# Integer flows and modulo orientations of signed graphs

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**Abstract**

This paper studies the fundamental relations among integer flows, modulo orientations, integer-valued and real-valued circular flows, and monotonicity of flows in signed graphs. A (signed) graph is modulo-$(2p+1)$-orientable if it has an orientation such that the indegree is congruent to the outdegree modulo $2p+1$ at each vertex. An integer-valued $\frac{2p+1}{p}$-flow is a flow taking integer values in $\{\pm p, \pm(p+1)\}$. Extending a fundamental result of Jaeger to signed graphs, we show that a bridgeless signed graph is modulo-$(2p+1)$-orientable if and only if it admits an integer-valued $\frac{2p+1}{p}$-flow. It was conjectured by Raspaud and Zhu that, for any signed graph, the admission of a circular $r$-flow implies the admission of an integer-valued $\lceil r \rceil$-flow. Although this conjecture has been disproved in general, it is confirmed in this paper for bridgeless signed graphs if $r = \frac{2p+1}{p}$ and $p \geq 3$.

**Keywords:** signed graph; nowhere-zero flow; circular flow; modulo orientation; integer flow

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# 1 Introduction

Graphs considered in this paper may have multiple edges or loops. A *signed graph* $(G, \sigma)$ is a graph $G$ associated with a *signature* $\sigma\colon\ E(G) \to \{\pm 1\}$. An edge $e$ is *positive* if $\sigma(e) = 1$ and *negative* otherwise. An ordinary graph can be considered as a signed graph with all edges positive.

## 1.1 Motivations

Integer flows of ordinary graphs were introduced by Tutte [28] as the dual of vertex coloring of graphs embedded on orientable surfaces. Bouchet [4] extended the concept of flows to signed graphs as dual notion to local tensions of graphs embedded on non-orientable surfaces. There are significant differences between the flows of signed graphs and that of ordinary graphs. Some fundamental results on flows of ordinary graphs no longer hold for signed graphs. In this paper we address those differences from the aspects related to circular flows and modulo orientations.

Firstly, for ordinary graphs, Jaeger [13] showed that the admission of a modulo $(2p+1)$-orientation is equivalent to the admission of an integer-valued $\frac{2p+1}{p}$-flow. The modulo orientation (and, more generally, modulo flow) technique is one of the most important tools in flow theory (see [13, 29]). It is well known that the above equivalence plays an important role in the proofs of some landmark flow theorems, such as [11, 13, 17, 18, 25, 26, 29] among others. However, this equivalence is not true for signed graphs in general (cf. [5, 24, 31, 33]). In this paper, this equivalence is established for all bridgeless signed graphs, which improves several previous results [5, 24, 31, 33] in this direction.

Secondly, for ordinary graphs, Goddyn, Tarsi and Zhang [9] showed that the admission of a circular $r$-flow implies the admission of an integer-valued $\lceil r \rceil$-flow. For signed graphs, this basic property was proposed as an open problem by Raspaud and Zhu [23]. Although many counterexamples have been discovered recently in [20, 24, 19, 14], in this paper, this open problem is verified for bridgeless signed graphs if $r = \frac{2p+1}{p}$ and $p \geq 3$. In fact, this result follows from a more general monotonicity property of circular flows of bridgeless signed graphs.

## 1.2 Notation and terminology

Every edge of a signed graph $(G, \sigma)$ is composed of two half-edges $h$ and $\hat{h}$, each of which is incident with one end. Denote the set of half-edges of $(G, \sigma)$ by $H(G)$ and the set of half-edges incident with $v$ by $H_G(v)$. For a half-edge $h \in H(G)$, we use $e_h$ to refer to the edge containing $h$. An *orientation* of a signed graph $(G, \sigma)$ is a mapping $\tau : H(G) \to \{-1, 1\}$ such that $\tau(h)\tau(\hat{h}) = -\sigma(e_h)$ for each $h \in H(G)$. We may consider $\tau$ as an assignment of orientations on $H(G)$ such that $h$ is a half-edge oriented away from its end if $\tau(h) = 1$ and otherwise towards its end. A signed graph $(G, \sigma)$ together with an orientation $\tau$ is called an *oriented signed graph*, denoted by $(G, \tau)$, with underlying signature $\sigma_\tau$.

**Definition 1.1** *Let $(G, \sigma)$ be a signed graph with an orientation $\tau$. Let $k$ be a positive integer and $f\colon\ E(G) \to \mathbb{Z}$ be a mapping such that $0 \leq |f(e)| \leq (k-1)$ for every edge $e \in E(G)$.*

*(1) The* support *of $f$, denoted by supp$(f)$, is the set of edges $e$ with $f(e) \neq 0$.*

*(2) The* boundary *of $f$ at a vertex $v$ is defined as $\partial f(v) = \sum_{h \in H(v)} f(e_h)\tau(h)$.*

*(3) The mapping $f$ is an* integer-valued $k$-flow *(or $k$-flow for short) of $(G, \sigma)$ if $\partial f(v) = 0$ for each vertex $v \in V(G)$.*

*(4) A flow $f$ is* nowhere-zero *if supp$(f) = E(G)$.*

For convenience, we usually shorten the notation of nowhere-zero integer-valued $k$-flow into $k$-NZF. For ordinary graphs, Goddyn, Tarsi and Zhang [9] introduced the concept of circular flows as a refinement of Tutte's integer flows, which allows flow values to be real numbers. The circular flows are extended from ordinary graphs to signed graphs.

**Definition 1.2** *Let $(G, \sigma)$ be a signed graph with an orientation $\tau$. Let $k$ and $d$ be two positive integers where $k \geq 2d > 0$.*

*(1) An* integer-valued $\frac{k}{d}$-flow *of $(G, \sigma)$ is an integer-valued flow $f$ with $d \leq |f(e)| \leq k-d$ for every edge $e \in E(G)$.*

*(2) A* real-valued $\frac{k}{d}$-flow *of $(G, \sigma)$ is a real-valued flow $f$ with $|f(e)| \in [d, k-d]$ for every edge $e \in E(G)$ (where $[d, k-d]$ denotes the real-valued interval from $d$ to $k-d$).*

Let $(G, \tau)$ be an oriented signed graph. Denote by $d^+_\tau(v)$ ($d^-_\tau(v)$, resp.) the number of half-edges incident with $v$ which are oriented away from $v$ (oriented toward $v$, resp.). An edge $e$ is a *source* (resp., *sink*) of $(\tau, f)$ if $\tau(h_1) = \tau(h_2) = -1$ (resp., $\tau(h_1) = \tau(h_2) = 1$), where $h_1$ and $h_2$ are the two half-edges of $e$. For a positive integer $p$, an orientation $\tau$ is called a *modulo* $(2p+1)$*-orientation* if $d^+_\tau(v) \equiv d^-_\tau(v) \pmod{2p+1}$ for every vertex $v \in V(G)$. A signed graph $(G, \sigma)$ is called *modulo-*$(2p+1)$*-orientable* if it has a modulo $(2p+1)$-orientation.

### 1.3 Main results

The circular $\frac{2p+1}{p}$-flows was introduced and studied by Jaeger [12, 13] even before Goddyn et al. [9] introduced the concept of general circular $\frac{k}{d}$-flow, and he proved that the following three statements are equivalent.

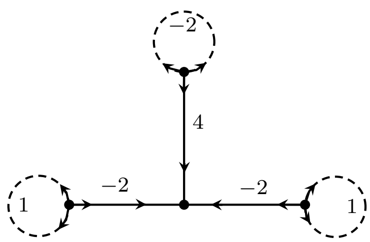
**Proposition 1.3** (Jaeger [12, 13]) *Let $G$ be an ordinary graph and $p$ be a positive integer. The following statements are equivalent.*

**(I)** *$G$ admits a modulo $(2p+1)$-orientation.*

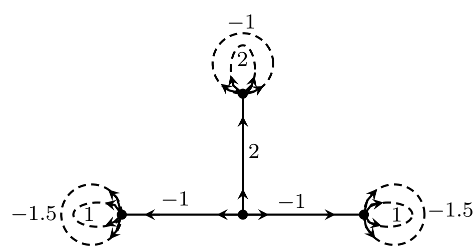
**(II)** *$G$ admits an integer-valued $\frac{2p+1}{p}$-flow.*

**(III)** *$G$ admits a real-valued $\frac{2p+1}{p}$-flow.*

Proposition 1.3 provides a fundamental tool to study $k$-NZFs and integer-valued $\frac{2p+1}{p}$-flows for ordinary graphs in terms of modulo orientations, which is technically easier to handle. Tutte's 3-flow conjecture asserts that every 4-edge-connected ordinary graph admits a 3-NZF. The weak 3-flow theorem, established by Lovász, Thomassen, Wu, Zhang [18] using modulo 3-orientations, states that every 6-edge-connected ordinary graph admits a 3-NZF. Applying some modulo $(2p+1)$-orientation techniques, Thomassen [26] and Lovász et al. [18] prove the weak circular flow conjecture of Jaeger [13] by showing that every $6p$-edge-connected graph admits an integer-valued $\frac{2p+1}{p}$-flow, while the circular flow conjecture was disproved in [10] for $p \geq 3$.




($a$) No real-valued or integer-valued $\frac{3}{1}$-flow.




($b$) There is a real-valued but no integer-valued $\frac{3}{1}$-flow.

Figure 1: Modulo-3-orientable signed graphs without integer-valued $\frac{3}{1}$-flow or real-valued $\frac{3}{1}$-flow.

How about Proposition 1.3 for signed graphs? It is not hard to see that **(II)** implies both **(I)** and **(III)** by the definitions. However all other directions of implication fail. The graph in Figure 1-(a) has a modulo 3-orientation but has no real-valued $\frac{3}{1}$-flow (of course no integer-valued $\frac{3}{1}$-flow). Thus **(I)** does not imply **(II)**. The graph in Figure 1-(b) has a modulo 3-orientation and has a real-valued $\frac{3}{1}$-flow but has no integer-valued $\frac{3}{1}$-flow. Hence **(I)** does not imply **(III)**. The equivalence of **(I)** and **(III)** and of **(II)** and **(III)** fails even for some signed graphs with high edge connectivity as shown in Proposition 5.3 for every positive integer $p$.

On the other hand high edge connectivity may still guarantee the equivalence of **(I)** and **(II)** for signed graphs. The following are some early results in this direction under some connectivity conditions due to Xu and Zhang [31], Schubert and Steffen [24], Zhu [33], and Cheng et al. [5], respectively.

**Theorem 1.4** *Let $(G, \sigma)$ be a signed graph and $p \geq 1$ be an integer. Then* **(I)** *and* **(II)** *are equivalent if one of the following conditions is satisfied:*

1. ([31]) *$p = 1$ and $(G, \sigma)$ is cubic and contains a perfect matching;*
2. ([24]) *$(G, \sigma)$ is $(2p + 1)$-regular and contains a $p$-factor;*
3. ([33]) *$(G, \sigma)$ is $(12p - 1)$-edge-connected;*
4. ([5]) *$(G, \sigma)$ is odd-$(2p + 1)$-edge-connected.*

Our first main result establishes the best possible edge connectivity condition for the equivalence of **(I)** and **(II)**.

**THEOREM A** **(I)** *and* **(II)** *are equivalent for all bridgeless signed graphs. That is, a bridgeless signed graph is modulo-$(2p + 1)$-orientable if and only if it admits an integer-valued $\frac{2p+1}{p}$-flow.*

**Remark 1.** The connectivity condition in Theorem A is necessary. Figure 1-(a) can be generalized for any positive integer $p$. For each integer $p \geq 1$, let $\mathcal{H}_p$ be the family of signed graphs obtained from a tree in which the degree of each vertex is either 1 or $2p+1$ by adding $p$ negative loops to each leaf vertex. Note that Figure 1-(a) is a graph in $\mathcal{H}_1$. One can see that every graph in $\mathcal{H}_p$ is modulo-$(2p + 1)$-orientable but has no integer-valued $\frac{2p+1}{p}$-flow.

For ordinary graphs, by the definitions and Proposition 1.3, we have the following monotonicity of circular flows.

**Proposition 1.5** ([9, 13]) *Let $G$ be an ordinary graph. Let $k, k', d, d'$ be positive integers such that $\frac{k'}{d'} \geq \frac{k}{d} \geq 2$. If $G$ admits a circular $\frac{k}{d}$-flow (integer-valued or real-valued, respectively), then $G$ admits a circular $\frac{k'}{d'}$-flow (integer-valued or real-valued, respectively).*

Obviously Proposition 1.5 still holds for real-valued circular flows of signed graphs. However it does not hold for integer-valued circular flows of signed graphs. There are even some signed graphs with high edge connectivity that admit integer-valued $\frac{2k}{2d}$-flows but no integer-valued $\frac{k}{d}$-flows (see Section 5 for more details).

On the other hand, Raspaud and Zhu [23] suggested a conjecture concerning circular flows and integer flows of signed graphs.

**Conjecture 1.6** (Raspaud and Zhu [23]) *For any positive integers $k, d$ with $k \geq 2d$, every integer-valued $\frac{k}{d}$-flow admissible signed graph admits a nowhere-zero $\lceil \frac{k}{d} \rceil$-flow.*

Raspaud and Zhu [23] showed that every integer-valued $\frac{k}{d}$-flow admissible signed graph admits a nowhere-zero $(2\lceil\frac{k}{d}\rceil-1)$-flow. Conjecture 1.6 has been disproved for signed graphs in general (see [20, 24]), and some bridgeless counterexamples are found in [19, 14] recently. In contrast, we confirm Conjecture 1.6 for certain integer-valued $\frac{2p+1}{p}$-flows of bridgeless signed graphs.

**THEOREM B** *For each positive integer $p \neq 2$, every bridgeless integer-valued $\frac{2p+1}{p}$-flow admissible signed graph admits a nowhere-zero 3-flow.*

The case when $p = 2$ remains open and will be further discussed in Sections 4 and 5.

The organization of the rest of the paper is as follows. In Section 2, we introduce additional notation and terminology and some basic lemmas needed for the proofs of the main results. Section 3 introduces a method to construct a new regular modulo-$(2p+1)$-orientable signed graph from an arbitrary modulo-$(2p+1)$-orientable signed graph, which allows to reduce our theorems to regular signed graphs. Then we apply those results to complete the proofs of Theorems A and B in Section 4. Some further remarks on the difference of integer-valued and real-valued circular flows, as well as a few open problems, will be presented in Section 5.

## 2 Preliminaries

In this section we first introduce additional notation and terminology needed for the rest of the paper and then present some basic properties of flows of signed graphs. For terminology and notation not defined here we follow [2, 3, 30].

Let $G$ be a graph with *vertex set* $V(G)$ and *edge set* $E(G)$. The degree of a vertex $v$ is the number of edges incident with $v$, where each loop is counted twice. Let $X$ and $Y$ be two disjoint vertex sets. We denote by $[X, Y]$ the set of edges with one end in $X$ and the other end in $Y$. Denote by $B(G)$ the set of bridges of $G$. The graph $G - B(G)$ consists of some components, called *blocks*, each of which is either 2-edge-connected or a single vertex. A block is called a *leaf block* if it is incident with exactly one bridge in $B(G)$. Note that leaf blocks always exist when $G$ contains bridges.

A signed graph is *flow-admissible* if it admits a nowhere-zero $k$-flow for some integer $k$. In a signed graph, *switching* at a vertex $u$ means reversing the signs of all edges incident with $u$. Two signed graphs are *equivalent* if one can be obtained from the other by a sequence of switching operations. A signed graph is *balanced* if and only if it is equivalent to a graph without negative edges. In particular, a circuit is *balanced* if it has an even number of negative edges and is *unbalanced* otherwise. A signed graph $(G, \sigma)$ is *antibalanced* if there is a bipartition $(A, B)$ of $V(G)$ such that an edge $e$ is positive if and only if $e$ belongs to

$[A,B]$. We use $K_1^{-p}$ to denote the signed graph consisting of $p$ negative loops sharing a common vertex.

Note that switching at a vertex does not change the parity of the number of negative edges in a circuit and it does not change the admission of flows either. Bouchet [4] provided a characterization for flow-admissible signed graphs.

**Proposition 2.1** (Bouchet [4]) *A connected signed graph $(G,\sigma)$ is flow-admissible if and only if it is not equivalent to a signed graph with exactly one negative edge and it has no bridge $b$ such that $(G-b,\sigma|_{G-b})$ has a balanced component.*

**Proposition 2.2** *Let $p\geq 1$ be an integer. Suppose that $(G,\sigma)$ is modulo-$(2p+1)$-orientable.*

*(i) If $e=uv$ is a bridge, then each component of $G-e$ has at least $p$ negative edges and thus each component of $G-e$ is unbalanced.*

*(ii) $(G,\sigma)$ is flow-admissible.*

*(iii) If $G$ is $(2p+1)$-regular, then $(G,\sigma)$ is antibalanced.*

**Proof.** Let $\tau$ be a modulo $(2p+1)$-orientation of $(G,\sigma)$.

(i) and (ii): Let $[U,U^c]$ be an edge cut of $(G,\sigma)$. Let $a$ and $b$ be the number of sink edges and the number of source edges with both endvertices in $U$ respectively. Denote by $u^+$ and $u^-$ the numbers of oriented-in and oriented-out half-edges of the edges in $[U,U^c]$ incident with a vertex in $U$, respectively. Since $\tau$ is a modulo $(2p+1)$-orientation, we have

$$2a+u^+\equiv 2b+u^- \pmod{2p+1}. \qquad (*)$$

If $[U,U^c]$ is a bridge, then $u^++u^-=1$ and thus $|a-b|\equiv p \pmod{2p+1}$. Therefore $a+b\geq |a-b|\geq p$. This proves (i).

If $U^c=\emptyset$, then $u^+=u^-=0$ and thus by $(*)$ we have $a\equiv b \pmod{2p+1}$. This implies $a+b\neq 1$. By (i), if $G$ has a bridge $e$, each component of $G-e$ is unbalanced. Therefore by Proposition 2.1, $(G,\sigma)$ is flow-admissible.

(iii): Since $G$ is $(2p+1)$-regular and $\tau$ is a modulo $(2p+1)$-orientation, either $d_\tau^+(v)=0$ or $d_\tau^-(v)=0$ for each vertex $v\in V(G)$. Let $A=\{v\in V(G)|d_\tau^-(v)=0\}$ and $B=\{v\in V(G)|d_\tau^+(v)=0\}$. Then $(A,B)$ is a bipartition of $V(G)$ and an edge $e$ is positive if and only if $e\in[A,B]$. This proves that $(G,\sigma)$ is antibalanced. ■

**Lemma 2.3** *Let $f$ be an integer-valued flow of a signed graph $(G,\sigma)$ with an orientation $\tau$. Then $f(e)$ must be even for each bridge $e\in B(G)$.*

**Proof.** Let $[U,U^c]$ be an edge cut. We use $U^+$ ($U^-$, resp.) to denote the set of oriented-out (oriented-in, resp.) half-edges in $[U,U^c]$ which are incident with a vertex in $U$. Then we have

$$\sum_{e\in U^+} f(e)-\sum_{e\in U^-} f(e)=\sum_{e\in E(U),\sigma(e)=-1} \pm 2f(e).$$

The lemma follows immediately from the above fact when $[U, U^c]$ is a bridge. ■

**Proposition 2.4** *Let $C$ be a circuit in a signed graph $(G, \sigma)$ with an orientation $\tau$. Let $v$ be a vertex in $C$. Then there is a mapping $f_C : E(G) \to \{0, 1, -1\}$ with supp$(f) = E(C)$ such that $\partial f(x) = 0$ for each vertex $x \neq v$.*

**Proof.** One may start to assign nonzero flow values to the edges $E(C)$ from $v$ clockwise until going back to $v$ so that the boundary at every vertex distinct from $v$ is 0, which gives a desired mapping. ■

## 3 Modulo orientable graphs and $(2p + 1)$-regular graphs

In the study of flows and orientations, one may often try to reduce the graphs to regular graphs. A classical method for this reduction is to apply some splitting results that preserves the edge connectivity (see [7, 8, 21, 22, 32]). Usually the classical splitting method does require high edge connectivity. In this section, we propose a new method to construct a regular graph from certain graphs such that the regular graph easily preserves the properties of orientations, flows and edge connectivity, and the original graph is the contraction of some positive edges in the new regular graph. We believe that this construction is of interest itself and will be useful in the future study of flows and orientations of (signed) graphs.

**Lemma 3.1** *For any two nonnegative integers $a, b$ with $a \equiv b \pmod{2p + 1}$, there exists a $2p$-edge-connected bipartite simple graph $B(a, b) = (X, Y)$ such that*

*(i) each vertex in $X \cup Y$ is of degree $2p$ or $2p + 1$,*

*(ii) the numbers of vertices of degree $2p$ in $X$ and in $Y$ are exactly $a$ and $b$ respectively.*

*For example, Figure 2 shows the construction of $B(a, b)$ when $p = 1, a = 1$ and $b = 4$.*

**Proof.** Without loss of generality, we assume that $b \geq a$ and $b - a = (2p + 1)t$, where $t \geq 0$. Let $n = a + b + 2p + 2$ and $\mathbb{Z}_{2n} = \{0, 1, 2, \ldots, 2n - 1\}$ be the additive cyclic group of order $2n$. We construct $B(a, b)$ in the following three steps.

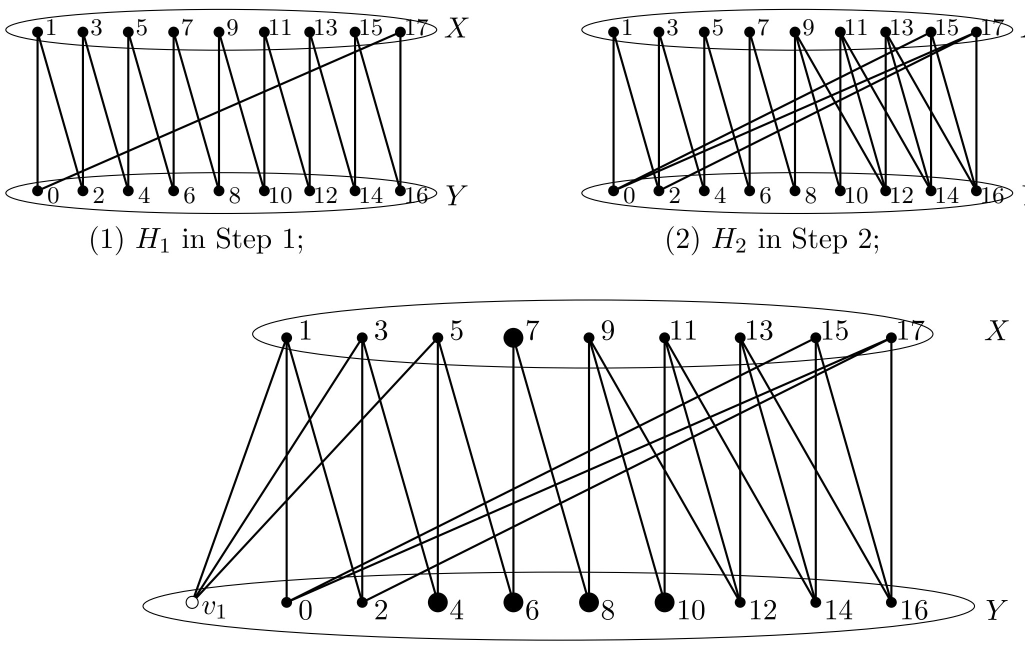
**Step 1:** Construct a $2p$-edge-connected circulant graph $H_1$.

(1-1) $V(H_1) = \mathbb{Z}_{2n}$ and bipartition $V(H_1)$ into $X$ and $Y$, where $X = \{1, 3, 5, \ldots, 2n - 1\}$ and $Y = \{0, 2, 4, \ldots, 2n - 2\}$.

(1-2) $E(H_1) = \{xy | x \in X,\ y \in Y,\ x - y \in \{\pm 1, \pm 3, \pm 5, \ldots, \pm(2p - 1)\}$.

Clearly, $H_1$ is a vertex-transitive graph and hence is $2p$-edge-connected (see Theorem 9.14 of [2]).

**Step 2:** Add more edges to $H_1$ to obtain a new graph $H_2$.




(1) $H_1$ in Step 1; (2) $H_2$ in Step 2;

(3) $B(1, 4)$ in Step 3.

Figure 2: The construction of $B(a, b)$ in Lemma 3.1 for $p = 1, a = 1$ and $b = 4$, where the larger circles are degree 2 vertices.

The graph $H_2$ is obtained from $H_1$ by adding the edges in

$$S = \bigcup_{i=b+1}^{n} \{xy : x = 2i - 1, y = 2i + 2p\}.$$

Then in $H_2$ both $X$ and $Y$ have $b$ vertices of degree $2p$.

**Step 3:** Add $t$ new vertices to $Y$ and more edges to finally obtain $B(a, b)$.

The graph $B(a, b)$ is obtained from $H_2$ by adding $t$ new vertices $v_1, v_2, \ldots, v_t$ (to $Y$) and adding the edges in

$$S' = \bigcup_{j=1}^{t} \bigcup_{i=1}^{2p+1} \{v_j x : x = 2(2p + 1)(j - 1) + 2i - 1\}.$$

It is easy to see that $B(a, b)$ is $2p$-edge-connected and satisfies (i) and (ii) as required. ■

**Construction 3.2** *Let $(H, \sigma)$ be a signed graph with a modulo $(2p + 1)$-orientation $\tau$. We construct a $(2p + 1)$-regular signed graph $(G, \sigma')$ from $(H, \sigma)$ as follows.*

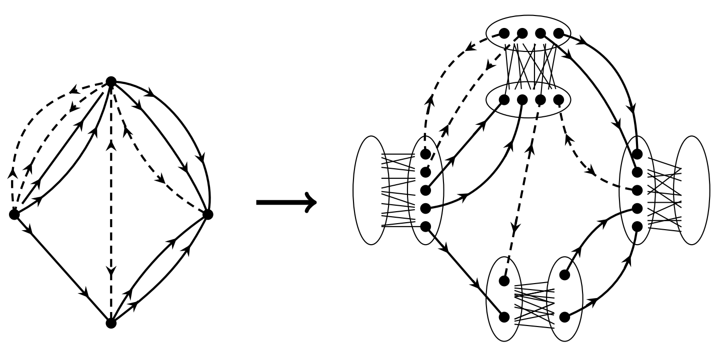
*(1) For each vertex $v \in V(H)$, let $B_v(d^+_\tau(v), d^-_\tau(v)) = (X, Y)$ be the $2p$-edge-connected bipartite graph constructed in Lemma 3.1.*

*(2) First split $v$ into $d^+_\tau(v) + d^-_\tau(v)$ vertices of degree 1 and then identify each degree 1 vertex of an out-arc with a vertex of degree $2p$ in $X$ and identify each degree 1 vertex of an in-arc with a vertex of degree $2p$ in $Y$.*

*(3) Let $G$ be the resulting graph from (2). The signature $\sigma'$ of $G$ is defined as follows, for each $e \in E(G)$,*

$$\sigma'(e) = \begin{cases} \sigma(e), & \text{if } e \in E(H) \subset E(G); \\ 1, & \text{if } e \in E(G) \setminus E(H). \end{cases}$$

*(4) The orientation $\tau$ of $(H, \sigma)$ can be extended to $(G, \sigma')$ to obtain a modulo $(2p+1)$-orientation by orienting the half edges of each edge in $B_v(d^+_\tau(v), d^-_\tau(v))$ away from $X$ and toward $Y$.*



$(H, \sigma)$ $\qquad$ $(G, \sigma') = anti(H, \sigma)$

Figure 3: An example of $(G, \sigma') = anti(H, \sigma)$ in Construction 3.2 when $p = 2$.

By Construction 3.2-(4) above, $(G, \sigma')$ is modulo-$(2p+1)$-orientable, and thus by Proposition 2.2, the graph $(G, \sigma')$ constructed above is antibalanced. We denote such a graph $(G, \sigma')$ by $anti(H, \sigma)$. See Figure 3 for an example of Construction 3.2.

The following proposition directly follows from the construction of $(G, \sigma')$.

**Proposition 3.3** *Let $(H, \sigma)$ be a modulo-$(2p+1)$-orientable signed graph. Then $(G, \sigma') = anti(H, \sigma)$ satisfies the following:*

*(i) $(G, \sigma')$ is $(2p+1)$-regular and is modulo-$(2p+1)$-orientable.*

*(ii) There is a set $T$ of positive edges in $G$ such that $H = G/T$ and $\sigma'$ agrees with $\sigma$ for all edges in $H$.*

*(iii) If $H$ is $k$-edge-connected, then $G$ is $t$-edge-connected, where $t = \min\{2p, k\}$. In particular, $G$ is bridgeless if $H$ is bridgeless.*

*(iv) Every bridge in $G$ is also a bridge in $H$.*

A classical result of Bäbler [1] shows that every bridgeless $(2p+1)$-regular graph contains a $k$-factor if $k$ is odd and $\frac{2p+1}{3} \le k \le 2p-1$. Using Tutte's $f$-factor theorem [27], Kano [15, 16] obtained an extension of Bäbler's result, allowing at most one bridge.

**Theorem 3.4** (Kano [16]) *Let $G$ be a $(2p+1)$-regular graph with at most one bridge. If $k$ is odd and $\frac{2p+1}{3} \le k \le 2p-1$, then $G$ has a $k$-factor.*

Note that the existence of one bridge is useful in our later inductive arguments.

**Corollary 3.5** *Let $p \ge 1$ be an integer and $G$ be a $(2p+1)$-regular graph with at most one bridge. Then each of the following holds.*

*(a) $E(G)$ can be partitioned into a $p$-factor and a $(p+1)$-factor;*

*(b) If $p \ge 3$, then $E(G)$ can be partitioned into a $(p-1)$-factor and a $(p+2)$-factor.*

**Proof.** Since $G$ is $(2p+1)$-regular, the complement of a $k$-factor is a $(2p+1-k)$-factor in $G$. Thus we only need to show that $G$ has a $p$-factor or a $(p+1)$-factor in (a) and has a $(p-1)$-factor or a $(p+2)$-factor in (b).

For (a), let

$$k_1 = \begin{cases} p+1, & \text{if } p \equiv 0 \pmod 2; \\ p, & \text{otherwise.} \end{cases}$$

Then $G$ has a $k_1$-factor by Theorem 3.4, since $3k_1 \ge 3p \ge 2p+1$.

For (b), let

$$k_2 = \begin{cases} p-1, & \text{if } p \equiv 0 \pmod 2; \\ p+2, & \text{otherwise.} \end{cases}$$

Then $G$ has a $k_2$-factor by Theorem 3.4, since $p \ge 3$ and $3k_2 \ge 2p+1$. ■

# 4 Modulo $(2p+1)$-orientations and integer-valued flows

We will present the proofs of our main results in this section.

### 4.1 Modulo $(2p+1)$-orientations and integer-valued $\frac{2p+1}{p}$-flows

In this subsection we will prove Theorem A. Actually, we shall show the following slightly stronger theorem instead, which will be also useful in the next subsection.

**Theorem 4.1** *Let $(H, \sigma)$ be a modulo-$(2p+1)$-orientable signed graph with at most one bridge. Then $(H, \sigma)$ admits an integer-valued $\frac{2p+1}{p}$-flow.*

By Proposition 3.3, there is a $(2p+1)$-regular modulo-$(2p+1)$-orientable signed graph $(G, \sigma')$ with at most one bridge such that $H = G/X$ for some set $X$ consisting of positive edges. Since the flow property is preserved under contraction, Theorem 4.1 follows directly from the lemma below.

**Lemma 4.2** *Let $(G, \sigma)$ be a $(2p+1)$-regular modulo-$(2p+1)$-orientable signed graph with at most one bridge. Then $G$ can be partitioned into a $p$-factor $M_1$ and a $(p+1)$-factor $M_2$ so that $(G, \sigma)$ has an integer-valued $\frac{2p+1}{p}$-flow $f$ where $f(e) = -(p+1)$ if $e \in M_1$ and $f(e) = p$ if $e \in M_2$.*

**Proof.** By Corollary 3.5, there is a partition of $E(G)$ into a $p$-factor $M_1$ and a $(p+1)$-factor $M_2$. Let $\tau$ be a modulo $(2p+1)$-orientation of $(G, \sigma)$ and let

$$f(e) = \begin{cases} -(p+1), & \text{if } e \in M_1; \\ p, & \text{otherwise.} \end{cases}$$

Therefore $f$ is a desired flow. ■

Taking $p = 1$, an integer-valued $\frac{3}{1}$-flow is indeed a 3-NZF. For the case when $p = 2$, an integer-valued $\frac{5}{2}$-flow is exactly a 4-NZF with flow values in $\{\pm 2, \pm 3\}$ by definition. Therefore we have the following corollary.

**Corollary 4.3** *(i) Every modulo-3-orientable signed graph with at most one bridge admits a nowhere-zero 3-flow.*

*(ii) Every modulo-5-orientable signed graph with at most one bridge admits a nowhere-zero 4-flow with flow values in $\{\pm 2, \pm 3\}$.*

The case when $p = 1$ slightly strengthens the result by Xu and Zhang [31] which claims that every bridgeless modulo-3-orientable signed graph admits a nowhere-zero 3-flow.

### 4.2 Modulo $(2p+1)$-orientations and integer flows

In this subsection, we will prove Theorem B. In fact, Theorem B is a corollary of Theorem A and the following theorem.

**THEOREM C** *Let $p \geq 3$ be an integer. If a signed graph $(G, \sigma)$ has a modulo $(2p+1)$-orientation, then $(G, \sigma)$ has a modulo 3-orientation.*

Note that the case when $p = 2$ and $G$ has bridges is excluded from Theorem B and from Corollary 4.3 which will be settled for 5-flows in the following two theorems.

**THEOREM D** *Every modulo-5-orientable signed graph admits a 5-NZF.*

DeVos et al. [6] showed that every modulo-3-orientable signed graph has a 5-NZF, which is one of the key steps in establishing a 11-flow theorem of signed graphs. This result together with Theorem C (for $p \neq 2$) and Theorem D (for $p = 2$) implies the following theorem.

**THEOREM E** *For each integer $p \geq 1$, every modulo-$(2p+1)$-orientable signed graph admits a 5-NZF.*

**Remark 2.** Here the flow number 5 in Theorem E is sharp as every signed graph in $\mathcal{H}_p$ (defined in Remark 1) has no 4-NZF by Proposition 2.3.

Now we start the proof of Theorem C.

**Proof of Theorem C.** Let $(H, \sigma)$ be a counterexample to the theorem with $|V(H)| + |E(H)|$ minimized. That is, $(H, \sigma)$ is modulo-$(2p+1)$-orientable but is not modulo-3-orientable.

**Claim 1** *$H$ contains at least two bridges.*

*Proof.* Suppose to the contrary that $H$ has at most one bridge. Let $(G, \sigma') = anti(H, \sigma)$ be the signed graph defined in Construction 3.2. Then $G$ is $(2p+1)$-regular and $(G, \sigma')$ has a modulo $(2p+1)$-orientation $\tau$. By Proposition 3.3, $G$ has at most one bridge. Since $p \geq 3$, by Corollary 3.5-(a), $G$ has a $(p-1)$-factor $M$. By reversing the direction of each edge in $M$, we obtain a new orientation $\tau'$ of $(G, \sigma')$ satisfying the following:

- $d^+_{\tau'}(v) = p+2, d^-_{\tau'}(v) = p-1$ if $d^+_{\tau}(v) = 2p+1$;
- $d^+_{\tau'}(u) = p-1, d^-_{\tau}(u) = p+2$ if $d^-_{\tau}(v) = 2p+1$.

Hence $d^+_{\tau'}(v) - d^-_{\tau'}(v) \equiv 0 \pmod 3$ for each $v \in V(G)$. Therefore $\tau'$ is a modulo 3-orientation of $(G, \sigma')$, which yields a modulo 3-orientation of $(H, \sigma)$, a contradiction. Thus $H$ contains at least two bridges. □

By possibly some switching operations, we assume that every bridge is positive.

**Claim 2** *Every leaf block of $H$ is a $K_1^{-p}$.*

*Proof.* Suppose to the contrary that there is a bridge $e = uv$ such that one of the components $H_1$ and $H_2$ of $G - e$, say $H_1$, is a leaf block and $H_1 \neq K_1^{-p}$. By Claim 1, we have $H_2 \neq K_1^{-p}$. For each $i = 1, 2$, let $G_i$ be the new graph obtained from $H$ by replacing $H_i$ with $p$ negative loops $K_1^{-p}$. Then both $G_1$ and $G_2$ are modulo-$(2p+1)$-orientable.

Since $H$ admits a modulo $(2p+1)$-orientation, by Proposition 2.2, $H_i$ contains at least $p$ negative edges for $i = 1, 2$. Since $H_1 \neq K_1^{-p}$ and $H_2$ contains a bridge which is a positive edge, we have $|E(G_i)| < |E(H)|$ for each $i = 1, 2$. Hence by the minimality of $H$, each $G_i$ admits a modulo 3-orientation $\tau_i$. One may choose $\tau_1$ and $\tau_2$ such that $e = uv$ has the same directions in both $\tau_1$ and $\tau_2$. Combining $H_1 + uv$ of $G_2$ under orientation $\tau_1$ and $H_2$ of $G_1$ under orientation $\tau_2$, we obtain a modulo 3-orientation of $(H, \sigma)$, which is a contradiction. This proves the claim. □

Now by Claim 2, each leaf block of $H$ is a $K_1^{-p}$. Let $t$ be the number of leaf blocks and $u_1, \ldots, u_t$ be the $t$ vertices of the leaf blocks. Let $G^*$ be a new signed graph obtained from $H$ by identifying $u_1, u_2, \ldots, u_t$ into a new vertex $u^*$. Then $G^*$ is bridgeless and is modulo-$(2p+1)$-orientable. By Theorem 4.1, $G^*$ admits a modulo 3-orientation. In the modulo 3-orientation of $G^*$, we split $u^*$ back to $u_1, u_2, \ldots, u_t$. Since $p \geq 3$, we can reverse the direction of some negative loops adjacent to $u_i$ for each $1 \leq i \leq t$ to obtain a modulo 3-orientation of $(H, \sigma)$. ■

Next, we will prove Theorem D. We first prove the following lemma.

**Lemma 4.4** *Let $(H, \sigma)$ be a modulo-5-orientable signed graph with exactly one bridge $e_0$. Then each of the following holds.*
*(i) $(H, \sigma)$ admits a 4-NZF $f_1$ such that $f_1(e) \in \{-3, 2\}$ for each edge $e \in E(H)$ and $f_1(e_0) = 2$.*
*(ii) If one of the two components of $H - e_0$ is a $K_1^{-2}$, then $(H, \sigma)$ admits a 5-NZF $f_2$ such that $f_2(e_0) = 4$.*

**Proof.** Let $(G, \sigma') = anti(H, \sigma)$ be the signed graph defined in Construction 3.2. Note that $G$ contains precisely one bridge, which is corresponding to the edge $e_0$ in $H$.

(i) By Lemma 4.2, $(G, \sigma')$ has an integer-valued $\frac{5}{2}$-flow $f_1$ such that $f_1(e) = -3$ if $e$ is in a 2-factor of $G$, and $f_1(e) = 2$ otherwise. Since $e_0$ is a bridge, it does not belong to a 2-factor of $G$, and hence $f_1(e_0) = 2$. By Proposition 3.3, $(H, \sigma)$ is obtained from $G$ by contracting a set of positive edges, and thus the flow $f_1$ is preserved in $(H, \sigma)$. Hence we actually obtain a 4-NZF $f_1$ of $(H, \sigma)$ such that $f_1(e) \in \{2, -3\}$ for each edge $e \in E(H)$ and $f_1(e_0) = 2$.

(ii) Denote $e_0 = xy$. Let $H_1$ and $H_2$ be the two components of $G - e_0$ where $H_2 = K_1^{-2}$, $x \in V(H_1)$ and $y \in V(H_2)$. By Proposition 2.2, $H_1$ is unbalanced since $H$ is modulo-5-orientable, and thus $H_1$ contains an unbalanced circuit.

First, suppose that there is an unbalanced circuit containing $x$. Then $C$ together with $e_0 = xy$ and a negative loop in $H_2$ forms a long barbell, which has a characteristic 3-flow $g_1$ such that $g_1(e_0) = 2$ and $g_1(e) \in \{\pm 1\}$ otherwise. Then $f_2 = f_1 + g_1$ is a 5-NZF with $f_2(e_0) = f_1(e_0) + g_1(e_0) = 4$.

Next, suppose that there is no unbalanced circuit containing $x$. Let $C$ be an unbalanced circuit $C$ in $H_1$. Then $C$ does not contain $x$. Since $H_1$ is bridgeless, by Menger's Theorem there are two edge-disjoint paths from $x$ to $C$, $P_1$ and $P_2$. We choose a pair of paths $P_1, P_2$ such that $|E(P_1)| + |E(P_2)|$ is the minimum. Denote by $u_1$ and $u_2$ be the other endvertices of $P_1$ and $P_2$ respetively. Then $u_1, u_2 \in V(C)$. If $u_1 \neq u_2$, let $P'$ be the $(u_1, u_2)$-segement of $C$ such that in $E(P_1) \cup E(P_2) \cup E(P')$ the circuit containing $E(P')$ is unbalanced; if $u_1 = u_2$, let $P' = C$. Therefore by the minimality of $|E(P_1)| + |E(P_2)|$, $E(P_1) \cup E(P_2) \cup E(P')$ consists of a chain of circuits $C_1, C_2, \ldots, C_s$ such that $x \in C_1$, $u_1 \in C_s$ and $|V(C_i) \cap V(C_{i+1})| = 1$ for each $i = 1, \ldots, s-1$ (see Figure 4 for an illustration of $H_1$ and $C$). Since $x$ is not contained in any unbalanced circuit of $H_1$, $C_1$ is balanced. Note that by the choice of $C_s$, $C_s$ is unbalanced. Let $t$ be the smallest integer $j$ such that $C_j$ is unbalanced. Then it is easy to see that the graph consists of $C_1, \ldots, C_t$ together with $e_0 = xy$ and a negative loop in $H_1$ has a 3-flow $g_2$ such that $g_2(e_0) = 2$ and $g_2(e) \in \{\pm 1\}$ otherwise. Therefore $f_2 = f_1 + g_2$ is a 5-NZF with $f_2(e_0) = f_1(e_0) + g_1(e_0) = 4$, as desired. ■

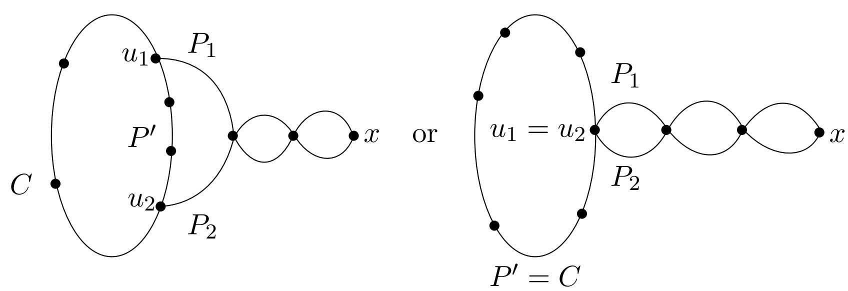



Figure 4: The structure of $H_1$ in the proof of Lemma 4.4.

Now we are ready to prove Theorem D.

**Proof of Theorem D.** The proof applies similar ideas to those of Theorem C. Let $(H, \sigma)$ be a counterexample of Theorem D with $|V(H)| + |E(H)|$ minimized. Since every modulo-5-orientable signed graph with at most one bridge has a 4-NZF by Corollary 4.3-(ii). Thus $H$ contains at least two bridges since $(H, \sigma)$ is a counterexample. As before, we assume each bridge is a positive edge by applying possibly some switching operations.

**Claim 3** *Every leaf block of $H$ is a $K_1^{-2}$.*

*Proof.* Suppose to the contrary that $H_1$ is a leaf block of $H$ and $H_1 \neq K_1^{-2}$. Let $e_0$ be the bridge adjacent to $H_1$. Denote by $H_2$ the other component of $G - e_0$. Let $G_i$ be the graph obtained from $H$ by replacing $H_i$ with a $K_1^{-2}$ for each $i = 1, 2$. Thus both $G_1$ and $G_2$ are modulo-5-orientable. Since $|V(G_1)| + |E(G_1)| < |V(H)| + |E(H)|$, by the the minimality of $H$, $G_1$ has a 5-NZF $f_1$. Let $e_1, e_2$ be the two negative loops adjacent to $e_0$ in $G_1$. Since $f_1$ is a 5-NZF and $e_0$ is a bridge of $G_1$, by Lemma 2.3, $f_1(e_0)$ is even, i.e., $f_1(xy) \in \{\pm 2, \pm 4\}$. Note that $G_2$ contains exactly one bridge. If $f_1(e_0) \in \{\pm 2\}$, we apply Lemma 4.4-(i), with possibly negating flow values of each edge, to obtain a 5-NZF $f_2$ of $G_2$ such that $f_2(e_0) = f_1(e_0) \in \{\pm 2\}$. If $f_1(e_0) \in \{\pm 4\}$, we apply Lemma 4.4-(ii), with possibly negating flow values of each edge, to obtain a 5-NZF $f_2$ of $G_2$ such that $f_2(e_0) = f_1(e_0) \in \{\pm 4\}$. Then in each case we combine those flows together to obtain a 5-NZF of $H$, a contradiction. This proves the claim. $\square$

By Claim 3, each leaf block of $H$ is a $K_1^{-2}$. Let $t$ be the number of leaf blocks and $u_1, \dots, u_t$ be the $t$ vertices of the leaf blocks. Let $u_i'$ be the neighbor of $u_i$ for each $i = 1, \dots, t$. Let $G^*$ be the new signed graph obtained from $H$ by identifying $u_1, u_2, \dots, u_t$ into a new vertex $u^*$. Then $G^*$ is bridgeless and is modulo-5-orientable. By Construction 3.2 and Proposition 3.3, $(G', \sigma') = anti(G^*, \sigma)$ has a modulo 5-orientation and $G'$ is bridgeless. By Theorem 4.1, $G'$ has a 4-NZF $f'$ and a 2-factor $M$ such that $f'(e) = 3$ if $e \in M$ and $f'(e) = -2$ otherwise.

We are going to obtain a contradiction by finding a 5-NZF of $(H, \sigma)$ from $f'$ in the following.

First, we modify the flow values of $u_i u_i'$ to be an even number in $\{2, 4\}$.

Let $M' \subset M$ be the set of circuits in $M$ containing at least one edge $u_i u_i'$ where $u_i u_i'$ is corresponding to the bridge in $H$ connecting a leaf block. By Proposition 2.4, for each circuit $C \in M'$, there is a vertex $u_i \in C$ and an edge weight $f_C : E(G^*) \to \{0, 1, -1\}$ with $supp(f_C) = E(C)$ such that $\partial f_C(x) = 0$ for each vertex $x \neq u_i$. Let $g = f' + \sum_{C \in M'} f_C$. Then $\partial g(x) = 0$ if $x \notin \{u_1, \dots, u_t\}$ and $g(e) \in \{-2, 2, 3, 4\}$ for each edge $e$ in $G^*$. In particular, $g(u_i u_i') \in \{2, 4\}$ for each $i = 1, \dots, t$.

Second, we further modify $g$ to obtain a 5-NZF of $(H, \sigma)$ by reassigning flows values to the negative loops adjacent to each $u_i$.

For each $i \in \{1, \dots, t\}$, we have $g(u_i u_i') = 2a$ where $a \in \{1, 2\}$. Without loss of generality, we assume that the half edge of $u_i u_i'$ with end $u_i$ is oriented toward $u_i$. We first orient the two negative loops such that one is a sink and the other one is a source. Then assign the flow values $a + 1$ and 1 to the sink and the source respetively.

In this way we extend $g$ to be a 5-NZF of $(H, \sigma)$, a contradiction. This completes the proof of the theorem. ■

## 5 The differences among modulo orientations, integer-valued and real-valued circular flows

Let $k, d$ be two integers with $k \geq 2d > 0$. It is known from [9, 13] that for an ordinary graph, it has a real-valued $\frac{k}{d}$-flow if and only if it has an integer-valued $\frac{k}{d}$-flow. Lu et al. [19] showed the following interesting result about circular flows of signed graphs.

**Lemma 5.1** *([19]) Let $k, d$ be two integers with $k \geq 2d > 0$. If $(G, \sigma)$ admits a real-valued $\frac{k}{d}$-flow, then it admits a real-valued $\frac{k}{d}$-flow $f$ such that $|f(e)| \in \{d, d+\frac{1}{2}, d+\frac{2}{2}, \ldots, k-d-\frac{1}{2}, k-d\}$.*

Lemma 5.1 implies the following relation between real-valued circular flows and integer-valued circular flows.

**Proposition 5.2** *Let $(G, \sigma)$ be a signed graph. Then $(G, \sigma)$ has a real-valued $\frac{k}{d}$-flow if and only if it has an integer-valued $\frac{2k}{2d}$-flow.*

**Proof.** If $f$ is an integer-valued $\frac{2k}{2d}$-flow, then $d \leq |\frac{1}{2}f(e)| \leq k-d$ and thus $\frac{1}{2}f(e)$ is a real-valued $\frac{k}{d}$-flow. This proves the sufficiency.

Now we show the necessity. Assume that $(G, \sigma)$ has a real-valued $\frac{k}{d}$-flow. Then by Lemma 5.1, $(G, \sigma)$ has a real-valued $\frac{k}{d}$-flow $(\tau, f)$ such that $|f(e)| \in \{d, d+\frac{1}{2}, d+\frac{2}{2}, \ldots, k-d-\frac{1}{2}, k-d\}$. Thus $|2f(e)| \in \{2d, 2d+1, \ldots, 2k-2d\}$. Therefore $2f$ is an integer-valued $\frac{2k}{2d}$-flow. ■

In fact, there are many signed graphs which have a real-valued $\frac{k}{d}$-flow but no integer-valued $\frac{k}{d}$-flow (see [14, 19, 24] and Proposition 5.3 below).

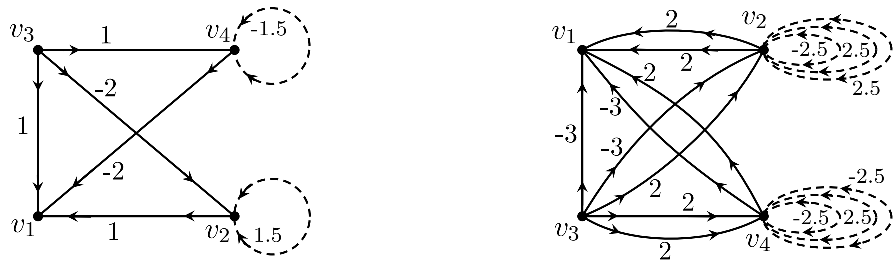
**Remark 3.** By Proposition 5.2, Conjecture 1.6 is equivalent to that every real-valued $\frac{k}{d}$-flow admissible signed graph admits a $\lceil \frac{k}{d} \rceil$-NZF (which is the original form in [23]).

In the following, for each integer $p \geq 1$, we will present a $2p$-edge-connected signed graph $G_p$ which shows that the equivalence of **(I)** and **(III)** and the equivalence of **(II)** and **(III)** both fail.

Let $C_4 = v_1v_2v_3v_4v_1$ be a circuit of length 4 and $pC_4 + v_1v_3$ be the graph obtained by replacing every edge in $C_4$ with $p$ parallel edges and then adding one edge $v_1v_3$ (the multiplicity of $v_1v_3$ is one).

Let $(G_p, \sigma)$ be the signed graph obtained from $pC_4 + v_1v_3$ by adding $(2p-1)$ negative loops at each of $v_2$ and $v_4$. An illustration for $p = 1, 2$ is shown in Figure 5.

**Proposition 5.3** *Let $p \geq 1$ be a positive integer. Then*
*(1) $(G_p, \sigma)$ admits a real-valued $\frac{2p+1}{p}$-flow.*




(1) $G_1$ when $p = 1$ (2) $G_2$ when $p = 2$

Figure 5: Graphs with real-valued $\frac{2p+1}{p}$-flow but no integer-valued $\frac{2p+1}{p}$-flow for $p = 1, 2$.

*(2) $(G_p, \sigma)$ does not admit an integer-valued $\frac{2p+1}{p}$-flow.*
*(3) $(G_p, \sigma)$ does not admit a modulo $(2p + 1)$-orientation.*
*(4) In particular, when $p = 1$, $(G_1, \sigma)$ admits an integer-valued $\frac{6}{2}$-flow but no integer-valued $\frac{3}{1}$-flow.*

**Proof.** It is clear that $G_p$ is $2p$-edge-connected. By Theorem A, (3) and (2) are equivalent. If (1) holds, then by Proposition 5.2 $(G_1, \sigma)$ has an integer-valued $\frac{6}{2}$-flow in (4). Hence (4) follows from (1) and (2). Therefore we only need to show (1) and (3).

We first prove (1) by finding a real-valued $\frac{2p+1}{p}$-flow. Let $t = p$ if $p$ is even, and $t = p-1$ otherwise.

We first define the orientation of $G_p$: all half edges incident with the end $v_1$ are oriented toward it and all half edges incident with the end $v_3$ are oriented away from it; all negative loops are oriented as sources.

Then the flow $f$ is defined as follows:

(i) $f(e) = p$ for each (parallel) edge $e$ between $v_1$ and $v_2$ and between $v_3$ and $v_4$.

(ii) Among $2p - 1$ negative loops incident with $v_2$, $2p - 1 - \frac{t}{2}$ loops have flow values $p + \frac{1}{2}$ and the remaining $\frac{t}{2}$ loops have flow values $-(p + \frac{1}{2})$.

(iii) Among $2p - 1$ negative loops incident with $v_4$, $2p - 1 - \frac{t}{2}$ loops have flow values $-(p + \frac{1}{2})$ and the remaining $\frac{t}{2}$ loops have flow values $p + \frac{1}{2}$.

(iv) If $p$ is odd, $f(v_1v_3) = p$ and $f(e) = -(p + 1)$ if $e \in [v_1, v_4] \cup [v_2, v_3]$. If $p$ is even, $f(v_1v_3) = -(p + 1)$, $f(e_1) = f(e_2) = p$ where $e_1$ is an edge in $[v_2, v_3]$ and $e_2$ is an edge in $[v_1, v_4]$, and $f(e) = -(p + 1)$ for each $e \in [v_2, v_3] \cup [v_1, v_4] \setminus \{e_1, e_2\}$.

One can easily check that $(f, \tau)$ is a real-valued $\frac{2p+1}{p}$-flow. This proves (1).

Next we show (3). Suppose to the contrary that $G_p$ has a modulo-$(2p + 1)$-orientation $\tau$. Since $d_{G_p}(v_1) = d_{G_p}(v_3) = 2p + 1$ and $v_1$ and $v_2$ are adjacent, we may assume that in $\tau$, all half edges incident with the end $v_1$ are oriented out and all half edges incident with the end $v_3$ are oriented in. Therefore exactly half of negative loops incident with $v_2$ must be oriented in and the other half must be oriented out. This is impossible since there are $(2p-1)$ negative loops incident with $v_2$. This proves (3). ■

We would like to point out that such signed graphs $(G_p, \sigma)$ can be modified to be $(2p+1)$-edge-connected as well. However, we are not aware of any such examples with higher edge connectivity.

Note that Theorems C does not include the case when $p = 2$. We propose the following conjecture.

**Conjecture 5.4** *Every modulo-5-orientable signed graph is modulo-3-orientable.*

Clearly, Conjecture 5.4 implies that every bridgeless modulo-5-orientable signed graph has a 3-NZF. By Corollary 4.3-(ii), every bridgeless modulo-5-orientable signed graph has a 4-NZF with flow values in $\{\pm 2, \pm 3\}$. Theorem D shows that every modulo-5-orientable signed graph admits a 5-NZF with flow values in $\{\pm 1, \pm 2, \pm 3, \pm 4\}$, and perhaps this could be strengthened to a special 5-NZF to prove Conjecture 5.4 provided that the values $\{\pm 3\}$ are forbidden. Also by Lemma 4.9 in [5] and Theorem 3.1, every odd-5-edge-connected modulo-5-orientable signed graph is modulo-3-orientable and thus admits a 3-NZF. Those observations provide some evidence to support Conjecture 5.4.

For ordinary graphs, Propositions 1.3 and 1.5 imply the following monotonicity of modulo orientations.

**Proposition 5.5** ([9, 13]) *Let $G$ be an ordinary graph. If $G$ has a modulo $(2p+1)$-orientation for some $p \geq 1$, then it has a modulo $(2p'+1)$-orientation for each integer $p'$ with $1 \leq p' \leq p$.*

It is unknown whether Proposition 5.5 remains true for signed graphs, and we can show that it is true whenever $p - p'$ is even for bridgeless signed graphs.

**Proposition 5.6** *Let $p$ and $p'$ be two positive integers with $p > p'$ and $p - p'$ even. If $G$ is bridgeless and $(G, \sigma)$ has a modulo $(2p+1)$-orientation, then $(G, \sigma)$ has a modulo $(2p'+1)$-orientation.*

**Proof.** It is sufficient to show the case when $p' = p - 2$ and $(G, \sigma)$ is $(2p+1)$-regular by Proposition 3.3. Let $\tau$ be a modulo $(2p+1)$-orientation of $(G, \sigma)$. Since $p' = p - 2 \geq 1$, we have $p \geq 3$. Thus $2p - 1 \geq \frac{2p+1}{3}$. Hence by Theorem 3.4, $G$ has a $(2p-1)$-factor, whose complement is a 2-factor, denoted by $M$. One may obtain a modulo-$(2(p-2)+1)$-orientation by reversing the directions of all edges in $M$. ■

By Theorem A, an equivalent form of Proposition 5.6 says that for positive integers $p, p'$ with $p > p'$ and $p - p'$ even, if $G$ is bridgeless and $(G, \sigma)$ admits an integer-valued $\frac{2p+1}{p}$-flow, then $(G,\sigma)$ admits an integer-valued $\frac{2p'+1}{p'}$-flow as well. However, Proposition 5.6 does not completely solve the monotonicity of modulo orientations and circular flows. We conclude the paper with the following problem.

**Problem 5.7** *Let $p \geq 2$ be an integer. Is it true that for any integer $p'$ with $1 \leq p' < p$, if $(G,\sigma)$ is modulo-$(2p+1)$-orientable, then it is also modulo-$(2p'+1)$-orientable?*

Conjecture 5.4 suggests a positive answer to Problem 5.7 for $p = 2$, but we are not sure in general.

## References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