ON ODD RANKS OF ODD DURFEE SYMBOLS

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Abstract. An odd Durfee symbol of n is an array of positive odd integers and a subscript D,

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_s \\ b_1 & b_2 & \cdots & b_t \end{pmatrix}_D$$

such that $2D+1 \geq a_1 \geq a_2 \geq \cdots \geq a_s > 0$, $2D+1 \geq b_1 \geq b_2 \geq \cdots \geq b_t > 0$, and $n = \sum_{i=1}^{s} a_i + \sum_{j=1}^{t} b_j + 2D^2 + 2D + 1$. Andrews defined the odd rank of an odd Durfee symbol as $(s-t)$. Let $N^0(a, M; n)$ be the number of odd Durfee symbols of n with odd rank congruent to a modulo M. We decompose the generating function of $N^0(a, M; n)$ into modular and mock modular parts. Specifically, we derive some special cases of the generating functions of $N^0(a, M; n)$ for $M \in \{3, 6, 12\}$. Some generating functions are related to classical mock theta functions.

1. Introduction

In this paper, we adopt the following standard q-series notation [10]. Let q denote a complex number with $0 < |q| < 1$. For any positive integer n,

$$(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k) \qquad \text{and} \qquad (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k).$$

Let

$$j(x;q) := (x;q)_\infty (q/x;q)_\infty (q;q)_\infty.$$

Then for integers a and m with m positive, define

$$J_{a,m} := j(q^a;q^m), \quad \overline{J}_{a,m} := j(-q^a;q^m), \quad \text{and} \quad J_m := J_{m,3m} = \prod_{i=1}^{\infty}(1-q^{mi}).$$

A partition of a positive integer n is a weakly decreasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of partitions of n. In order to give a combinatorial explanation of the following congruences of Ramanujan

$$p(5n+4) \equiv 0 \pmod 5 \qquad \text{and} \qquad p(7n+5) \equiv 0 \pmod 7,$$

Dyson [8] introduced the rank of a partition as the largest part minus the number of parts. Let $N(m, n)$ denote the number of partitions of n with rank m, and let

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$N(a,M,n)$ be the number of partitions of n with rank congruent to a modulo M. Then he showed that

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}N(m,n)q^nx^m=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(qx;q)_n(qx^{-1};q)_n}.$$

In addition, Dyson [8] conjectured that

$$N(i,5,5n+4)=\frac{p(5n+4)}{5},\quad i=0,1,2,3,4,$$
$$N(i,7,7n+5)=\frac{p(7n+5)}{7},\quad i=0,1,2,3,4,5,6.$$

In 1954, Atkin and Swinnerton-Dyer [3] first proved the conjecture by using generalized Lambert series. For more properties of $N(m,n)$, see, for example, [5,6,16].

Hickerson and Mortenson [13] defined Appell-Lerch sums as follows.

Definition 1.1. *Let $x,z\in\mathbb{C}^*:=\mathbb{C}\backslash\{0\}$ with neither z nor xz an integral power of q. Then*

$$m(x,q,z):=\frac{1}{j(z;q)}\sum_{r=-\infty}^{\infty}\frac{(-1)^rq^{\binom{r}{2}}z^r}{1-q^{r-1}xz}.$$

For the study of generalized Lambert series/Appell-Lerch sums, the articles [7,13,21] contribute to the modern theory of this subject. With this theory, results are much more accessible than they were 20 years ago. In [12], Hickerson and Mortenson considered the deviation of the ranks of partitions from the average:

$$D(a,M):=\sum_{n=0}^{\infty}\left(N(a,M,n)-\frac{p(n)}{M}\right)q^n.$$

Then in view of Appell–Lerch sum properties, they expressed $D(a,M)$ in terms of modular and mock modular parts. Recently, Zhang [20] established the similar results for the deviation of the ranks of overpartitions from the average where an overpartition [14] is a partition in which the first occurrence of a part may be overlined. The object of this paper is to use properties of Appell-Lerch sums to study odd ranks of odd Durfee symbols. Odd Durfee symbols were first introduced by Andrews [1] to give a new partition-theoretic interpretation of $q\omega(q)$, where the third order mock theta function $\omega(q)$ was defined by Watson [19]

$$q\omega(q):=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)+1}}{(q;q^2)_{n+1}^2}. \tag{1.1}$$

Definition 1.2. *An odd Durfee symbol of n is a two-rowed array with a subscript of the form*

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_s \\ b_1 & b_2 & \cdots & b_t \end{pmatrix}_D$$

where all the entries are odd numbers such that

(1) $2D+1\geq a_1\geq a_2\geq\cdots\geq a_s>0$;

(2) $2D+1 \geq b_1 \geq b_2 \geq \cdots \geq b_t > 0$;
(3) $n = \sum_{i=1}^{s} a_i + \sum_{j=1}^{t} b_j + 2D^2 + 2D + 1$.

Fine [9] showed that

$$q\omega(q) = \sum_{n=0}^{\infty} \frac{q^{n+1}}{(q;q^2)_{n+1}} = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^{1+0})(1-q^{2+1})\cdots(1-q^{n+(n-1)})}. \tag{1.2}$$

From the above identity, it is easy to give another partition-theoretic interpretation of $q\omega(q)$. Let $p_\omega(n)$ be the number of partitions of n, where 0 is allowed as a part and the subpartition obtaining from removing one occurrence of the largest part can be grouped into pairs of consecutive integers. For example, there are 6 such partitions of 5: 5, $4+(1+0)$, $3+(1+0)+(1+0)$, $2+(2+1)$, $2+(1+0)+(1+0)+(1+0)$, $1+(1+0)+(1+0)+(1+0)+(1+0)$. Then (1.2) can be restated as

$$q\omega(q) = \sum_{n=0}^{\infty} p_\omega(n) q^n. \tag{1.3}$$

In view of MacMahon's modular partitions [15] with modulus 2, Andrews [1] showed that each partition enumerated by $p_\omega(n)$ has associated with it an odd Durfee symbol of n.

In analogy with ranks of partitions, Andrews [1] defined the odd rank of an odd Durfee symbol as the number of entries in the top row minus the number of entries in the bottom row. Let $N^0(m,n)$ denote the number of odd Durfee symbols of n with odd rank m, and let $N^0(a,M;n)$ denote the number of odd Durfee symbols of n with odd rank congruent to a modula M. By interchanging the rows of the symbol, it is clear that

$$\begin{aligned} N^0(m,n) &= N^0(-m,n), \\ N^0(a,M;n) &= N^0(-a,M;n). \end{aligned} \tag{1.4}$$

Andrews [1] showed that

$$p_\omega(n) = \sum_{m=-\infty}^{\infty} N^0(m,n), \tag{1.5}$$

$$\sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N^0(m,n) z^m q^n = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)+1}}{(zq;q^2)_{n+1}(z^{-1}q;q^2)_{n+1}}.$$

Meanwhile, Andrews [1, Corollary 27] provided that

$$\sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N^0(m,n) z^m q^n = \frac{1}{J_2} \sum_{n=0}^{\infty} \frac{(-1)^n (1-q^{4n+2}) q^{3n^2+3n+1}}{(1-z^{-1}q^{2n+1})(1-zq^{2n+1})}. \tag{1.6}$$

Recently, Wang [18] gave some explicit formulas of generating functions associated with $N^0(a,M;\ell n+r)$ where $0 \leq a < M$, $0 \leq r < \ell$, and $M, \ell \in \{2,4,8\}$. Moreover, letting $\zeta_M := e^{2\pi i/M}$, Wang [18] showed that

$$\sum_{n=0}^{\infty} N^0(a,M;n) q^n = \frac{q}{M} \sum_{j=0}^{M-1} \zeta_M^{-aj} g(\zeta_M^j q, q^2), \tag{1.7}$$

where the universal mock theta function $g(x,q)$ is given by Ramanujan [17],

$$g(x;q) := x^{-1}\left(-1+\sum_{n=0}^{\infty}\frac{q^{n^2}}{(x;q)_{n+1}(qx^{-1};q)_n}\right).$$

For convenience, set $N^0(a,M;0)=0$.

In this paper, we give the following generating function of $N^0(a,M;n)$.

Theorem 1.3. *We have*

$$\sum_{n=0}^{\infty}N^0(a,M;n)q^n=\frac{1}{J_2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n^2+3n+(2n+1)a+1}}{1-q^{(2n+1)M}}.$$

Based on Theorem 1.3, we deduce the following corollary.

Corollary 1.4. *Let M be a positive integer and a be an integer with $0\le a<M$. Then*

$$N^0(2a,2M;2n)=0,$$
$$N^0(2a+1,2M;2n+1)=0.$$

Furthermore, we decompose the generating function of $N^0(a,M;n)$ into modular and mock modular parts.

Theorem 1.5. *Let M be a positive integer and a be an integer with $0\le a<M$. Then*

$$\sum_{n=0}^{\infty}N^0(a,M;n)q^n=\frac{3q}{M}\sum_{t=0}^{M/3-1}(-1)^{\frac{a}{3}}q^{-\frac{a^2}{3}}\frac{J_6^3}{J_2 j(q^{3-2a}\omega_{\frac{M}{3}}^t;q^6)} \tag{1.8}$$

if $a\equiv 0 \pmod 3$ and $M\equiv 0 \pmod 3$;

$$\sum_{n=0}^{\infty}N^0(a,M;n)q^n=\frac{3q}{M}\sum_{t=0}^{M/3-1}(-1)^{\frac{a+2}{3}}q^{\frac{-a^2-2}{3}}m(q^{3-2a}\omega_{\frac{M}{3}}^t,q^6,q^{2a}) \tag{1.9}$$

if $a\equiv 1 \pmod 3$ and $M\equiv 0 \pmod 3$.

Set

$$\Delta(x;q):=\frac{x^{-2}J_1J_3^3 j(-x^2;q)}{j(x;q)j(-qx^3;q^3)j(-q^2x^3;q^3)\overline{J}_{0,3}} \tag{1.10}$$

and

$$\Psi_k^n(x,z,z';q):=-\frac{x^kz^{k+1}J_{n^2}^3}{j(z;q)j(z';q^{n^2})}\times\sum_{t=0}^{n-1}\frac{q^{\binom{t+1}{2}+kt}(-z)^t j(-q^{\binom{n+1}{2}+nk+nt}(-z)^n/z';q^{n^2})j(q^{nt}x^nz^nz';q^{n^2})}{j(-q^{\binom{n}{2}-nk}(-x)^nz',q^{nt}x^nz^n;q^{n^2})}. \tag{1.11}$$

Theorem 1.6. *Let M be a positive integer and a be an integer with $0\le a<M$. Then*

$$\sum_{n=0}^{\infty}N^0(a,M;n)q^n=d(a,M)+T_{a,M}(q),$$

where

$$d(a,M)=\begin{cases}(-1)^{M+a}q^{-\frac{M^2+2Ma+a^2-1}{3}}m((-1)^{M+1}q^{M^2-2Ma},q^{6M^2},z)-\Psi^M_{\frac{M+a-1}{3}}(q,-1,z;q^6)\\ +(-1)^{M+a}q^{-\frac{M^2-2Ma+a^2-1}{3}}m((-1)^{M+1}q^{M^2+2Ma},q^{6M^2},z)-\Psi^M_{\frac{M-a-1}{3}}(q,-1,z;q^6)\\ \text{if } a\equiv 0 \pmod 3 \text{ and } M\equiv 1 \pmod 3\\ \hline (-1)^{a}q^{-\frac{a^2-1}{3}}m((-1)^{M+1}q^{3M^2-2Ma},q^{6M^2},z)-\Psi^M_{\frac{a-1}{3}}(q,-1,z;q^6)\\ +(-1)^{M+a+1}q^{-\frac{M^2+2Ma+a^2-1}{3}}m((-1)^{M+1}q^{M^2-2Ma},q^{6M^2},z)-\Psi^M_{\frac{2M-a-1}{3}}(q,-1,z;q^6)\\ \text{if } a\equiv 1 \pmod 3 \text{ and } M\equiv 1 \pmod 3\\ \hline (-1)^{M+a+1}q^{-\frac{M^2-2Ma+a^2-1}{3}}m((-1)^{M+1}q^{M^2+2Ma},q^{6M^2},z)-\Psi^M_{\frac{2M+a-1}{3}}(q,-1,z;q^6)\\ +(-1)^{M+a+1}q^{-\frac{M^2+2Ma+a^2-1}{3}}m((-1)^{M+1}q^{M^2-2Ma},q^{6M^2},z)-\Psi^M_{\frac{2M-a-1}{3}}(q,-1,z;q^6)\\ \text{if } a\equiv 0 \pmod 3 \text{ and } M\equiv 2 \pmod 3\\ \hline (-1)^{a}q^{-\frac{a^2-1}{3}}m((-1)^{M+1}q^{3M^2-2Ma},q^{6M^2},z)-\Psi^M_{\frac{a-1}{3}}(q,-1,z;q^6)\\ +(-1)^{M+a}q^{-\frac{M^2-2Ma+a^2-1}{3}}m((-1)^{M+1}q^{M^2+2Ma},q^{6M^2},z)-\Psi^M_{\frac{M-a-1}{3}}(q,-1,z;q^6)\\ \text{if } a\equiv 1 \pmod 3 \text{ and } M\equiv 2 \pmod 3\end{cases}$$

and

$$T_{a,M}(q)=\frac{q}{M}\sum_{j=0}^{M-1}\xi_M^{-aj}\Delta(\xi_M^j q;q^2).$$

Specifically, by considering some special cases of the generating functions of $N^0(a,M;n)$, we relate some generating functions with the third order mock theta functions $\omega(q)$ and $\rho(q)$ where $\rho(q)$ is defined as [19]

$$\rho(q):=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}(q;q^2)_{n+1}}{(q^3;q^6)_{n+1}}. \tag{1.12}$$

Theorem 1.7. *We have*

$$\sum_{n=0}^{\infty}\left(N^0(0,3;n)-N^0(1,3;n)\right)q^n=q\rho(q), \tag{1.13}$$

$$\sum_{n=0}^{\infty}\left(N^0(0,6;n)-N^0(3,6;n)\right)q^n=q\frac{J_3^2J_{12}^2}{J_2J_6^2}, \tag{1.14}$$

$$\sum_{n=0}^{\infty}\left(N^0(0,6;n)-N^0(2,6;n)\right)q^n=\frac{q}{2}\left(\rho(q)+\rho(-q)\right), \tag{1.15}$$

$$\sum_{n=0}^{\infty}\left(N^0(1,6;n)-N^0(3,6;n)\right)q^n=\frac{q}{2}\left(\rho(-q)-\rho(q)\right), \tag{1.16}$$

$$\sum_{n=0}^{\infty}\left(N^0(1,6;n)-N^0(2,6;n)\right)q^n=\frac{q}{3}\left(\rho(-q)-\omega(-q)\right). \tag{1.17}$$

Recall the following two Ramanujan's theta functions:

$$\varphi(q):=\sum_{n=-\infty}^{\infty}q^{n^2}=(-q;q^2)_\infty^2(q^2;q^2)_\infty,$$

$$\psi(q) := \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

In addition, the following theta function identities are frequently used in this paper. Let n and m be integers with m positive. Then

$$j(x;q) = j(qx^{-1};q),$$
$$j(x;q) = (-1)^n q^{\binom{n}{2}} x^n j(q^n x;q). \tag{1.18}$$

This paper is organized as follows. In Section 2, we introduce some preliminary lemmas. In Section 3, we prove Theorems 1.3, 1.5, and 1.6. In Section 4, we give some examples of the generating functions of $N^0(a,M;n)$ for $M \in \{3,6,12\}$, and then prove Theorem 1.7.

2. Preliminaries

In this section, we give some preliminary lemmas which are used in the proofs of the main results.

Lemma 2.1. *[13, Lemma 3.10] Let n and k be integers with $0 \le k < n$. Let ω be a primitive nth root of unity, and suppose that $x^n \neq 1$. Then*

$$\frac{x^k}{1-x^n} = \frac{1}{n}\sum_{t=0}^{n-1} \frac{\omega^{-kt}}{1-\omega^t x}.$$

Lemma 2.2. *[2, Entry 12.2.2] If z is not an integral power of q, then*

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}}}{1-q^n z} = \frac{J_1^3}{j(z;q)}. \tag{2.1}$$

Lemma 2.3. *([11, Theorem 2.2], [13, Eq. (4.5)]) For generic $x, z \in \mathbb{C}^*$,*

$$g(x,q) = -x^{-2}m(qx^{-3},q^3,x^3z) - x^{-1}m(q^2x^{-3},q^3,x^3z) + \frac{J_1^2 j(xz;q) j(z;q^3)}{j(x;q)j(z;q)j(x^3z;q^3)}. \tag{2.2}$$

Following [13], the term "generic" means that the parameters do not cause poles in the Appell-Lerch sums or in the quotients of theta functions. The above lemma can be found in the lost notebook [17] which was first proved by Hickerson [11], and then rewritten in terms of Appell-Lerch sums by Hickerson and Mortenson [13].

Setting $z = -x^{-3}$ in (2.2), we derive the following special case,

$$g(x,q) = -x^{-2}m(qx^{-3},q^3,-1) - x^{-1}m(q^2x^{-3},q^3,-1) + \Delta(x;q), \tag{2.3}$$

where $\Delta(x;q)$ is defined in (1.10).

Lemma 2.4. *[13] For generic $x, z \in \mathbb{C}^*$,*

$$m(x,q,z) = m(x,q,qz),$$
$$m(x,q,z) = x^{-1}m(x^{-1},q,z^{-1}), \tag{2.4}$$

$$m(qx, q, z) = 1 - xm(x, q, z).$$

Lemma 2.5. *[13, Theorem 3.9] Let n and k be integers with $0 \leq k < n$. Let ω be a primitive nth root of unity. Then*

$$\sum_{t=0}^{n-1} \omega^{-kt} m(\omega^t x, q, z) = nq^{-\binom{k+1}{2}}(-x)^k m(-q^{\binom{n}{2}-nk}(-x)^n, q^{n^2}, z') + n\Psi_k^n(x, z, z'; q),$$

where $\Psi_k^n(x, z, z'; q)$ is defined in (1.11).

3. Main results

In this section, we prove Theorems 1.3, 1.5, and 1.6.

Proof of Theorem 1.3. First, for $|q^{-1}| < |z| < |q|$,

$$\begin{aligned}\frac{1}{(1-z^{-1}q^{2n+1})(1-zq^{2n+1})} &= \frac{1}{1-q^{4n+2}} + \frac{z^{-1}q^{2n+1}}{(1-q^{4n+2})(1-z^{-1}q^{2n+1})} \\ &\quad + \frac{zq^{2n+1}}{(1-q^{4n+2})(1-zq^{2n+1})} \\ &= \frac{1}{1-q^{4n+2}} + \sum_{m=1}^{\infty} \frac{(zq^{2n+1})^m + (z^{-1}q^{2n+1})^m}{1-q^{4n+2}}.\end{aligned}$$

Then from (1.6) and the above identity, we derive

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} N^0(m,n)z^m q^n &= \frac{1}{J_2}\sum_{n=0}^{\infty}(-1)^n q^{3n^2+3n+1} \\ &\quad + \frac{1}{J_2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(-1)^n q^{3n^2+3n+1}\left((zq^{2n+1})^m + (z^{-1}q^{2n+1})^m\right).\end{aligned}$$

Thus,

$$\sum_{n=1}^{\infty} N^0(m,n)q^n = \frac{1}{J_2}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}(-1)^n q^{3n^2+3n+(2n+1)|m|+1}. \tag{3.1}$$

Setting $m = kM + a$ with $0 \leq a < M$ in (3.1), we prove the theorem. □

Proof of Theorem 1.5. For $M \equiv 0 \pmod 3$, in view of Theorem 1.3 and Lemma 2.1 with $n = M/3$, $k = 0$, and $x = q^{6n+3}$, we have

$$\sum_{n=0}^{\infty} N^0(a, M; n)q^n = \frac{3q}{M}\frac{1}{J_2}\sum_{t=0}^{M/3-1}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n^2+3n+(2n+1)a}}{1-q^{(6n+3)}\omega_{\frac{M}{3}}^t}. \tag{3.2}$$

Case I: $a \equiv 0 \pmod 3$.

Changing n to $n - a/3$ in the sum on the right-hand side of (3.2) gives that

$$\sum_{t=0}^{M/3-1}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n^2+3n+(2n+1)a}}{1-q^{(6n+3)}\omega_{\frac{M}{3}}^t} = \sum_{t=0}^{M/3-1}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+\frac{a}{3}} q^{3n^2+3n-\frac{a^2}{3}}}{1-q^{(6n-2a+3)}\omega_{\frac{M}{3}}^t}$$

$$= \sum_{t=0}^{M/3-1} (-1)^{\frac{a}{3}} q^{-\frac{a^2}{3}} \frac{J_6^3}{j(q^{3-2a}\omega_{\frac{M}{3}}^t; q^6)}, \tag{3.3}$$

where the the last equality follows from (2.1). Then in view of (3.2) and (3.3), we obtain (1.8).

Case II: $a \equiv 1 \pmod 3$.

According to Definition 1.1, we obtain that

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+3n+(2n+1)a}}{1-q^{(6n+3)}\omega_{\frac{M}{3}}^t} &= -q^{-a} j(q^{2a}; q^6) m(q^{3-2a}\omega_{\frac{M}{3}}^t, q^6, q^{2a}) \\ &= -q^{-a} j(q^{6\frac{a-1}{3}+2}; q^6) m(q^{3-2a}\omega_{\frac{M}{3}}^t, q^6, q^{2a}) \\ &= -q^{-a}(-1)^{\frac{a-1}{3}} q^{\frac{-a^2+3a-2}{3}} j(q^2; q^6) m(q^{3-2a}\omega_{\frac{M}{3}}^t, q^6, q^{2a}) \\ &= (-1)^{\frac{a+2}{3}} q^{\frac{-a^2-2}{3}} J_2 m(q^{3-2a}\omega_{\frac{M}{3}}^t, q^6, q^{2a}),\end{aligned}$$

where the third step follows from (1.18). Then combining (3.2) and the above identity yields (1.9). □

Proof of Theorem 1.6. First, by means of (1.7) and (2.3), we have

$$\begin{aligned}&\sum_{n=0}^{\infty} N^0(a, M; n) q^n \\ &= \frac{q}{M} \sum_{j=0}^{M-1} \xi_M^{-aj} g(\xi_M^j q, q^2) \\ &= \frac{q}{M} \sum_{j=0}^{M-1} \xi_M^{-aj} \left(-\xi_M^{-2j} q^{-2} m(\xi_M^{-3j} q^{-1}, q^6, -1) - \xi_M^{-j} q^{-1} m(\xi_M^{-3j} q, q^6, -1)\right) \\ &\quad + \frac{q}{M} \sum_{j=0}^{M-1} \xi_M^{-aj} \Delta(\xi_M^j q; q^2) \\ &= -\frac{1}{M} \sum_{j=0}^{M-1} \xi_M^{(-a+1)j} m(\xi_M^{3j} q, q^6, -1) - \frac{1}{M} \sum_{j=0}^{M-1} \xi_M^{(-a-1)j} m(\xi_M^{-3j} q, q^6, -1) \\ &\quad + \frac{q}{M} \sum_{j=0}^{M-1} \xi_M^{-aj} \Delta(\xi_M^j q; q^2),\end{aligned} \tag{3.4}$$

where the first summand in the third equality follows from (2.4). Then for given a and M, let $k_\pm$ be an integer such that $\mp 3k_\pm \equiv -a \pm 1 \pmod M$. Notice that such $k_\pm$ must exist since $gcd(3, M) = 1$. Then applying Lemma 2.5 with $(n, k, \omega, x, q, z, z') \to (M, k_\pm, \xi_M^{\pm 3}, q, q^6, -1, z)$ yields

$$-\frac{1}{M} \sum_{j=0}^{M-1} \xi_M^{(-a\pm1)j} m(\xi_M^{\pm 3j} q, q^6, -1)$$

$$= -\frac{1}{M}\sum_{j=0}^{M-1}\xi_M^{\mp 3jk_\pm} m(\xi_M^{\pm 3j}q, q^6, -1)$$
$$= -q^{-3k_\pm(k_\pm+1)}(-q)^{k_\pm} m(-q^{3M(M-1)-6Mk_\pm}(-q)^M, q^{6M^2}, z) - \Psi_{k_\pm}^M(q,-1,z;q^6)$$
$$= (-1)^{k_\pm+1}q^{-3k_\pm^2-2k_\pm} m((-1)^{M+1}q^{3M^2-(6k_\pm+2)M}, q^{6M^2}, z) - \Psi_{k_\pm}^M(q,-1,z;q^6). \quad (3.5)$$

Hence, by combining (3.4) and (3.5), we prove the theorem. □

4. Examples

In this section, we provide some examples which are the generating functions of $N^0(a,M;n)$ for $M \in \{3,6,12\}$. Then we prove Theorem 1.7.

Case M=3:

$$\sum_{n=0}^{\infty} N^0(0,3;n)q^n = \frac{qJ_6^4}{J_2J_3^2}, \quad (4.1)$$
$$\sum_{n=0}^{\infty} N^0(1,3;n)q^n = \sum_{n=0}^{\infty} N^0(2,3;n)q^n = -m(q,q^6,q^2). \quad (4.2)$$

Proof. From Theorem 1.5, we obtain the case for $M=3$. □

Case M=6:

$$\sum_{n=0}^{\infty} N^0(0,6;n)q^n = \frac{qJ_{24}^5}{J_2J_6J_{48}^2}, \quad (4.3)$$
$$\sum_{n=0}^{\infty} N^0(3,6;n)q^n = 2\cdot\frac{q^4J_{12}^2J_{48}^2}{J_2J_6J_{24}}, \quad (4.4)$$
$$\sum_{n=0}^{\infty} N^0(1,6;n)q^n = \sum_{n=0}^{\infty} N^0(5,6;n)q^n = -\frac{1}{2}\left(m(q,q^6,q^2)+m(-q,q^6,q^2)\right), \quad (4.5)$$
$$\sum_{n=0}^{\infty} N^0(2,6;n)q^n = \sum_{n=0}^{\infty} N^0(4,6;n)q^n = -\frac{1}{2}\left(m(q,q^6,q^2)-m(-q,q^6,q^2)\right). \quad (4.6)$$

Proof. In view of (1.8), we have

$$\sum_{n=0}^{\infty} N^0(0,6;n)q^n = \frac{1}{2}\cdot\frac{qJ_6^3}{J_2\varphi(-q^3)} + \frac{1}{2}\cdot\frac{qJ_6^3}{J_2\varphi(q^3)} \quad (4.7)$$
$$= \frac{1}{2}\cdot\frac{qJ_6^3\left(\varphi(q^3)+\varphi(-q^3)\right)}{J_2\varphi(-q^3)\varphi(q^3)}, \quad (4.8)$$
$$\sum_{n=0}^{\infty} N^0(3,6;n)q^n = \frac{1}{2}\cdot\frac{qJ_6^3}{J_2\varphi(-q^3)} - \frac{1}{2}\cdot\frac{qJ_6^3}{J_2\varphi(q^3)} \quad (4.9)$$
$$= \frac{1}{2}\cdot\frac{qJ_6^3\left(\varphi(q^3)-\varphi(-q^3)\right)}{J_2\varphi(-q^3)\varphi(q^3)}. \quad (4.10)$$

In addition, from Entry 25 (i)-(iii) in [4, p. 40], we have

$$\varphi(q)+\varphi(-q)=2\varphi(q^4), \tag{4.11}$$
$$\varphi(q)-\varphi(-q)=4q\psi(q^8), \tag{4.12}$$
$$\varphi(-q)\varphi(q)=\varphi^2(-q^2). \tag{4.13}$$

Substituting (4.11) and (4.13) into (4.8), we arrive at (4.3). Similarly, combining (4.10), (4.12), and (4.13) yields (4.4). The last two equalities follow from (1.9) and Lemma 2.4. □

Case M=12:

$$\sum_{n=0}^{\infty} N^0(0,12;n)q^n=\frac{1}{2}\cdot\frac{qJ_{24}^5}{J_2J_6J_{48}^2}+\frac{1}{2}\cdot\frac{qJ_6^3J_{24}}{J_2J_{12}^2}, \tag{4.14}$$
$$\sum_{n=0}^{\infty} N^0(3,12;n)q^n=\frac{q^4J_{12}^2J_{48}^2}{J_2J_6J_{24}},$$
$$\sum_{n=0}^{\infty} N^0(6,12;n)q^n=\frac{1}{2}\cdot\frac{qJ_{24}^5}{J_2J_6J_{48}^2}-\frac{1}{2}\cdot\frac{qJ_6^3J_{24}}{J_2J_{12}^2}.$$

Proof. In light of (1.8), we derive

$$\begin{aligned}\sum_{n=0}^{\infty} N^0(0,12;n)q^n&=\frac{1}{4}\cdot\frac{qJ_6^3}{J_2\varphi(-q^3)}+\frac{1}{4}\cdot\frac{qJ_6^3}{J_2\varphi(q^3)}\\&\quad+\frac{1}{4}\cdot\frac{qJ_6^3}{J_2j(iq^3;q^6)}+\frac{1}{4}\cdot\frac{qJ_6^3}{J_2j(-iq^3;q^6)}\\&=\frac{1}{4}\cdot\frac{qJ_6^3\left(\varphi(q^3)+\varphi(-q^3)\right)}{J_2\varphi(-q^3)\varphi(q^3)}+\frac{1}{2}\cdot\frac{qJ_6^3}{J_2j(iq^3;q^6)}.\end{aligned} \tag{4.15}$$

Moreover,

$$j(iq^3;q^6)=(iq^3;q^6)_\infty(-iq^3;q^6)_\infty(q^6;q^6)_\infty=(-q^6;q^{12})_\infty(q^6;q^6)_\infty=\varphi(-q^{12}).$$

Then substituting (4.11), (4.13), and the above equality into (4.15), we arrive at (4.14). Similarly, by means of (1.8), we prove the rest of the equalities. □

Proof of Theorem 1.7. Watson [19] showed that

$$\omega(q)+2\rho(q)=3\cdot\frac{J_6^3}{J_2\varphi(-q^3)}, \tag{4.16}$$

where $\omega(q)$ and $\rho(q)$ are defined in (1.1) and (1.12), respectively. By combining (4.1) and (4.16), we prove

$$\sum_{n=0}^{\infty} N^0(0,3;n)q^n=\frac{q}{3}\omega(q)+\frac{2q}{3}\rho(q). \tag{4.17}$$

From (1.3), (1.4), and (1.5), it follows that

$$2\sum_{n=0}^{\infty} N^0(1,3;n)q^n=q\omega(q)-\sum_{n=0}^{\infty} N^0(0,3;n)q^n=\frac{2q}{3}\omega(q)-\frac{2q}{3}\rho(q),$$

where we derive the last equality by using (4.17). Therefore, we arrive at

$$\sum_{n=0}^{\infty} N^0(1,3;n)q^n = \frac{q}{3}\omega(q) - \frac{q}{3}\rho(q). \tag{4.18}$$

Combining (4.17) and (4.18), we complete the proof of (1.13).

Furthermore, with the aid of (4.7) and (4.9), we prove (1.14).

From (4.2) and (4.18), it can be seen that

$$m(q,q^6,q^2) = -\frac{q}{3}\omega(q) + \frac{q}{3}\rho(q). \tag{4.19}$$

Thus,

$$m(-q,q^6,q^2) = \frac{q}{3}\omega(-q) - \frac{q}{3}\rho(-q). \tag{4.20}$$

Then substituting (4.19) and (4.20) into (4.6), we obtain

$$\sum_{n=0}^{\infty} N^0(2,6;n)q^n = \frac{q}{6}\left(\omega(q)+\omega(-q)\right) - \frac{q}{6}\left(\rho(q)+\rho(-q)\right). \tag{4.21}$$

Moreover, using (4.7) and (4.16) yields

$$\sum_{n=0}^{\infty} N^0(0,6;n)q^n = \frac{q}{6}\left(\omega(q)+\omega(-q)\right) + \frac{q}{3}\left(\rho(q)+\rho(-q)\right).$$

Hence, combining the above two identities, we derive (1.15).

Similarly, by using (4.5), (4.9), (4.16), (4.19), and (4.20), we have

$$\sum_{n=0}^{\infty} N^0(1,6;n)q^n = \frac{q}{6}\left(\omega(q)-\omega(-q)\right) - \frac{q}{6}\left(\rho(q)-\rho(-q)\right), \tag{4.22}$$

$$\sum_{n=0}^{\infty} N^0(3,6;n)q^n = \frac{q}{6}\left(\omega(q)-\omega(-q)\right) + \frac{q}{3}\left(\rho(q)-\rho(-q)\right),$$

which imply (1.16).

Finally, combining (4.21) and (4.22), we deduce (1.17). □

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