Restricted intersecting families on simplicial complex

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Abstract

Chvátal's conjecture on the intersecting family of the faces of the simplicial complex is a long-standing problem in combinatorics. Snevily gave an affirmative answer to this conjecture for near-cone complex. Woodroofe gave Erdős-Ko-Rado type theorem for near-cone complex by using algebraic shift method. Motivated by these results, we concern with the restricted intersecting family for the simplicial complex. First, we give an upper bound for the cardinality of the restricted intersecting family of the faces of the simplicial complex, which is a generalization of Frankl-Wilson theorem. Furthermore, we prove that if $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ is a set of s positive integers, suppose that $\triangle$ is a near-cone simplicial complex with an apex vertex v and $\mathcal{F} = \{F_1, \ldots, F_m\}$ is a family of the faces of $\triangle$ such that $|F_i \cap F_j| \in \mathcal{L}$ for every $1 \leq i \neq j \leq m$, then

$$m \leq \sum_{i=-1}^{s-1} f_i(\mathrm{link}_{\triangle}(v)),$$

which generalizes Snevily's two theorems. We also propose a conjecture that this upper bound holds for all simplicial complexes. Finally, applying our theorems to certain simplicial complex, we can deduce the upper bounds for the cardinalities of the restricted intersecting families of the independent set of the graph, the set partition, the r-separated sets and the King Arthur and his Knight Table.

Keywords: simplicial complex, Frankl-Wilson theorem, Chvátal's conjecture, near-cone complex, Erdős-Ko-Rado theorem

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1 Introduction

The objective of this paper is to give upper bounds for the restricted intersecting families of the faces of the simplicial complex.

Let us recall some notions first. Denote X for the set $[n] = \{1, 2, \ldots, n\}$. A family $\mathcal{F}$ of subsets of $X = [n]$ is called *intersecting* if every pair of distinct subsets $E, F \in \mathcal{F}$ have a nonempty intersection. Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s nonnegative integers. A family $\mathcal{F}$ of subsets of $[n]$ is called *$\mathcal{L}$-intersecting* if $|E \cap F| \in \mathcal{L}$ for every pair of distinct subsets $E, F \in \mathcal{F}$. A family $\mathcal{F}$ is k-uniform if it is a collection of k-subsets of X. Thus, a k-uniform intersecting family is $\mathcal{L}$-intersecting for $\mathcal{L} = \{1, 2, \ldots, k-1\}$.

Erdős-Ko-Rado theorem and Frankl-Wilson theorem are two celebrated theorems in extremal set theory. In 1961, Erdős, Ko and Rado [7] prove that if $n \geq 2k$ and $\mathcal{F}$ is a k-uniform intersecting family of $[n]$, then

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

A recent work for generalizing the Erdős-Ko- Rado Theorem, due to Holroyd and Talbot [13], defines the Erdős-Ko-Rado property for a graph in terms of the graph's independent sets. Since the family of all independent sets of a graph forms a simplicial complex, Woodroofe [23] used the algebraic shift method to generalize Erdős-Ko-Rado property to near-cone complex. Borg [2] gave multi-level solution of the simplicial complex generalization of the conjecture proposed by Holroyd and Talbot. Recently, Olarte, Santos, Spreer and Stump [16] proved the family of facets of a pure simplicial complex of dimension up to three satisfies the Erdős-Ko-Rado property whenever it is flag and has no boundary ridges.

In 1981, Frankl and Wilson [11] obtained the following tight upper bound for the restricted intersecting family.

Theorem 1.1 (Theorem 11, [11]) *Let $\mathcal{L}$ be an ordered set of s distinct nonnegative integers less than n. If $\mathcal{F}$ is an $\mathcal{L}$-intersecting family of subsets of $[n]$, then*

$$|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}.$$

In 1991, Alon, Babai, and Suzuki [1] considered the problem of how large a set system with specific intersection sizes and subset sizes can be.

Theorem 1.2 (Alon-Babai-Suzuki) *let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s nonnegative integers and $\mathcal{K} = \{k_1, k_2 \ldots, k_r\}$ be a set of integers satisfying $k_i > s - r$ for every i. Let $\mathcal{F}$ be an $\mathcal{L}$-intersecting family of subsets of $[n]$ such that $|F| \in \mathcal{K}$ for every $F \in \mathcal{F}$. Then*

$$|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}.$$

For $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ of s positive integers, Snevily [19] proposed the following conjecture and proved it by himself in 2003 [20].

Theorem 1.3 (Snevily) *Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ a set of s positive integers. If $\mathcal{F}$ is an $\mathcal{L}$-intersecting family of subsets of $[n]$, then*

$$|\mathcal{F}| \leq \sum_{i=0}^{s} \binom{n-1}{i}.$$

These results have been considered to be extended to simplicial complexes. Recall that by a *simplicial complex* we mean a collection of sets $\triangle$ with the property that if $A \in \triangle$ and $B \subseteq A$ then $B \in \triangle$. We call the elements of $\triangle$ the *faces* of $\triangle$. For $S \in \triangle$, the dimension of S is $|S| - 1$. The *dimension* of $\triangle$ is $\dim(\triangle) \stackrel{\text{def}}{=} \max\{|A| - 1 : A \in \triangle\}$. Given a simplicial complex $\triangle$ of dimension $d-1$ we let $f_{i-1}(\triangle) \stackrel{\text{def}}{=} |\{A \in \triangle : |A| = i\}|$, for $i = 0, 1, \ldots, d$, and call $\mathbf{f}(\triangle) \stackrel{\text{def}}{=} (f_0(\triangle), f_1(\triangle), \ldots, f_{d-1}(\triangle),)$ the *f-vector* of $\triangle$. For all $\triangle$, $f_{-1} = 1$. We call the 0-dimensional faces the vertices of $\triangle$. We call F is a *facet* of $\triangle$ if there do not exist $F' \in \triangle$ such that $F \subsetneq F'$. For a vertex v of $\triangle$, denote the *link* of v in $\triangle$ to be

$$\mathrm{link}_{\triangle}(v) := \{E \colon E \cup v \in \triangle,\ v \notin E\},$$

that is it is the star at v, with v itself removed from each set thereof. Obviously, $\mathrm{link}_{\triangle}(v)$ is also a simplicial complex. A simplicial complex $\triangle$ is called a *near-cone* with respect to an *apex vertex* v if for every face F, the set $(F \backslash \{w\}) \cup \{v\}$ is also a face for each vertex $w \in F$.

A long-standing problem in extremal set theory is the following **Chvátal's conjecture** [5].

Conjecture 1.4 (Chvátal's conjecture, [5]) *Let $\mathcal{F}$ be any family of subsets of $[n]$ such that $S \in \mathcal{F}$, $T \subset S$ implies $T \in \mathcal{F}$, then some largest intersecting subfamily of $\mathcal{F}$ has the form*

$$\{A \in \mathcal{F} \colon x \in A\} \text{ for some } x \in [n].$$

Usually, we consider the following version of Chvátal's conjecture. If $\mathcal{F}$ is an intersecting family of the faces (of possibly differing dimensions), then

$$|\mathcal{F}| \leq \max_{v \in V(\triangle)} \left(\sum_{r} f_r(\mathrm{link}_{\triangle}(v)) \right).$$

Erdős [6] mentioned that this conjecture is one of the combinatorial problems which he would most like to see solved and remarked "It is surprising that this attractive conjecture is probably very difficult". In 1992, Snevily [18] showed the positivity of Chvátal conjecture when $\triangle$ is a near-cone complex, which is the most exciting improvement for this conjecture.

Theorem 1.5 (Snevily) *If $\mathcal{F}$ is an intersecting family of the faces of a near-cone complex, then*

$$|\mathcal{F}| \leq \max_{v \in V(\triangle)} \left(\sum_r f_r(link_\triangle(v)) \right).$$

In this paper, we will give the simplicial complex versions of Frankl-Wilson type theorem and Snevily's theorem. This paper is organized as follow. In Section 2, we shall give simplicial complex version of Frankl-Wilson theorem [11] and Alon-Suzuki-Babai theorem [1]. In Section 3, we will give an upper bound for the restricted intersecting family of faces of a near cone complex, which is a generalization of Snevily's two theorems. We also propose a general conjecture. Finally, applying our theorems to some special simplicial complexes, we shall deduce some upper bounds for the cardinality of the restricted intersecting families on independent set of the graph, the set partition, the r-separated sets and the King Arthur and his Knight Table.

2 Frankl-Wilson type theorem for simplicial complex

In this section, we will show a Frankl-Wilson type theorem for the simpicial complex. First, let us introduce some notations which will be used in the proof. Denote $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ a vector of n variables with each variable x_j taking values 0 or 1. A polynomial $p(x)$ in variables x_i, $1 \leq i \leq n$, is called *multilinear* if the power of each variable x_i in each term is at most one. Clearly, if each variable x_i takes only the values 0 and 1, then any polynomial in variables x_i, $1 \leq i \leq n$, is multilinear since $x_i^k = x_i$ for $k \in \mathbb{N}^+$. For a subset F of $[n]$, that is, let *characteristic vector* of F to be the vector $u_F = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$ with $u_j = 1$ if $j \in F$ and $u_j = 0$ otherwise.

As warming up, we first state a simplicial complex version of Frankl-Wilson theorem. Note that we just make some slight changes in the proof due to Frankl and Wilson [11]. .

Theorem 2.1 *Let $\mathcal{L}$ be an ordered set of s distinct non-negative integers less that n. Suppose that $\triangle$ is a simplicial complex and $\mathcal{F} = \{F_1, \ldots, F_m\}$ is a family of faces of $\triangle$ such that $|F_i \cap F_j| \in \mathcal{L}$ for every $1 \leq i, j \leq m$. Then*

$$m \leq \sum_{i=-1}^{s-1} f_i(\triangle).$$

Proof. For $1 \leq i \leq m$, define

$$\phi_{F_i}(\mathbf{x}) = \prod_{l_j < |F_i|} (\sum_{t \in F_i} x_t - l_j),$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ with each x_j taking values 0 or 1. Recall that u_{F_j} be the characteristic vector of F_j. Then we have $\phi_{F_i}(u_{F_j}) \neq 0$ if and only if $F_i \subseteq F_j$ for every pair $1 \leq i \neq j \leq m$.

Now we proceed to show the polynomials $\{\phi_{F_1}(\mathbf{x}), \phi_{F_2}(\mathbf{x}), \ldots, \phi_{F_m}(\mathbf{x})\}$ are linearly independent. Suppose that we have the following linear combination of these polynomials that equals zero

$$\sum_{i=1}^{m} \alpha_i \phi_{F_i}(\mathbf{x}) = 0. \tag{2.1}$$

If $\alpha_1, \alpha_2, \ldots, \alpha_m$ are not all zero, then let F_{i_0} be the minimal face for which $\alpha_{F_{i_0}} \neq 0$. It means that for each $F \subsetneq F_{i_0}$, $\alpha_F = 0$. On the other hand, for each $F \in \mathcal{F}$ satisfying that F_{i_0} do not contain F, we have that $|F \cap F_{i_0}| \in \mathcal{L}$ and $|F \cap F_{i_0}| < |F|$. It follows that $\phi_F(u_{F_{i_0}}) = 0$. Thus, from (2.1) we get that $\alpha_{F_{i_0}} \phi_{F_{i_0}}(u_{F_{i_0}}) = 0$. Since $\phi_{F_{i_0}}(u_{F_{i_0}}) \neq 0$, we have $\alpha_{Fi_0} = 0$, it is a contradiction. Hence the polynomials $\{\phi_{F_1}(\mathbf{x}), \phi_{F_2}(\mathbf{x}), \ldots, \phi_{F_m}(\mathbf{x})\}$ are linearly independent.

By the definition of simplicial complex and the definition of $\phi_{F_i}(\mathbf{x})$, we see that the monomial $x_{j_1} x_{j_2} \cdots x_{j_t}$ appears in $\phi_{F_i}(x)$ only if $\{j_1, j_2, \ldots, j_t\}$ is a face of $\triangle$. It follows that

$$m \leq \sum_{i=-1}^{s-1} f_i(\triangle)$$

This completes the proof. ∎

In fact, using the same approach, we can also deduce a simpicial complex version of Alon-Babai-Suzuki-type theorem.

Theorem 2.2 *let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s nonnegative integers and $\mathcal{K} = \{k_1, k_2 \ldots, k_r\}$ be a set of integers satisfying* $\min k_i > \max l_j$ *for $1 \leq i \leq r$ and $1 \leq j \leq s$. Suppose that $\triangle$ is a simplicial complex and $\mathcal{F} = \{F_1, \ldots, F_m\}$ is a family of faces of $\triangle$ such that $|F_i \cap F_j| \in \mathcal{L}$ for every $1 \leq i \neq j \leq m$ and $|F| \in \mathcal{K}$ for every $F \in \mathcal{F}$. Then*

$$m \leq \sum_{i=s-r}^{s-1} f_i(\triangle).$$

Proof. For $1 \leq i \leq m$, define

$$\varphi_{F_i}(\mathbf{x}) = \prod_{j=1}^{s} (\sum_{t \in F_i} x_t - l_j),$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ with each x_j taking values 0 or 1. From the condition $\min k_i > \max l_j$ for $1 \leq i \leq r$ and $1 \leq j \leq s$, we have $\mathcal{L} \cap \mathcal{K} = \emptyset$. It implies that $\varphi_{F_i}(u_{F_i}) \neq 0$ for $1 \leq i \leq m$ and $\varphi_{F_i}(u_{F_j}) = 0$ for every pair $1 \leq i \neq j \leq m$.

Let Q be the family of the faces I of $\triangle$ with at most $s-r$ vertices, define

$$x_I = \prod_{i\in I} x_i \quad \text{and} \quad p_I(\mathbf{x}) = x_I P(\mathbf{x}),$$

where $P(\mathbf{x})$ is the sum of all the monomials $x_{i_1}x_{i_2}\cdots x_{i_t}$ in the expansion of $\prod_{j=1}^{r}\left(\sum_{i=1}^{n} x_i - k_j\right)$, such that $\{x_{i_1}, x_{i_2}, \ldots, x_{i_t}\}$ are the faces of $\triangle$. It means that for each $F_i \in \mathcal{F}$, since the intersection of I and F_i is still a face of $\triangle$, we have $p_I(u_{F_i}) = 0$.

We aim to prove the polynomials φ_{F_i} and p_I are linearly independent. Assume that we have a linear combination of these polynomials that equals zero:

$$\sum_{i}^{m} \alpha_i \varphi_{F_i}(\mathbf{x}) + \sum_{I\in Q} \beta_I p_I(\mathbf{x}) = 0 \tag{2.2}$$

We need to prove that α_i and β_I are all zero.

For any $1 \le i \le m$, subscribing $x = u_{F_i}$ in (2.2) leads to $p_I(u_{F_i}) = 0$ for all $I \in Q$ and $\varphi_{F_j}(u_{F_i}) = 0$ for $i \ne j$. It follows that $\alpha_i \varphi_{F_i}(u_{F_i}) = 0$. Since $\varphi_{F_i}(u_{F_i}) \ne 0$, we have $\alpha_i = 0$. Thus (2.2) reduces to $\sum_{I\in Q} \beta_I p_I(\mathbf{x}) = 0$. It is easily seen that p_I are linearly independent. It implies that $\beta_I = 0$ for all $I \in Q$. Hence, we attain

$$m \le \sum_{i=s-r}^{s-1} f_i(\triangle).$$

∎

3 Near-cone complexes

In this section, we will study the restricted intersecting family on the near-cone complex and give an upper bound for the cardinality of the $\mathcal{L}$-intersecting family on the near-cone complex when $\mathcal{L}$ is a set of positive integers. Recall that a simplicial complex $\triangle$ is called a near-cone with respect to an apex vertex v if for every face F, the set $(F\backslash\{w\}) \cup \{v\}$ is also a face for each vertex $w \in F$. Let $X = \{x_1.x_2, \ldots, x_n\}$ be a collection of n integer variables. Following the notation in [20], we denote $\binom{X}{k}$ the sum of all k-term multilinear monomials from X and

$$\sum \binom{X}{k} = \sum_{x_{i_1}x_{i_2}\cdots x_{i_k} \in \binom{X}{k}} x_{i_1}x_{i_2}\cdots x_{i_k}.$$

Let $\binom{X}{0} = 1$. Define

$$g_{\mathcal{L}}(y) = \prod_{1\le i\le k} (y - l_i).$$

Since $g_{\mathcal{L}}(y)$ is a polynomial in y of degree k, we can rewrite it in the form $g_{\mathcal{L}}(y) = \sum_{h=0}^{k} c_h \binom{y}{h}$, where $c_0, c_1, \ldots, c_k$ are rational numbers independent of y, which we will call the *coefficients* of $\mathcal{L}$.

Let

$$g^*_{\mathcal{L}}(\mathbf{x}) = c_k \sum \binom{X}{k} + c_{k-1} \sum \binom{X}{k-1} + \cdots + c_0,$$

where the coefficients c_i are the coefficients of $\mathcal{L}$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. With each set $F_i \in \triangle$, we associate its characteristic vector $v_i = (v_{i_1}, v_{i_2}, \ldots, v_{i_n}) \in R^n$, where $v_{i_j} = 1$ if $j \in F_i$, and $v_{i_j} = 0$ otherwise. Note that $g^*_{\mathcal{L}}(v_i) = g_{\mathcal{L}}(|F_i|)$.

For each $F_i = (i_1, i_2, \ldots, i_t)$ which is a member of $\mathcal{F}$, let $F_i^* = \{x_{i_1}, x_{i_2}, \ldots, x_{i_t}\}$ be a collection of $|F_i|$ variables where $x_{i_j} \in F_i^*$ if and only if $i_j \in F_i$. Let $\binom{F_i^*}{k}$ $(k \geq 1)$ denote the set of all k-term multilinear monomials from F_i^* and $\binom{F_i^*}{0} = 1$.

Using the same coefficients as in $g^*_{\mathcal{L}}(x)$ define

$$g^*_{F_i}(\mathbf{x}) = c_k \sum \binom{F_i^*}{k} + c_{k-1} \sum \binom{F_i^*}{k-1} + \cdots + c_0.$$

Note that $g^*_{F_i}(v_i) = g_{\mathcal{L}}(v_i) = g_{\mathcal{L}}(|F_i|)$ and that $g^*_{F_i}(v_j) = g_{\mathcal{L}}(|F_i \cap F_j|) = 0$ for all $i \neq j$.

Before the statement of our result, let us recall that $f_{i-1}(\triangle)$ is the number of i-dimension faces of $\triangle$ and for vertex v of $\triangle$, $\mathrm{link}_{\triangle}(v) := \{E \colon E \cup v \in \triangle,\ v \notin E\}$. Now we proceed to prove our main result.

Theorem 3.1 *Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s positive integers. Suppose that $\triangle$ is a near-cone simplicial complex with an apex vertex v and $\mathcal{F} = \{F_1, \ldots, F_m\}$ is a family of the faces of $\triangle$ such that $|F_i \cap F_j| \in \mathcal{L}$ for every $1 \leq i \neq j \leq m$. Then*

$$m \leq \sum_{i=-1}^{s-1} f_i(link_{\triangle}(v)).$$

Proof. Let

$$\mathcal{F}_1 = \{F_i \colon F_i \in \mathcal{F} \text{ and } v \in F_i\}$$

and

$$\mathcal{F}_2 = \{F_i \colon F_i \in \mathcal{F},\ v \notin F_i \text{ and } F_i \cup v \in \triangle\}.$$

Then, denote $\mathcal{F}_3 = \mathcal{F} \backslash \{\mathcal{F}_1 \cup \mathcal{F}_2\}$, that is,

$$\mathcal{F}_3 = \{F_i \colon F_i \in \mathcal{F},\ v \notin F_i \text{ and } F_i \cup v \notin \triangle\}.$$

For $F_i \in \mathcal{F}_1$ or $F_i \in \mathcal{F}_3$, define

$$\phi_{F_i}(\mathbf{x}) = \prod_{l_k < |F_i|} (\sum_{t \in F_i} x_t - l_k),$$

and for $F_i \in \mathcal{F}_2$, define

$$\phi_{F_i}(\mathbf{x}) = g^*_{F_i}(x),$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ with each x_j taking values 0 or 1. Then we have $\phi_{F_i}(u_{F_j}) \neq 0$ if and only if $F_i \subseteq F_j$ for every pair $1 \leq i, j \leq m$. By the definition of $\phi_{F_i}(x)$, we see that each the monomial $x_{j_1}x_{j_2}\cdots x_{j_t}$ appearing in $\phi_{F_i}(\mathbf{x})$ satisfies that $(j_1, j_2, \ldots, j_t)$ is a face of $\triangle$.

Let Q be the family of the faces of $\triangle$ with dimension at most $s-1$ which contain the apex vertex v. For each $L \in Q$, define

$$q_L(\mathbf{x}) = (x_v - 1) \prod_{j \in L \setminus \{v\}} x_j.$$

Let H be the family of the faces of $\triangle$ with dimension at most $s-1$ satisfying that for each $R \in H$, we have that $v \notin R$ and $R \cup \{v\} \notin \triangle$. For each $R \in H$, define

$$h_R(\mathbf{x}) = \prod_{j \in R} x_j.$$

We aim to show these three polynomials $\phi_{F_i},\ q_L,\ h_R$ are linearly independent. Suppose that we have a linear combination of these polynomials that equals zero:

$$\sum_{i=1}^{m} \alpha_{F_i}\phi_{F_i}(\mathbf{x}) + \sum_{L \in Q} \beta_L q_L(\mathbf{x}) + \sum_{R \in H} \gamma_R h_R(\mathbf{x}) = 0. \tag{3.1}$$

We need to prove that α_{F_i}, β_L and γ_R are all zero.

Claim 1. $\gamma_R = 0$ for $R \in H$.

If not, there exists a face R_0 such that $\gamma_R \neq 0$. We consider the coefficient of the monomial $\prod\limits_{j \in R_0} x_j$ in (3.1). Since $\triangle$ is a near-cone complex with apex vertex v and $R_0 \cup \{v\}$ is not a face of $\triangle$, we claim that R_0 is a facet of $\triangle$. If not, assume that there exist $F \in \triangle$ such that $R_0 \subsetneq F$. If $v \in F$, then we find that $v \cup R_0 \in \triangle$, it is a contradiction. If $v \notin F$, since $\triangle$ is a near cone complex with v, we have that for any subset S of F, $S \cup \{v\}$ is a face of $\triangle$. Thus $v \cup R_0 \in \triangle$. It is a contradiction. Hence R_0 is a facet of $\triangle$. By the definition of $\phi_{F_i}(\mathbf{x})$ and $q_L(\mathbf{x})$, it is easily seen that the monomial $\prod\limits_{j \in R_0} x_j$ do not appear in $\phi_{F_i}(\mathbf{x})$ and $q_L(\mathbf{x})$. It follows that the monomial $\prod\limits_{j \in R_0} x_j$ only appear in $h_{R_0}(\mathbf{x})$. Thus the coefficient of $\prod\limits_{j \in R_0} x_j$ in (3.1) is γ_R. It implies that $\gamma_{R_0} = 0$, it is a contradiction. Hence $\gamma_R = 0$ for $R \in H$.

Claim 2. $\alpha_{F_i} = 0$ for $F_i \in \mathcal{F}_1$.

If not, let F_{i_0} be the minimal face for which $\alpha_{F_{i_0}} \neq 0$. It means that for each $F \subsetneq F_{i_0}$ and $F \in \mathcal{F}_1$, $\alpha_F = 0$. On the other hand, for each $F \in \mathcal{F}_1$ satisfying that F_{i_0} do not contain F, we have that $|F \cap F_{i_0}| \in \mathcal{L}$ and $|F \cap F_{i_0}| < F$. It follows that $\phi_F(u_{F_{i_0}})=0$ for $F\in\mathcal{F}_1$. For $F_i\in\mathcal{F}_2$, by the definition of $g^*_{F_i}(\mathbf{x})$, we have $\phi_{F_i}(u_{F_{i_0}})=0$. For $F_i\in\mathcal{F}_3$, it follows from the definition of $\mathcal{F}_3$ that F_{i_0} does not contain F_i. Thus $\phi_{F_i}(u_{F_{i_0}})=0$. At last, since $v\in F_{i_0}$, we have $q_L(\mathbf{x})=0$ for each $L\in G$. Summing up, we deduce from (3.1) that $\alpha_{F_{i_0}}\phi_{F_{i_0}}(u_{F_{i_0}})=0$. In view of $\phi_{F_{i_0}}(u_{F_{i_0}})\neq 0$, we arrive at that $\alpha_{F_{i_0}}=0$, it is a contradiction. Hence $\alpha_{F_i}=0$ for $F_i\in\mathcal{F}_1$.

Combining Claim 1 and Claim 2 reduces (3.1) to

$$\sum_{F_i\in\mathcal{F}_2\cup\mathcal{F}_3}\alpha_{F_i}\phi_{F_i}(\mathbf{x})+\sum_{L\in Q}\beta_L q_L(\mathbf{x})=0 \tag{3.2}$$

Claim 3. $\beta_L=0$ for $L\in Q$.

Rewrite (3.2) as

$$\left[\sum_{F_i\in\mathcal{F}_2\cup\mathcal{F}_3}\alpha_{F_i}\phi_{F_i}(\mathbf{x})+\sum_{L\in Q}\beta_L q'_L(\mathbf{x})\right]+x_v\left(\sum_{L\in Q}\beta_L q'_L(\mathbf{x})\right)=0, \tag{3.3}$$

where

$$q'_L(\mathbf{x})=\prod_{j\in L,\ j\neq v}x_j.$$

Notice that x_v does not appear in the first parentheses of equation (3.2). It follows that

$$\sum_{L\in Q}\beta_L q'_L(\mathbf{x})=0.$$

It is easily checked that the polynomials $q'_L(\mathbf{x})$ for $L\in Q$ are linearly independent. Therefore, we conclude that $\beta_L=0$ for $L\in Q$.

The above claims lead to the conclusion that we need only to show that $\phi_{F_i}(\mathbf{x})$ are linearly independent for $F_i\in\mathcal{F}_2\cup\mathcal{F}_3$.

Claim 4. $\alpha_{F_i}=0$ for $F_i\in\mathcal{F}_3$.

If not, let F_{i_0} be the minimal face for which $\alpha_{F_{i_0}}\neq 0$. Note that for each $F\in\mathcal{F}_3$ satisfying that F_{i_0} do not contain F, we have $\phi_F(u_{F_{i_0}})=0$. For $F_i\in\mathcal{F}_2$, by the definition of $g^*_{F_i}(\mathbf{x})$, we have $\phi_{F_i}(u_{F_{i_0}})=0$. Summing up, we obtain that $\alpha_{F_{i_0}}\phi_{F_{i_0}}(u_{F_{i_0}})=0$ and $\phi_{F_{i_0}}(u_{F_{i_0}})\neq 0$. It follows that $\alpha_{F_{i_0}}=0$, it is a contradiction. Hence $\alpha_{F_i}=0$ for $F_i\in\mathcal{F}_3$.

By the above argument, in order to prove the polynomials $\phi_{F_i}(\mathbf{x})$, $q_L(\mathbf{x})$ and $h_R(\mathbf{x})$ are linearly independent, we aim to show that $\phi_{F_i}(\mathbf{x})$ are linearly independent for $F_i\in\mathcal{F}_2$. In [20], the polynomials $g^*_{F_i}(\mathbf{x})$ have been proven to be linearly independent. Hence we conclude that the polynomials $\phi_{F_i}(\mathbf{x})$, $q_L(\mathbf{x})$ and $h_R(\mathbf{x})$ are linearly independent.

By the definition of simplicial complex and the definitions of the polynomials $\phi_{F_i}(\mathbf{x})$, $q_L(\mathbf{x})$ and $h_R(\mathbf{x})$, each monomial $x_{i_1}x_{i_2}\cdots x_{i_t}$ appearing in the above polynomials satisfies that $\{i_1, i_2, \ldots, i_t\}$ is a face of $\triangle$. Thus we have

$$m + |Q| + |H| \le \sum_{i=-1}^{s-1} f_i.$$

By the definition of Q and H, we have $|Q| = \sum_{i=-1}^{s-1} f_i'$ and $|H| = \sum_{i=-1}^{s-1} f_i''$, where f_i' and f_i'' denote the number of the i-dimensional faces which contain v and the number of the i-dimensional faces which do not contain v and unite v is not a face, respectively. Hence, we get

$$\begin{aligned} m &\le \sum_{i=-1}^{s-1} f_i - \sum_{i=-1}^{s-1} f_i' - \sum_{i=-1}^{s-1} f_i'' \\ &= \sum_{i=-1}^{s-1} f_i(\mathrm{link}_{\triangle}(v)). \end{aligned}$$

This completes the proof. ∎

Remark that the above theorem implies Theorem 1.3 when $\triangle$ is the family of all subsets of $[n]$ and Theorem 1.5 when $\mathcal{L} = \{1, 2, \ldots, d(\triangle)\}$.

Moreover, we propose a conjecture that Theorem 3.1 holds for all simplicial complexes.

Conjecture 3.2 *let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$. If $\mathcal{F} = \{F_1, \ldots, F_m\}$ is a family of the faces of $\triangle$ such that $|F_i \cap F_j| \in \mathcal{L}$ for every $1 \le i, j \le m$. Then*

$$|\mathcal{F}| \le \max_{v \in V(\triangle)} \left(\sum_{i=-1}^{s-1} f_i(link_{\triangle}(v)) \right).$$

Note that Conjecture 3.2 implies Chvátal's conjecture when $\mathcal{L} = \{1, 2, \ldots, d(\triangle)\}$.

4 Applications

In this section, we will apply our results to some special simplicial complexes and give upper bounds on cardinalities of certain intersecting families of independent sets of graphs.

4.1 Independece Complex

Hurlbert and Kamat [14] considered the Erdős-Ko-Rado type property of the independent set of the chordal and the bipartite graph. As known, the independent sets of a graph can be considered as a simplicial complex. Given a graph

$G = (V, E)$, follow Hurlbert and Kamat's notions, we denote the family of independent r-sets of V by $J^{(r)}(G)$ and the subfamily $\{A \in J^{(r)}(G),\ v \in A\}$ by $J_v^{(r)}(G)$. Applying Theorem 2.1, we get the Frankl-Wilson type theorem for the independent sets of a graph.

Theorem 4.1 *Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s nonnegative integers. Suppose that $\triangle$ is a simplicial complex and $\mathcal{F} = \{F_1, \ldots, F_m\}$ is a family of the independent sets of G such that $|F_i \cap F_j| \in \mathcal{L}$ for every $1 \leq i \neq j \leq m$. Then*

$$m \leq \sum_{i=-1}^{s-1} |J^{(i)}(G)|.$$

It is easily seen that if graph G has an isolated vertex v, then the independence complex of G is a near cone complex with apex vertex v. Note that $|J_v^{(r)}(G)|$ also enumerates the number of $(r-2)$-dimensional faces F of the independence complex of G satisfying that $v \notin F$ and $v \cup F$ is a face of the independence complex of G. Thus, by Theorem 3.1, we obtain the following theorem.

Theorem 4.2 *Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s positive integers. Suppose that $\mathcal{F} = \{F_1, \ldots, F_m\}$ is a family of the independent sets of a graph G with an isolated vertex v such that $|F_i \cap F_j| \in \mathcal{L}$ for every $1 \leq i \neq j \leq m$. Then*

$$m \leq \sum_{i=1}^{s} J_v^{(i)}(G).$$

4.2 Set Partition

Intersecting problems for the set partition have also received some attention. Recall that a *set partition* of $[n]$ is a collection of pairwise disjoint non-empty subsets (called *blocks*) of $[n]$ whose union is $[n]$. Let $\mathcal{B}(n)$ denote the set of all set partitions of $[n]$. Then $|\mathcal{B}(n)|$ is the nth Bell number B_n. A family $\mathcal{F} \subset \mathcal{B}(n)$ is said to be $\mathcal{L}$ -intersecting if any two elements of A have exactly l blocks in common, where $l \in \mathcal{L}$.

Denote P_k^n the set of all set partitions of $[n]$ with k blocks. Then $|P_k^n|$ is the Stirling number of the second kind, denoted by $S(n, k)$. Erdős and Székely [8] gave the following theorem.

Theorem 4.3 (Erdős and Székely) *Let $n \geq k \geq t \geq 1$. Suppose that $\mathcal{F} \subset P_k^n$ is $\{1, 2, \ldots, t\}$-intersecting. If $n \geq n_0(k, t)$, then $|\mathcal{F}| \leq |\mathcal{P}|$, where $\mathcal{P} = \{P \in P_k^n \colon \{1\}, \ldots, \{t\} \in P\}$.*

In 2008, Ku and Renshaw [15] obtained EKR type theorem for set partition.

Theorem 4.4 (Ku and Renshaw) *Let $n \geq 2$. Suppose $\mathcal{F} \subset \mathcal{B}_n$ is intersecting. Then $|\mathcal{F}| \leq B_{n-1}$ with equality if and only if $\mathcal{F}$ consists of all set partitions with a fixed singleton.*

Theorem 4.5 (Ku and Renshaw) *Let $t \geq 2$. Suppose $\mathcal{F} \subset \mathcal{B}_n$ is t-intersecting. Then there exists a positive number $n_0(t)$ such that $|\mathcal{F}| \leq B_{n,t}$ with equality if and only if $\mathcal{F}$ consists of all set partitions with t fixed singletons.*

From the structure of set partition, Brenti [4] constructed the following simplicial complex

$$\triangle_p = \{F \subset V\colon S \cap T = \emptyset \text{ for all } S, T \in F \text{ such that } S \neq T, \text{ and } \sum_{S \in F} |S| \leq n-1\},$$

where $V = \{S \subset [n-1]\colon 1 \leq |S| \leq n-1\}$. Note that each $k-1$-dimensional face of $\triangle_p$ is a set partition of $[n]$ with $k+1$ blocks. From the definition of $\triangle_p$, it is easy to see that if two partition F_1 and F_2 share s blocks, then the faces of $\triangle_p$ corresponding to F_1 and F_2 have s or $s-1$ same vertices. Hence, by Theorem 2.1 we get the following theorem.

Theorem 4.6 *Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s nonnegative integers. Suppose that $\mathcal{F} = \{F_1, \ldots, F_m\}$ is a family of the set partitions of $[n]$ such that any two elements of $\mathcal{F}$ share l_i blocks, $1 \leq i \leq s$. Then*

$$m \leq \sum_{i=0}^{s'} S(n, i),$$

where s' denotes the cardinality of the set $\{l_1, \ldots, l_s\} \cup \{l_1 - 1, \ldots, l_s - 1\}$.

Furthermore, if $\mathcal{L}$ consists of consecutive integers, Theorem 4.6 leads to the following two corollaries.

Corollary 4.7 *For $l \geq 1$, let $\mathcal{L} = \{l, l+1, \ldots, l+s-1\}$ be a set of s positive integers. Suppose that $\mathcal{F} = \{F_1, \ldots, F_m\}$ is a family of the set partitions of $[n]$ such that any two elements of $\mathcal{F}$ have $l+i-1$ same blocks, $1 \leq i \leq s$. Then*

$$m \leq \sum_{i=0}^{s+1} S(n, i).$$

Corollary 4.8 *Suppose that $\mathcal{F} = \{F_1, \ldots, F_m\}$ is a family of the set partitions of $[n]$ such that $|F_i \cap F_j| \leq t$ for every $1 \leq i \neq j \leq m$. Then*

$$m \leq \sum_{i=0}^{t} S(n, i).$$

4.3 r-separated sets

For the circle, $F \subseteq \{1, 2, \ldots, n\}$ is said to be *k-separated* if any two elements of F are separated by a gap of size at least k. Let $\triangle_{sp}$ be the collection of all k-separated set of $[n]$, it is obvious that $\triangle_{sp}$ is a simplicial complex. It is known [12] that the number of the $(i-1)$-dimensional faces is $\binom{n-ki}{i}$.

In 1997, Holroyd [12] posed the following question about King Arthur and the Knights of the Round Table. There were altogether n knights and they each had their own place at the Round Table. King Arthur needed to send out excursion parties $r(\leq n/2)$ to each party, on different days, but he did not wish to invite trouble by sending out the same party more than once, nor did he wish to send out two knights in the same party who occupied adjacent seats. King Arthur also wished the information found by different parties to be correlated, and to aid in this he required that any two parties should have at least one knight in common. Holroyd's question was: how long could this go on for, in other words, how many different parties could be made up? It seemed likely that the number of different parties would be $\binom{n-r-1}{r-1}$.

This problem can be restated as follow. Let G be a graph with vertex set V where $|V| = n$, and there exist edge between $v_1,\ v_2 \in V$ if the distance between v_1 and v_2 is no more than $k-1$. Let $\mathcal{A}$ be an intersecting family of independent r-subsets of V. Thus the above problem can be represented as follow.

If $1 \leq r \leq n/2$ and G is a cycle with n vertices, and $\mathcal{A}$ is an intersecting family of independent r-subsets of $V(G)$, then

$$|\mathcal{A}| \leq \binom{n-r-1}{r-1}.$$

In [22], Talbot proved a more general result as follow.

Theorem 4.9 (Talbot) *Let $1 \leq r \leq n/2$ and G be the kth power of a cycle C_n^k. (The power of a graph G is the graph obtained from G by adding an extra edge joining two vertices u and w whenever u and w are a distance $\leq k$ apart in G.). If $\mathcal{A}$ is an intersecting family of independent r-subsets of $V(G)$, then*

$$|\mathcal{A}| \leq \binom{n-kr-1}{r-1}.$$

From Theorem 2.2, we can deduce the following result.

Theorem 4.10 *Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ and $\mathcal{K} = \{k_1, k_2 \ldots, k_r\}$ satisfying $\min k_i > \max l_j$ for $1 \leq i \leq r$ and $1 \leq j \leq s$. Let G be the kth power of a cycle C_n^k. If $\mathcal{F}$ is a family of independent k_i-subsets of $V(G)$ for $1 \leq i \leq r$ such that $|F_i \cap F_j| \in \mathcal{L}$*

for every $1 \leq i, j \leq m$. *Then*

$$|\mathcal{F}| \leq \sum_{i=s-r}^{s-1} \binom{n-ki}{i}.$$

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