# SOME NEW MOCK THETA FUNCTIONS

SU-PING CUI AND NANCY S.S. GU

Abstract. Mock theta functions can be represented by Eulerian forms, Hecke-type double sums, Appell–Lerch sums, and Fourier coefficients of meromorphic Jacobi forms. In view of some $q$-series identities, we establish three two-parameter mock theta functions, and express them in terms of Appell-Lerch sums. In addition, we find three mock theta functions in Eulerian forms. Then in light of their Hecke-type double sums, we provide the Hecke-type double sums for some third order mock theta functions, and establish the relations between these functions and some classical mock theta functions.

## 1. INTRODUCTION

Mock theta functions were first introduced by Ramanujan [30] in his last letter to Hardy. He found that these functions have certain asymptotic properties as $q$ approaches a root of unity, which are similar to theta functions, but they are not really theta functions. Historically, mock theta functions can be represented by Eulerian forms, Hecke-type double sums, Appell–Lerch sums, and Fourier coefficients of meromorphic Jacobi forms. In these years, it has received a great deal of attention to seek new mock theta functions and find different forms for mock theta functions. In this paper, we find the following two-parameter mock theta functions:

$$U_1(x,q) := (1+x)(1+xq)(1+x^{-1}q)\sum_{n=0}^{\infty}\frac{(-1;q)_{2n}q^{3n}}{(xq,x^{-1}q;q^2)_{n+1}},$$

$$U_2(x,q) := (1+q)(1+xq)(1+x^{-1}q)\sum_{n=0}^{\infty}\frac{(-1,-q^3;q^2)_n q^n}{(xq,x^{-1}q;q^2)_{n+1}},$$

$$U_3(x,q) := (1+q)(1-xq)(1-x^{-1}q)\sum_{n=0}^{\infty}\frac{(q,-q^3;q^2)_n(-1)^n q^{2n+1}}{(xq^2,x^{-1}q^2;q^2)_{n+1}}.$$

In addition, by finding the following functions:

$$E_1(q) := \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q^2)_{n+1}}, \tag{1.1}$$

$$E_2(q) := \sum_{n=0}^{\infty}\frac{(1-q)q^{n^2+2n}}{(-q^2;q^2)_n}, \tag{1.2}$$

---

*Date*: August 18, 2021.

2010 *Mathematics Subject Classification.* 33D15, 11F27.

*Key words and phrases.* mock theta functions, two-parameter mock theta functions, Hecke-type double sums, Appell–Lerch sums.

$$E_3(q) := \sum_{n=0}^{\infty} \frac{(1-q)q^{n^2+n}}{(-q;q^2)_n}, \tag{1.3}$$

we establish some relations between these functions and some third order mock theta functions, and then provide the Hecke-type double sums for these third order mock theta functions.

Here and throughout the paper, we use the standard $q$-series notation [16]. Let $q$ denote a complex number with $|q|<1$. Then for positive integers $n$ and $m$,

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k),$$
$$(a;q)_{-n} := \frac{q^{\binom{n}{2}}(-q/a)^n}{(q/a;q)_n}, \quad (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k),$$
$$(a_1,a_2,\cdots,a_m;q)_\infty := (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty.$$

The (unilateral) basic hypergeometric series ${}_r\phi_s$ is defined as

$${}_r\phi_s\left(\begin{matrix} a_1,a_2,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,x\right) = \sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(q,b_1,\ldots,b_s;q)_n}\left((-1)^n q^{\binom{n}{2}}\right)^{1+s-r} x^n.$$

Jacobi's triple product identity is given by

$$j(x;q) := (x;q)_\infty(q/x;q)_\infty(q;q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{\binom{n}{2}} x^n. \tag{1.4}$$

Let $a$ and $m$ be integers with $m$ positive. Then define

$$J_{a,m} := j(q^a;q^m), \quad \overline{J}_{a,m} := j(-q^a;q^m), \quad J_m := J_{m,3m} = \prod_{i=1}^{\infty}(1-q^{mi}).$$

Ramanujan listed 17 mock theta functions which were assigned orders 3, 5, and 7 in the last letter to Hardy. The following two third order mock theta functions are included in the list.

$$\phi(q) = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q^2;q^2)_n}, \tag{1.5}$$
$$\psi(q) = \sum_{n=1}^{\infty}\frac{q^{n^2}}{(q;q^2)_n}. \tag{1.6}$$

In 1936, Waston [32] found another three third order mock theta functions. For example,

$$\nu(q) = \sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(-q;q^2)_{n+1}}. \tag{1.7}$$

Then Waston [33] proved some linear relations for the fifth order mock theta functions given by Ramanujan. In 1981, by means of Bailey chains, Andrews [2] established the

Hecke-type double sums for the fifth and seventh order mock theta functions. Hecke-type double sums have the following form:

$$\sum_{(m,n)\in D}(-1)^{H(m,n)}q^{Q(m,n)+L(m,n)},$$

where $H(m,n)$ and $L(m,n)$ are linear forms, $Q(m,n)$ is an indefinite quadratic form, and $D$ is some subset of $\mathbb{Z}\times\mathbb{Z}$ such that $Q(m,n)\geq 0$ for all $(m,n)\in D$. For example, Andrews gave the following Hecke-type identities:

$$J_1f_0(q)=\sum_{n=0}^{\infty}(1-q^{4n+2})q^{\frac{5n^2+n}{2}}\sum_{j=-n}^{n}(-1)^jq^{-j^2}, \tag{1.8}$$

$$J_1f_1(q)=\sum_{n=0}^{\infty}(1-q^{2n+1})q^{\frac{5n^2+3n}{2}}\sum_{j=-n}^{n}(-1)^jq^{-j^2}, \tag{1.9}$$

where the fifth order mock theta functions $f_0(q)$ and $f_1(q)$ are stated as

$$f_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n}\quad\text{and}\quad f_1(q)=\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(-q;q)_n}.$$

Subsequently, Hecke-type double sums were widely used to prove identities related to mock theta functions. Based on (1.8) and (1.9), Hickerson [21] introduced the universal mock theta function $g(x,q)$,

$$g(x,q)=x^{-1}\left(-1+\sum_{n=0}^{\infty}\frac{q^{n^2}}{(x;q)_{n+1}(x^{-1}q;q)_n}\right)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(x,x^{-1}q;q)_{n+1}},$$

and then proved

$$f_0(q)=\frac{J_{5,10}J_{2,5}}{J_1}-2q^2g(q^2,q^{10}),$$

$$f_1(q)=\frac{J_{5,10}J_{4,5}}{J_1}-2q^3g(q^4,q^{10}).$$

The above identities are called as "mock theta conjectures" which express the fifth order mock theta functions in terms of $g(x,q)$ and theta functions. It is customary to refer to the analogous identities involving other mock theta functions as mock theta conjectures. Later, based on $g(x,q)$ and the Hecke-type double sums given by Andrews [2], three mock theta conjectures related to the seventh order mock theta functions were proved in [20]. Furthermore, McIntosh [28] proved the mock theta conjectures for the third order mock theta functions. For example,

$$\phi(q)=-2qg_3(q,q^4)+\frac{J_2^7}{J_1^3J_4^3},$$

where he used $g_3(x,q)$ to denote $g(x,q)$. In 2012, in view of $q$-orthogonal polynomials, Andrews [3] established some Hecke-type double sums related to some known and new third order mock theta functions. For example,

$$1+\psi(q)=\frac{1}{J_1}\sum_{n=0}^{\infty}(1-q^{6n+6})(-1)^nq^{2n^2+n}\sum_{j=0}^{n}q^{-\binom{j+1}{2}}.$$

In [22], Hickerson and Mortenson gave the following definition for a special type of Hecke-type double sums.

**Definition 1.1.** *Let $x$, $y \in \mathbb{C}^* := \mathbb{C}\backslash\{0\}$ and define $sg(r) := 1$ for $r \geq 0$ and $sg(r) := -1$ for $r < 0$. Then*
$$f_{a,b,c}(x,y,q) := \sum_{sg(r)=sg(s)} sg(r)(-1)^{r+s}x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}.$$

It is clear that
$$f_{a,b,a}(x,y,q) = f_{a,b,a}(y,x,q). \tag{1.10}$$
In [29], Mortenson obtained different Hecke-type double sums for some third order mock theta functions. For example,
$$1+2\psi(q) = \frac{1}{J_1}\sum_{n=0}^{\infty}(1+q^{2n+1})(-1)^n q^{2n^2+n}\sum_{j=-n}^{n} q^{-\binom{j+1}{2}},$$
which can be rewritten as
$$\psi(q)+1 = \frac{1}{J_1}f_{3,5,3}(q^2,q^3,q).$$

For the even order mock theta functions, in view of Bailey's Lemma, Andrews and Hickerson [6] established the Hecke-type double sums for the sixth order mock theta functions, and proved the linear relations for them given by Ramanujan. In [7], Berndt and Chan found two more sixth order mock theta functions. Furthermore, by means of some classical $q$-series identities, Lovejoy [25] proved some relations for the sixth order mock theta functions. In view of the half-shifted method, Gordon and McIntosh [17] found eight eighth order mock theta functions. Later, McIntosh [26] studied three second order mock theta functions. In [14], the authors provided the Hecke-type double sums for the second and eighth order mock theta functions. For the tenth order mock theta functions appearing in the Lost Notebook [31], Choi [10–13] established the Hecke-type double sums for these functions, and proved eight linear relations given by Ramanujan.

In [19], it shows that the mock theta functions with odd order can be expressed by the universal mock theta function $g_3(x,q)$. Similarly, the mock theta functions with even order are related to another universal mock theta function $g_2(x,q)$,
$$g_2(x,q) = \sum_{n=0}^{\infty}\frac{(-q;q)_n q^{n(n+1)/2}}{(x,x^{-1}q;q)_{n+1}}.$$

In 2014, Hickerson and Mortenson [22] provided the following definition of Appell–Lerch sums.

**Definition 1.2.** *Let $x$, $z \in \mathbb{C}^* := \mathbb{C}\backslash\{0\}$ with neither $z$ nor $xz$ an integral power of $q$. Then*
$$m(x,q,z) := \frac{1}{j(z;q)}\sum_{r=-\infty}^{\infty}\frac{(-1)^r q^{\binom{r}{2}} z^r}{1-q^{r-1}xz}.$$

Changing $r$ to $r+1$ in the above series gives another useful form for $m(x,q,z)$ [22, Eq. (3.1)]:

$$m(x,q,z) = \frac{-z}{j(z;q)} \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{\binom{r+1}{2}} z^r}{1-q^r xz}. \tag{1.11}$$

Hickerson and Mortenson showed that all of Ramanujan's classical mock theta functions as well as those of Gordon and McIntosh can be expressed in terms of Appell–Lerch sums. In light of Appell-Lerch sums, results for mock theta functions are much more accessible than they were 20 years ago. For more on mock theta functions, one can see [1, 4, 5, 8, 15, 18, 27].

In this paper, we establish the Appell–Lerch sums for $U_1(x,q)$, $U_2(x,q)$, and $U_3(x,q)$.

**Theorem 1.3.** *We have*

$$U_1(x,q) = -2xq^{-1}(1-q)m(x,q^2,q) - 2(1+x)m(-q,q^2,q).$$

**Theorem 1.4.** *We have*

$$U_2(x,q) = -2(1+x)m(x,q^2,q) + 2(1-q)m(-q,q^2,q).$$

**Theorem 1.5.** *We have*

$$U_3(x,q) = 2(1-x)m(-x,q^2,-1) - 2(1-q)m(-q,q^2,-1).$$

Notice that by using Corollary 3.2 and Theorem 3.3 in [22], we can evaluate the following terms.

$$m(-q,q^2,q) = m(-q,q^2,-1) = \frac{1}{2}.$$

In addition, in view of some $q$-series identities given by Liu [23, 24] and Chen and Wang [9], we obtain the following Hecke-type double sums related to $E_1(q)$, $E_2(q)$, and $E_3(q)$.

**Theorem 1.6.** *We have*

$$E_1(q) - 1 = -\frac{2q^5}{J_{1,4}} f_{1,2,1}(q^9, q^{11}, q^4).$$

**Theorem 1.7.** *We have*

$$E_2(q) - 2 = -\frac{1}{J_{1,4}} f_{1,2,1}(iq, -iq, q).$$

**Theorem 1.8.** *We have*

$$E_3(q) - 1 = -\frac{q}{J_{2,4}} f_{1,2,1}(iq^{\frac{3}{2}}, -iq^{\frac{3}{2}}, q).$$

Meanwhile, we provide some relations between these functions and the third order mock theta functions.

**Theorem 1.9.** *We have*

$$E_1(q)-2\psi(q)=1, \tag{1.12}$$
$$E_2(q)+\phi(q)=2, \tag{1.13}$$
$$E_3(q)+q\nu(q)=1.$$

Then according to Theorems 1.6-1.9, we derive the following Hecke-type double sums for $\psi(q)$, $\phi(q)$, and $\nu(q)$.

**Corollary 1.10.** *We have*

$$\psi(q)=-\frac{q^5}{J_{1,4}}f_{1,2,1}(q^9,q^{11},q^4),$$
$$\phi(q)=\frac{1}{J_{1,4}}f_{1,2,1}(iq,-iq,q),$$
$$\nu(q)=\frac{1}{J_{2,4}}f_{1,2,1}(iq^{\frac{3}{2}},-iq^{\frac{3}{2}},q). \tag{1.14}$$

Notice that the Hecke-type double sum of $\nu(q)$ [9, Equation (6.30)] can be converted to (1.14).

This paper is organized as follows. In Section 2, we state some lemmas which are used to prove the main theorems. In Section 3, we prove Theorems 1.3-1.5. In Section 4, we prove Theorems 1.6-1.9.

## 2. Preliminaries

In this paper, we need the following identities given by Liu [23, 24].

**Lemma 2.1.** *[23, Theorem 1.7] [24, p. 2089] For* $\max\{|uab/q|,|ua|,|ub|,|c|,|d|\}<1$,

$$\frac{(uq,uab/q;q)_\infty}{(ua,ub;q)_\infty}{}_3\phi_2\left(\begin{matrix}q/a, & q/b, & v\\ & c, & d\end{matrix};q,\frac{uab}{q}\right)$$
$$=\sum_{n=0}^{\infty}\frac{(1-uq^{2n})(u,q/a,q/b;q)_n}{(1-u)(q,ua,ub;q)_n}(-uab)^n q^{\frac{n^2-3n}{2}}{}_3\phi_2\left(\begin{matrix}q^{-n}, & uq^n, & v\\ & c, & d\end{matrix};q,q\right). \tag{2.1}$$

**Lemma 2.2.** *[24, Eq. (3.14)] For any nonnegative integer n,*

$${}_3\phi_2\left(\begin{matrix}q^{-n}, & \alpha q^n, & \beta\\ & c, & d\end{matrix};q,q\right)=(-c)^n q^{\frac{n^2-n}{2}}\frac{(q\alpha/c;q)_n}{(c;q)_n}$$
$$\times{}_3\phi_2\left(\begin{matrix}q^{-n}, & \alpha q^n, & d/\beta\\ & d, & q\alpha/c\end{matrix};q,q\beta/c\right). \tag{2.2}$$

Chen and Wang [9] established the following identities.

**Lemma 2.3.** *[9, Lemma 2.2] For any nonnegative integer n,*

$${}_3\phi_2\left(\begin{matrix}q^{-n}, & \alpha q^{n+1}, & q/c\\ & \alpha d, & q^2/c\end{matrix};q,d\right)=\left(\frac{q}{c}\right)^n\frac{(\alpha c,q;q)_n}{(q^2/c,\alpha q;q)_n}$$

$$\times \sum_{j=0}^{n}(-1)^j \frac{(1-\alpha q^{2j})(\alpha, q/c, q/d; q)_j}{(1-\alpha)(q, \alpha c, \alpha d; q)_j} \left(\frac{cd}{q}\right)^j q^{\frac{-j^2-j}{2}}. \tag{2.3}$$

**Lemma 2.4.** *[16, p. 40, Eq. (2.2.1)] The q-Pfaff-Saalschütz summation formula is stated as*

$$ {}_3\phi_2\left(\begin{matrix} q^{-n}, & aq^n, & aq/bc \\ & aq/b, & aq/c \end{matrix}; q, q\right) = \frac{(b, c; q)_n}{(aq/b, aq/c; q)_n}\left(\frac{aq}{bc}\right)^n. \tag{2.4}$$

We also need some properties of Appell–Lerch sums and Hecke-type double sums.

**Proposition 2.5.** *[34] For generic $x, z \in \mathbb{C}^*$,*

$$\begin{aligned} m(x, q, z) &= m(x, q, qz), \\ m(x, q, z) &= x^{-1} m(x^{-1}, q, z^{-1}), \\ m(qx, q, z) &= 1 - x m(x, q, z). \end{aligned}$$

Following [22], the term "generic" means that the parameters do not cause poles in the Appell–Lerch sums or in the quotients of theta functions.

**Lemma 2.6.** *[22] For $x, y \in \mathbb{C}^*$,*

$$\begin{aligned} f_{a,b,c}(x, y, q) = {} & f_{a,b,c}(-x^2q^a, -y^2q^c, q^4) - x f_{a,b,c}(-x^2q^{3a}, -y^2q^{c+2b}, q^4) \\ & - y f_{a,b,c}(-x^2q^{a+2b}, -y^2q^{3c}, q^4) \\ & + xyq^b f_{a,b,c}(-x^2q^{3a+2b}, -y^2q^{3c+2b}, q^4), \end{aligned} \tag{2.5}$$

$$f_{a,b,c}(x, y, q) = -\frac{q^{a+b+c}}{xy} f_{a,b,c}(q^{2a+b}/x, q^{2c+b}/y, q), \tag{2.6}$$

$$f_{a,b,c}(x, y, q) = -y f_{a,b,c}(q^b x, q^c y, q) + j(x; q^a), \tag{2.7}$$

$$f_{a,b,c}(x, y, q) = -x f_{a,b,c}(q^a x, q^b y, q) + j(y; q^c). \tag{2.8}$$

## 3. Proofs of Theorems 1.3-1.5

In this section, using Lemma 2.1, Lemma 2.4, and the properties of Appell-Lerch sums, we prove Theorems 1.3-1.5.

**Proof of Theorem 1.3.** Setting $(a, b, c, d, u, v, q) \to (-q^2, -q, xq^3, x^{-1}q^3, q^2, q^2, q^2)$ in (2.1), we deduce

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1; q)_{2n} q^{3n}}{(xq, x^{-1}q; q^2)_{n+1}} = {} & \frac{2(1-q)}{(1-xq)(1-x^{-1}q)J_{1,2}} \sum_{n=0}^{\infty} \frac{(1-q^{2n+1})(-1)^n q^{n^2+2n}}{(1+q^{2n})(1+q^{2n+2})} \\ & \times {}_3\phi_2\left(\begin{matrix} q^{-2n}, & q^{2n+2}, & q^2 \\ & xq^3, & x^{-1}q^3 \end{matrix}; q^2, q^2\right). \end{aligned} \tag{3.1}$$

Then applying (2.4) with $(a, b, c, q) \to (q^2, xq, x^{-1}q, q^2)$ yields that

$$ {}_3\phi_2\left(\begin{matrix} q^{-2n}, & q^{2n+2}, & q^2 \\ & xq^3, & x^{-1}q^3 \end{matrix}; q^2, q^2\right) = \frac{(xq, x^{-1}q; q^2)_n q^{2n}}{(xq^3, x^{-1}q^3; q^2)_n}. \tag{3.2}$$

Substituting (3.2) into (3.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1;q)_{2n}q^{3n}}{(xq,x^{-1}q;q^2)_{n+1}}&=\frac{2(1-q)}{J_{1,2}}\sum_{n=0}^{\infty}\frac{(1-q^{2n+1})(-1)^nq^{n^2+4n}}{(1+q^{2n})(1+q^{2n+2})(1-xq^{2n+1})(1-x^{-1}q^{2n+1})}\\&=\frac{(1-q)}{J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(1-q^{2n+1})(-1)^nq^{n^2+4n}}{(1+q^{2n})(1+q^{2n+2})(1-xq^{2n+1})(1-x^{-1}q^{2n+1})}.\end{aligned}$$

Notice that

$$\frac{(1-q)(1-q^{2n+1})}{(1+q^{2n})(1+q^{2n+2})}=\frac{1}{1+q^{2n}}-\frac{q}{1+q^{2n+2}}.$$

Then

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(-1;q)_{2n}q^{3n}}{(xq,x^{-1}q;q^2)_{n+1}}\\&=\frac{1}{J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+4n}}{(1+q^{2n})(1-xq^{2n+1})(1-x^{-1}q^{2n+1})}\\&\quad-\frac{1}{J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+4n+1}}{(1+q^{2n+2})(1-xq^{2n+1})(1-x^{-1}q^{2n+1})}\\&=\frac{2}{J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+4n}}{(1+q^{2n})(1-xq^{2n+1})(1-x^{-1}q^{2n+1})}\\&=\frac{2}{(x-x^{-1})J_{1,2}}\sum_{n=-\infty}^{\infty}\left(\frac{(-1)^nxq^{n^2+4n}}{(1+q^{2n})(1-xq^{2n+1})}-\frac{(-1)^nx^{-1}q^{n^2+4n}}{(1+q^{2n})(1-x^{-1}q^{2n+1})}\right)\\&=\frac{2}{(x-x^{-1})(1+xq)J_{1,2}}\sum_{n=-\infty}^{\infty}\left(\frac{(-1)^nxq^{n^2+4n}}{1+q^{2n}}+\frac{(-1)^nx^2q^{n^2+4n+1}}{1-xq^{2n+1}}\right)\\&\quad-\frac{2}{(x-x^{-1})(1+x^{-1}q)J_{1,2}}\sum_{n=-\infty}^{\infty}\left(\frac{(-1)^nx^{-1}q^{n^2+4n}}{1+q^{2n}}+\frac{(-1)^nx^{-2}q^{n^2+4n+1}}{1-x^{-1}q^{2n+1}}\right)\\&=\frac{2}{(x-x^{-1})(1+xq)J_{1,2}}\left(-xq^{-3}J_{3,2}m(-q^{-3},q^2,q^3)-x^2q^{-2}J_{3,2}m(xq^{-2},q^2,q^3)\right)\\&\quad-\frac{2}{(x-x^{-1})(1+x^{-1}q)J_{1,2}}\\&\quad\times\left(-x^{-1}q^{-3}J_{3,2}m(-q^{-3},q^2,q^3)-x^{-2}q^{-2}J_{3,2}m(x^{-1}q^{-2},q^2,q^3)\right)\\&=\frac{2}{(x-x^{-1})(1+xq)}\left(xq^{-4}m(-q^{-3},q^2,q^3)+x^2q^{-3}m(xq^{-2},q^2,q^3)\right)\\&\quad-\frac{2}{(x-x^{-1})(1+x^{-1}q)}\left(x^{-1}q^{-4}m(-q^{-3},q^2,q^3)+x^{-2}q^{-3}m(x^{-1}q^{-2},q^2,q^3)\right)\\&=\frac{2x^2q^{-3}}{(x-x^{-1})(1+xq)}m(xq^{-2},q^2,q^3)-\frac{2x^{-2}q^{-3}}{(x-x^{-1})(1+x^{-1}q)}m(x^{-1}q^{-2},q^2,q^3)\\&\quad+\frac{2q^{-4}}{(1+xq)(1+x^{-1}q)}m(-q^{-3},q^2,q^3),\end{aligned}\tag{3.3}$$

where we use (1.11) to obtain the fifth equality. From Proposition 2.5, it can be seen that

$$m(xq^{-2},q^2,q^3)=x^{-1}q^2-x^{-1}q^2m(x,q^2,q), \tag{3.4}$$

$$m(x^{-1}q^{-2},q^2,q^3)=xq^2m(xq^2,q^2,q)=xq^2-x^2q^2m(x,q^2,q), \tag{3.5}$$

$$m(-q^{-3},q^2,q^3)=-q^3-q^4m(-q,q^2,q). \tag{3.6}$$

Substituting the above three identities into (3.3) yields that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1;q)_{2n}q^{3n}}{(xq,x^{-1}q;q^2)_{n+1}}&=-\frac{2xq^{-1}(1-q)}{(1+x)(1+xq)(1+x^{-1}q)}m(x,q^2,q)\\&\quad-\frac{2}{(1+xq)(1+x^{-1}q)}m(-q,q^2,q).\end{aligned}$$

Hence, we complete the proof. □

**Proof of Theorem 1.4.** Setting $(a,b,c,d,u,v,q)\to(-q^2,-q^{-1},xq^3,x^{-1}q^3,q^2,q^2,q^2)$ in (2.1), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(-1,-q^3;q^2)_nq^n}{(xq,x^{-1}q;q^2)_{n+1}}\\&=\frac{2}{(1+q)(1-xq)(1-x^{-1}q)J_{1,2}}\sum_{n=0}^{\infty}\frac{(1+q^{2n+1})(1-q^{4n+2})(-1)^nq^{n^2}}{(1+q^{2n})(1+q^{2n+2})}\\&\quad\times{}_3\phi_2\left(\begin{matrix}q^{-2n}, & q^{2n+2}, & q^2\\ & xq^3, & x^{-1}q^3\end{matrix};q^2,q^2\right)\end{aligned} \tag{3.7}$$

Then substituting (3.2) into (3.7) yields that

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(-1,-q^3;q^2)_nq^n}{(xq,x^{-1}q;q^2)_{n+1}}\\&=\frac{2}{(1+q)J_{1,2}}\sum_{n=0}^{\infty}\frac{(1+q^{2n+1})(1-q^{4n+2})(-1)^nq^{n^2+2n}}{(1+q^{2n})(1+q^{2n+2})(1-xq^{2n+1})(1-x^{-1}q^{2n+1})}\\&=\frac{1}{(1+q)J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(1+q^{2n+1})(1-q^{4n+2})(-1)^nq^{n^2+2n}}{(1+q^{2n})(1+q^{2n+2})(1-xq^{2n+1})(1-x^{-1}q^{2n+1})}\\&=\frac{1}{(1+q)J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(1+q^{2n+1})(-1)^nq^{n^2+2n}}{(1+q^{2n+2})(1-xq^{2n+1})(1-x^{-1}q^{2n+1})}\\&\quad-\frac{1}{(1+q)J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(1+q^{2n+1})(-1)^nq^{n^2+4n}}{(1+q^{2n})(1-xq^{2n+1})(1-x^{-1}q^{2n+1})}.\end{aligned} \tag{3.8}$$

Changing $n$ to $-n-1$ in the first summand on the right-hand side of (3.8), we have

$$\sum_{n=0}^{\infty}\frac{(-1,-q^3;q^2)_nq^n}{(xq,x^{-1}q;q^2)_{n+1}}$$

$$
\begin{aligned}
&= \frac{-2}{(1+q)J_{1,2}} \sum_{n=-\infty}^{\infty} \frac{(1+q^{2n+1})(-1)^n q^{n^2+4n}}{(1+q^{2n})(1-xq^{2n+1})(1-x^{-1}q^{2n+1})} \\
&= \frac{-2}{(1+q)(x-1)J_{1,2}} \sum_{n=-\infty}^{\infty} \left( \frac{(-1)^n x q^{n^2+4n}}{(1+q^{2n})(1-xq^{2n+1})} - \frac{(-1)^n q^{n^2+4n}}{(1+q^{2n})(1-x^{-1}q^{2n+1})} \right) \\
&= \frac{-2}{(1+q)(x-1)(1+xq)J_{1,2}} \sum_{n=-\infty}^{\infty} \left( \frac{(-1)^n x^2 q^{n^2+4n+1}}{1-xq^{2n+1}} + \frac{(-1)^n x q^{n^2+4n}}{1+q^{2n}} \right) \\
&\quad + \frac{2}{(1+q)(x-1)(1+x^{-1}q)J_{1,2}} \sum_{n=-\infty}^{\infty} \left( \frac{(-1)^n x^{-1} q^{n^2+4n+1}}{1-x^{-1}q^{2n+1}} + \frac{(-1)^n q^{n^2+4n}}{1+q^{2n}} \right) \\
&= \frac{1}{(1+q)(x-1)(1+xq)} \left( -2x^2q^{-3} m(xq^{-2}, q^2, q^3) - 2xq^{-4} m(-q^{-3}, q^2, q^3) \right) \\
&\quad + \frac{1}{(1+q)(x-1)(1+x^{-1}q)} \left( 2x^{-1}q^{-3} m(x^{-1}q^{-2}, q^2, q^3) + 2q^{-4} m(-q^{-3}, q^2, q^3) \right) \\
&= -\frac{2x^2q^{-3}}{(1+q)(x-1)(1+xq)} m(xq^{-2}, q^2, q^3) - \frac{2q^{-4}(1-q)}{(1+q)(1+xq)(1+x^{-1}q)} m(-q^{-3}, q^2, q^3) \\
&\quad + \frac{2x^{-1}q^{-3}}{(1+q)(x-1)(1+x^{-1}q)} m(x^{-1}q^{-2}, q^2, q^3),
\end{aligned}
$$

where the fourth step follows from (1.11). Then with the help of (3.4)-(3.6), we deduce

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-1, -q^3; q^2)_n q^n}{(xq, x^{-1}q; q^2)_{n+1}} \\
&= -\frac{2(1+x)}{(1+q)(1+xq)(1+x^{-1}q)} m(x, q^2, q) + \frac{2(1-q)}{(1+q)(1+xq)(1+x^{-1}q)} m(-q, q^2, q).
\end{aligned}
$$

Therefore, we arrive at the theorem. $\square$

**Proof of Theorem 1.5.** Setting $(a, b, c, d, u, v, q) \to (q, -q^{-1}, xq^4, x^{-1}q^4, q^4, q^2, q^2)$ in (2.1), we find

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(q, -q^3; q^2)_n (-1)^n q^{2n}}{(xq^2, x^{-1}q^2; q^2)_{n+1}} \\
&= \frac{2}{(1+q)(1-xq^2)(1-x^{-1}q^2)\overline{J}_{0,2}} \sum_{n=0}^{\infty} \frac{(1-q^{2n+2})(1-q^{4n+4}) q^{n^2+n}}{(1-q^{2n+1})(1-q^{2n+3})} \\
&\quad \times {}_3\phi_2 \left( \begin{matrix} q^{-2n}, & q^{2n+4}, & q^2 \\ & xq^4, & x^{-1}q^4 \end{matrix} ; q^2, q^2 \right).
\end{aligned} \tag{3.9}
$$

Then applying (2.4) with $(a, b, c, q) \to (q^4, xq^2, x^{-1}q^2, q^2)$ yields that

$$
{}_3\phi_2 \left( \begin{matrix} q^{-2n}, & q^{2n+4}, & q^2 \\ & xq^4, & x^{-1}q^4 \end{matrix} ; q^2, q^2 \right) = \frac{(xq^2, x^{-1}q^2; q^2)_n}{(xq^4, x^{-1}q^4; q^2)_n} q^{2n}. \tag{3.10}
$$

Substituting (3.10) into (3.9), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(q,-q^3;q^2)_n(-1)^nq^{2n}}{(xq^2,x^{-1}q^2;q^2)_{n+1}}\\
&=\frac{2}{(1+q)\overline{J}_{0,2}}\sum_{n=0}^{\infty}\frac{(1-q^{2n+2})(1-q^{4n+4})q^{n^2+3n}}{(1-q^{2n+1})(1-q^{2n+3})(1-xq^{2n+2})(1-x^{-1}q^{2n+2})}\\
&=\frac{1}{(1+q)\overline{J}_{0,2}}\sum_{n=-\infty}^{\infty}\frac{(1-q^{2n+2})(1-q^{4n+4})q^{n^2+3n}}{(1-q^{2n+1})(1-q^{2n+3})(1-xq^{2n+2})(1-x^{-1}q^{2n+2})}\\
&=\frac{1}{(1+q)\overline{J}_{0,2}}\sum_{n=-\infty}^{\infty}\frac{(1-q^{2n+2})q^{n^2+3n}}{(1-q^{2n+1})(1-xq^{2n+2})(1-x^{-1}q^{2n+2})}\\
&\quad+\frac{1}{(1+q)\overline{J}_{0,2}}\sum_{n=-\infty}^{\infty}\frac{(1-q^{2n+2})q^{n^2+5n+3}}{(1-q^{2n+3})(1-xq^{2n+2})(1-x^{-1}q^{2n+2})}. \qquad (3.11)
\end{aligned}$$

Changing $n$ to $-n-2$ in the second summand on the right-hand side of (3.11), we have

$$\sum_{n=0}^{\infty}\frac{(q,-q^3;q^2)_n(-1)^nq^{2n}}{(xq^2,x^{-1}q^2;q^2)_{n+1}}=\frac{2}{(1+q)\overline{J}_{0,2}}\sum_{n=-\infty}^{\infty}\frac{(1-q^{2n+2})q^{n^2+3n}}{(1-q^{2n+1})(1-xq^{2n+2})(1-x^{-1}q^{2n+2})}.$$

Then in view of the fact

$$\frac{(1+x)(1-q^{2n+2})}{(1-xq^{2n+2})(1-x^{-1}q^{2n+2})}=\frac{x}{1-xq^{2n+2}}+\frac{1}{1-x^{-1}q^{2n+2}},$$

we find

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(q,-q^3;q^2)_n(-1)^nq^{2n}}{(xq^2,x^{-1}q^2;q^2)_{n+1}}\\
&=\frac{2}{(1+q)(1+x)\overline{J}_{0,2}}\sum_{n=-\infty}^{\infty}\left(\frac{xq^{n^2+3n}}{(1-q^{2n+1})(1-xq^{2n+2})}+\frac{q^{n^2+3n}}{(1-q^{2n+1})(1-x^{-1}q^{2n+2})}\right)\\
&=\frac{2}{(1+q)(1+x)(1-xq)\overline{J}_{0,2}}\sum_{n=-\infty}^{\infty}\left(\frac{xq^{n^2+3n}}{1-q^{2n+1}}-\frac{x^2q^{n^2+3n+1}}{1-xq^{2n+2}}\right)\\
&\quad+\frac{2}{(1+q)(1+x)(1-x^{-1}q)\overline{J}_{0,2}}\sum_{n=-\infty}^{\infty}\left(\frac{q^{n^2+3n}}{1-q^{2n+1}}-\frac{x^{-1}q^{n^2+3n+1}}{1-x^{-1}q^{2n+2}}\right)\\
&=\frac{2q^{-2}(1-q)}{(1+q)(1-xq)(1-x^{-1}q)}m(-q^{-1},q^2,-q^2)-\frac{2x^2q^{-1}}{(1+q)(1+x)(1-xq)}m(-x,q^2,-q^2)\\
&\quad-\frac{2x^{-1}q^{-1}}{(1+q)(1+x)(1-x^{-1}q)}m(-x^{-1},q^2,-q^2), \qquad (3.12)
\end{aligned}$$

where we apply (1.11) to obtain (3.12). From Proposition 2.5, it can be seen that

$$\begin{aligned}
-q^{-1}m(-q^{-1},q^2,-q^2)&=m(-q,q^2,-1),\\
m(-x,q^2,-q^2)&=m(-x,q^2,-1),
\end{aligned}$$

$$-x^{-1}m(-x^{-1},q^2,-q^2)=m(-x,q^2,-1).$$

Substituting the above identities into (3.12) yields that

$$\sum_{n=0}^{\infty}\frac{(q,-q^3;q^2)_n(-1)^nq^{2n}}{(xq^2,x^{-1}q^2;q^2)_{n+1}}=\frac{-2q^{-1}(1-q)}{(1+q)(1-xq)(1-x^{-1}q)}m(-q,q^2,-1)$$
$$+\frac{2q^{-1}(1-x)}{(1+q)(1-xq)(1-x^{-1}q)}m(-x,q^2,-1).$$

Therefore, we obtain the desirable result. □

## 4. Proofs of Theorems 1.6-1.9

In this section, by using the lemmas in Section 2, we prove Theorems 1.6-1.9.

**Proof of Theorem 1.6.** Setting $(a,b,c,d,u,v,q)\to(0,-q^{-1},q^3,-q^3,q^2,q^2,q^2)$ in (2.1), we find

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q^2)_{n+1}}&=\frac{1}{(1-q)J_{1,4}}\sum_{n=0}^{\infty}\frac{(1-q^{4n+2})(-q^3;q^2)_n(-1)^nq^{2n^2-n}}{(-q;q^2)_n}\\&\quad\times{}_3\phi_2\left(\begin{matrix}q^{-2n}, & q^{2n+2}, & q^2\\ & q^3, & -q^3\end{matrix};q^2,q^2\right)\\&=\frac{1}{(1-q)J_{1,4}}\sum_{n=0}^{\infty}(1-q^{4n+2})(-1)^nq^{3n^2+n}\\&\quad\times{}_3\phi_2\left(\begin{matrix}q^{-2n}, & q^{2n+2}, & q\\ & q^3, & -q\end{matrix};q^2,-q\right),\end{aligned}\tag{4.1}$$

where the second equality follows from (2.2) with $(\alpha,\beta,c,d,q)\to(q^2,q^2,-q^3,q^3,q^2)$. Then setting $(\alpha,c,d,q)\to(1,q,-q,q^2)$ in (2.3) yields that

$${}_3\phi_2\left(\begin{matrix}q^{-2n}, & q^{2n+2}, & q\\ & q^3, & -q\end{matrix};q^2,-q\right)=q^n\frac{(q;q^2)_n}{(q^3;q^2)_n}\sum_{j=-n}^{n}q^{-j^2-j}.\tag{4.2}$$

Substituting (4.2) into (4.1), we obtain that

$$E_1(q)=\frac{1}{J_{1,4}}\sum_{n=0}^{\infty}(1+q^{2n+1})(-1)^nq^{3n^2+2n}\sum_{j=-n}^{n}q^{-j^2-j}.$$

Based on the above identity, we find that

$$\begin{aligned}E_1(q)&=\frac{1}{J_{1,4}}\left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^nq^{3n^2+2n-j^2-j}+\sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^nq^{3n^2+4n+1-j^2-j}\right)\\&=\frac{1}{J_{1,4}}\left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^nq^{3n^2+2n-j^2-j}+\sum_{n=1}^{\infty}\sum_{j=-n+1}^{n-1}(-1)^{n-1}q^{3n^2-2n-j^2-j}\right)\\&=\frac{1}{J_{1,4}}\left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^nq^{3n^2+2n-j^2-j}-\sum_{n=-\infty}^{-1}\sum_{j=n+1}^{-n-1}(-1)^nq^{3n^2+2n-j^2-j}\right)\end{aligned}$$

$$= \frac{1}{J_{1,4}} \left( \sum_{\substack{n+j\geq 0\\ n-j\geq 0}} - \sum_{\substack{n+j<0\\ n-j<0}} \right) (-1)^n q^{3n^2+2n-j^2-j}. \tag{4.3}$$

Then setting $n=(r+s)/2$ and $j=(r-s)/2$ in (4.3) yields that

$$\begin{aligned} E_1(q) &= \frac{1}{J_{1,4}} \left( \sum_{\substack{r,s\geq 0\\ r\equiv s \pmod 2}} - \sum_{\substack{r,s<0\\ r\equiv s \pmod 2}} \right) (-1)^{\frac{r+s}{2}} q^{\frac{r^2+s^2+4rs+r+3s}{2}} \\ &= \frac{1}{J_{1,4}} \sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}} sg(r)(-1)^{\frac{r+s}{2}} q^{\frac{r^2+s^2+4rs+r+3s}{2}} \\ &= \frac{1}{J_{1,4}} \left( f_{1,2,1}(q^3,q^5,q^4) - q^5 f_{1,2,1}(q^9,q^{11},q^4) \right). \end{aligned} \tag{4.4}$$

From (1.10) and (2.7), it can be seen that

$$\begin{aligned} f_{1,2,1}(q^3,q^5,q^4) &= -q^5 f_{1,2,1}(q^{11},q^9,q^4) + J_{3,4} \\ &= -q^5 f_{1,2,1}(q^9,q^{11},q^4) + J_{3,4}. \end{aligned}$$

Therefore, substituting the above identity into (4.4) yields the desirable result. □

**Proof of Theorem 1.7.** Setting $(a,b,c,d,u,v,q) \to (0,-q,-q,-q^2,q^2,q^2,q^2)$ in (2.1), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(-q^2;q^2)_n} &= \frac{1}{J_{1,4}} \sum_{n=0}^{\infty} (1-q^{2n+1})(-1)^n q^{2n^2+n} {}_3\phi_2 \left( \begin{matrix} q^{-2n}, & q^{2n+2}, & q^2 \\ & -q, & -q^2 \end{matrix} ; q^2, q^2 \right) \\ &= \frac{1}{J_{1,4}} \sum_{n=0}^{\infty} (1-q^{2n+1})(-1)^n q^{3n^2+n} \frac{(-q^3;q^2)_n}{(-q;q^2)_n} \\ &\quad \times {}_3\phi_2 \left( \begin{matrix} q^{-2n}, & q^{2n+2}, & -1 \\ & -q^3, & -q^2 \end{matrix} ; q^2, -q^3 \right), \end{aligned} \tag{4.5}$$

where we apply (2.2) with $(\alpha,\beta,c,d,q) \to (q^2,q^2,-q,-q^2,q^2)$ to obtain the second equality. Next, setting $(\alpha,c,d,q) \to (1,-q^2,-q^3,q^2)$ in (2.3) yields that

$$\begin{aligned} &{}_3\phi_2 \left( \begin{matrix} q^{-2n}, & q^{2n+2}, & -1 \\ & -q^3, & -q^2 \end{matrix} ; q^2, -q^3 \right) \\ &= (-1)^n \left( 1 + 2(1+q^{-1})(1+q) \sum_{j=1}^{n} \frac{(-1)^j q^{-j^2+2j}}{(1+q^{2j-1})(1+q^{2j+1})} \right) \qquad (4.6) \\ &= (-1)^n \left( 1 + 2\frac{1+q}{1-q} \sum_{j=1}^{n} \left( \frac{(-1)^j q^{-j^2+2j-1}}{1+q^{2j-1}} - \frac{(-1)^j q^{-j^2+2j+1}}{1+q^{2j+1}} \right) \right) \\ &= (-1)^n \left( 1 + 2\frac{1+q}{1-q} \sum_{j=0}^{n-1} \frac{(-1)^{j+1} q^{-j^2}}{1+q^{2j+1}} - 2\frac{1+q}{1-q} \sum_{j=1}^{n} \frac{(-1)^j q^{-j^2+2j+1}}{1+q^{2j+1}} \right) \end{aligned}$$

$$
\begin{aligned}
&= (-1)^n \left(1 + 2\frac{1+q}{1-q}\sum_{j=1}^{n-1}(-1)^{j+1}q^{-j^2} - \frac{2}{1-q} - 2\frac{1+q}{1-q}\frac{(-1)^n q^{-n^2+2n+1}}{1+q^{2n+1}}\right)\\
&= (-1)^n \frac{1+q}{1-q}\left(\sum_{j=-n+1}^{n-1}(-1)^{j+1}q^{-j^2} - 2\frac{(-1)^n q^{-n^2+2n+1}}{1+q^{2n+1}}\right). \qquad (4.7)
\end{aligned}
$$

Then substituting (4.7) into (4.5), we obtain

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(-q^2;q^2)_n} &= \frac{1}{(1-q)J_{1,4}}\sum_{n=0}^{\infty}(1-q^{4n+2})q^{3n^2+n}\sum_{j=-n+1}^{n-1}(-1)^{j+1}q^{-j^2}\\
&\quad - \frac{2}{(1-q)J_{1,4}}\sum_{n=0}^{\infty}(1-q^{2n+1})(-1)^n q^{2n^2+3n+1}\\
&= \frac{1}{(1-q)J_{1,4}}\sum_{n=0}^{\infty}(1-q^{4n+2})q^{3n^2+n}\sum_{j=-n}^{n}(-1)^{j+1}q^{-j^2}\\
&\quad + \frac{2}{(1-q)J_{1,4}}\sum_{n=0}^{\infty}(1-q^{2n+1})(-1)^n q^{2n^2+n}\\
&= \frac{1}{(1-q)J_{1,4}}\sum_{n=0}^{\infty}(1-q^{4n+2})q^{3n^2+n}\sum_{j=-n}^{n}(-1)^{j+1}q^{-j^2}\\
&\quad + \frac{2}{(1-q)J_{1,4}}\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+n}\\
&= \frac{1}{(1-q)J_{1,4}}\sum_{n=0}^{\infty}(1-q^{4n+2})q^{3n^2+n}\sum_{j=-n}^{n}(-1)^{j+1}q^{-j^2} + \frac{2}{1-q},
\end{aligned}
$$

where the last step follows from (1.4). Then based on (1.2) and the above identity, we have

$$
\begin{aligned}
E_2(q) - 2 &= \frac{1}{J_{1,4}}\sum_{n=0}^{\infty}(1-q^{4n+2})q^{3n^2+n}\sum_{j=-n}^{n}(-1)^{j+1}q^{-j^2}\\
&= \frac{1}{J_{1,4}}\left(\sum_{\substack{n+j\ge 0\\ n-j\ge 0}} - \sum_{\substack{n+j<0\\ n-j<0}}\right)(-1)^{j+1}q^{3n^2+n-j^2}\\
&= \frac{1}{J_{1,4}}\sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}} sg(r)(-1)^{\frac{r-s+2}{2}}q^{\frac{r^2+s^2+4rs+r+s}{2}}\\
&= \frac{1}{J_{1,4}}\left(-f_{1,2,1}(q^3,q^3,q^4) - q^4 f_{1,2,1}(q^9,q^9,q^4)\right). \qquad (4.8)
\end{aligned}
$$

In view of (1.10) and (2.5), we deduce that

$$
\begin{aligned}
&f_{1,2,1}(iq,-iq,q)\\
&= f_{1,2,1}(q^3,q^3,q^4) - iqf_{1,2,1}(q^5,q^7,q^4) + iqf_{1,2,1}(q^7,q^5,q^4) + q^4 f_{1,2,1}(q^9,q^9,q^4)
\end{aligned}
$$

$$= f_{1,2,1}(q^3,q^3,q^4) + q^4 f_{1,2,1}(q^9,q^9,q^4). \tag{4.9}$$

Substituting (4.9) into (4.8), we prove the theorem. □

**Proof of Theorem 1.8.** Setting $(a,b,c,d,u,v,q) \to (0,-1,-q,-q^2,q^2,q^2,q^2)$ in (2.1), we deduce

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q^2)_n} &= \frac{1}{J_{2,4}} \sum_{n=0}^{\infty} (1-q^{4n+2})(-1)^n q^{2n^2} {}_3\phi_2 \left( \begin{matrix} q^{-2n}, & q^{2n+2}, & q^2 \\ & -q, & -q^2 \end{matrix} ; q^2, q^2 \right) \\ &= \frac{1}{J_{2,4}} \sum_{n=0}^{\infty} (1-q^{4n+2})(-1)^n q^{3n^2} \frac{(-q^3;q^2)_n}{(-q;q^2)_n} \\ &\quad \times {}_3\phi_2 \left( \begin{matrix} q^{-2n}, & q^{2n+2}, & -1 \\ & -q^2, & -q^3 \end{matrix} ; q^2, -q^3 \right), \end{aligned} \tag{4.10}$$

where we use (2.2) with $(\alpha,\beta,c,d,q) \to (q^2,q^2,-q,-q^2,q^2)$ to derive the last equality. Then substituting (4.6) into (4.10), we obtain

$$\begin{aligned}(1-q)\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q^2)_n} &= \frac{1}{J_{2,4}} \sum_{n=0}^{\infty} (1+q^{2n+1})(1-q^{4n+2}) q^{3n^2} \sum_{j=-n+1}^{n-1} (-1)^{j+1} q^{-j^2} \\ &\quad - \frac{2}{J_{2,4}} \sum_{n=0}^{\infty} (1-q^{4n+2})(-1)^n q^{2n^2+2n+1} \\ &= \frac{1}{J_{2,4}} \sum_{n=0}^{\infty} (1+q^{2n+1})(1-q^{4n+2}) q^{3n^2} \sum_{j=-n}^{n} (-1)^{j+1} q^{-j^2} \\ &\quad + \frac{2}{J_{2,4}} \sum_{n=0}^{\infty} (1-q^{4n+2})(-1)^n q^{2n^2} \\ &= \frac{1}{J_{2,4}} \sum_{n=0}^{\infty} (1+q^{2n+1})(1-q^{4n+2}) q^{3n^2} \sum_{j=-n}^{n} (-1)^{j+1} q^{-j^2} + 2. \end{aligned}$$

So we obtain that

$$\begin{aligned} &E_3(q) - 2 \\ &= \frac{1}{J_{2,4}} \sum_{n=0}^{\infty} (1+q^{2n+1})(1-q^{4n+2}) q^{3n^2} \sum_{j=-n}^{n} (-1)^{j+1} q^{-j^2} \\ &= \frac{1}{J_{2,4}} \left( \sum_{\substack{n+j\geq 0 \\ n-j\geq 0}} - \sum_{\substack{n+j<0 \\ n-j<0}} \right) (-1)^{j+1} \left( q^{3n^2-j^2} + q^{3n^2+2n+1-j^2} \right) \\ &= \frac{1}{J_{2,4}} \sum_{\substack{sg(r)=sg(s) \\ r\equiv s \pmod 2}} sg(r)(-1)^{\frac{r-s+2}{2}} \left( q^{\frac{r^2+s^2+4rs}{2}} + q^{\frac{r^2+s^2+4rs+2r+2s+2}{2}} \right) \\ &= \frac{1}{J_{2,4}} \left( -f_{1,2,1}(q^2,q^2,q^4) - q f_{1,2,1}(q^4,q^4,q^4) - q^3 f_{1,2,1}(q^8,q^8,q^4) - q^6 f_{1,2,1}(q^{10},q^{10},q^4) \right). \end{aligned} \tag{4.11}$$

Next, in view of (2.8), we find that

$$f_{1,2,1}(q^2,q^2,q^4) = -q^2 f_{1,2,1}(q^6,q^{10},q^4) + J_{2,4}. \tag{4.12}$$

Then applying (2.6) yields that

$$f_{1,2,1}(q^6,q^{10},q^4) = -f_{1,2,1}(q^{10},q^6,q^4), \tag{4.13}$$

$$f_{1,2,1}(q^8,q^8,q^4) = -f_{1,2,1}(q^8,q^8,q^4). \tag{4.14}$$

According to (1.10) and (4.13), we have

$$f_{1,2,1}(q^6,q^{10},q^4) = 0.$$

Then combining (4.12) and the above identity, we derive that

$$f_{1,2,1}(q^2,q^2,q^4) = J_{2,4}. \tag{4.15}$$

In addition, the identity (4.14) implies that

$$f_{1,2,1}(q^8,q^8,q^4) = 0. \tag{4.16}$$

Thus, substituting (4.15) and (4.16) into (4.11), we arrive at

$$\begin{aligned} E_3(q) - 2 &= \frac{1}{J_{2,4}} \left(-q f_{1,2,1}(q^4,q^4,q^4) - q^6 f_{1,2,1}(q^{10},q^{10},q^4) - J_{2,4}\right) \\ &= -\frac{q}{J_{2,4}} f_{1,2,1}(iq^{\frac{3}{2}}, -iq^{\frac{3}{2}}, q) - 1, \end{aligned}$$

where the last step follows from (1.10) and (2.5). Hence, we complete the proof. □

**Proof of Theorem 1.9.** By combining (1.1) and (1.6), we deduce

$$\begin{aligned} E_1(q) - \psi(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q^2)_{n+1}} - \sum_{n=0}^{\infty} \frac{q^{n^2+2n+1}}{(q;q^2)_{n+1}} \\ &= \sum_{n=0}^{\infty} \frac{q^{n^2}(1-q^{2n+1})}{(q;q^2)_{n+1}} \\ &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q^2)_n} \\ &= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n} \\ &= 1 + \psi(q). \end{aligned}$$

So, we obtain (1.12).

Then combining (1.2) and (1.5) yields that

$$\begin{aligned} \phi(q) + \frac{E_2(q)}{1-q} &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n} + \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(-q^2;q^2)_n} \\ &= 2 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n} + \sum_{n=1}^{\infty} \frac{q^{n^2+2n}}{(-q^2;q^2)_n} \end{aligned}$$

$$
\begin{aligned}
&= 2 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_{n-1}} \\
&= 2 + \sum_{n=0}^{\infty} \frac{q^{n^2+2n+1}}{(-q^2;q^2)_{n}} \\
&= 2 + \frac{qE_2(q)}{1-q}.
\end{aligned}
$$

Hence, we prove (1.13).

Finally, based on (1.3) and (1.7), we have

$$
\begin{aligned}
\frac{q^{-1}E_3(q)}{1-q} + \nu(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2+n-1}}{(-q;q^2)_n} + \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q^2)_{n+1}} \\
&= q^{-1} + \sum_{n=1}^{\infty} \frac{q^{n^2+n-1}}{(-q;q^2)_n} + \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q^2)_{n+1}} \\
&= q^{-1} + \sum_{n=0}^{\infty} \frac{q^{n^2+3n+1}}{(-q;q^2)_{n+1}} + \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q^2)_{n+1}} \\
&= q^{-1} + \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q^2)_n} \\
&= q^{-1} + \frac{E_3(q)}{1-q}.
\end{aligned}
$$

Therefore, we complete the proof. □

**Acknowledgements:** We would like to thank the referee for valuable comments and suggestions. This work was supported by the National Natural Science Foundation of China (Grant Nos. 11771222 and 12001309), the Fundamental Research Funds for the Central Universities (Nankai University), the Foreign Youth Exchange Program of China Association of Science and Technology, and the Natural Science Foundation Youth Fund of Qinghai Normal University.

## References

[1] G.E. Andrews, Mordell integrals and Ramanujan's "lost" notebook, Analytic number theory (Philadelphia, Pa., 1980), pp. 10–18, Lecture Notes in Math., 899, Springer, Berlin-New York, 1981.
[2] G.E. Andrews, The fifth and seventh order mock theta functions, Trans. Amer. Math. Soc. 293 (1986) 113–134.
[3] G.E. Andrews, $q$-orthogonal polynomials, Rogers-Ramanujan identities, and mock theta functions, Proc. Steklov Inst. Math. 276 (2012) 21–32.
[4] G.E. Andrews, B.C. Berndt, Ramanujan's Lost Notebook, Part V, Springer, New York, 2018.
[5] G.E. Andrews, F.G. Garvan, Ramanujan's "lost" notebook VI: The mock theta conjectures, Adv. Math. 73 (1989) 242–255.
[6] G.E. Andrews, D. Hickerson, Ramanujan's "lost" notebook VII: The sixth order mock theta functions, Adv. Math. 89 (1991) 60–105.
[7] B.C. Berndt, S.H. Chan, Sixth order mock theta functions, Adv. Math. 216 (2007) 771–786.

[8] K. Bringmann, K. Hikami, J. Lovejoy, On the modularity of the unified WRT invariants of certain Seifert manifolds, Adv. in Appl. Math. 46 (2011) 86–93.

[9] D. Chen, L. Wang, Representations of mock theta functions, Adv. Math. 365 (2020) 107037.

[10] Y.-S. Choi, Tenth order mock theta functions in Ramanujan's lost notebook, Invent. Math. 136 (1999) 497–569.

[11] Y.-S. Choi, Tenth order mock theta functions in Ramanujan's lost notebook. II, Adv. Math. 156 (2000) 180–285.

[12] Y.-S. Choi, Tenth order mock theta functions in Ramanujan's lost notebook. IV, Trans. Amer. Math. Soc. 354 (2002) 705–733.

[13] Y.-S. Choi, Tenth order mock theta functions in Ramanujan's lost notebook. III, Proc. London Math. Soc. (3) 94 (2007) 26–52.

[14] S.-P. Cui, N.S.S. Gu, L.-J. Hao, On second and eighth order mock theta functions, Ramanujan J. 50 (2019) 393–422.

[15] F.G. Garvan, Universal mock theta functions and two-variable Hecke-Rogers identities, Ramanujan J. 36 (2015) 267–296.

[16] G. Gasper, M. Rahman, Basic Hypergeometric Series, Second Ed., Cambridge University Press, Cambridge, 2004.

[17] B. Gordon, R.J. McIntosh, Some eighth order mock theta functions, J. London Math. Soc. (2) 62 (2000) 321–335.

[18] B. Gordon, R.J. McIntosh, Modular transformations of Ramanujan's fifth and seventh order mock theta functions, Ramanujan J. 7 (2003), 193–222.

[19] B. Gordon, R.J. McIntosh, A survey of classical mock theta functions, in "Partitions, $q$-series, and modular forms", Dev. Math. 23 (2012) 95–144.

[20] D. Hickerson, On the seventh order mock theta functions, Invent. Math. 94 (1988) 661–677.

[21] D. Hickerson, A proof of the mock theta conjectures, Invent. Math. 94 (1988) 639–660.

[22] D.R. Hickerson, E.T. Mortenson, Hecke-type double sums, Appell–Lerch sums, and mock theta functions, I, Proc. London Math. Soc. (3) 109 (2014) 382–422.

[23] Z.-G. Liu, A $q$-series expansion formula and the Askey-Wilson polynomials, Ramanujan J. 30 (2013) 193–210.

[24] Z.-G. Liu, On the $q$-derivative and $q$-series expansions, Int. J. Number Theory 9 (2013) 2069–2089.

[25] J. Lovejoy, On identities involving the sixth order mock theta functions, Proc. Amer. Math. Soc. 138 (2010) 2547–2552.

[26] R.J. McIntosh, Second order mock theta functions, Canad. Math. Bull. 50 (2007) 284–290.

[27] R.J. McIntosh, The $H$ and $K$ family of mock theta functions, Canad. J. Math. 64 (2012) 935–960.

[28] R.J. McIntosh, New mock theta conjectures Part I, Ramanujan J. 46 (2018) 593–604.

[29] E.T. Mortenson, On three third order mock theta functions and Hecke-type double sums, Ramanujan J. 30 (2013) 279–308.

[30] S. Ramanujan, Collected Papers, Cambridge Univ. Press, 1927; Reprinted, Chelsea, New York, 1962.

[31] S. Ramanujan, The Lost Notebook and Other Unpublished Papers, Narosa, New Delhi, 1988.

[32] G.N. Watson, The final problem: An account of the mock theta functions, J. London Math. Soc. 11 (1936) 55–80.

[33] G.N. Watson, The mock theta functions (2), Proc. London Math. Soc. (2) 42 (1936) 274–304.

[34] S.P. Zwegers, Mock theta functions, Ph.D. Thesis, Universiteit Utrecht, 2002.

(S.-P. Cui) School of Mathematics and Statistics, Qinghai Normal University, Xining, Qinghai 810008, P.R. China; Academy of Plateau Science and Sustainability, Xining, Qinghai 810008, P.R. China

*E-mail address*: jiayoucui@163.com

(N.S.S. Gu) Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, P.R. China

*E-mail address*: gu@nankai.edu.cn