different colors; then let $c(u_1u_2) = c(vv')$ and $c(u_1u_3) = c(ww')$. Thus, we obtain a PRC-coloring of G with at most $e(G) - 2$ colors, a contradiction completing the proof. $\square$

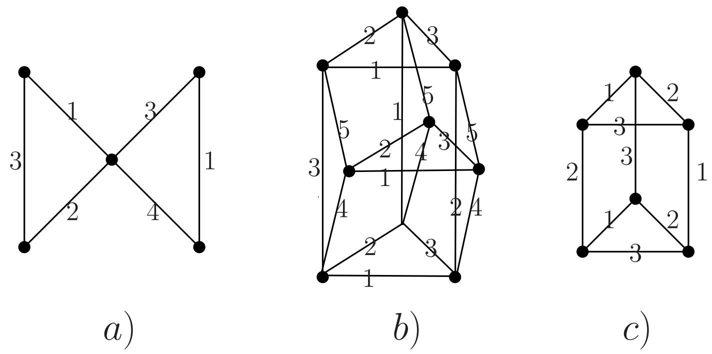

Figure 2: Example edge-colorings of $G_0, K_3 \square K_3$ and $K_2 \square K_3$, respectively.

3 Cartesian products of complete graphs

Suppose that $n \geq 1$, and $p_1, \cdots, p_n > 1$ are integers. Let $G = K_{p_1} \square \cdots \square K_{p_n}$. In this section, we further study the class of graphs considered in [3]. We first state Vizing's theorem.

Theorem 4 ([1]). *If G is a simple graph, then either $\chi'(G) = \Delta(G)$ or $\Delta(G) + 1$.*

Lemma 1 ([1]). *For every loopless graph G,*

$$\chi'(G) \geq \max\left\{\left\lceil \frac{2e(H)}{v(H) - 1} \right\rceil : H \subseteq G, v(H) \text{ odd}, \ v(H) \geq 3\right\}.$$

By Lemma 1, it follows that $\chi'(K_3 \Box K_3) = 5$. Moreover, we can give $K_3 \Box K_3$ a $PSRC$-coloring with 5 colors as shown in Figure 2. Hence, $prc(K_3 \Box K_3) = psrc(K_3 \Box K_3) = 5 = \chi'(K_3 \Box K_3) = \Delta(K_3 \Box K_3) + 1$. Does $prc(G) = psrc(G) = \chi'(G)$ always hold when all the p_is are odd?

What happens for the other cases ? If $p_1 = ... = p_n = 2$, then G is an n-cube Q_n, and $prc(Q_n) = psrc(Q_n) = \chi'(Q_n) = \Delta(Q_n) = n$ as showed in [3]. Moreover, we know that $prc(K_3 \Box K_2) = psrc(K_3 \Box K_2) = 3 = \chi'(K_3 \Box K_2) = \Delta(K_3 \Box K_2)$ from Figure 2. In the following, we list two properties concerning the Cartesian product.

Proposition 5. *For two simple connected graphs G and H, $G \Box H \cong H \Box G$.*

Lemma 2 ([1]). *If H is a nontrivial graph with $\chi'(H) = \Delta(H)$, then $\chi'(G \Box H) = \Delta(G \Box H)$ for any simple graph G.*

Now, we state the main result of this section, also a complete solution to Question 2.

Theorem 5. *Suppose that $n \geq 1$, and $p_1, \cdots, p_n > 1$ are integers. Let $G = K_{p_1} \Box \cdots \Box K_{p_n}$. Then $prc(G) = psrc(G) = \chi'(G)$.*

Proof. Since $\chi'(G) \leq prc(G) \leq psrc(G)$, we only need to prove that $psrc(G) = \chi'(G)$. By Theorem 1, we have $psrc(G) = \chi'(G) = \Delta(G)$ when $n = 1$ and p_1 is odd; $psrc(G) = \chi'(G) = \Delta(G) + 1$ when $n = 1$ and p_1 is even. In the following, we assume that $n \geq 2$. We need to consider the following two cases according to the parities of the p_is.

Case 1. Suppose $p_1, \cdots, p_n$ are all odd.

Note that $v(G) = p_1 p_2 \cdots p_n$ and $e(G) = \frac{p_1 \cdots p_n}{2} \sum_{i=1}^{n} (p_i - 1)$. By Lemma 1, we have

$$\chi'(G) \geq \left\lceil \frac{2e(G)}{v(G) - 1} \right\rceil > \sum_{i=1}^{n} (p_i - 1) = \Delta(G).$$

Moreover, $\chi'(G) \leq \Delta(G) + 1$ by Theorem 4. Thus $psrc(G) \geq \chi'(G) = \Delta(G) + 1$.

Next we show that $psrc(G) \leq \Delta(G) + 1$ by induction on n. Let $H_k = K_{p_1} \Box \cdots \Box K_{p_k}$, where $1 < k \leq n$. Thus $H_k = H_{k-1} \Box K_{p_k}$. It follows from Theorem 1 that $psrc(K_{p_i}) = \chi'(K_{p_i}) = p_i = \Delta(K_{p_i}) + 1$ for each i with $1 \leq i \leq n$. If $n = 2$, that is, $G = H_2$, we use (a_i, b_j) to denote the vertex in H_2, where $a_i \in V(K_{p_1})$ $(1 \leq i \leq p_1)$ and $b_j \in V(K_{p_2})$ $(1 \leq$

$j \le p_2$). Then the vertex sets $U_j = \{(a, b_j) : a \in V(K_{p_1})\}$ and $V_i = \{(a_i, b) : b \in V(K_{p_2})\}$ form complete graphs K_{p_1} and K_{p_2}, respectively.

Now we give a $PSRC$-coloring of H_2 with $\Delta(H_2) + 1$ colors as follows. Let each $H_2[V_i]$ be edge colored with p_2 colors so that the coloring is proper. Let each $H_2[U_j]$ be edged colored with $p_1 - 1$ colors and one color that does not appear on the edges incident with b_j so that the coloring is proper. Note that this coloring of H_2 is also proper. Since the diameter of H_2 is 2, H_2 is proper strong rainbow connected with $p_1 + p_2 - 1 = \Delta(H_2) + 1$ colors, implying that $psrc(G) \le \Delta(G) + 1$ in this case.

We assume that there exists a $PSRC$-coloring of H_{n-1} with $\Delta(H_{n-1}) + 1$ colors. Now we consider the graph $G = H_n$. Note that $H_n = H_{n-1} \Box K_{p_n}$. We use (a_i, b_j) to denote the vertex in H_n, where $a_i \in V(H_{n-1})$ $(1 \le i \le v(H_{n-1}))$ and $b_j \in V(K_{p_n})$ $(1 \le j \le p_n)$. Then the vertex sets $U_j = \{(a, b_j) : a \in V(H_{n-1})\}$ and $V_i = \{(a_i, b) : b \in V(K_{p_n})\}$ form graphs H_{n-1} and K_{p_n}, respectively.

Now we provide a $PSRC$-coloring of H_n as follows. Let each $H_n[V_i]$ be edge colored with p_n colors so that the coloring is proper. Let each $H_n[U_j]$ be edged colored with $\Delta(H_{n-1})$ colors and one color that does not appear on the edges incident with b_j so that the coloring is proper strong rainbow. Obviously, this coloring of H_n is also proper. For two vertices $u = (a_i, b_j)$ and $v = (a_{i'}, b_{j'})$ $(1 \le i \ne i' \le v(H_{n-1})$ and $1 \le j \ne j' \le p_n)$, there exists a rainbow geodesic P' between u and $(a_{i'}, b_j)$ in $H_n[U_j]$. Thus the path $P = uP'(a_{i'}, b_j)v$ is a rainbow geodesic from u to v since the color of the edge between $(a_{i'}, b_j)$ and v does not appear on P'. For the other two vertices, it can be easily checked that there exists a rainbow geodesic between them. Thus this coloring defined above with $\Delta(G) + 1$ colors makes H_n proper strong rainbow connected. So $psrc(G) \le \Delta(G) + 1$, and hence, $psrc(G) = \chi'(G) = \Delta(G) + 1$ in this case.

Case 2. Suppose that at least one number of $p_1, \cdots, p_n$ is even.

By Theorem 1, Proposition 5 and Lemma 2, it follows that $psrc(G) \ge \chi'(G) = \Delta(G)$.

Next we show that $psrc(G) \le \Delta(G)$. If $p_1, \cdots, p_n$ are all even, then $psrc(G) = \Delta(G)$ by Proposition 2. Otherwise, we prove that $psrc(G) \le \Delta(G)$ by induction on n. Without loss of generality, suppose that $p_1, \cdots, p_d$ $(d \ge 1)$ are even and $p_{d+1}, \cdots, p_n$ $(n \ge 2)$ are odd. By Theorem 1, we have $\chi'(K_{p_i}) = p_i - 1$ if $1 \le i \le d$, and $\chi'(K_{p_i}) = p_i$ if $d + 1 \le i \le n$. If $n = 2$, we give a coloring of $G = H_2$ which is similar to that defined in Case 1, only with the difference that each $H_2[U_j]$ is edged colored with $p_1 - 2$ colors and one color that does not appear on the edges incident with b_j so that the coloring is proper. Similarly, we can prove that under this coloring with $p_1 + p_2 - 2 = \Delta(H_2)$ colors, H_2 is proper strong rainbow connected, implying that $psrc(G) \le \Delta(G)$ in this case.

We assume that there exists a $PSRC$-coloring of H_{n-1} with $\Delta(H_{n-1})$ colors. Now we consider the graph $G = H_n = H_{n-1}\Box K_{p_n}$. We provide a coloring of H_n which is analogous to that defined in Case 1, only with the difference that each $H_n[U_j]$ is edged colored with $\Delta(H_{n-1}) - 1$ colors and one color that does not appear on the edges incident with b_j so that the coloring is proper strong rainbow. Analogously, we show that this coloring with $\Delta(G)$ colors makes H_n proper strong rainbow connected. So $psrc(G) \leq \Delta(G)$, and hence, $psrc(G) = \chi'(G) = \Delta(G) + 1$ in this case.

The proof is thus complete. □

4 Forcing $prc(G)$ to equal $rc(G)$

In the previous section, we have considered Cartesian products of complete graphs. How about Cartesian products of general graphs? We consider two kinds of simple graphs, paths and cycles. Since $prc(G) \geq rc(G) \geq diam(G)$, the lower bounds of the following observations are immediate. For the upper bounds, color every copy within each dimension to be rainbow connected (and therefore strong rainbow connected) and for each dimension, use a disjoint set of colors.

Observation 1. Given integers $d \geq 1$ and $t_1, t_2, \ldots, t_d \geq 3$, if $G = P_{t_1}\Box P_{t_2}\Box \ldots \Box P_{t_d}$, then $psrc(G) = prc(G) = rc(G) = \sum_{i=1}^{d}(t_i - 1)$.

Observation 2. Given integers $d \geq 1$ and $t_1, t_2, \ldots, t_d \geq 4$, if $G = C_{t_1}\Box C_{t_2}\Box \ldots \Box C_{t_d}$, then $\sum_{i=1}^{d}\lfloor \frac{t_i}{2} \rfloor \leq rc(G) \leq prc(G) \leq psrc(G) \leq \sum_{i=1}^{d}\lceil \frac{t_i}{2} \rceil$.

Based on these observations, the following question is natural.

Question 3. *Is it possible to classify the class of graphs G satisfying $prc(G) = rc(G)$, or satisfying $psrc(G) = src(G)$?*

When $\Delta(G)$ is large, certainly $prc(G)$ must be large. On the other hand, when $diam(G)$ is large, $rc(G)$ must be large. Many rainbow connection colorings are proper edge-colorings, especially for strong rainbow connection colorings. Based on this, it might be tempting to ask if $diam(G) \gg \Delta(G)$ might imply $prc(G) = rc(G)$ or $psrc(G) = src(G)$. Here "$\gg$" is used to mean "is sufficiently larger than" so here the assumption is that $diam(G)$ is much larger than $\Delta(G)$. Unfortunately, it turns out that this question has a negative answer, as seen in the following result.

Theorem 6. *Let $G = P_t\Box K_k$ where $t \gg k \geq 4$. Then $rc(G) = src(G) = t$, $prc(G) > rc(G)$ and $psrc(G) > src(G)$.*

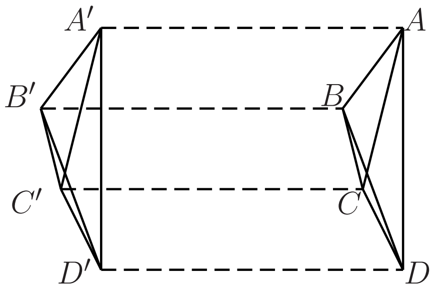

Figure 3: The illustration of Theorem 6.

Proof. We first point out that $diam(G) = t$ and $\Delta(G) = k + 1$. It can be easily checked that $rc(G) = src(G) = t$. Suppose that $prc(G) = rc(G)$. Let A, B, C, D denote four vertices in the copy of K_k representing an end-vertex of the P_t, respectively. And let A', B', C', D' be the corresponding vertices of the K_k representing the opposite end-vertex of the P_t, respectively. Consider the assumed rainbow path between A and B'. Without loss of generality, assume the rainbow path connecting them is $A...A'B'$ ($A...A'$ means the dashed lines corresponding to the copy of P_t between A and A' in Figure 3). Note that $X...X'(X \in \{A, B, C, D\})$ must be a rainbow path, otherwise, the length of a rainbow path connecting X and X' is at least $t+1$, contradicting the assumption $prc(G) = t$. Let $c(A...A') = \{1, 2, ..., t-1\}$ and $c(A'B') = t$.

First suppose that the color t does not appear on the path $B'...B$. Then consider the rainbow path P connecting B' and C. If P is $B'...BC$, then $c(BC) = t$. Thus the rainbow path connecting C' and B must be $C'...CB$ since the coloring is proper. This yields a contradiction since there does not exist a rainbow path connecting C' and A. If P is $B'C'...C$, then $c(B'C') \in \{1, 2, ..., t-1\}$, say $c(B'C') = 1$. Moreover, the color 1 does not appear on the path $C'...C$. Note that the rainbow path connecting A' and C is $A'...AC$ and $c(AC) = t$ since the coloring must be proper. Similarly, the rainbow path connecting C' and B is $C'...CB$ and $c(CB) = 1$. Thus, there does not exist a rainbow path connecting C' and A, a contradiction.

Now, assume that the color t does appear on the path $B'...B$ and the color 1 does not. Then the rainbow path connecting B to A' is $BA...A'$ and $c(AB) = t$. Note that the rainbow path connecting C to A' is $C...C'A'$ and let $c(C'A') = x$ $(1 \le x \le t-1)$. Next consider the rainbow path P connecting B' and C. If P is $B'C' \dots C$, then $c(B'C') = x$, contradicting the assumption that the coloring is proper. If P is $B'...BC$, then $c(BC) = 1$. Suppose that the rainbow path from C' to B is $C'...CB$, then $x = 1$. But there does not exist a rainbow path connecting C' and A, a contradiction. So the rainbow path from C' to B is $C'B'...B$, and then $c(C'B') = 1$. Since the coloring is proper, the rainbow path connecting B' and D is $B'D'...D$. Let $c(B'D') = 2(\neq x)$. Hence the rainbow path from D' to B is $D'...DB$ and $c(DB) = 2$. This yields a contradiction since there does not exist a rainbow path connecting D' and C. Therefore, $prc(G) > t$ and $psrc(G) > t$. $\square$

It turns out that using a restriction on the girth yields slightly more. If $g(G) \geq 5$, then every strong rainbow connection coloring is also a proper coloring so $psrc(G) = src(G)$. This restriction still does not quite achieve the goal for $prc(G)$ though. For example, let G be a graph obtained from s cycles C_t with a common vertex. Then $g(G) = t$ and $prc(G) \geq \Delta(G) = 2s$. Moreover, $rc(G) \leq t$. If $s \gg t$, then $prc(G) \gg rc(G)$.

We conclude with some slightly positive progress by using a relation between $rc(G)$ and the girth of the graph.

Proposition 6. *If $rc(G) < g(G) - 2$, then any minimum rainbow connection coloring of G is also proper.*

This means that if $rc(G) < g(G) - 2$, then $prc(G) = rc(G)$. In particular, sufficiently long cycles satisfy this restriction.

Proof. For a contradiction, consider a rainbow connection coloring of G using $rc(G)$ colors, and suppose this coloring is not proper. Let uvw be a monochromatic copy of P_3. Since u cannot be connected to w by a rainbow path through v, such a path, say P, must go elsewhere in the graph. Then $C = uPwvu$ is a cycle in G, meaning that $|C| \geq g(G)$ so P must use at least $|C| - 2 \geq g(G) - 2$ colors, contradicting the assumption that $rc(G) < g(G) - 2$ colors were used. $\square$

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