ON EDGE-PRIMITIVE GRAPHS WITH SOLUBLE EDGE-STABILIZERS

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Abstract

A graph is edge-primitive if its automorphism group acts primitively on the edge set, and 2-arc-transitive if its automorphism group acts transitively on the set of 2-arcs. In this paper, we present a classification for those edge-primitive graphs which are 2-arc-transitive and have soluble edge-stabilizers.

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1. Introduction

In this paper, all graphs are assumed to be finite and simple, and all groups are assumed to be finite.

A graph is a pair $\Gamma = (V, E)$ of a nonempty set V and a set E of 2-subsets of V. The elements in V and E are called the vertices and edges of Γ, respectively. For $v \in V$, the set $\Gamma(v) = \{u \in V \mid \{u, v\} \in E\}$ is called the neighborhood of v in Γ, while $|\Gamma(v)|$ is called the valency of v. We say that the graph Γ has valency d or Γ is d-regular if its vertices have equal valency d. For an integer $s \geq 1$, an s-arc in Γ is an $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices such that $\{v_{i-1}, v_i\} \in E$ for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. A 1-arc is also called an arc.

Let $\Gamma = (V, E)$ be a graph. A permutation g on V is called an automorphism of Γ if $\{u^g, v^g\} \in E$ for all $\{u, v\} \in E$. All automorphisms of Γ form a subgroup of the symmetric group $\mathrm{Sym}(V)$, denoted by $\mathsf{Aut}\Gamma$, which is called the automorphism group of Γ. The group $\mathsf{Aut}\Gamma$ has a natural action on E,

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namely, $\{u,v\}^g = \{u^g, v^g\}$ for $\{u,v\} \in E$ and $g \in \mathsf{Aut}\Gamma$. If this action is transitive, that is, for each pair of edges there exists some $g \in \mathsf{Aut}\Gamma$ mapping one edge to the other one, then Γ is called *edge-transitive*. Similarly, we may define the *vertex-transitivity*, *arc-transitivity* and *s-arc-transitivity* of Γ. The graph Γ is called *edge-primitive* if $\mathsf{Aut}\Gamma$ acts primitively on E, that is, Γ is edge-transitive and the stabilizer $(\mathsf{Aut}\Gamma)_{\{u,v\}}$ of some (and hence every) edge $\{u,v\}$ in $\mathsf{Aut}\Gamma$ is a maximal subgroup.

The class of edge-primitive graphs includes may famous graphs such as the Heawood graph, the Tutte's 8-cage, the Biggs-Smith graph, the Hoffman-Singleton graph, the Higman-Sims graph and the rank 3 graphs associated with the sporadic simple groups M_{22}, J_2, McL, Ru, Suz and Fi_{23}, and so on. In 1973, Weiss [34] determined all edge-primitive graphs of valency three. Up to isomorphism, all edge-primitive cubic graphs consist of the complete bipartite graph $\mathsf{K}_{3,3}$ and the first three graphs mentioned above. After that, edge-primitive graphs had received little attention until Giudici and Li [9] systematically investigated the existence and the general structure of such graphs in 2000. Giudici and Li's work has stimulated a lot of progress in the study of edge-primitive graphs, see [8, 11, 12, 18, 22, 25] for example. Also, their work reveals that those graphs associated with almost simple groups play an important role in the study of edge-primitive graphs. This is one of the main motivations of [22] and the present paper.

Let $\Gamma = (V, E)$ be an edge-primitive graph of valency no less than 3. Then, as observed in [9], Γ is also arc-transitive. If Γ is 2-arc-transitive then Praeger's reduction theorems [26, 27] will be effective tools for us to investigate the group-theoretic and graph-theoretic properties of Γ. However, Γ is not necessarily 2-arc-transitive; for example, by the Atlas [3], the sporadic Rudvalis group Ru is the automorphism group of a rank 3 graph, which is edge-primitive and of valency 2304 but not 2-arc-transitive. Using O'Nan-Scott Theorem for (quasi)primitive groups [26], Giudici and Li [9] gave a reduction theorem on the automorphism group of Γ. They proved that, as a primitive group on E, only four of the eight O'Nan-Scott types for primitive groups may occur for $\mathsf{Aut}\Gamma$, say SD, CD, PA and AS. They also considered the possible O'Nan-Scott types for $\mathsf{Aut}\Gamma$ acting on V, and presented constructions or examples to verify the existence of corresponding graphs. Then what will happen if we assume further that Γ is 2-arc-transitive? The third author of this paper showed that either $\mathsf{Aut}\Gamma$ is almost simple or Γ is a complete bipartite graph if Γ is 2-arc-transitive, see [22]. This stimulate our interest in classifying those edge-primitive graphs which are 2-arc-transitive.

In this paper, we present a classification result stated as follows.

THEOREM 1.1. *Let $\Gamma = (V, E)$ be a graph of valency $d \geq 6$, and let $G \leq \mathsf{Aut}\Gamma$ such that G acts primitively on the edge set and transitively on the 2-arc set of Γ. Assume further that G is almost simple and, for $\{u,v\} \in E$,* *the edge-stabilizer $G_{\{u,v\}}$ is soluble. Then either Γ is $(G,4)$-arc-transitive, or G, $G_{\{u,v\}}$, G_v and d are listed as in Table 1.*

REMARK. If Γ is edge-primitive and either 4-arc-transitive or of valency less than 6, then the edge-stabilizers must be soluble. The reader may find a complete list of such graphs in [11, 12, 18, 34]. For each triple $(G, G_v, G_{\{u,v\}})$ listed in Table 1, the coset graph $\mathsf{Cos}(G, G_v, G_{\{u,v\}})$, see Section 2 for the definition, is both $(G,2)$-arc-transitive and G-edge-primitive. □

G	$G_{\{u,v\}}$	G_v	d	Remark
$\mathrm{PSL}_4(2).2$	$2^4{:}\mathrm{S}_4$	$2^3{:}\mathrm{SL}_3(2)$	7	
$\mathrm{PSL}_5(2).2$	$[2^8]{:}\mathrm{S}_3^2.2$	$2^6{:}(\mathrm{S}_3\times\mathrm{SL}_3(2))$	7	
$\mathrm{F}_4(2).2$	$[2^{22}]{:}\mathrm{S}_3^2.2$	$[2^{20}].(\mathrm{S}_3\times\mathrm{SL}_3(2))$	7	
$\mathrm{PSL}_4(3).2$	$3_+^{1+4}{:}(2\mathrm{S}_4\times 2)$	$3^3{:}\mathrm{SL}_3(3)$	13	
$\mathrm{PSL}_4(3).2^2$	$3_+^{1+4}{:}(2\mathrm{S}_4\times\mathbb{Z}_2^2)$	$3^3{:}(\mathrm{SL}_3(3)\times\mathbb{Z}_2)$	13	
$\mathrm{PSL}_5(3).2$	$[3^8]{:}(2\mathrm{S}_4)^2.2$	$3^6.2\mathrm{S}_4.\mathrm{SL}_3(3)$	13	
S_p	$\mathbb{Z}_p{:}\mathbb{Z}_{p-1}$	$\mathrm{PSL}_2(p)$	$p{+}1$	$p\in\{7,11\}$
M_{11}	$3^2{:}\mathrm{Q}_8.2$	M_{10}	10	K_{11}
J_1	$\mathbb{Z}_{11}{:}\mathbb{Z}_{10}$	$\mathrm{PSL}_2(11)$	12	
$\mathrm{J}_3.2$	$\mathbb{Z}_{19}{:}\mathbb{Z}_{18}$	$\mathrm{PSL}_2(19)$	20	
$\mathrm{O'N}.2$	$\mathbb{Z}_{31}{:}\mathbb{Z}_{30}$	$\mathrm{PSL}_2(31)$	32	
B	$\mathbb{Z}_{19}{:}\mathbb{Z}_{18}\times\mathbb{Z}_2$	$\mathrm{PGL}_2(19)$	20	
B	$\mathbb{Z}_{23}{:}\mathbb{Z}_{11}\times\mathbb{Z}_2$	$\mathrm{PSL}_2(23)$	24	
M	$\mathbb{Z}_{41}{:}\mathbb{Z}_{40}$	$\mathrm{PSL}_2(41)$	42	
$\mathrm{PSL}_2(19)$	D_{20}	$\mathrm{PSL}_2(5)$	6	
$\mathrm{A}_6.2, \mathrm{A}_6.2^2$	$\mathbb{Z}_5{:}[4], \mathbb{Z}_{10}{:}\mathbb{Z}_4$	$\mathrm{PSL}_2(5), \mathrm{PGL}_2(5)$	6	$\mathsf{K}_{6,6}$, $G\not\cong\mathrm{S}_6$
$\mathrm{PGL}_2(11)$	D_{20}	$\mathrm{PSL}_2(5)$	6	
$\mathrm{PSL}_3(r)$ $\mathrm{PSL}_3(r).2$	$3^2{:}\mathrm{Q}_8$ $3^2{:}\mathrm{Q}_8.2$	$\mathrm{PSL}_2(9)$ $\mathrm{PGL}_2(9)$	10	r is a prime with $r\equiv 4,16,31,34 \mod 45$
$\mathrm{PSU}_3(r)$ $\mathrm{PSU}_3(r).2$	$3^2{:}\mathrm{Q}_8$ $3^2{:}\mathrm{Q}_8.2$	$\mathrm{PSL}_2(9)$ $\mathrm{PGL}_2(9)$	10	r is a prime with $r\equiv 11,14,29,41 \mod 45$
$\mathrm{HS}.2$	$[5^3]{:}[2^5]$	$\mathrm{PSU}_3(5){:}2$	126	
Ru	$[5^3]{:}[2^5]$	$\mathrm{PSU}_3(5){:}2$	126	
M_{10}	$\mathbb{Z}_8{:}\mathbb{Z}_2$	$3^2{:}\mathrm{Q}_8$	9	K_{10}
$\mathrm{PSL}_3(3).2$	$\mathrm{GL}_2(3){:}2$	$3^2{:}\mathrm{GL}_2(3)$	9	
J_1	$\mathbb{Z}_7{:}\mathbb{Z}_6$	$\mathbb{Z}_2^3{:}\mathbb{Z}_7{:}\mathbb{Z}_3$	8	
$\mathrm{PSL}_2(p^f).[o]$	$\mathrm{D}_{\frac{2(p^f-1)}{(2,p-1)}}.[o]$	$\mathbb{Z}_p^f{:}\mathbb{Z}_{\frac{p^f-1}{(2,p-1)}}.[o]$	p^f	K_{p^f+1}, $o\mid(2,p-1)f$
$\mathrm{Sz}(2^f).o$	$\mathrm{D}_{2(2^f-1)}.o$	$\mathbb{Z}_2^f{:}\mathbb{Z}_{2^f-1}.o$	2^f	f is odd, $o\mid f$

TABLE 1. Graphs.

2. Preliminaries

Let G be a finite group and $H, K \leq G$ with $|K : (H\cap K)| = 2$ and $\cap_{g\in G} H^g = 1$, and let $[G:H] = \{Hx \mid x\in G\}$. We define a graph $\mathsf{Cos}(G,H,K)$ on $[G:H]$ such that $\{Hx,Hy\}$ is an edge if and only if $yx^{-1}\in HKH\setminus H$. The group G can be viewed as a subgroup of $\mathsf{Aut}\mathsf{Cos}(G,H,K)$, where G acts on $[G:H]$ by right multiplication. Then $\mathsf{Cos}(G,H,K)$ is G-arc-transitive and, for $x\in K\setminus H$, the edge $\{H,Hx\}$ has stabilizer K in G. Thus $\mathsf{Cos}(G,H,K)$ is G-edge-primitive if and only if K is maximal in G.

Assume that $\Gamma=(V,E)$ is a G-edge-primitive graph of valency $d\geq 3$. Then Γ is G-arc-transitive by [9, Lemma 3.4]. Take an edge $\{u,v\}\in E$, let $H=G_v$ and $K=G_{\{u,v\}}$. Then K is maximal in G, and $H\cap K=G_{uv}$, which has index 2 in K. Noting that $\cap_{g\in G}H^g$ fixes V pointwise, $\cap_{g\in G}H^g=1$. Further, $v^g\mapsto G_vg$, $\forall g\in G$ gives an isomorphism from Γ to $\mathsf{Cos}(G,H,K)$. Then, by [5, Theorem 2.1], the following lemma holds.

LEMMA 2.1. *Let $\Gamma=(V,E)$ be a connected graph of valency $d\geq 3$, and $G\leq\mathsf{Aut}\Gamma$. Then Γ is both $(G,2)$-arc-transitive and G-edge-primitive if and only if $\Gamma\cong\mathsf{Cos}(G,H,K)$ for some subgroups H and K of G satisfying*

(1) *$|K:(H\cap K)|=2$, $\cap_{g\in G}H^g=1$ and K is maximal in G;*

(2) *H acts 2-transitively on $[H:(H\cap K)]$ by right multiplication.*

Let $\Gamma=(V,E)$ be a connected graph of valency at least 3, $\{u,v\}\in E$ and $G\leq\mathsf{Aut}\Gamma$. Assume that Γ is (G,s)-arc-transitive for some $s\geq 1$, that is, G acts transitively on the s-arc set of Γ. Then G_v acts transitively on the neighborhood $\Gamma(v)$ of v in Γ. Let $\mathrm{G}_v^{\Gamma(v)}$ be the transitive permutation group induced by G_v on $\Gamma(v)$, and let $G_v^{[1]}$ be the kernel of G_v acting on $\Gamma(v)$. Then $G_v^{\Gamma(v)}\cong G_v/G_v^{[1]}$. Considering the action of G_{uv} on $\Gamma(v)$, we have

$$(G_v^{\Gamma(v)})_u=G_{uv}^{\Gamma(v)}\cong G_{uv}/G_v^{[1]}.$$

Similarly, $(G_u^{\Gamma(u)})_v=G_{uv}^{\Gamma(u)}\cong G_{uv}/G_u^{[1]}$. Since G is transitive on the arcs of Γ, there is some element in G interchanging u and v. This implies that

$$|G_{\{u,v\}}{:}G_{uv}|=2 \text{ and } (G_v^{\Gamma(v)})_u\cong(G_u^{\Gamma(u)})_v.$$

Set $G_{uv}^{[1]}=G_u^{[1]}\cap G_v^{[1]}$. Then $G_{uv}^{[1]}$ is the kernel of G_{uv} acting on $\Gamma(u)\cup\Gamma(v)$ and, noting that $G_{uv}/(G_u^{[1]}\cap G_v^{[1]})\lesssim(G_{uv}/G_u^{[1]})\times(G_{uv}/G_v^{[1]})$, we have

$$G_{uv}/G_{uv}^{[1]}=G_{uv}/(G_u^{[1]}\cap G_v^{[1]})\lesssim(G_v^{\Gamma(v)})_u\times(G_u^{\Gamma(u)})_v.$$

Since $G_v^{[1]}\trianglelefteq G_{uv}$, we know that $G_v^{[1]}$ induces a normal subgroup $(G_v^{[1]})^{\Gamma(u)}$ of $(G_u^{\Gamma(u)})_v$. In particular,

$$G_v^{[1]}/G_{uv}^{[1]}\cong(G_v^{[1]})^{\Gamma(u)}\trianglelefteq(G_u^{\Gamma(u)})_v.$$

Writing $G_v^{[1]}$, G_{uv} and G_v in group extensions, the next lemma follows.

LEMMA 2.2. (1) *$G_v^{[1]}=G_{uv}^{[1]}.(G_v^{[1]})^{\Gamma(u)}$, $(G_v^{[1]})^{\Gamma(u)}\trianglelefteq(G_u^{\Gamma(u)})_v$.*

(2) $G_{uv} = (G_{uv}^{[1]}.(G_v^{[1]})^{\Gamma(u)}).(G_v^{\Gamma(v)})_u$, $G_v = (G_{uv}^{[1]}.(G_v^{[1]})^{\Gamma(u)}).G_v^{\Gamma(v)}$.

(3) *If* $G_{uv}^{[1]} = 1$ *then* $G_{uv} \lesssim (G_v^{\Gamma(v)})_u \times (G_u^{\Gamma(u)})_v$.

By [32], $s \leq 7$, and if $s \geq 2$ then $G_{uv}^{[1]}$ is a p-group for some prime p, refer to [7]. Thus Lemma 2.2 yields a fact as follows.

COROLLARY 2.3. *Let* $\Gamma = (V, E)$ *be a connected* $(G, 2)$*-arc-transitive graph, and* $\{u, v\} \in E$. *Then* $G_{\{u,v\}}$ *is soluble if and only if* $(G_v^{\Gamma(v)})_u$ *is soluble, and* G_v *is soluble if and only if* $G_v^{\Gamma(v)}$ *is soluble.*

Choose s maximal as possible, that is, Γ is (G, s)-arc-transitive but not $(G, s+1)$-arc-transitive. In this case, Γ is said to be (G, s)-transitive. If further $G_{uv}^{[1]} \neq 1$, then one can read out the vertex-stabilizer G_v from [31, 33] for $s \geq 4$ and from [29] for $2 \leq s \leq 3$. In particular, we have the following result from [29, 33].

THEOREM 2.4. *Let* $\Gamma = (V, E)$ *be a connected* (G, s)*-transitive graph of valency at least* 3, *and* $\{u, v\} \in E$. *Assume that* $s \geq 2$.

(1) *If* $G_{uv}^{[1]} = 1$ *then* $s = 2$ *or* 3.

(2) *If* $G_{uv}^{[1]} \neq 1$ *then* $G_{uv}^{[1]}$ *is a p-group for some prime p*, $\mathrm{PSL}_n(q) \trianglelefteq G_v^{\Gamma(v)}$, $|\Gamma(v)| = \frac{q^n-1}{q-1}$ *and* $6 \neq s \leq 7$, *where* $n \geq 2$ *and* $q = p^f$ *for some integer* $f \geq 1$; *moreover, either*

  (i) $n = 2$ *and* $s \geq 4$; *or*

  (ii) $n \geq 3$, $s \leq 3$ *and* $\mathbf{O}_p(G_v)$ *is given as in Table 2, where* $\mathbf{O}_p(G_v)$ *is the maximal normal p-subgroup of* G_v.

$\mathbf{O}_p(G_v)$	$G_{uv}^{[1]}$	s	n	q	G_v
$\mathbb{Z}_p^{n(n-1)f}$	$\mathbb{Z}_p^{(n-1)^2f}$	3			$\mathrm{SL}_{n-1}(q)\times\mathrm{SL}_n(q) \trianglelefteq G_v/\mathbf{O}_p(G_v)$
$\mathbb{Z}_p^{nf}$	$\mathbb{Z}_p^{f}$	2			$a.\mathrm{PSL}_n(q) \trianglelefteq G_v/\mathbf{O}_p(G_v)$ with $a \mid q-1$
$\mathbb{Z}_p^{\frac{n(n-1)f}{2}}$	$\mathbb{Z}_p^{\frac{(n-1)(n-2)f}{2}}$	2			$a.\mathrm{PSL}_n(q) \trianglelefteq G_v/\mathbf{O}_p(G_v)$ with $a \mid q-1$
$[q^{20}]$	$[q^{18}]$	3	3	even	$\mathrm{SL}_2(q)\times\mathrm{SL}_3(q) \trianglelefteq G_v/\mathbf{O}_p(G_v)$
$[3^6]$	$\mathbb{Z}_3^4$	2	3	3	$[3^6]{:}\mathrm{SL}_3(3)$
$\mathbb{Z}_2^{n+1}$	$\mathbb{Z}_2^2$	2		2	$\mathbb{Z}_2^{n+1}{:}\mathrm{SL}_n(2)$
$\mathbb{Z}_2^{11}, \mathbb{Z}_2^{14}$	$\mathbb{Z}_2^{8}, \mathbb{Z}_2^{11}$	2	4	2	$\mathbb{Z}_2^{11}{:}\mathrm{SL}_4(2), \mathbb{Z}_2^{14}{:}\mathrm{SL}_4(2)$
$[2^{30}]$	$[2^{26}]$	2	5	2	$[2^{30}]{:}\mathrm{SL}_5(2)$

TABLE 2.

LEMMA 2.5. *Let* $\Gamma = (V, E)$ *be a connected* $(G, 2)$*-arc-transitive graph, and* $\{u, v\} \in E$. *If* r *is a prime divisor of* $|\Gamma(v)|$ *then* $\mathbf{O}_r(G_v^{[1]}) = 1$, $\mathbf{O}_r(G_{uv}) = 1$, *and either* $\mathbf{O}_r(G_v) = 1$, *or* $\mathbf{O}_r(G_v) \cong \mathbb{Z}_r^e \cong \mathrm{soc}(G_v^{\Gamma(v)})$ *and* $|\Gamma(v)| = r^e$ *for some integer* $e \geq 1$.

PROOF. Since Γ is $(G,2)$-arc-transitive, $G_v^{\Gamma(v)}$ is a 2-transitive group, and thus G_{uv} is transitive on $\Gamma(v)\setminus\{u\}$. Since $\mathbf{O}_r(G_{uv})\trianglelefteq G_{uv}$, all $\mathbf{O}_r(G_{uv})$-orbits on $\Gamma(v)\setminus\{u\}$ have the same size. Noting that $|\Gamma(v)\setminus\{u\}|$ is coprime to r, it follows that $\mathbf{O}_r(G_{uv})\leq G_v^{[1]}$. Since $G_v^{[1]}\trianglelefteq G_{uv}$, we have $\mathbf{O}_r(G_v^{[1]})\leq \mathbf{O}_r(G_{uv})$, and so $\mathbf{O}_r(G_v^{[1]})=\mathbf{O}_r(G_{uv})$. Similarly, considering the action of G_{uv} on $\Gamma(u)\setminus\{v\}$, we get $\mathbf{O}_r(G_u^{[1]})=\mathbf{O}_r(G_{uv})$. Then $\mathbf{O}_r(G_u^{[1]})=\mathbf{O}_r(G_{uv})=\mathbf{O}_r(G_v^{[1]})\leq G_{uv}^{[1]}$. By Theorem 2.4, either $G_{uv}^{[1]}=1$, or $G_{uv}^{[1]}$ is a nontrivial p-group for a prime divisor p of $|\Gamma(v)|-1$. It follows that $\mathbf{O}_r(G_u^{[1]})=\mathbf{O}_r(G_{uv})=\mathbf{O}_r(G_v^{[1]})=1$.

Note that $\mathbf{O}_r(G_v)G_v^{[1]}/G_v^{[1]}\cong\mathbf{O}_r(G_v)/(\mathbf{O}_r(G_v)\cap G_v^{[1]})$. Clearly, $\mathbf{O}_r(G_v)\cap G_v^{[1]}\leq\mathbf{O}_r(G_v^{[1]})$, we have $\mathbf{O}_r(G_v)\cap G_v^{[1]}=1$. It follows that $\mathbf{O}_r(G_v)\cong\mathbf{O}_r(G_v)G_v^{[1]}/G_v^{[1]}\trianglelefteq G_v/G_v^{[1]}\cong \mathrm{G}_v^{\Gamma(v)}$. Thus $\mathbf{O}_r(G_v)$ is isomorphic to a normal r-subgroup of $\mathrm{G}_v^{\Gamma(v)}$. This implies that either $\mathbf{O}_r(G_v)=1$, or $\mathrm{G}_v^{\Gamma(v)}$ is an affine 2-transitive group of degree r^e for some e. Thus the lemma follows. □

Let $a\geq 2$ and $f\geq 1$ be integers. A prime divisor r of a^f-1 is primitive if r is not a divisor of a^e-1 for all $1\leq e<f$. By Zsigmondy's theorem [37], if $f>1$ and a^f-1 has no primitive prime divisor then $a^f=2^6$, or $f=2$ and $a=2^t-1$ for some prime t. Assume that a^f-1 has a primitive prime divisor r. Then a has order f modulo r. Thus f is a divisor of $r-1$, and if r is a divisor of $a^{f'}-1$ for some $f'\geq 1$ then f is a divisor of f'. Thus we have the following lemma.

LEMMA 2.6. *Let $a\geq 2$, $f\geq 1$ and $f'\geq 1$ be integers. If a^f-1 has a primitive prime divisor r then f is a divisor of $r-1$, and r is a divisor of $a^{f'}-1$ if and only if f is a divisor of f'. If $f\geq 3$ then a^f-1 has a prime divisor no less than* 5.

We end this section with a fact on finite primitive groups.

LEMMA 2.7. *Assume that G is a finite primitive group with a point-stabilizer H. If H has a normal Sylow subgroup $P\neq 1$, then P is also a Sylow subgroup of G.*

PROOF. Assume that $P\neq 1$ is a normal Sylow subgroup of H. Clearly, P is not normal in G. Take a Sylow subgroup Q of G with $P\leq Q$. Then $H\leq\langle\mathbf{N}_Q(P),H\rangle\leq\mathbf{N}_G(P)\neq G$. Since H is maximal in G, we have $H=\langle\mathbf{N}_Q(P),H\rangle$ and so $\mathbf{N}_Q(P)\leq H$. It follows that $\mathbf{N}_Q(P)=P$, and hence $P=Q$. Then the lemma follows. □

3. Some restrictions on stabilizers

In Sections 4 and 5, we shall prove Theorem 1.1 using the result given in [18] which classifies finite primitive groups with soluble point-stabilizers. Let

$\Gamma=(V,E)$ be a graph of valency $d\ge 6$, $\{u,v\}\in E$ and $G\le \mathsf{Aut}\Gamma$. Assume that G is almost simple, $G_{\{u,v\}}$ is soluble, Γ is G-edge-primitive and $(G,2)$-arc-transitive. Clearly, each nontrivial normal subgroup of G acts transitively on the edge set E. Choose a minimal X among the normal subgroups of G which act primitively on E. By the choice of X, we have $\mathrm{soc}(X)=\mathrm{soc}(G)$, $X_{\{u,v\}}=X\cap G_{\{u,v\}}$, $G=XG_{\{u,v\}}$ and $G/X=XG_{\{u,v\}}/X\cong G_{\{u,v\}}/X_{\{u,v\}}$. Then, considering the restrictions on both $X_{\{u,v\}}$ and X_v caused by the 2-arc-transitivity of Γ, we may work out the pair $(X,X_{\{u,v\}})$ from [18, Theorem 1.1], and then determine the group G and the graph Γ. Thus we make the following assumptions.

Hypothesis 3.1. Let $\Gamma=(V,E)$ be a G-edge-primitive graph of valency $d\ge 6$, and $\{u,v\}\in E$, where G is an almost simple group with socle T. Assume that

(1) Γ is $(G,2)$-arc-transitive, and the edge-stabilizer $G_{\{u,v\}}$ is soluble;
(2) G has a normal subgroup X such that $\mathrm{soc}(X)=T$, $X_{\{u,v\}}$ is maximal in X, and $(X,X_{\{u,v\}})$ is one of the pairs (G_0,H_0) listed in [18, Tables 14-20].

□

For the group X in Hypothesis 3.1, we have $1\ne X_v^{\Gamma(v)}\trianglelefteq G_v^{\Gamma(v)}$. Note that $G_v^{\Gamma(v)}$ is 2-transitive (on $\Gamma(v)$). Then $G_v^{\Gamma(v)}$ is affine or almost simple, see [4, Theorem 4.1B] for example. It follows that $\mathrm{soc}(G_v^{\Gamma(v)})=\mathrm{soc}(X_v^{\Gamma(v)})$.

3.1. Assume that G_v is insoluble. Then $G_v^{\Gamma(v)}$ is an almost simple 2-transitive group (on $\Gamma(v)$). Recall that $\mathrm{soc}(G_v^{\Gamma(v)})=\mathrm{soc}(X_v^{\Gamma(v)})$. Checking the point-stabilizers of almost simple 2-transitive groups (see [17, Table 2.1] for example), since $(G_v^{\Gamma(v)})_u$ is soluble, we conclude that either $X_v^{\Gamma(v)}$ is 2-transitive, or $G_v^{\Gamma(v)}\cong \mathrm{PSL}_2(8).3$ and $d=28$. (For a complete list of finite 2-transitive groups, the reader may refer to [2, Tables 7.3 and 7.4].)

Lemma 3.2. *Suppose that Hypothesis 3.1 holds. If $d=28$ then $G_v^{\Gamma(v)}$ is not isomorphc to* $\mathrm{PSL}_2(8).3$.

Proof. Suppose that $G_v^{\Gamma(v)}\cong \mathrm{PSL}_2(8).3$ and $d=28$. Note that $X_{uv}^{[1]}\le G_{uv}^{[1]}=1$, see Theorem 2.4. Thus $X_{uv}\lesssim (X_v^{\Gamma(v)})_u\times(X_u^{\Gamma(u)})_v$ by Lemma 2.2.

Assume that $X_v^{\Gamma(v)}\cong \mathrm{PSL}_2(8)$. Then $(X_v^{\Gamma(v)})_u\cong \mathrm{D}_{18}$, and $X_{uv}\cong \mathrm{D}_{18}$, $(\mathbb{Z}_3\times\mathbb{Z}_9){:}\mathbb{Z}_2$, $(\mathbb{Z}_9\times\mathbb{Z}_9){:}\mathbb{Z}_2$ or $\mathrm{D}_{18}\times\mathrm{D}_{18}$. In particular, the unique Sylow 3-subgroup of $X_{\{u,v\}}=X_{uv}.2$ is isomorphic to $\mathbb{Z}_m\times\mathbb{Z}_9$, where $m=1$, 3 or 9. Checking the primitive groups listed in [18, Tables 14-20], we know that only the pairs $(\mathrm{PSL}_2(q),\mathrm{D}_{\frac{2(q\pm1)}{(2,q-1)}})$ possibly meet our requirements on $X_{\{u,v\}}$, yielding $X_{\{u,v\}}\cong \mathrm{D}_{\frac{2(q\pm1)}{(2,q-1)}}$. Then $\mathrm{D}_{36}\cong X_{\{u,v\}}\cong \mathrm{D}_{\frac{2(q\pm1)}{(2,q-1)}}$. Calculation shows that $q=37$; however, $\mathrm{PSL}_2(37)$ has no subgroup which has a quotient $\mathrm{PSL}_2(8)$, a contradiction.

Now let $X_v^{\Gamma(v)} = G_v^{\Gamma(v)} \cong \mathrm{PSL}_2(8).3$. Then $(X_v^{\Gamma(v)})_u \cong (X_u^{\Gamma(u)})_v \cong \mathbb{Z}_9{:}\mathbb{Z}_6$ and $X_{uv} \lesssim \mathbb{Z}_9{:}\mathbb{Z}_6{\times}\mathbb{Z}_9{:}\mathbb{Z}_6$. In particular, a Sylow 2-subgroup of $X_{\{u,v\}} = X_{uv}.2$ is not a cyclic group of order 8, and the unique Sylow 3-subgroup of $X_{\{u,v\}}$ is nonabelian and contains elements of order 9. Since $X_{\{u,v\}} = X_{uv}.2 = X_v^{[1]}.(X_v^{\Gamma(v)})_u.2$ and $X_v^{[1]} \cong (X_v^{[1]})^{\Gamma(u)} \trianglelefteq (X_u^{\Gamma(u)})_v$, we have $|X_{\{u,v\}}| = 2^2\cdot 3^3$, $2^2\cdot 3^4$, $2^2\cdot 3^5$, $2^2\cdot 3^6$, $2^3\cdot 3^5$ or $2^3\cdot 3^6$. Checking the Tables 14-20 given in [18], we conclude that $X = \mathrm{G}_2(3).2$, and $X_{\{u,v\}} \cong [3^6]{:}\mathrm{D}_8$. In this case, $X_v^{[1]} \cong \mathbb{Z}_9{:}\mathbb{Z}_6$ and $X_v \cong \mathbb{Z}_9{:}\mathbb{Z}_6.\mathrm{PSL}_2(8).3$; however, X has no such subgroup by the Atlas [3], a contradiction. This completes the proof. □

By Lemma 3.2, combining with Theorem 2.4, the next lemma follows from checking the point-stabilizers of finite almost simple 2-transitive groups, refer to [17, Table 2.1].

LEMMA 3.3. *Suppose that Hypothesis 3.1 holds and $G_v^{\Gamma(v)}$ is almost simple. Then one of the following holds:*

(1) $G_v^{\Gamma(v)} = X_v^{\Gamma(v)} = \mathrm{PSL}_3(2)$ *or* $\mathrm{PSL}_3(3)$*, and* $d = 7$ *or* 13*, respectively;*

(2) $\mathrm{soc}(X_v^{\Gamma(v)}) = \mathrm{PSL}_2(q)$ *with* $q > 4$*, and* $d = q{+}1$*;*

(3) $G_{uv}^{[1]} = 1$*,* $\mathrm{soc}(X_v^{\Gamma(v)}) = \mathrm{PSU}_3(q)$ *with* $q > 2$*, and* $d = q^3{+}1$*;*

(4) $G_{uv}^{[1]} = 1$*,* $\mathrm{soc}(X_v^{\Gamma(v)}) = \mathrm{Sz}(q)$ *with* $q = 2^{2n+1} > 2$*, and* $d = q^2{+}1$*;*

(5) $G_{uv}^{[1]} = 1$*,* $\mathrm{soc}(X_v^{\Gamma(v)}) = \mathrm{Ree}(q)$ *with* $q = 3^{2n+1} > 3$*, and* $d = q^3{+}1$*.*

In particular, Γ is $(X,2)$-arc-transitive.

Recall that the Fitting subgroup $\mathrm{Fit}(H)$ of a finite group H is the direct product of $\mathbf{O}_r(H)$, where r runs over the set of prime divisors of $|H|$.

LEMMA 3.4. *Suppose that Hypothesis 3.1 holds and (2) or (5) of Lemma 3.3 occurs. Let $q = p^f$ for some prime p. Assume that $X_{uv}^{[1]} = 1$. Then* $\mathrm{Fit}(X_{uv}) = \mathbf{O}_p(X_{uv})$*, and either* $\mathrm{Fit}(X_{uv}) = \mathrm{Fit}(X_{\{u,v\}})$ *or* $\mathrm{Fit}(X_{\{u,v\}}) = \mathrm{Fit}(X_{uv}).2$*; in particular,* $|\mathrm{Fit}(X_{\{u,v\}}) : \mathbf{O}_p(X_{\{u,v\}})| \le 2$*.*

PROOF. Let r be a prime divisor of $|X_{uv}|$. Then $\mathbf{O}_r(X_{uv})$ is normal in X_{uv}. Since Γ is $(X,2)$-arc-transitive, X_{uv} acts transitively on $\Gamma(v)\setminus\{u\}$. Thus all $\mathbf{O}_r(X_{uv})$-orbits (on $\Gamma(v)\setminus\{u\}$) have equal size, which is a power of r and a divisor of $|\Gamma(v)\setminus\{u\}|$. Note that $|\Gamma(v)\setminus\{u\}| = d-1$, which is a power of p. It follows that either $r = p$ or $\mathbf{O}_r(X_{uv}) = 1$. Then $\mathrm{Fit}(X_{uv}) = \mathbf{O}_p(X_{uv})$.

Note that X_{uv} is normal in $X_{\{u,v\}}$ as $|X_{\{u,v\}} : X_{uv}| = 2$. Since $\mathbf{O}_p(X_{uv})$ is a characteristic subgroup of X_{uv}, it follows that $\mathbf{O}_p(X_{uv})$ is normal in $X_{\{u,v\}}$, and so $\mathbf{O}_p(X_{uv}) \le \mathbf{O}_p(X_{\{u,v\}}) \le \mathrm{Fit}(X_{\{u,v\}})$. For each odd prime divisor r of $|X_{\{u,v\}}|$, since $|X_{\{u,v\}} : X_{uv}| = 2$, we have $\mathbf{O}_r(X_{\{u,v\}}) \le X_{uv}$, and so $\mathbf{O}_r(X_{\{u,v\}}) = \mathbf{O}_r(X_{uv})$. It follows that

$$\mathrm{Fit}(X_{\{u,v\}}) = \mathrm{Fit}(X_{uv})\mathbf{O}_2(X_{\{u,v\}}) = \mathbf{O}_p(X_{uv})\mathbf{O}_2(X_{\{u,v\}}).$$

In particular, $\mathbf{O}_p(X_{uv}) = \mathbf{O}_p(X_{\{u,v\}})$ if $p \neq 2$.

It is easily shown that $X_{uv} \cap \mathbf{O}_2(X_{\{u,v\}}) = \mathbf{O}_2(X_{uv})$. If $X_{uv} \geq \mathbf{O}_2(X_{\{u,v\}})$ then $p = 2$, $\mathrm{Fit}(X_{\{u,v\}}) = \mathbf{O}_2(X_{\{u,v\}}) = \mathrm{Fit}(X_{uv})$, and the lemma is true. Assume that $\mathbf{O}_2(X_{\{u,v\}}) \not\leq X_{uv}$. Since $|X_{\{u,v\}} : X_{uv}| = 2$, we have $X_{\{u,v\}} = X_{uv}\mathbf{O}_2(X_{\{u,v\}})$. Then

$$\begin{aligned} 2|X_{uv}| = |X_{\{u,v\}}| &= |X_{uv}||\mathbf{O}_2(X_{\{u,v\}}) : (X_{uv} \cap \mathbf{O}_2(X_{\{u,v\}}))| \\ &= |X_{uv}||\mathbf{O}_2(X_{\{u,v\}}) : \mathbf{O}_2(X_{uv})|, \end{aligned}$$

yielding $|\mathbf{O}_2(X_{\{u,v\}}) : \mathbf{O}_2(X_{uv})| = 2$. If $p = 2$ then $\mathrm{Fit}(X_{\{u,v\}}) = \mathbf{O}_2(X_{\{u,v\}})$ and $\mathrm{Fit}(X_{uv}) = \mathbf{O}_2(X_{uv})$. If $p \neq 2$ then $\mathbf{O}_2(X_{uv}) = 1$, $|\mathbf{O}_2(X_{\{u,v\}})| = 2$, and so $\mathrm{Fit}(X_{\{u,v\}}) = \mathbf{O}_p(X_{uv}) \times \mathbb{Z}_2$. This completes the proof. □

3.2. Assume that Hypothesis 3.1 holds and G_v is soluble. Then $G_v^{\Gamma(v)}$ is an affine 2-transitive group. Let $\mathrm{soc}(G_v^{\Gamma(v)}) = \mathbb{Z}_p^f$. Then $d = p^f$. Recalling that $d \geq 6$, we have $G_{uv}^{[1]} = 1$ by Theorem 2.4, and so $G_{uv} \lesssim (G_v^{\Gamma(v)})_u \times (G_u^{\Gamma(u)})_v$. If G_{uv} is abelian then Γ is known by [22]. Thus we assume further that G_{uv} is not abelian. Then $(G_v^{\Gamma(v)})_u$ is nonabelian, and so $(G_v^{\Gamma(v)})_u \not\leq \mathrm{GL}_1(p^f)$; in particular, $f > 1$. Since $(G_v^{\Gamma(v)})_u$ is soluble, by [2, Table 7.3], we have the following lemma.

LEMMA 3.5. *Suppose that Hypothesis 3.1 holds, G_v is soluble and G_{uv} is not abelian. Let $\mathrm{soc}(G_v^{\Gamma(v)}) = \mathbb{Z}_p^f$, where p is a prime. Then $f > 1$, and one of the following holds:*

(1) *$f = 2$, and either $\mathrm{SL}_2(3) \trianglelefteq (G_v^{\Gamma(v)})_u \leq \mathrm{GL}_2(p)$ and $p \in \{3, 5, 7, 11, 23\}$, or $p = 3$ and $(G_v^{\Gamma(v)})_u = \mathrm{Q}_8$;*

(2) *$2_+^{1+4}{:}\mathbb{Z}_5 \leq (G_v^{\Gamma(v)})_u \leq 2_+^{1+4}.(\mathbb{Z}_5{:}\mathbb{Z}_4) < 2_+^{1+4}.\mathrm{S}_5$, and $p^f = 3^4$;*

(3) *$(G_v^{\Gamma(v)})_u \not\leq \mathrm{GL}_1(p^f)$, $(G_v^{\Gamma(v)})_u \leq \Gamma\mathrm{L}_1(p^f)$ and $|(G_v^{\Gamma(v)})_u|$ is divisible by $p^f - 1$.*

Consider the case (3) in Lemma 3.5. Write

$$\Gamma\mathrm{L}_1(p^f) = \langle \tau, \sigma \mid \tau^{p^f-1} = 1 = \sigma^f, \sigma^{-1}\tau\sigma = \tau^p \rangle.$$

Let $\langle \tau \rangle \cap (G_v^{\Gamma(v)})_u = \langle \tau^m \rangle$, where $m \mid (p^f - 1)$. Then

$$(G_v^{\Gamma(v)})_u / \langle \tau^m \rangle \cong \langle \tau \rangle (G_v^{\Gamma(v)})_u / \langle \tau \rangle \lesssim \langle \sigma \rangle.$$

Set $(G_v^{\Gamma(v)})_u / \langle \tau^m \rangle \cong \langle \sigma^e \rangle$ for some divisor e of f. Then

$$(G_v^{\Gamma(v)})_u \cong \mathbb{Z}_{\frac{p^f-1}{m}}.\mathbb{Z}_{\frac{f}{e}}.$$

Choose $\tau^l\sigma^k \in (G_v^{\Gamma(v)})_u$ with $(G_v^{\Gamma(v)})_u = \langle \tau^m \rangle \langle \tau^l\sigma^k \rangle$. Then $(\tau^l\sigma^k)^{\frac{f}{e}} \in \langle \tau^m \rangle$ but $(\tau^l\sigma^k)^j \notin \langle \tau^m \rangle$ for $1 \leq j < \frac{f}{e}$. It follows that σ^k has order $\frac{f}{e}$. Then

$\sigma^k = \sigma^{ie}$ for some i with $(i, \frac{f}{e}) = 1$, and then $(\sigma^k)^{i'} = \sigma^e$ for some i'. Thus, replacing $\tau^l\sigma^k$ by a power of it if necessary, we may let $k = e$. Then

$$(G_v^{\Gamma(v)})_u = \langle \tau^m \rangle \langle \tau^l \sigma^e \rangle.$$

Further, $(G_v^{\Gamma(v)})_u = \langle \tau^m \rangle \langle (\tau^m)^i \tau^l \sigma^e \rangle$ for an arbitrary integer i, thus we may assume further $0 \le l < m$. By [6, Proposition 15.3], letting $\pi(n)$ be the set of prime divisors of a positive integer n, we have

(⊛) $\pi(m) \subseteq \pi(p^e - 1)$, $me \mid f$ and $(m, l) = 1$; in particular, $m = 1$ if $l = 0$.

Suppose that X_{uv} is nonabelian. (The case where X_{uv} is abelian is left in Section 5.) Since $X_{uv}^{[1]} \le \mathrm{G}_{uv}^{[1]} = 1$, we have

$$X_v^{[1]} \trianglelefteq (X_u^{\Gamma(u)})_v \cong (X_v^{\Gamma(v)})_u, \ X_{uv} \lesssim (X_u^{\Gamma(u)})_v \times (X_v^{\Gamma(v)})_u.$$

This yields that $(X_v^{\Gamma(v)})_u$ is nonabelian. Then a limitation on $\pi(|X_{uv}|)$ is given as follows.

Lemma 3.6. *Assume that Lemma 3.5 (3) holds and X_{uv} is nonabelian. Then $(X_v^{\Gamma(v)})_u \cong \mathbb{Z}_{m'}.\mathbb{Z}_{\frac{f}{e'}}$, where m' and e' satisfy*

(1) $\mathbb{Z}_{m'} \cong (X_v^{\Gamma(v)})_u \cap \langle \tau^m \rangle$, $mm' \mid p^f - 1$, $e \mid e' \mid f$; *and*

(2) $m' > 1$, $e' < f$, $\pi(p^f - 1) \setminus \pi(p^{e'} - 1) \subseteq \pi(m') \subseteq \pi(|X_{uv}|)$.

Proof. Recall that $(X_v^{\Gamma(v)})_u \trianglelefteq (G_v^{\Gamma(v)})_u = \langle \tau^m \rangle \langle \tau^l \sigma^e \rangle \cong \mathbb{Z}_{\frac{p^f-1}{m}}.\mathbb{Z}_{\frac{f}{e}}$. Then

$$(X_v^{\Gamma(v)})_u / ((X_v^{\Gamma(v)})_u \cap \langle \tau^m \rangle) \cong (X_v^{\Gamma(v)})_u \langle \tau^m \rangle / \langle \tau^m \rangle \lesssim \mathbb{Z}_{\frac{f}{e}},$$

yielding $(X_v^{\Gamma(v)})_u \cong \mathbb{Z}_{m'}.\mathbb{Z}_{\frac{f}{e'}}$ with m' and e' satisfying (1). Since X_{uv} is nonabelian, $(X_v^{\Gamma(v)})_u$ is nonabelian, and so $m' > 1$ and $e' < f$.

By the above (⊛), each prime $r \in \pi(p^f - 1) \setminus \pi(p^{e'} - 1)$ is a divisor of $|\langle \tau^m \rangle| = \frac{p^f-1}{m}$. Let R be the unique subgroup of order r of $\langle \tau^m \rangle$. Then, since R is normal in $(G_v^{\Gamma(v)})_u$, either $R \le (X_v^{\Gamma(v)})_u$ or $R(X_v^{\Gamma(v)})_u = R \times (X_v^{\Gamma(v)})_u$. Suppose that the latter case occurs. Since $e' < f$, we may let $\tau^n \sigma^{e'} \in (X_v^{\Gamma(v)})_u \setminus \langle \tau^m \rangle$. Then $\sigma^{e'}$ centralizes R. Thus $x^{p^{e'}} = x$ for $x \in R$, yielding $r \mid (p^{e'} - 1)$, a contradiction. Then $R \le (X_v^{\Gamma(v)})_u \cap \langle \tau^m \rangle \cong \mathbb{Z}_{m'}$. Noting that m' is a divisor of $|X_{uv}|$, the result follows. □

4. Graphs with insoluble vertex-stabilizers

In this and next sections, we prove Theorem 1.1. Thus, we let G, T, X and $\Gamma = (V, E)$ be as in Hypothesis 3.1. Our task is to determine which pair (G_0, H_0) listed in [18, Tables 14-20] is a possible candidate for $(X, X_{\{u,v\}})$, and determine whether or not the resulting triple $(G, G_v, G_{\{u,v\}})$ meets the conditions (1) and (2) in Lemma 2.1.

In this section, we deal with the case where G_v is insoluble, that is, X_v is described as in Lemma 3.3. First, by the following lemma, (4) and (5) of Lemma 3.3 are excluded.

LEMMA 4.1. *(4) and (5) of Lemma 3.3 do not occur.*

PROOF. Suppose that Lemma 3.3 (4) or (5) holds. By Theorem 2.4, $X_{uv}^{[1]} = 1$. Then $X_v = X_v^{[1]}.X_v^{\Gamma(v)}$, $X_v^{[1]} \cong (X_v^{[1]})^{\Gamma(u)} \trianglelefteq (X_u^{\Gamma(u)})_v \cong (X_v^{\Gamma(v)})_u$, and $X_{uv} \lesssim (X_v^{\Gamma(v)})_u \times (X_u^{\Gamma(u)})_v$. Set $q = p^f$ with p a prime. Then the pair $(X_v^{\Gamma(v)}, (X_v^{\Gamma(v)})_u)$ is given as follows:

$X_v^{\Gamma(v)}$	$(X_v^{\Gamma(v)})_u$	
$\mathrm{Sz}(q).e$	$p^{f+f}{:}(q-1).e$	e a divisor of f, $p = 2$, odd $f > 1$
$\mathrm{Ree}(q).e$	$p^{f+2f}{:}(q-1).e$	e a divisor of f, $p = 3$, odd $f > 1$

In particular, $\mathbf{O}_p(X_{\{u,v\}})$ is not abelian.

We next show that none of the pairs (G_0, H_0) in [18, Tables 14-20] gives a desired pair $(X, X_{\{u,v\}})$. Since $\mathbf{O}_p(X_{\{u,v\}})$ is nonabelian, those pairs (G_0, H_0) with $\mathbf{O}_p(H_0)$ abelian are not in our consideration. In particular, $\mathrm{soc}(X)$ is not isomorphic to an alternating group. Also, noting that $X_{\{u,v\}}$ has a subgroup of index 2, those H_0 having no subgroup of index 2 are excluded.

Case 1. Suppose that $\mathrm{soc}(X_v^{\Gamma(v)}) = \mathrm{Ree}(q)$. Then $p = 3$, $\mathbf{O}_3(X_{\{u,v\}})$ is nonabelian and of order 3^{3f}, 3^{4f}, 3^{5f} or 3^{6f}, $|X_{\{u,v\}}|$ is a divisor of $2 \cdot 3^{6f} \cdot (q-1)^2 f^2$ and divisible by $2(q-1)$. Checking the orders of those H_0 given in [18, Tables 15], we conclude that $\mathrm{soc}(X)$ is not a sporadic simple group.

Suppose that $\mathrm{soc}(X)$ is a simple exceptional group of Lie type. By [18, Table 20], we conclude that $(X, X_{\{u,v\}})$ is one of $(\mathrm{Ree}(3^t), [3^{3t}]{:}\mathbb{Z}_{3^t-1})$ and $(\mathrm{G}_2(3^t).\mathbb{Z}_{2^{l+1}}, [3^{6t}]{:}\mathbb{Z}_{3^t-1}^2.\mathbb{Z}_{2^{l+1}})$, where 2^l is the 2-part of t. Recall that $|X_{\{u,v\}}|$ is a divisor of $2 \cdot 3^{6f} \cdot (q-1)^2 f^2$ and divisible by $2(q-1)$. It follows that $f = t$, $X = \mathrm{G}_2(q).\mathbb{Z}_{2^{l+1}}$ and $X_{\{u,v\}} \cong [q^6]{:}\mathbb{Z}_{q-1}^2.\mathbb{Z}_{2^{l+1}}$. This implies that $X_v^{[1]} \neq 1$, in fact, $|\mathbf{O}_3(X_v^{[1]})| = q^3$. Thus $\mathbf{O}_3(X_v) \neq 1$ and X_v has a quotient $\mathrm{Ree}(q).e$. Checking the maximal subgroups of $\mathrm{G}_2(q).\mathbb{Z}_{2^{l+1}}$, refer to [15, Theorems A and B], we conclude that $\mathrm{G}_2(q).\mathbb{Z}_{2^{l+1}}$ has no maximal subgroup containing such X_v as a subgroup, a contradiction.

Suppose that $\mathrm{soc}(X)$ is a simple classical group over a finite field of order r^t, where r is a prime. Since $f > 1$ is odd, $3^f - 1$ has an odd prime divisor, and so $X_{\{u,v\}}$ is not a $\{2,3\}$-group as $|X_{\{u,v\}}|$ is divisible by $3^f - 1$. Recall that $\mathbf{O}_3(X_{\{u,v\}})$ is nonabelian and of order 3^{3f}, 3^{4f}, 3^{5f} or 3^{6f}. Checking the groups H_0 given in [18, Table 16-19], we conclude that $\mathrm{soc}(X) = \mathrm{PSL}_n(r^t)$ or $\mathrm{PSU}_n(r^t)$, where $n \in \{3,4\}$. Take a maximal subgroup M of X such that $X_v \leq M$. Then M has a simple section (that is, a quotient of some subgroup) $\mathrm{Ree}(q)$. Recall that $q > 3$. Checking Tables 8.3-8.6 and 8.8-8.11

given in [1], we conclude that none of $\mathrm{PSL}_3(r^t)$, $\mathrm{PSL}_4(r^t)$, $\mathrm{PSU}_3(r^t)$ and $\mathrm{PSU}_4(r^t)$ has such maximal subgroups, a contradiction.

Case 2. Suppose that $\mathrm{soc}(X_v^{\Gamma(v)}) = \mathrm{Sz}(q)$. Then $q = 2^f$, $|\mathbf{O}_2(X_{\{u,v\}})| = 2^{2f}a$, $2^{3f}a$ or $2^{4f}a$, where $f > 1$ is odd, and $a = 1$ or 2. Noting that $|X_{\{u,v\}}|$ is divisible by $2(2^f-1)$, by Lemma 2.6, we conclude that $X_{\{u,v\}}$ is not a $\{2,3\}$-group. Since $X_{\{u,v\}}$ is nonabelian, it follows from [18, Table 15-20] that either $(X, X_{\{u,v\}})$ is one of $({}^2\mathrm{F}_4(2)', [2^9]{:}5{:}4)$, $(\mathrm{Sz}(2^t), [2^{2t}]{:}\mathbb{Z}_{2^t-1})$ and $(\mathrm{PSp}_4(2^t).\mathbb{Z}_{2^{l+1}}, [2^{4t}]{:}\mathbb{Z}_{2^t-1}^2.\mathbb{Z}_{2^{l+1}})$, or $\mathrm{soc}(X)$ is one of $\mathrm{PSL}_n(r^t)$ and $\mathrm{PSU}_n(r^t)$, where $n \in \{3,4\}$, 2^l is the 2-part of t, and r is odd if $n = 4$. The first pair leads to $q = 2^3$, and so $|X_{\{u,v\}}|$ is divisible by 7, a contradiction. Checking the maximal subgroups of $\mathrm{soc}(X)$ (refer to [1, Tables 8.3-8.6, 8.8-8.14]), the groups $\mathrm{PSL}_3(r^t)$, $\mathrm{PSU}_3(r^t)$, $\mathrm{PSL}_4(r^t)$ and $\mathrm{PSU}_4(r^t)$ are excluded as they have no maximal subgroup with a simple section $\mathrm{Sz}(q)$. Thus $(X, X_{\{u,v\}}) = (\mathrm{PSp}_4(2^t).\mathbb{Z}_{2^{l+1}}, [2^{4t}]{:}\mathbb{Z}_{2^t-1}^2.\mathbb{Z}_{2^{l+1}})$ or $(\mathrm{Sz}(2^t), [2^{2t}]{:}\mathbb{Z}_{2^t-1})$. Note that $|X_{\{u,v\}}|$ is a divisor of $2 \cdot 2^{4f} \cdot (q-1)^2 f^2$ and divisible by $2^{2f+1}(2^f-1)$. It follows that $X = \mathrm{PSp}_4(q).\mathbb{Z}_{2^{l+1}}$, and $X_v^{[1]} \cong [q^2]{:}\mathbb{Z}_{q-1}$. However, by [1, Table 8.14], $\mathrm{PSp}_4(q).\mathbb{Z}_{2^{l+1}}$ has no maximal subgroup containing $[q^2]{:}\mathbb{Z}_{q-1}.\mathrm{Sz}(q)$, a contradiction. This completes the proof. □

LEMMA 4.2. *Assume that (1) of Lemma 3.3 occurs. Then G, X, $X_{\{u,v\}}$ and X_v are listed as in Table 3.*

G	X	$X_{\{u,v\}}$	X_v	s	d
X	$\mathrm{PSL}_4(2).2, \mathrm{S}_8$	$2^4{:}\mathrm{S}_4$	$2^3{:}\mathrm{SL}_3(2)$	2	7
X	$\mathrm{PSL}_5(2).2$	$[2^8]{:}\mathrm{S}_3^2.2$	$2^6{:}(\mathrm{S}_3\times\mathrm{SL}_3(2))$	3	7
X	$\mathrm{F}_4(2).2$	$[2^{22}]{:}\mathrm{S}_3^2.2$	$[2^{20}].(\mathrm{S}_3\times\mathrm{SL}_3(2))$	3	7
$X, X.2$	$\mathrm{PSL}_4(3).2$	$3_+^{1+4}{:}(2\mathrm{S}_4\times 2)$	$3^3{:}\mathrm{SL}_3(3)$	2	13
X	$\mathrm{PSL}_5(3).2$	$[3^8]{:}(2\mathrm{S}_4)^2.2$	$3^6.2\mathrm{S}_4.\mathrm{SL}_3(3)$	3	13

TABLE 3.

PROOF. Assume first that $X_{uv}^{[1]} = 1$. Then $X_v = X_v^{[1]}.X_v^{\Gamma(v)}$, $X_v^{[1]} \cong (X_v^{[1]})^{\Gamma(u)} \trianglelefteq (X_u^{\Gamma(u)})_v \cong (X_v^{\Gamma(v)})_u$, and $X_{uv} \lesssim (X_v^{\Gamma(v)})_u \times (X_u^{\Gamma(u)})_v$.

Suppose that $X_v^{\Gamma(v)} = \mathrm{PSL}_3(2)$. Then $(X_v^{\Gamma(v)})_u \cong \mathrm{S}_4$, and thus $X_v^{[1]}$ and $X_{\{u,v\}}$ are given as follows:

$X_v^{[1]}$	1	2^2	A_4	S_4
$X_{\{u,v\}}$	$2^2{:}\mathrm{S}_3.2$	$2^4.\mathrm{S}_3.2$	$2^4{:}3^2.[4]$	$2^4{:}\mathrm{S}_3^2.2$

In particular, $2^2 \le |\mathbf{O}_2(X_{\{u,v\}})| \le 2^5$. Check all possible pairs $(X, X_{\{u,v\}})$ in [18, Tables 14-20]. Noting that $\mathrm{A}_8 \cong \mathrm{PSL}_4(2)$ and $\mathrm{PSU}_4(2) \cong \mathrm{PSp}_4(3)$, we conclude that $X \cong \mathrm{A}_8$, $X_{\{u,v\}} \cong 2^4{:}\mathrm{S}_3^2$ and $X_v^{[1]} \cong \mathrm{A}_4$; or $X = \mathrm{M}_{12}$ with $X_{\{u,v\}} \cong 2_+^{1+4}{:}\mathrm{S}_3$; or $X \cong \mathrm{PSU}_4(2)$ with $X_{\{u,v\}} \cong 2\mathrm{A}_4^2.2$. The group A_8 is excluded as it has no subgroup of the form of $X_v^{[1]}.\mathrm{PSL}_3(2)$. The groups M_{12} and $\mathrm{PSU}_4(2)$ are excluded as their orders are not divisible by $d = 7$.

Suppose that $X_v^{\Gamma(v)} = \mathrm{PSL}_3(3)$. Then $(X_v^{\Gamma(v)})_u \cong 3^2{:}2\mathrm{S}_4$. Thus $X_v^{[1]}$ and $X_{\{u,v\}}$ are given as follows:

$X_v^{[1]}$	1	3^2	$3^2{:}2$	$3^2\mathrm{Q}_8$	$3^2{:}2\mathrm{A}_4$	$3^2{:}2\mathrm{S}_4$
$X_{\{u,v\}}$	$3^2{:}2\mathrm{S}_4.2$	$3^4{:}2\mathrm{S}_4.2$	$3^4{:}([4].\mathrm{S}_4).2$	$3^4{:}\mathrm{Q}_8^2.\mathrm{S}_3.2$	$3^4{:}(2\mathrm{A}_4)^2.[4]$	$3^4{:}(2\mathrm{S}_4)^2.2$

Note that $\mathbf{O}_3(X_{\{u,v\}}) \cong 3^2$ or 3^4. Checking the possible pairs $(X, X_{\{u,v\}})$, we have $X_{\{u,v\}} \cong 3^4{:}2^3.\mathrm{S}_4$ and $X = \mathrm{A}_{12}$ or $\mathrm{P\Omega}_8^+(2)$; in this case, $d = 13$ is not a divisor of $|X|$, a contradiction.

Now let $X_{uv}^{[1]}$ be a nontrivial p-group. Then, by Theorem 2.4, X_v and $X_{\{u,v\}}$ are given as follows:

X_v	$X_{\{u,v\}}$	s	d	p
$2^6.(\mathrm{S}_3\times\mathrm{SL}_3(2))$	$[2^8].\mathrm{S}_3^2.2$	3	7	2
$[2^{20}].(\mathrm{S}_3\times\mathrm{SL}_3(2))$	$[2^{22}].\mathrm{S}_3^2.2$	3	7	2
$2^3.\mathrm{SL}_3(2)$	$[2^5].\mathrm{S}_3.2$	2	7	2
$2^4{:}\mathrm{SL}_3(2)$	$[2^6].\mathrm{S}_3.2$	2	7	2
$3^6.(2\mathrm{A}_4\times\mathrm{SL}_3(3))$	$[3^8].(2\mathrm{A}_4\times 2\mathrm{S}_4).2$	3	13	3
$3^6.(2\mathrm{S}_4\times\mathrm{SL}_3(3))$	$[3^8].(2\mathrm{S}_4)^2.2$	3	13	3
$3^3.\mathrm{SL}_3(3)$	$[3^5].2\mathrm{S}_4.2$	2	13	3
$3^3.(2\times\mathrm{SL}_3(3))$	$[3^5].(2\times 2\mathrm{S}_4).2$	2	13	3
$3^6{:}\mathrm{SL}_3(3)$	$[3^8].2\mathrm{S}_4.2$	2	13	3

Suppose that $p = 2$. Then $|X_{\{u,v\}}|$ is divisible by 9 if and only if $|\mathbf{O}_2(X_{\{u,v\}})| \geq 8$, and $\mathbf{O}_2(X_{\{u,v\}})$ contains no elements of order 8 unless $|\mathbf{O}_2(X_{\{u,v\}})| \geq 2^{22}$. Check the pairs (G_0, H_0) given in [18, Tables 14-20] by estimating $|H_0|$ and $|\mathbf{O}_2(H_0)|$. We conclude that one of the following holds:

(i) $X = \mathrm{PSL}_4(2).2 \cong \mathrm{S}_8$ and $X_{\{u,v\}} = 2^4{:}\mathrm{S}_4$;

(ii) $X = \mathrm{PSL}_5(2).2$ and $X_{\{u,v\}} = [2^8].\mathrm{S}_3^2.2$;

(iii) $X = \mathrm{F}_4(2).2$ and $X_{\{u,v\}} = [2^{22}].\mathrm{S}_3^2.2$;

(iv) $\mathrm{soc}(X) = \mathrm{PSL}_3(4)$ and $|\mathbf{O}_2(X_{\{u,v\}})| = 2^6$;

(v) $X = \mathrm{PSU}_4(3).2_3$ and $|\mathbf{O}_2(X_{\{u,v\}})| = 2^7$;

(vi) $X = \mathrm{He}.2$ and $X_{\{u,v\}} = [2^8]{:}\mathrm{S}_3^2.2$.

Case (iv) yields that $X_v \cong 2^3{:}\mathrm{SL}_3(2)$ or $2^4{:}\mathrm{SL}_3(2)$; however, X has no such subgroup by the Atlas [3]. Similarly, cases (v) and (vi) are excluded. For (i), $G = X$ and Γ is (isomorphic to) the point-plane incidence graph of the projective geometry $\mathrm{PG}(3, 2)$. For (ii), $G = X$ and Γ is (isomorphic to) the line-plane incidence graph of the projective geometry $\mathrm{PG}(4, 2)$. If (iii) holds then $G = X$ and Γ is the line-plane incidence graph of the metasymplectic space associated with $\mathrm{F}_4(2)$, see [30].

Now let $p = 3$. Then $|\mathbf{O}_3(X_{\{u,v\}})| = 3^5$ or 3^8, and $X_{\{u,v\}}$ has no normal Sylow subgroup. Checking all possible pairs $(X, X_{\{u,v\}})$ in [18, Tables 14-20], we know that $(X, X_{\{u,v\}})$ is one of the following pairs:

$$(\mathrm{F}_4(8).2, 9^4.(2_+^{1+4}{:}\mathrm{S}_3^2).2),$$
$$(\mathrm{PSL}_5(3).2, [3^8]{:}(2\mathrm{S}_4)^2.2),\ (\mathrm{PSL}_4(3).2, 3_+^{1+4}{:}(2{\times}2\mathrm{S}_4)).$$

Note that $\mathbf{O}_3(X_v) \le \mathbf{O}_3(X_{\{u,v\}})$. Then, for the first pair, $\mathbf{O}_3(X_{\{u,v\}}) \cong \mathbb{Z}_9^4$ has no subgroup isomorphic to $\mathbb{Z}_3^6$, which is impossible. For the second pair, $G = X$ and Γ is (isomorphic to) the line-plane incidence graph of the projective geometry $\mathrm{PG}(4, 3)$. The last pair implies that $X \cong \mathrm{PGL}_4(3)$, $G = X$ or $X.2$, and Γ is (isomorphic to) the line-plane incidence graph of the projective geometry $\mathrm{PG}(3, 3)$. This completes the proof. □

LEMMA 4.3. *Assume that Lemma 3.3 (2) holds. Then $d = q + 1$, and either Γ is $(X, 4)$-arc-transitive, or G, X, $X_{\{u,v\}}$ and X_v are listed as in Table 4.*

PROOF. Let $X_v^{\Gamma(v)} = \mathrm{PSL}_2(q).[o]$, and $q = p^f > 4$, where p is a prime and $o \mid (2, q-1)f$. Note that Γ is $(X, 2)$-arc-transitive, see Lemma 3.3. By Theorem 2.4, if $X_{uv}^{[1]} \ne 1$ then Γ is $(X, 4)$-arc-transitive. Thus we assume next that $X_{uv}^{[1]} = 1$, and then Lemma 3.4 works.

Note that $X_v = X_v^{[1]}.X_v^{\Gamma(v)}$, $X_v^{[1]} \cong (X_v^{[1]})^{\Gamma(u)} \trianglelefteq (X_u^{\Gamma(u)})_v \cong (X_v^{\Gamma(v)})_u = p^f{:}\frac{q-1}{(2,q-1)}.[o]$, and $X_{uv} \lesssim (X_v^{\Gamma(v)})_u \times (X_u^{\Gamma(u)})_v$. We have $\mathbf{O}_p(X_{\{u,v\}}) = \mathbb{Z}_p^{if}.a$, where $i \in \{1, 2\}$ and a is a divisor of $(2, p)$. It is easily shown that $i = 2$ if and only if $\mathbf{O}_p(X_v^{[1]}) = \mathbb{Z}_p^f$. Combining with Lemma 3.4, we need only consider those pairs (G_0, H_0) in [18, Tables 14-20] which satisfy

(a) $\mathbf{O}_p(H_0) = \mathbb{Z}_p^{if}.a$, where $i \in \{1, 2\}$ and a is a divisor of $(2, p)$; $|\mathrm{Fit}(H_0) : \mathbf{O}_p(H_0)| \le 2$; G_0 has a subgroup, say M_0, such that $|M_0 : (M_0 \cap H_0)| = q + 1$, $|H_0 : (M_0 \cap H_0)| = 2$, and M_0 has a simple section $\mathrm{PSL}_2(q)$;

(b) $|H_0{:}\mathbf{O}_p(H_0)|$ is a divisor of $2(q-1)^2 f^2$ and divisible by $q - 1$; if $i = 1$ then $|H_0{:}\mathbf{O}_p(H_0)|$ is a divisor of $2(q-1)f$.

Case 1. Assume that $\mathrm{soc}(X)$ is an alternating group. Using [18, Table 14], we have $G = X = \mathrm{S}_p$ and $X_{\{u,v\}} \cong \mathbb{Z}_p{:}\mathbb{Z}_{p-1}$, where $p \in \{7, 11, 17, 23\}$. Then $X_v = \mathrm{PSL}_2(p)$ and $d = p{+}1$. In particular, Γ is a bipartite graph with two parts being the orbits of A_p on the vertex set V. For $p = 17$ or 23, the group $\mathrm{PSL}_2(p)$ has no transitive permutation representation of degree p, and thus it cannot occur as a subgroup of S_p. Therefore, $p = 7$ or 11, and G, X and $X_{\{u,v\}}$ are listed in Table 4. In fact, X_{uv} and $X_{\{u,v\}}$ are the normalizers of some Syolw p-subgroup in $\mathrm{PSL}_2(p)$ and S_p, respectively. (Note that A_7 can be embedded in $\mathrm{PSL}_4(2)$ acting on the projective points or the hyperplanes of the projective geometry $\mathrm{PG}(3, 2)$, see [19, Table III] for example. Then, for $p = 7$, it is easily shown that the resulting graph is the point-plane nonincidence graph of $\mathrm{PG}(3, 2)$.)

Case 2. Assume that $\mathrm{soc}(X)$ is a simple sporadic group. By [18, Table 15],

G	X	$X_{\{u,v\}}$	X_v	d	Remark
S_p	S_p	$\mathbb{Z}_p{:}\mathbb{Z}_{p-1}$	$\mathrm{PSL}_2(p)$	$p{+}1$	$p\in\{7,11\}$, $\varGamma$ bipartite
M_{11}	M_{11}	$3^2{:}\mathrm{Q}_8.2$	M_{10}	10	K_{11}
J_1	J_1	$\mathbb{Z}_{11}{:}\mathbb{Z}_{10}$	$\mathrm{PSL}_2(11)$	12	
$\mathrm{J}_3.2$	$\mathrm{J}_3.2$	$\mathbb{Z}_{19}{:}\mathbb{Z}_{18}$	$\mathrm{PSL}_2(19)$	20	$\varGamma$ bipartite
$\mathrm{O'N}.2$	$\mathrm{O'N}.2$	$\mathbb{Z}_{31}{:}\mathbb{Z}_{30}$	$\mathrm{PSL}_2(31)$	32	$\varGamma$ bipartite
B	B	$\mathbb{Z}_{19}{:}\mathbb{Z}_{18}{\times}\mathbb{Z}_2$ $\mathbb{Z}_{23}{:}\mathbb{Z}_{11}{\times}\mathbb{Z}_2$	$\mathrm{PGL}_2(19)$ $\mathrm{PSL}_2(23)$	20 24	$X_v<\mathrm{Th}<\mathrm{B}$ $X_v<\mathrm{Fi}_{23}<B$
M	M	$\mathbb{Z}_{41}{:}\mathbb{Z}_{40}$	$\mathrm{PSL}_2(41)$	42	see [24] for X_v
$\mathrm{PSL}_2(19)$	$\mathrm{PSL}_2(19)$	D_{20}	$\mathrm{PSL}_2(5)$	6	
$X, X.2$	$\mathrm{PGL}_2(9)$	D_{20}	$\mathrm{PSL}_2(5)$	6	$\mathsf{K}_{6,6}$
$X, X.2$	M_{10}	$\mathbb{Z}_5{:}\mathbb{Z}_4$	$\mathrm{PSL}_2(5)$	6	$\mathsf{K}_{6,6}$
$\mathrm{PGL}_2(11)$	$\mathrm{PGL}_2(11)$	D_{20}	$\mathrm{PSL}_2(5)$	6	$\varGamma$ bipartite
$X, X.2$	$\mathrm{PSL}_3(r)$	$3^2{:}\mathrm{Q}_8$	$\mathrm{PSL}_2(9)$	10	r prime, [1, Tables 8.3, 8.4] $r\equiv 4,16,31,34 \mod 45$
$X, X.2$	$\mathrm{PSU}_3(r)$	$3^2{:}\mathrm{Q}_8$	$\mathrm{PSL}_2(9)$	10	r prime, [1, Tables 8.5, 8.6] $r\equiv 11,14,29,41 \mod 45$

TABLE 4.

with the restrictions (a) and (b), the only pairs (G_0,H_0) are listed as follows:

$$(\mathrm{M}_{11},3^2{:}\mathrm{Q}_8.2),\ (\mathrm{J}_1,\mathbb{Z}_{11}{:}\mathbb{Z}_{10}),\ (\mathrm{J}_1,\mathbb{Z}_7{:}\mathbb{Z}_6),\ (\mathrm{J}_3.2,\mathbb{Z}_{19}{:}\mathbb{Z}_{18}),\ (\mathrm{J}_4,\mathbb{Z}_{29}{:}\mathbb{Z}_{28}),$$
$$(\mathrm{O'N}.2,\mathbb{Z}_{31}{:}\mathbb{Z}_{30}),\ (\mathrm{B},\mathbb{Z}_{19}{:}\mathbb{Z}_{18}{\times}\mathbb{Z}_2),\ (\mathrm{B},\mathbb{Z}_{23}{:}\mathbb{Z}_{11}{\times}\mathbb{Z}_2),$$
$$(\mathrm{M},\mathbb{Z}_{41}{:}\mathbb{Z}_{40}),(\mathrm{M},\mathbb{Z}_{47}{:}\mathbb{Z}_{23}{\times}\mathbb{Z}_2).$$

In particular, $\mathbf{O}_p(H_0)$ is a Sylow p-subgroup of G_0. This yields that $X_v^{[1]}=1$, and so $\mathrm{soc}(X_v)=\mathrm{PSL}_2(p^f)$.

If $(X,X_{\{u,v\}})$ is one of $(\mathrm{J}_1,\mathbb{Z}_7{:}\mathbb{Z}_6)$, $(\mathrm{J}_4,\mathbb{Z}_{29}{:}\mathbb{Z}_{28})$ and $(\mathrm{M},\mathbb{Z}_{47}{:}\mathbb{Z}_{23}{\times}\mathbb{Z}_2)$, then $X_v=\mathrm{PSL}_2(p)$ for $p=7$, 29 and 47, respectively; however, by the Altas [3] and [36, Tables 5.6 and 5.11], X has no subgroup $\mathrm{PSL}_2(p)$, a contradiction. Thus G, X and $X_{\{u,v\}}$ are listed in Table 4. (Note that the Monster M has a maximal subgroup $\mathrm{PSL}_2(41)$ by [24].)

Case 3. Assume that $\mathrm{soc}(X)$ is a simple group of Lie type over a finite field of order r^t, where r is a prime. We first show $r\neq p$.

Suppose that $r=p$. Then, by (a), either $\mathbf{O}_p(H_0)$ is abelian or $r=p=2$. For $r=p>2$, noting that $|H_0|$ has a divisor $q-1$, there does not exist H_0 in [18, Tables 16-20] such that $\mathbf{O}_p(H_0)$ is abelian. Thus we have $r=p=2$. Recalling that $p^f>4$ and $|H_0/\mathbf{O}_p(H_0)|$ is divisible by 2^f-1, it follows from Lemma 2.6 that $H_0/\mathbf{O}_p(H_0)$ is not a $\{2,3\}$-group. Checking those H_0 given in [18, Tables 16-20], we conclude that (G_0,H_0) is one of the following pairs:

$$(\mathrm{PSL}_2(2^t),\mathbb{Z}_2^t{:}\mathbb{Z}_{2^t-1}),\ (\mathrm{PSL}_3(2^t),[2^{3t}]{:}[\tfrac{(2^t-1)^2}{(3,2^t-1)}].2),$$
$$(\mathrm{PSU}_3(2^t),[2^{3t}]{:}\mathbb{Z}_{\frac{2^{2t}-1}{(3,2^t+1)}}),$$
$$(\mathrm{PSp}_4(2^t).\mathbb{Z}_{2^{l+1}},[2^{4t}]{:}\mathbb{Z}_{2^t-1}^2.\mathbb{Z}_{2^{l+1}}),\ \text{where } 2^l \text{ is the 2-part of } t,$$
$$(\mathrm{Sz}(2^t),[2^{2t}]{:}\mathbb{Z}_{2^t-1}),(^3\mathrm{D}_4(2),[2^{11}]{:}(\mathbb{Z}_7\times\mathrm{S}_3)),\ (^2\mathrm{F}_4(2)',[2^9]{:}5{:}4).$$

First, the pair $(\mathrm{Sz}(2^t), [2^{2t}]{:}\mathbb{Z}_{2^t-1})$ is excluded as $\mathrm{Sz}(2^t)$ has no subgroup with a section $\mathrm{PSL}_2(2^f)$. For the last two pairs, we have $f=5$ and 4 respectively, which yields that 2^f-1 is not a divisor of $|H_0|$, a contradiction. For the three pairs after the first one, we have $t<f$, thus G_0 has no maximal subgroup with a section $\mathrm{PSL}_2(2^f)$, a contradiction. Suppose finally that $(X, X_{\{u,v\}}) = (\mathrm{PSL}_2(2^t), \mathbb{Z}_2^t{:}\mathbb{Z}_{2^t-1})$. Then $3 \le f < t \le 2f+1$. Noting that 2^f-1 is a divisor of 2^t-1, it follows that f is a divisor of t, and so $t=2f$. Then $\mathbf{O}_2(X_{\{u,v\}}) = 2^{2f}$, yielding $|\mathbf{O}_2(X_v^{[1]})| = 2^f$. Thus $\mathbf{O}_2(X_v) \ne 1$ and X_v has a section $\mathrm{PSL}_2(2^f)$. Check the subgroups of $\mathrm{PSL}_2(2^{2f})$, refer to [13, II.8.27]. We conclude that $\mathrm{PSL}_2(2^{2f})$ has no subgroup isomorphic to X_v, a contradiction.

We assume that $r \ne p$ in the following.

Subcase 3.1. We first deal with those pairs (G_0, H_0) such that H_0 is included in some infinite families in [18, Tables 16-20]. Note that $r \ne p$, and we consider only those H_0 having subgroups of index 2. It follows that either $H_0/\mathrm{Fit}(H_0)$ is a $\{2,3\}$-group, or $G_0 = \mathrm{E}_8(q')$ and $|H_0| = 30(q'^8 \pm q'^7 \mp q'^5 - q'^4 \mp q'^3 \pm q' + 1)$, where $q' = r^t$. Suppose the latter case occurs. It is easily shown that $q'^8 \pm q'^7 \mp q'^5 - q'^4 \mp q'^3 \pm q' + 1$ is divisible by some primitive prime divisor s of $q'^{15}-1$ or of $q'^{30}-1$. Noting that $s \ge 17$, we know that H_0 has normal cyclic Sylow s-subgroup. It follows from (a) that $17 \le p = s = q'^8 \pm q'^7 \mp q'^5 - q'^4 \mp q'^3 \pm q' + 1$. In particular, $\mathbf{O}_p(H_0) = \mathbb{Z}_p$ and $f=1$. By (b), $|H_0|$ is divisible by $p-1$, and then 30 is divisible by $p-1$. This implies that $30 = p-1 = q'^8 \pm q'^7 \mp q'^5 - q'^4 \mp q'^3 \pm q'$, which is impossible. Therefore, $H_0/\mathrm{Fit}(H_0)$ is a $\{2,3\}$-group.

By (a), $\mathrm{Fit}(H_0)$ a $\{2,p\}$-group. Then $|H_0|$ has no prime divisor other than 2, 3 and p. Since p^f-1 is a divisor of $|H_0|$, by Lemma 2.6, we have $f<3$. Recall that $(X_u^{\Gamma(u)})_v \cong (X_v^{\Gamma(v)})_u = p^f{:}\frac{q-1}{(2,q-1)}.[o]$, and $X_{uv} \lesssim (X_v^{\Gamma(v)})_u \times (X_u^{\Gamma(u)})_v$, where o is a divisor of $(2,q-1)f$. Then $X_{uv}/\mathbf{O}_p(X_{uv})$ has an abelian Hall $2'$-subgroup. Note that $X_{uv}\mathbf{O}_p(X_{\{u,v\}})/\mathbf{O}_p(X_{\{u,v\}}) \cong X_{uv}/(\mathbf{O}_p(X_{\{u,v\}} \cap X_{uv}) = X_{uv}/\mathbf{O}_p(X_{uv})$, and $|X_{\{u,v\}} : X_{uv}\mathbf{O}_p(X_{\{u,v\}})| \le 2$. It follows that $X_{\{u,v\}}/\mathbf{O}_p(X_{\{u,v\}})$ has an abelian Hall $2'$-subgroup. Thus, as a possible candidate for $X_{\{u,v\}}$, the quotient of H_0 over $\mathbf{O}_p(H_0)$ has abelian Hall $2'$-subgroups. In particular, $H_0/\mathbf{O}_p(H_0)$ has no section A_4.

Considering the restrictions on H_0, r and f, we conclude that (G_0, H_0) can only be one of the following pairs:

$(\mathrm{PSL}_2(r^t), \mathbb{Z}_{\frac{r^t\pm1}{(2,r^t-1)}}{:}\mathbb{Z}_2)$, $(\mathrm{PSL}_3(r^t), [\frac{(r^t-1)^2}{(3,r^t-1)}].\mathrm{S}_3)$, $(\mathrm{PSU}_3(r^t), [\frac{(r^t+1)^2}{(3,r^t+1)}].\mathrm{S}_3)$;
$(\mathrm{PSp}_4(2^t).\mathbb{Z}_{2^{l+1}}, \mathbb{Z}_{2^t\pm1}^2.[2^{l+4}])$, $(\mathrm{PSp}_4(2^t).\mathbb{Z}_{2^{l+1}}, \mathbb{Z}_{2^{2t}+1}.[2^{l+3}])$, $t \ge 3$;
$(\mathrm{Sz}(2^t), \mathbb{Z}_{2^t-1}{:}\mathbb{Z}_2)$, $(\mathrm{Sz}(2^t), \mathbb{Z}_{2^t\pm\sqrt{2^{t+1}}+1}{:}\mathbb{Z}_4)$, $t \ge 3$;
$(\mathrm{Ree}(3^t), \mathbb{Z}_{3^t\pm\sqrt{3^{t+1}}+1}.\mathbb{Z}_6)$, $(\mathrm{Ree}(3^t), \mathbb{Z}_{3^t+1}.\mathbb{Z}_6)$, $t \ge 3$;
$(\mathrm{G}_2(3^t).\mathbb{Z}_{2^{l+1}}, \mathbb{Z}_{3^t\pm1}^2.[3\cdot 2^{l+3}])$, $(\mathrm{G}_2(3^t).\mathbb{Z}_{2^{l+1}}, \mathbb{Z}_{3^{2t}\pm3^t+1}.[3\cdot 2^{l+2}])$, $t \ge 2$;
$({}^3\mathrm{D}_4(r^t), \mathbb{Z}_{r^{4t}-r^{2t}+1}{:}\mathbb{Z}_4)$, $({}^2\mathrm{F}_4(2^t), \mathbb{Z}_{2^{2t}\pm\sqrt{2^{3t+1}}+2^t\pm\sqrt{2^{t+1}}+1}.\mathbb{Z}_{12})$, $t \ge 3$;
$(\mathrm{F}_4(2^t).\mathbb{Z}_{2^{l+1}}, \mathbb{Z}_{2^{4t}-2^{2t}+1}.[3\cdot 2^{l+3}])$, $t \ge 2$;

where the power 2^l appeared means the 2-part of t. Recall that $|\mathrm{Fit}(H_0):\mathbf{O}_p(H_0)|\leq 2$ and $|H_0:\mathbf{O}_p(H_0)|$ is divisible by p^f-1. This allows us determine the values of p^f and r^t. As an example, we only deal with the second pair. Suppose that $(G_0,H_0)=(\mathrm{PSL}_3(r^t),[\frac{(r^t-1)^2}{(3,r^t-1)}].\mathrm{S}_3)$. Considering the structures of $\mathrm{Fit}(H_0)$ and $\mathbf{O}_p(H_0)$, either $(3,r^t-1)=1$, $p=r^t-1$ and $f\in\{1,2\}$, or $f=1$ and $p=r^t-1=3$. The latter implies that $\mathrm{PSL}_2(q)$ is soluble, which is not the case. Assume that the former case holds. Then $|\mathrm{S}_3|$ is divisible by r^t-1-1 or $(r^t-1)^2-1$. Then the only possibility is that $(p^f,r^t)=(7,8)$. The other pairs can be fixed out in a similar way, the details is omitted here. Eventually, we conclude that (G_0,H_0,p,f) is one of $(\mathrm{PSL}_2(19),\mathrm{D}_{20},5,1)$, $(\mathrm{PSL}_3(8),7^2{:}\mathrm{S}_3,7,1)$ and $(\mathrm{Sz}(8),\mathbb{Z}_5{:}\mathbb{Z}_4,5,1)$. By the Atlas [3], neither $\mathrm{PSL}_3(8)$ nor $\mathrm{Sz}(8)$ has subgroup with a section $\mathrm{PSL}_2(p)$. Thus, in this case, G, X and $X_{\{u,v\}}$ are given as in Table 4.

Subcase 3.2. For the pairs (G_0,H_0) not appearing in *Subcase 3.1*, we check the finite number of H_0 one by one. We observed that either $p=2$, or $H_0/\mathbf{O}_p(H_0)$ is a $\{2,3\}$-group. Recall that $r\neq p$.

Suppose that $p=2$. Recalling that $q=2^f>4$, we have $f\geq 3$. In particular, since $|H_0|$ is divisible by 2^f-1, H_0 is not a $\{2,3\}$-group by Lemma 2.6. Then the only possibility is that $G_0={}^2\mathrm{F}_4(2)'$ and $H_0=[2^9]{:}5{:}4$. Thus $|\mathbf{O}_2(H_0)|=2^9$, it follows from (a) that $f=4$ or 9, and then G_0 has a section $\mathrm{PSL}_2(2^4)$ or $\mathrm{PSL}_2(2^9)$, which is impossible by checking the (maximal) subgroups of ${}^2\mathrm{F}_4(2)'$. Thus $p>2$, and $H_0/\mathbf{O}_p(H_0)$ is a $\{2,3\}$-group; in particular, by (a), $\mathbf{O}_p(H_0)=\mathbb{Z}_p^{if}$ for some $i\in\{1,2\}$.

Suppose that H_0 has a section A_4. Then H_0 has no normal Sylow 3-subgroup. Further, H_0 has no quotient A_4 as H_0 has a subgroup of index 2. If $(3,(q-1)f)=1$ then, by (b), we conclude that $p=3$ and $\mathbf{O}_p(H_0)$ is the unique Sylow 3-subgroup of H_0, a contradiction. Thus 3 is a divisor of $(q-1)f$. Check those H_0 in [18, Table 16-20] which have a section A_4 and do not appear in *Subcase 3.1*. Recalling $r\neq p>2$ and $\mathbf{O}_p(H_0)=\mathbb{Z}_p^{if}$, it follows that either $\mathbf{O}_p(H_0)=\mathbb{Z}_3^2$ or $(G_0,H_0)=(\mathrm{F}_4(2).4,\mathbb{Z}_7^2{:}(3\times\mathrm{SL}_2(3)).4)$. Since 3 is a divisor of $(q-1)f$, we get $G_0=\mathrm{F}_4(2).4$ and $q=p^f=7$ or 7^2. By (b), for $q=7$ or 7^2, the order of H_0 should be a divisor of 72 or 192 respectively, which is impossible.

The above argument allows us ignore many cases without further inspection. Inspecting carefully the remaining pairs, the possible candidates for $(X,X_{\{u,v\}})$ are as follows:

$(\mathrm{PGL}_2(9),\mathrm{D}_{20})$, $(\mathrm{M}_{10},\mathbb{Z}_5{:}\mathbb{Z}_4)$, $(\mathrm{PGL}_2(11),\mathrm{D}_{20})$;
$(\mathrm{PSL}_3(r),3^2{:}\mathrm{Q}_8)$, where $r\equiv 4,7 \mod 9$;
$(\mathrm{PSp}_4(4).4,\mathbb{Z}_{17}{:}\mathbb{Z}_{16})$, $(\mathrm{PSp}_4(4).4,5^2{:}[2^5])$;
$(\mathrm{PSU}_3(r),3^2{:}\mathrm{Q}_8)$, where $5<r\equiv 2,5 \mod 9$;
$(\mathrm{PSU}_3(2^t),3^2{:}\mathrm{Q}_8)$, where t is a prime no less than 5;
$({}^2\mathrm{F}_4(2),\mathbb{Z}_{13}{:}\mathbb{Z}_{12})$.

For the first three pairs, G, X and $X_{\{u,v\}}$ are easily determined and given as in Table 4. The pair $(\mathrm{PSp}_4(4).4, \mathbb{Z}_{17}{:}\mathbb{Z}_{16})$ is excluded as $\mathrm{PSp}_4(4).4$ has no subgroup $\mathrm{PSL}_2(17)$, and the pair $({}^2\mathrm{F}_4(2), \mathbb{Z}_{13}{:}\mathbb{Z}_{12})$ is excluded as ${}^2\mathrm{F}_4(2)$ has no subgroup $\mathrm{PSL}_2(13)$. Suppose that $X \cong \mathrm{PSU}_3(2^t)$ and $X_{\{u,v\}} \cong 3^2{:}\mathrm{Q}_8$. Then we have $X_v \cong \mathrm{PSL}_2(9)$; however, by [1, Tables 8.3, 8.4], $\mathrm{PSU}_3(2^t)$ has no subgroup $\mathrm{PSL}_2(9)$, a contradiction. Suppose that $(X, X_{\{u,v\}}) = (\mathrm{PSp}_4(4).4, 5^2{:}[2^5])$. Then X_v contains a Sylow 5-subgroup P of X and has a section $\mathrm{PSL}_2(5)$ or $\mathrm{PSL}_2(25)$. By the information for $\mathrm{PSp}_4(4).4$ given in the Atlas [3], we conclude that $X_v \le M \cong (\mathrm{A}_5 \times \mathrm{A}_5){:}2^2 < \mathrm{PSp}_4(4).2 < \mathrm{PSp}_4(4).4$. Note that $X_{uv} = 5^2{:}[2^4]$, which should be the normalizer of P in X_v. Using GAP [10], computation shows that $|\mathbf{N}_L(P)| \le 200$ for any maximal subgroup L of M with $P \le L$. It follows that $X_v = M \cong (\mathrm{A}_5 \times \mathrm{A}_5){:}2^2$, yielding $d = |X_v : X_{uv}| = 36 \ne q+1$, a contradiction.

Let $(X, X_{\{u,v\}}) = (\mathrm{PSL}_3(r), 3^2{:}\mathrm{Q}_8)$. Then $X_{uv} \cong 3^2{:}4$. It is easily shown that $p = 3$ and $X_v \cong \mathrm{PSL}_2(9)$. Since $r \equiv 4, 7 \mod 9$, we know that $\mathrm{PSL}_3(r)$ has a Sylow 3-subgroup $\mathbb{Z}_3^2$. By [1, Tables 8.3, 8.4], $\mathrm{PSL}_3(r)$ has a subgroup $\mathrm{PSL}_2(9)$ if and only if $r \equiv 1, 4 \mod 15$. Thus, in this case, we have $r \equiv 11, 14, 29, 41 \mod 45$. For a subgroup $\mathrm{PSL}_2(9)$ of $\mathrm{PSL}_3(r)$, taking a Sylow 3-subgroup Q of $\mathrm{PSL}_2(9)$, the normalizers of Q in $\mathrm{PSL}_2(9)$ and $\mathrm{PSL}_3(r)$ are (isomorphic to) $3^2{:}4$ and $3^2{:}\mathrm{Q}_8$, respectively. Then these two normalizers of Q can serve as the roles of X_{uv} and $X_{\{u,v\}}$, respectively. Thus X and $X_{\{u,v\}}$ are given as in Table 4. Noting that $G = XG_{\{u,v\}}$, we have $G_{\{u,v\}}/X_{\{u,v\}} \cong G/X \lesssim \mathsf{Out}(\mathrm{PSL}_3(r)) \cong \mathrm{S}_3$, and so $G = X.[m]$ and $G_{\{u,v\}} = X_{\{u,v\}}.[m]$, where m is a divisor of 6. Thus $|G_{uv}{:}X_{uv}| = m$, since $|G_v{:}G_{uv}| = 10 = |X_v{:}X_{uv}|$, we have $|G_v{:}X_v| = m$. By [1, Table 8.4], $\mathbf{N}_{\mathsf{Aut}(\mathrm{PSL}_3(r))}(X_v) = X_v.2$. Since $X_v \trianglelefteq G_v$, it follows that $m \le 2$. Thus $G = X$ or $X.2$, and if $G = X.2$ then $G_v = X_v.2 \cong \mathrm{PGL}_2(9)$ and $G_{\{u,v\}} \cong 3^2{:}\mathrm{Q}_8.2$. The pair $(\mathrm{PSU}_3(r), 3^2{:}\mathrm{Q}_8)$ is similarly dealt with, the details are omitted. This completes the proof. □

Lemma 4.4. *If (3) of Lemma 3.3 holds then G, X, $X_{\{u,v\}}$ and X_v are listed in Table 5.*

G	X	$X_{\{u,v\}}$	X_v	d	Remark
HS.2	HS.2	$[5^3]{:}[2^5]$	$\mathrm{PSU}_3(5){:}2$	126	Γ bipartite
Ru	Ru	$[5^3]{:}[2^5]$	$\mathrm{PSU}_3(5){:}2$	126	

Table 5.

Proof. Let $X_v^{\Gamma(v)} = \mathrm{PSU}_3(q).[o]$ and $q = p^f > 2$, where p is a prime and $o \mid 2(3, q+1)f$. Then $(X_v^{\Gamma(v)})_u = p^{f+2f}{:}\frac{q^2-1}{(3,q+1)}.[o]$, and $X_{uv}^{[1]} = 1$ by Theorem 2.4. Thus $|\mathbf{O}_p(X_{\{u,v\}})| = p^{3f}.a$, $p^{4f}.a$, $p^{5f}.a$ or $p^{6f}.a$, where a is a divisor of $(2, p)$. Moreover, $\mathbf{O}_p(X_{\{u,v\}})$ is nonabelian, and $X_{\{u,v\}}/\mathbf{O}_p(X_{\{u,v\}})$ has a subgroup $\mathbb{Z}_{\frac{q^2-1}{(3,q+1)}}$. We next determine which pair (G_0, H_0) in [18, Tables

14-20] is a possible candidate for $(X, X_{\{u,v\}})$. Note that we may ignore those H_0 which either has no subgroup of index 2 or has abelian maximal normal p-subgroup. In particular, $\mathrm{soc}(X)$ is not an alternating group.

Case 1. Let (G_0, H_0) be a pair with H_0 included in some infinite families given in [18, Table 16-20]. Since $\mathbf{O}_p(X_{\{u,v\}})$ is nonabelian, we conclude that $(X, \mathbf{O}_p(X_{\{u,v\}}))$ is one of the following pairs:

$$(\mathrm{PSL}_3(p^t).2, [p^{3t}]),\ (\mathrm{PGL}_3(p^t).2, [p^{3t}])\ (\text{with } p = 2),$$
$$(\mathrm{PSU}_3(p^t), [p^{3t}]),\ (\mathrm{PSp}_4(p^t).\mathbb{Z}_{2^{l+1}}, [p^{4t}])\ (\text{with } p = 2),$$
$$(\mathrm{Sz}(p^t), [p^{2t}]),\ (\mathrm{Ree}(p^t), [p^{3t}]) \text{ and } (\mathrm{G}_2(p^t).\mathbb{Z}_{2^{l+1}}, [p^{6t}]),$$

where 2^l is the 2-part of t. Check the maximal subgroups of $\mathrm{PSp}_4(p^t).\mathbb{Z}_{2^{l+1}}$, $\mathrm{Sz}(p^t)$ and $\mathrm{Ree}(p^t)$, refer to [1, Table 8.14], [28, Theorem 9] and [15, Theorem C], respectively. We conclude that none of $\mathrm{PSp}_4(t^f).\mathbb{Z}_{2^{l+1}}$, $\mathrm{Sz}(p^t)$ and $\mathrm{Ree}(p^t)$ has maximal subgroups with a simple section $\mathrm{PSU}_3(q)$, and they are excluded. For the first three and the last pairs, $|X/\mathbf{O}_p(X_{\{u,v\}})|$ is a divisor of $2(p^t-1)^2$, and $\mathbf{O}_p(X_{\{u,v\}}) = [p^{3t}]$ or $[p^{6t}]$. Clearly, $t \le 2f$.

Suppose that $t = 2f$. Then $\mathrm{soc}(X) = \mathrm{PSL}_3(q^2)$ or $\mathrm{PSU}_3(q^2)$, and $\mathbf{O}_p(X_{\{u,v\}}) = [q^6]$. It follows that $\mathbf{O}_p(X_v^{[1]}) = [q^3]$. Thus $\mathbf{O}_p(X_v) \ne 1$ and X_v has an almost simple quotient $\mathrm{PSU}_3(q).[o]$. Checking Tables 8.3 and 8.5 given in [1], we conclude that X has no maximal subgroup containing X_v, a contradiction. If $t = f$ then we have $(X, \mathbf{O}_p(X_{\{u,v\}})) = (\mathrm{G}_2(p^t).\mathbb{Z}_{2^{l+1}}, [q^6])$, and we get a similar contradiction by checking the maximal subgroups of $\mathrm{G}_2(p^t).\mathbb{Z}_{2^{l+1}}$.

Suppose that $f \ne t < 2f$. Then $f > 1$. Recalling that $X_{\{u,v\}}/\mathbf{O}_p(X_{\{u,v\}})$ has a subgroup $\mathbb{Z}_{\frac{q^2-1}{(3,q+1)}}$, we know that $p^{2f}-1$ is a divisor of $2(3,q+1)(p^t-1)^2$. If $p^{2f}-1$ has a primitive prime divisor say s, then $s \ge 2f+1 \ge 5$, and s is not a divisor of $2(3,q+1)(p^t-1)^2$, a contradiction. It follows from Zsigmondy's theorem that $2f = 6$ and $p = 2$, and so $t = 1$ or 2. Then 7 is a divisor of $p^{2f}-1$ but not a divisor of $2(3,q+1)(p^t-1)^2$, a contradiction.

Case 2. Let (G_0, H_0) be one of the pairs in [18, Table 15-20] which is not considered in Case 1. Assume that $X_{\{u,v\}}/\mathbf{O}_p(X_{\{u,v\}})$ is a $\{2,3\}$-group. Then $p^{2f}-1$ has no prime divisor other than 2 and 3. It follows that $f = 1$, and so $p = q > 2$. Calculation shows that $p \in \{3,5,7\}$. For $q = p = 3$, it is easily shown that $X_{\{u,v\}}/\mathbf{O}_p(X_{\{u,v\}})$ is a 2-group. These observations yield that either $q = p = 3$ and $X_{\{u,v\}}/\mathbf{O}_p(X_{\{u,v\}})$ is a 2-group, or $X_{\{u,v\}}$ is not a $\{2,3\}$-group.

Recall that $X_{\{u,v\}}/\mathbf{O}_p(X_{\{u,v\}})$ has a subgroup $\mathbb{Z}_{\frac{q^2-1}{(3,q+1)}}$, and $\mathbf{O}_p(X_{\{u,v\}})$ has order $p^{if}.a$, where $3 \le i \le 6$. It follows that $(X, X_{\{u,v\}})$ is one of the following pairs:

$$(\mathrm{HS}.2, [5^3]{:}[2^5]),\ (\mathrm{Ru}, [5^3]{:}[2^5]),\ (\mathrm{McL}, [5^3]{:}3{:}8),\ (\mathrm{Co}_2, [5^3]{:}4\mathrm{S}_4),$$
$$(\mathrm{Th}, [5^3]{:}4\mathrm{S}_4),\ (\mathrm{J}_4, [11^3]{:}(5 \times 2\mathrm{S}_4)).$$

Then $q = p \in \{5, 11\}$ and $X_v^{[1]} = 1$. In particular, $\mathrm{soc}(X_v) = \mathrm{PSU}_3(p)$, and $X_{\{u,v\}}$ is the normalizer $\mathbf{N}_X(P)$ of some Sylow p-subgroup P of X. Thus $X_{uv} = X_v \cap X_{\{u,v\}} \le \mathbf{N}_{X_v}(P)$. For the pairs $(\mathrm{HS}.2, [5^3]{:}[2^5])$ and $(\mathrm{Ru}, [5^3]{:}[2^5])$, by the Atlas [3], $X_{\{u,v\}}$ is a normalizer of some Sylow 5-subgroup, which intersects a maximal subgroup $\mathrm{PSU}_3(5){:}2$ of $\mathrm{soc}(X)$ at $[5^3]{:}8{:}2$, thus G, X and $X_{\{u,v\}}$ are listed in Table 5. The other pairs are excluded as follows.

First, the group Th is excluded as it has no maximal subgroup with a simple section $\mathrm{PSU}_3(5)$, refer to [36, Table 5.8]. For the pair $(\mathrm{McL}, [5^3]{:}3{:}8)$, by the Atlas [3], we have $X_v = \mathrm{PSU}_3(5)$, and so $X_{uv} \le \mathbf{N}_{\mathrm{PSU}_3(5)}(P) = [5^3]{:}8$, which contradicts that $|X_{\{u,v\}} : X_{uv}| = 2$. For the pair $(\mathrm{J}_4, [11^3]{:}(5 \times 2\mathrm{S}_4))$, by [36, Table 5.8], $X_v = \mathrm{PSU}_3(11).2$, yielding $X_{uv} \le \mathbf{N}_{X_v}(P) = [11^3]{:}(5 \times 8{:}2)$, we get a similar contradiction. For the pair $(X, X_{\{u,v\}}) = (\mathrm{Co}_2, [5^3]{:}4\mathrm{S}_4)$, by the Atlas [3], $X_v < \mathrm{HS}.2 < \mathrm{Co}_2$. Checking the maximal subgroups of HS.2, we have $X_v = \mathrm{PSU}_3(5)$ or $X_v = \mathrm{PSU}_3(5){:}2$. It follows that $X_{uv} \le \mathbf{N}_{X_v}(P) = [5^3]{:}8$ or $[5^3]{:}[2^5]$, and then $|X_{\{u,v\}} : X_{uv}| \ne 2$, a contradiction. This completes the proof. □

5. Graphs with soluble vertex-stabilizers

Let G, T, X and $\Gamma = (V, E)$ be as in Hypothesis 3.1. The following lemma says that if Γ is a complete bipartite graph then $\Gamma \cong \mathsf{K}_{6,6}$ and $G_v^{\Gamma(v)}$ is insoluble.

Lemma 5.1. *Assume that $\Gamma \cong \mathsf{K}_{d,d}$. Then $T \cong \mathrm{A}_6$, $d = 6$, $T_v = \mathrm{PSL}_2(5)$ and $T_{uv} \cong \mathrm{D}_{10}$. In particular, X_{uv} is nonabelian.*

Proof. Let G^+ be the subgroup of G fixing the bipartition of Γ. Then $G_v \le G^+$, and G_v is 2-transitive on the partite set which does not contain v. Thus G^+ acts 2-transitively on each partite set, and these two actions are not equivalent. Check the almost simple 2-transitive groups, refer to [2, Table 7.4]. We conclude that $T \cong \mathrm{A}_6$ or M_{12}, $T_v \cong \mathrm{A}_5$ or M_{11}, and $T_{uv} \cong \mathrm{D}_{10}$ or $\mathrm{PSL}_2(11)$, respectively. Since T_{uv} is soluble, the lemma follows. □

Assume that G_v is soluble, and let $\mathrm{soc}(G_v^{\Gamma(v)}) = \mathbb{Z}_p^f$, where p is a prime. By Lemma 5.1, since G_v is soluble, Γ is not a complete bipartite graph. Then we have the following result by [22, Theorem 3.3].

Lemma 5.2. *Assume that X_{uv} is abelian. Then one of the following holds:*

(1) $T \cong \mathrm{PSL}_2(p^f)$, $T_{\{u,v\}} \cong \mathrm{D}_{\frac{2(p^f-1)}{(2,p-1)}}$, $T_v \cong \mathbb{Z}_p^f{:}\mathbb{Z}_{\frac{p^f-1}{(2,p-1)}}$ *and* $\Gamma \cong \mathsf{K}_{p^f+1}$;

(2) $T = \mathrm{Sz}(2^f)$, $T_{\{u,v\}} \cong \mathrm{D}_{2(2^f-1)}$, $T_v \cong \mathbb{Z}_2^f{:}\mathbb{Z}_{2^f-1}$ *and Γ is $(T, 2)$-arc-transitive, where $f \ge 3$ is odd.*

REMARK. In Lemma 5.2, $T_{\{u,v\}}$ is soluble and maximal in T, and thus $X=T$ by the choice of X. For part (1), since Γ is $(G,2)$-arc-transitive, G is a 3-transitive group of degree p^f+1, and thus $X\neq G$ if p is odd. The graphs satisfying part (2) are determined by [5, Construction 5.4 and Proposition 5.5]; in particular, for any given odd $f\geq 3$, there is a unique $(\mathrm{Sz}(2^f),2)$-arc-transitive of valency 2^f, which has automorphism group $\mathsf{Aut}(\mathrm{Sz}(2^f))$. □

LEMMA 5.3. *Assume that (1) or (2) of Lemma 3.5 holds, and X_{uv} is nonabelian. Then one of the following holds:*

(1) $G=X$ *or* $X.2$, $X=\mathrm{M}_{10}$, $X_{\{u,v\}}\cong\mathbb{Z}_8{:}\mathbb{Z}_2$, $X_v\cong 3^2{:}\mathrm{Q}_8$ *and* $\Gamma\cong\mathsf{K}_{10}$;

(2) $G=X=\mathrm{PSL}_3(3).2$, $X_{\{u,v\}}\cong\mathrm{GL}_2(3){:}2$, $X_v\cong 3^2{:}\mathrm{GL}_2(3)$, *and* Γ *is the point-line nonincidence graph of* $\mathrm{PG}(2,3)$.

PROOF. *Case 1.* Assume that Lemma 3.5 (1) holds. Suppose first that $(X_v^{\Gamma(v)})_u=\mathrm{Q}_8$. Then $X_{uv}\lesssim\mathrm{Q}_8{\times}\mathrm{Q}_8$. This implies that $|X_{\{u,v\}}|$ is a divisor of 2^7 and divisible by 2^4. Checking the Tables 14-20 in [18], we have $X\cong\mathrm{PSL}_2(9).2=\mathrm{M}_{10}$ and $X_{\{u,v\}}\cong\mathbb{Z}_8{:}\mathbb{Z}_2$. In this case, $X_v\cong 3^2{:}\mathrm{Q}_8$, and $d=9$. Since Γ has valency 9 and order $|X:X_v|=10$, we have $\Gamma\cong\mathsf{K}_{10}$, desired as in part (1).

Suppose that $(X_v^{\Gamma(v)})_u\neq\mathrm{Q}_8$. If $p=3$ and $(G_v^{\Gamma(v)})_u=\mathrm{Q}_8$, then $(X_v^{\Gamma(v)})_u$ is abelian, it follows that X_{uv} is abelian, a contradiction. Thus we have $\mathrm{SL}_2(3)\trianglelefteq(G_v^{\Gamma(v)})_u\leq\mathrm{GL}_2(p)$, and $p\in\{3,5,7,11,23\}$. Then $(G_v^{\Gamma(v)})_u\leq\mathbf{N}_{\mathrm{GL}_2(p)}(\mathrm{SL}_2(3))=\mathbb{Z}_{p-1}\circ\mathrm{GL}_2(3)$. Since $(X_v^{\Gamma(v)})_u$ is nonabelian and normal in $(G_v^{\Gamma(v)})_u$, we have $\mathrm{Q}_8\trianglelefteq(X_v^{\Gamma(v)})_u$, and hence $\mathrm{SL}_2(3)\trianglelefteq(X_v^{\Gamma(v)})_u$. Moreover, $|X_{\{u,v\}}|$ is a divisor of $2^7\cdot 3^2\cdot(p-1)^2$ and divisible by 2^4. Let M be an arbitrary normal abelian subgroup of $X_{\{u,v\}}$. Then $M\cap X_{uv}$ has index at most 2 in M, and $(M\cap X_{uv})X_v^{[1]}/X_v^{[1]}$ is isomorphic to a normal subgroup of $(X_v^{\Gamma(v)})_u$. Thus $(M\cap X_{uv})X_v^{[1]}/X_v^{[1]}\lesssim\mathbb{Z}_{p-1}$. Since $M\cap X_v^{[1]}\trianglelefteq X_v^{[1]}$ and $X_v^{[1]}$ is isomorphic to a normal subgroup of $(X_v^{\Gamma(v)})_u$, we have $M\cap X_v^{[1]}\lesssim\mathbb{Z}_{p-1}$. Noting that $(M\cap X_{uv})X_v^{[1]}/X_v^{[1]}\cong M\cap X_{uv}/(M\cap X_v^{[1]})$, it follows that $|M\cap X_{uv}|$ is a divisor of $(p-1)^2$. Thus $|M|$ is a divisor of $2(p-1)^2$.

The above observations allow us to consider only the pairs (G_0,H_0) in [18, Tables 14-20] which satisfy the following conditions:

(c1) $|H_0|$ is a divisor of $2^7\cdot 3^2\cdot(p-1)^2$ and divisible by 2^4; H_0 has a factor (a quotient of some subnormal subgroup) Q_8; and H_0 has no element of order 3^2, 5^2 or 11^2;

(c2) If M is a normal abelian subgroup of H_0 then $|M|$ is a divisor of $2(p-1)^2$; if $p\in\{7,11,23\}$, the order of $\mathbf{O}_{\frac{p-1}{2}}(H_0)$ is a divisor of $\frac{(p-1)^2}{4}$.

Checking the those H_0 which satisfy conditions (c1) and (c2), we conclude that the possible pairs $(X, X_{\{u,v\}})$ are listed as follows:

$(\mathrm{M}_{11}, 3^2{:}\mathrm{Q}_8.2)$, $(\mathrm{M}_{11}, 2\mathrm{S}_4)$, $(\mathrm{M}_{12}, [2^5].\mathrm{S}_3)$, $(\mathrm{M}_{12}, 3^2{:}2\mathrm{S}_4)$,
$(\mathrm{J}_2, [2^6]{:}(3{\times}\mathrm{S}_3))$, $(\mathrm{J}_3, [2^6]{:}(3{\times}\mathrm{S}_3))$, $(\mathrm{Co}_3, [2^9].3^2.\mathrm{S}_3))$,
$(\mathrm{He}.2, [2^8]{:}3^2.\mathrm{D}_8))$, $(\mathrm{McL}.2, [2^6]{:}\mathrm{S}_3^2)$,
$(\mathrm{PSL}_3(3), 3^2{:}2\mathrm{S}_4)$, $(\mathrm{PSL}_3(3).2, 2\mathrm{S}_4{:}2)$, $(\mathrm{PSL}_3(4).2, 2^{2+4}.3.2)$,
$(\mathrm{PGL}_3(4).2, [2^6].3.\mathrm{S}_3)$, $(\mathrm{PSL}_4(3).2, 2.\mathrm{S}_4^2.2)$, $(\mathrm{PSL}_5(2).2, [2^8].\mathrm{S}_3^2.2)$,
$(\mathrm{PSp}_4(4).4, [2^8]{:}3.12)$, $(\mathrm{PSp}_4(4).4, 5^2{:}[2^5])$, $(\mathrm{PSp}_6(2), [2^7]{:}\mathrm{S}_3^2)$,
$(\mathrm{PSp}_6(3), [2^8]{:}3^3.S_3)$, $(\mathrm{PSU}_3(3), 4.\mathrm{S}_4)$, $(\mathrm{PSU}_4(2), 2.\mathrm{A}_4^2.2)$,
$(\mathrm{PSU}_4(3), 2.\mathrm{A}_4^2.4)$, $(\mathrm{PSU}_4(3).2, [2^5].\mathrm{S}_4)$, $(\mathrm{P\Omega}_8^+(3).\mathrm{A}_4, 10^2{:}4\mathrm{A}_4)$,
$(\mathrm{G}_2(2)', 4.\mathrm{S}_4)$, $(\mathrm{G}_2(3), \mathrm{SL}_2(3)\circ\mathrm{SL}_2(3){:}2)$, $({}^2\mathrm{F}_4(2)', 5^2{:}4\mathrm{A}_4)$.

Note these groups X are included in the Atlas [3]. Inspecting the subgroups of X, only the pair $(\mathrm{PSL}_3(3).2, 2\mathrm{S}_4{:}2)$ gives a desired $X_v \cong 3^2{:}\mathrm{GL}_2(3)$, and then the desired graph Γ has valency $d = 9$. In this case, the socle $\mathrm{PSL}_3(3)$ of X has two orbits on the vertex set of Γ, each of them has size 13 and can be viewed as the point set or the line set of the projective plane $\mathrm{PG}(2, 3)$. This forces that Γ is (isomorphic to) one of the following graphs: $\mathsf{K}_{13,13}{-}13\mathsf{K}_2$, the point-line incidence graph and the point-line nonincidence graph of $\mathrm{PG}(2, 3)$. Since Γ has valency 9, the graph Γ is the point-line nonincidence graph of $\mathrm{PG}(2, 3)$. Then part (2) of this lemma follows.
Case 2. Let $2_+^{1+4}{:}\mathbb{Z}_5 \le (G_v^{\Gamma(v)})_u \le 2_+^{1+4}.(\mathbb{Z}_5{:}\mathbb{Z}_4)$. Then $2_+^{1+4} \trianglelefteq (X_v^{\Gamma(v)})_u$, and so, $|X_{\{u,v\}}|$ is a divisor of $2^{15}\cdot 5^2$ and divisible by 2^6. Further, if M is a normal abelian subgroup of $X_{\{u,v\}}$ then a similar argument as in *Case 1* yields that $|M|$ is a divisor of 2^5. It is easily shown that $\mathbf{O}_2(X_{uv}) \ne 1$, and hence $\mathbf{O}_2(X_{\{u,v\}}) \ne 1$. Checking the pairs (G_0, H_0) in [18, Tables 14-20], either $\mathbf{O}_2(H_0) = 1$ or $|H_0|$ has an odd prime divisor other than 5. Thus, in this case, no desired pair $(X, X_{\{u,v\}})$ exists. This completes the proof. □

We assume next that Lemma 3.5 (3) occurs. Thus $(G_v^{\Gamma(v)})_u \not\le \mathrm{GL}_1(p^f)$ and $(G_v^{\Gamma(v)})_u \le \Gamma\mathrm{L}_1(p^f)$. Then $f > 1$ and $(G_v^{\Gamma(v)})_u \lesssim \mathbb{Z}_{p^f-1}{:}\mathbb{Z}_f$. Recalling $X_{uv} \lesssim (X_u^{\Gamma(u)})_v{\times}(X_v^{\Gamma(v)})_u \le (G_u^{\Gamma(u)})_v{\times}(G_v^{\Gamma(v)})_u$, we have the following simple fact.

Lemma 5.4. *If (3) of Lemma 3.5 occurs then $X_{\{u,v\}}$ has no section $\mathbb{Z}_t^3$, $\mathbb{Z}_r^5$ or $\mathbb{Z}_2^6$, where t is a primitive prime divisor of $p^f - 1$ and r is an arbitrary odd prime.*

Lemma 5.5. *Assume that X_{uv} is nonabelian and (3) of Lemma 3.5 occurs. Then $p^f \ne 2^6$.*

Proof. Suppose that $p^f = 2^6$. Then X has order divisible by 2^6, $X_{uv} \lesssim \mathbb{Z}_{63}{:}\mathbb{Z}_6 \times \mathbb{Z}_{63}{:}\mathbb{Z}_6$, and thus $X_{\{u,v\}}$ has a normal Hall $2'$-subgroup and $|X_{\{u,v\}}|$ is indivisible by 2^4. Checking Tables 14-20 given in [18], $(X, X_{\{u,v\}})$ is one of the following pairs:

$(\mathrm{S}_7, \mathbb{Z}_7{:}\mathbb{Z}_6)$, $(\mathrm{M}_{12}.2, 3_+^{1+2}{:}\mathrm{D}_8)$, $(\mathrm{PSL}_2(2^6), \mathrm{D}_{126})$, $(\mathrm{PSL}_2(5^3), \mathrm{D}_{126})$, $(\mathrm{PSL}_2(7937), \mathrm{D}_{7938})$, $(\mathrm{PSL}_3(8), 7^2{:}\mathrm{S}_3)$, $(\mathrm{Sz}(8), \mathrm{D}_{14})$, $(\mathrm{G}_2(3).2, [3^6]{:}\mathrm{D}_8)$.

The pair $(\mathrm{PSL}_2(2^6), \mathrm{D}_{126})$ yields that $X_v \cong 2^6{:}\mathbb{Z}_{63}$, and thus X_{uv} is abelian, this is not the case. The other pairs are easily excluded as none of them gives a desired X_v. This completes the proof. □

LEMMA 5.6. *Assume that X_{uv} is nonabelian and (3) of Lemma 3.5 occurs. Suppose that X_{uv} has a normal abelian Hall $2'$-subgroup. Then $G = X$ or $X.2$, $X = \mathrm{M}_{10}$, $X_{\{u,v\}} \cong \mathbb{Z}_8{:}\mathbb{Z}_2$, $X_v \cong 3^2{:}\mathrm{Q}_8$ and $\Gamma \cong \mathsf{K}_{10}$.*

PROOF. Note that $X_{\{u,v\}} = X_{uv}.2$. The unique Hall $2'$-subgroup of X_{uv} is also the Hall $2'$-subgroup of $X_{\{u,v\}}$. Checking Tables 14-20 given in [18], we know that $(X, X_{\{u,v\}})$ is one of the following pairs:

(i) $(\mathrm{PGL}_2(7), \mathrm{D}_{16})$, $(\mathrm{PSL}_3(2).2, \mathrm{D}_{16})$, $(\mathrm{PGL}_2(9), \mathrm{D}_{16})$, $(\mathrm{M}_{10}, \mathbb{Z}_8{:}\mathbb{Z}_2)$, $(\mathrm{A}_5, \mathrm{D}_{10})$, $(\mathrm{A}_6, 3^2{:}\mathbb{Z}_4)$, $(\mathrm{M}_{11}, 3^2{:}\mathrm{Q}_8.2)$, $(\mathrm{J}_1, \mathrm{D}_6 \times \mathrm{D}_{10})$, $(\mathrm{PGL}_2(7), \mathrm{D}_{12})$, $(\mathrm{PGL}_2(9), \mathrm{D}_{20})$, $(\mathrm{M}_{10}, \mathbb{Z}_5{:}\mathbb{Z}_4)$, $(\mathrm{PGL}_2(11), \mathrm{D}_{20})$, $(\mathrm{PSL}_2(t^a), \mathrm{D}_{\frac{2(t^a\pm1)}{(2,t-1)}})$, $(\mathrm{PSp}_4(4).4, \mathbb{Z}_{17}{:}\mathbb{Z}_{16})$;

(ii) $(\mathrm{PSL}_2(t^a), \mathbb{Z}_t^a{:}\mathbb{Z}_{\frac{t^a-1}{2}})$, t is a prime $a \le 4$ and $t^a - 1$ is a power of 2;
$(\mathrm{PSL}_3(t), \mathbb{Z}_3^2{:}\mathrm{Q}_8)$, t is a prime with $t \equiv 4, 7 \mod 9$;
$(\mathrm{PSU}_3(t), \mathbb{Z}_3^2{:}\mathrm{Q}_8)$, t is a prime with $t \equiv 2, 5 \mod 9$;
$(\mathrm{PSU}_3(2^a), \mathbb{Z}_3^2{:}\mathrm{Q}_8)$ with prime $a > 3$;
$(\mathrm{PSp}_4(2^a).\mathbb{Z}_{2^{b+1}}, \mathrm{D}_{2(q\pm1)}^2{:}2.\mathbb{Z}_{2^{b+1}})$, $(\mathrm{PSp}_4(2^a).\mathbb{Z}_{2^{b+1}}, \mathbb{Z}_{2^{2a}+1}.4.\mathbb{Z}_{2^{b+1}})$, 2^b is the 2-part of a;
$(\mathrm{Sz}(2^{2a+1}), \mathrm{D}_{2(2^{2a+1}-1)})$, $(\mathrm{Sz}(2^{2a+1}), \mathbb{Z}_{2^{2a+1}\pm2^{a+1}+1}{:}\mathbb{Z}_4)$;
$({}^3\mathrm{D}_4(t^a), \mathbb{Z}_{t^{4a}-t^{2a}+1}{:}\mathbb{Z}_4)$, t is a prime.

The pair $(\mathrm{M}_{10}, \mathbb{Z}_8{:}\mathbb{Z}_2)$ yields that $X_v \cong 3^2{:}\mathrm{Q}_8$ and $d = 9$. The third pair in (i) implies that $X_v \cong \mathbb{Z}_3^2{:}\mathbb{Z}_8$; however, X_{uv} is abelian, which is not the case. For $(\mathrm{PSL}_2(t^a), \mathrm{D}_{\frac{2(t^a\pm1)}{(2,t^a-1)}})$, checking the subgroups of $\mathrm{PSL}_2(t^a)$, we have $t^a = p^f$ and $X_v \cong \mathbb{Z}_p^f{:}\mathbb{Z}_{\frac{p^f-1}{(2,p-1)}}$, and then X_{uv} is abelian, a contradction. The other pairs in (i) are also excluded as $|X|$ is indivisible by p^f. (Note that $f > 1$.)

Now we deal with the pairs in (ii). Note that, for an odd prime r, the edge-stabilizer $X_{\{u,v\}}$ has a unique Sylow r-subgroup $\mathbf{O}_r(X_{\{u,v\}})$. Then $\mathbf{O}_r(X_{\{u,v\}})$ is a Sylow subgroup of X by Lemma 2.7. This implies that the unique Hall $2'$-subgroup of $X_{\{u,v\}}$, say K, is a Hall subgroup of X. Since $X_{\{u,v\}} = X_{uv}.2$, we have $K \le X_{uv}$. Note that $|X_v{:}X_{uv}| = d = p^f$ and X_v is contained in a maximal subgroup of X. We now check the maximal subgroups of X which contain K, refer to [13, II.8.27], [1, Tables 8.3-8.6, 8.14, 8.15] and [14, 16, 28]. Then one of the following occurs:

(iii) $X = \mathrm{Sz}(2^{2a+1})$ and $X_v \cong \mathbb{Z}_2^{2a+1}{:}\mathbb{Z}_{2^{2a+1}-1}$;

(iv) $X = \mathrm{PSp}_4(2^a).\mathbb{Z}_{2^{b+1}}$ and $X_v \lesssim \mathrm{Sp}_2(2^{2a}){:}2.\mathbb{Z}_{2^b}$;

(v) $X = \mathrm{PSp}_4(2^a).\mathbb{Z}_{2^{b+1}}$ and $X_v \lesssim \mathrm{Sp}_2(2^a) \wr \mathrm{S}_2.\mathbb{Z}_{2^b}$.

Item (iii) yields that X_{uv} is abelian, which is not the case. Item (iv) gives $X_{uv} = X_v$, a contradiction. Suppose that (v) occurs, we have $X_v \cong (\mathbb{Z}_2^a{:}\mathbb{Z}_{2^a-1})^2{:}2.\mathbb{Z}_{2^b}$. Then $1 \neq \mathbf{O}_2(X_v) \leq \mathbf{O}_2(G_v)$, and hence $d = |\mathbf{O}_2(G_v)|$ by Lemma 2.5. Since X_v is transitive on $\Gamma(v)$, it follows that $p^f = d = 2^{2a}$. Thus $|X_{uv}| = (2^a-1)^2 2^{b+1}$, and so $|X_{\{u,v\}}{:}X_{uv}| = 8 > 2$, a contradiction. □

Corollary 5.7. *Assume that X_{uv} is nonabelian and (3) of Lemma 3.5 occurs. If $f = 2$ then $G = X$ or $X.2$, $X = \mathrm{M}_{10}$, $X_{\{u,v\}} \cong \mathbb{Z}_8{:}\mathbb{Z}_2$, $X_v \cong 3^2{:}\mathrm{Q}_8$ and $\Gamma \cong \mathsf{K}_{10}$.*

Proof. Let $f = 2$. Then $(X_v^{\Gamma(v)})_u \lesssim \mathbb{Z}_{p^2-1}.\mathbb{Z}_2$. Note that $X_{\{u,v\}} = X_{uv}.2$ and $X_{uv} \lesssim \mathbb{Z}_{p^2-1}.\mathbb{Z}_2 \times \mathbb{Z}_{p^2-1}.\mathbb{Z}_2$. Then Lemma 5.6 is applicable, and the result follows. □

Let $\pi_0(p^f-1)$ be the set of primitive primes of p^f-1. By Zsigmondy's theorem, if $\pi_0(p^f-1) = \emptyset$ and $f > 1$ then $p^f = 2^6$, or $f = 2$ and $p = 2^t - 1$, where t is a prime. Thus, in view of Lemma 5.5 and Corollary 5.7, we assume next that $\pi_0(p^f-1) \neq \emptyset$.

Lemma 5.8. *Assume that $\pi{:} = \pi_0(p^f-1) \neq \emptyset$, X_{uv} is nonabelian and (3) of Lemma 3.5 occurs. Then $f \geq 3$, and*

(1) *$\pi \neq \pi(|X_{\{u,v\}}|) \setminus \{2\}$, $\min(\pi) \geq \max\{5, f{+}1\}$;*
(2) *$p \not\equiv \pm 1 \mod r$ and $\mathbf{O}_r(X_{\{u,v\}}) \neq 1$ for each $r \in \pi$;*
(3) *$X_{\{u,v\}}$ has a unique (nontrivial) Hall π-subgroup, which is either cyclic or a direct product of two cyclic subgroups.*

Proof. By the assumptions in this lemma and Lemma 3.6, we have that $(X_v^{\Gamma(v)})_u \cong \mathbb{Z}_{m'}.\mathbb{Z}_{\frac{f}{e'}}$, and $\emptyset \neq \pi = \pi_0(p^f-1) \subseteq \pi(m')$. For $r \in \pi$, since $p^{r-1} \equiv 1 \mod r$, we have $f \leq r-1$, and so $r \geq f+1$. In particular, $r \geq 5$ and $p \not\equiv \pm 1 \mod r$. Recall that $X_{\{u,v\}} = X_{uv}.2$ and $X_{uv} \lesssim \mathbb{Z}_{m'}.\mathbb{Z}_{\frac{f}{e'}} \times \mathbb{Z}_{m'}.\mathbb{Z}_{\frac{f}{e'}}$. It follows that $\mathbf{O}_r(X_{\{u,v\}}) \neq 1$, and $\mathbf{O}_r(X_{\{u,v\}})$ is the unique Sylow r-subgroup of $X_{\{u,v\}}$. Clearly, $\mathbf{O}_r(X_{\{u,v\}})$ is either cyclic or a direct product of two cyclic subgroups. Then $X_{\{u,v\}}$ has a unique Hall π-subgroup F, which is either cyclic or a direct product of two cyclic subgroups. Clearly, $F \neq 1$ and, by Lemma 5.6, $X_{\{u,v\}}$ has no normal abelian Hall $2'$-subgroup. Then $\pi \neq \pi(|X_{\{u,v\}}|) \setminus \{2\}$, and the lemma follows. □

Recall that $X_{\{u,v\}}$ has no section $\mathbb{Z}_2^6$ or $\mathbb{Z}_3^5$, see Lemma 5.4. Combining with Lemma 5.8, we next check the pairs (G_0, H_0) listed in [18, Tables 14-20].

Lemma 5.9. *Assume that $\pi_0(p^f-1) \neq \emptyset$, X_{uv} is nonabelian and (3) of Lemma 3.5 occurs. Then $T = \mathrm{soc}(X)$ is not a simple group of Lie type.*

Proof. Suppose that T is a simple group of Lie type over a finite field of order $q' = t^a$, where t is a prime. Since $T \trianglelefteq G$, we know that T is transitive on the edge set of Γ. Then $T_v^{\Gamma(v)} \neq 1$. Noting that $T_v^{\Gamma(v)} \trianglelefteq G_v^{\Gamma(v)}$, we have $\mathrm{soc}(G_v^{\Gamma(v)}) \leq T_v^{\Gamma(v)}$. In particular, T_v is transitive on $\Gamma(v)$, and so $|T_v| = p^f|T_{uv}|$. In view of this, noting that $T_v = T \cap X_v = T \cap G_v$ and $T_{\{u,v\}} = T \cap X_{\{u,v\}} = T \cap G_{\{u,v\}}$, we sometimes work on the triple $(T, T_v, T_{\{u,v\}})$ instead of $(X, X_v, X_{\{u,v\}})$.

By Lemmas 5.6 and 5.8, $X_{\{u,v\}}$ is not a $\{2,3\}$-group and has no normal abelian Hall $2'$-subgroup. Assume that $t \in \pi_0(p^f - 1)$. By Lemmas 5.4 and 5.8, $t \geq 5$, $X_{\{u,v\}}$ has no section $\mathbb{Z}_t^3$ and $\mathbf{O}_t(X_{\{u,v\}}) \neq 1$ is abelian. Checking the pairs (G_0, H_0) listed in [18, Tables 16-20], we have $X = \mathrm{PSL}_2(t^2)$ and $X_{\{u,v\}} \cong \mathbb{Z}_t^2{:}\mathbb{Z}_{\frac{t^2-1}{2}}$. For this case, checking the subgroups of $\mathrm{PSL}_2(t^2)$, no desired X_v arises, a contradiction. Therefore, $t \notin \pi_0(p^f - 1)$.

By Lemma 5.8, $\mathbf{O}_r(X_{\{u,v\}}) \neq 1$ for each $r \in \pi_0(p^f - 1)$. Recall that $X_{\{u,v\}}$ is not a $\{2,3\}$-group and has a subgroup of index 2. Checking the pairs (G_0, H_0) listed in [18, Tables 16-20], we conclude that $\mathbf{O}_t(X_{\{u,v\}}) = 1$. Further, we observe that a desired $X_{\{u,v\}}$ if exists has the form of $N.K$, where N is an abelian subgroup of T and either K is a $\{2,3\}$-group or $(X, K) = (\mathrm{E}_8(q'), \mathbb{Z}_{30})$. For the case where $K \not\cong \mathbb{Z}_{30}$, by Lemma 3.6, $\pi_0(p^f - 1) \subseteq \pi(|N|)$, and thus, by Lemma 5.4, N has no subgroup $\mathbb{Z}_r^3$ for $r \in \pi_0(p^f - 1)$. With these restrictions, only one of the following Cases 1-4 occurs.

Case 1. Either $X = \mathrm{PSL}_3(q')$ and $X_{\{u,v\}} \cong \frac{1}{(3,q'-1)}\mathbb{Z}_{q'-1}^2.\mathrm{S}_3$ with $q' \neq 2$, 4, or $X = \mathrm{PSU}_3(q')$ and $X_{\{u,v\}} \cong \frac{1}{(3,q'+1)}\mathbb{Z}_{q'+1}^2.\mathrm{S}_3$. Then $|X_v| = \frac{3}{(3,q'\mp1)}p^f(q'\mp1)^2$. Checking Tables 8.3-8.6 given in [1], we have $X = \mathrm{PSL}_3(q')$ and $X_v \lesssim [q'^3]{:}\frac{1}{(3,q'-1)}\mathbb{Z}_{q'-1}^2$. It follows that $p = t = 3$, and $|\mathbf{O}_3(X_v)| = 3^{f+1} = 3d$, which contradicts Lemma 2.5.

Case 2. $T = \mathrm{soc}(X) = \mathrm{P\Omega}_8^+(q')$ and $T_{\{u,v\}} \cong \mathrm{D}^2_{\frac{2(q'^2+1)}{(2,q'-1)}}.[2^2]$. In this case, noting that $|T_{\{u,v\}}{:}T_{uv}| \leq 2$, we have $|T_v| = 2^4p^f\frac{(q'^2+1)^2}{(2,q'-1)^2}$ or $2^3p^f\frac{(q'^2+1)^2}{(2,q'-1)^2}$. Let M be a maximal subgroup of T with $T_v \leq M$. By [14], since $|M|$ is divisible by $(q'^2+1)^2$, we have $M \cong \mathrm{PSL}_2(q'^2)^2.2^2$. It is easily shown that $\mathrm{PSL}_2(q'^2)^2.2^2$ does not have subgroups of order $2^4p^f\frac{(q'^2+1)^2}{(2,q'-1)^2}$ or $2^3p^f\frac{(q'^2+1)^2}{(2,q'-1)^2}$, a contradiction.

Case 3. $(X, X_{\{u,v\}})$ is one of $({}^2\mathrm{F}_4(2)', 5^2{:}4\mathrm{A}_4)$ and $({}^2\mathrm{F}_4(2), 13{:}12)$. For the first pair, we have $\pi_0(p^f - 1) = \{5\}$ and, since p^f is a divisor of $|{}^2\mathrm{F}_4(2)'|$, we conclude that $p^f = 2^4$ or 3^4. The second pair implies that $\pi_0(p^f - 1) = \{13\}$, and then $p^f = 2^{12}$ or 3^3. By the Atlas [3], X has no maximal subgroup containing X_{uv} as a subgroup of index divisible by p^f, a contradiction.

Case 4. $T_{\{u,v\}}$ has a normal abelian subgroup N listed as follows:

T	N	$\lvert T_{\{u,v\}}:N\rvert$	Remark
$\mathrm{Ree}(3^a)$	$\mathbb{Z}_{3^a\pm 3^{\frac{a+1}{2}}+1}$	6	odd $a \geq 3$
	$\mathbb{Z}_2\times\mathbb{Z}_{\frac{3^a+1}{2}}$	6	& $X = T$
$\mathrm{G}_2(3^a)$	$\mathbb{Z}_{3^a\pm1}^2$	12	odd $a \geq 2$
	$\mathbb{Z}_{3^{2a}\pm3^a+1}$	6	
${}^2\mathrm{F}_4(2^a)$	$\mathbb{Z}_{2^a+1}^2$	48	odd $a \geq 3$
	$\mathbb{Z}^2_{2^a\pm2^{\frac{a+1}{2}}+1}$	96	& $X = T$
	$\mathbb{Z}_{2^{2a}\pm2^{\frac{3a+1}{2}}+2^a\pm2^{\frac{a+1}{2}}+1}$	12	& $2^a\pm2^{\frac{a+1}{2}}+1 > 5$
$\mathrm{F}_4(2^a)$	$\mathbb{Z}^2_{2^{2a}\pm2^a+1}$	72	
	$\mathbb{Z}^2_{2^{2a}+1}$	96	$a \geq 2$
	$\mathbb{Z}_{2^{4a}-2^{2a}+1}$	12	
$\mathrm{E}_8(q')$	$\mathbb{Z}^2_{q'^4-q'^2+1}$	288	$X = T$
	$\mathbb{Z}_{q'^8\pm q'^7\mp q'^5-q'^4\mp q'^3\pm q'+1}$	30	

Let M be a maximal subgroup of T with $T_v \leq M$. Then $|M|$ is divisible by $p^f|N|$. Check the maximal subgroups of T of order divisible by $|N|$, refer to [15, 21, 23]. Then we may deduce a contradiction. First, by [15, Theorem C], we conclude that $\mathrm{Ree}(3^a)$ has no maximal subgroup of order divisible by $p^f|N|$. Similarly, by [23], the group ${}^2\mathrm{F}_4(2^a)$ is excluded. We next deal with the remaining cases.

Suppose that $T = \mathrm{G}_2(3^a)$. For $|N| = 3^{2a}\pm3^a+1$, by [15, Theorems A and B]. Since $|M|$ is divisible by $3^{2a}\pm3^a+1$, we have $M \cong \mathrm{SL}_3(3^a){:}2$ or $\mathrm{SU}_3(3^a){:}2$. By [1, Tables 8.3-8.6], we conclude that $T_v \lesssim \mathbb{Z}_{3^{2a}\pm3^a+1}{:}[6]$, which is impossible. Similarly, for $|N| = (3^a\pm1)^2$, we have that $T_v \lesssim (\mathrm{SL}_2(3^a)\circ\mathrm{SL}_2(3^a)).2$, $\mathrm{SL}_3(3^a){:}2$ or $\mathrm{SU}_3(3^a){:}2$. Since $|T_v|$ is divisible by $\frac{1}{2}|T_{\{u,v\}}|p^f = 6p^f(3^a\pm1)^2$, checking the maximal subgroups of $\mathrm{SL}_2(3^a)$, $\mathrm{SL}_3(3^a)$ and $\mathrm{SU}_3(3^a)$, we have $p = 3$ and $T_v \lesssim [3^{ba}]{:}\mathbb{Z}^2_{3^a-1}.2$ for $b = 2$ or 3. Since T_{uv} has order divisible by 3, it follows that $\mathbf{O}_3(T_{uv}) \neq 1$, which contradicts Lemma 2.5.

Suppose that $T = \mathrm{F}_4(2^a)$. By [20, 21], noting that $|M|$ is divisible by $p^f|N|$, we conclude that $M \cong \mathrm{Sp}_8(2^a)$ or $\mathrm{P}\Omega_8^+(2^a).\mathrm{S}_3$ with $|N| = (2^{2a}+1)^2$, or $M \cong c.\mathrm{PSL}_3(2^a)^2.c.2$ or $c.\mathrm{PSU}_3(2^a)^2.c.2$ with $|N| = (2^{2a}\pm2^a+1)^2$, where $c = (3, 2^a \pm 1)$. Then a contradiction follows from checking the maximal subgroups of $\mathrm{Sp}_8(2^a)$, $\mathrm{P}\Omega_8^+(2^a)$, $\mathrm{PSL}_3(2^a)$ and $\mathrm{PSU}_3(2^a)$, refer to [1, Tables 8.3-8.6, 8.48-8.50].

Finally, suppose that $T = \mathrm{E}_8(q')$. Then $|N| = (q'^4 - q'^2+1)^2$ and $M \cong \mathrm{PSU}_3(q'^2)^2.8$. For this case, checking the maximal subgroups $\mathrm{PSU}_3(q'^2)$, we get a contradiction. This completes the proof. □

Lemma 5.10. *Assume that $\pi_0(p^f - 1) \neq \emptyset$, X_{uv} is not abelian and (3) of Lemma 3.5 occurs. Then $G = X = \mathrm{J}_1$, $X_{\{u,v\}} \cong \mathbb{Z}_7{:}\mathbb{Z}_6$, $X_v \cong \mathbb{Z}_2^3{:}\mathbb{Z}_7{:}\mathbb{Z}_3$ and $d = 8$.*

Proof. By Lemma 5.9, $T = \mathrm{soc}(X)$ is either an alternating group or a sporadic simple group. Note that $X_{\{u,v\}}$ is not a $\{2, 3\}$-group and has no normal abelian Hall $2'$-subgroup.

Assume that T is an alternating group. Then, by [18, Table 14], either $X = \mathrm{A}_r$ and $X_{\{u,v\}} \cong \mathbb{Z}_r{:}\mathbb{Z}_{\frac{r-1}{2}}$ for $r \notin \{7, 11, 17, 23\}$, or $X = \mathrm{S}_r$ and $X_{\{u,v\}} \cong \mathbb{Z}_r{:}\mathbb{Z}_{r-1}$ for $r \in \{7, 11, 17, 23\}$. For these two cases, X_v is a transitive subgroup of S_r in the natural action of S_r. Then either X_v is almost simple or $X_v \lesssim \mathbb{Z}_r{:}\mathbb{Z}_{r-1}$ (refer to [4, page 99, Corollary 3.5B]), a contradiction.

Assume that T is a sporadic simple group, and let $r \in \pi_0(p^f - 1)$. Then $(X, X_{\{u,v\}}, r)$ is one of the following triples:

$(\mathrm{J}_1, \mathbb{Z}_7{:}\mathbb{Z}_6, 7)$, $(\mathrm{J}_1, \mathbb{Z}_{11}{:}\mathbb{Z}_{10}, 11)$, $(\mathrm{J}_1, \mathbb{Z}_{19}{:}\mathbb{Z}_6, 19)$, $(\mathrm{J}_2, \mathbb{Z}_5^2{:}\mathrm{D}_{12}, 5)$,
$(\mathrm{J}_3.2, \mathbb{Z}_{19}{:}\mathbb{Z}_{18}, 19)$, $(\mathrm{J}_4, \mathbb{Z}_{29}{:}\mathbb{Z}_{28}, 29)$, $(\mathrm{J}_4, \mathbb{Z}_{37}{:}\mathbb{Z}_{12}, 37)$, $(\mathrm{J}_4, \mathbb{Z}_{43}{:}\mathbb{Z}_{14}, 43)$,
$(\mathrm{O'N}.2, \mathbb{Z}_{31}{:}\mathbb{Z}_{30}, 31)$, $(\mathrm{He}, \mathbb{Z}_5^2{:}4\mathrm{A}_4, 5)$, $(\mathrm{Co}_1, \mathbb{Z}_7^2{:}(3{\times}2\mathrm{A}_4), 7)$,
$(\mathrm{Ly}, \mathbb{Z}_{37}{:}\mathbb{Z}_{18}, 37)$, $(\mathrm{Ly}, \mathbb{Z}_{67}{:}\mathbb{Z}_{22}, 67)$, $(\mathrm{Fi}_{24}', \mathbb{Z}_{29}{:}\mathbb{Z}_{14}, 29)$,
$(\mathrm{B}, \mathbb{Z}_{13}{:}\mathbb{Z}_{12}{\times}\mathrm{S}_4, 13)$, $(\mathrm{B}, \mathbb{Z}_{19}{:}\mathbb{Z}_{18}{\times}\mathbb{Z}_2, 19)$, $(\mathrm{B}, \mathbb{Z}_{23}{:}\mathbb{Z}_{11}{\times}2, 23)$,
$(\mathrm{M}, \mathbb{Z}_{23}{:}\mathbb{Z}_{11}{\times}\mathrm{S}_4, 23)$, $(\mathrm{M}, (\mathbb{Z}_{29}{:}\mathbb{Z}_{14}{\times}3).2, 29)$, $(\mathrm{M}, \mathbb{Z}_{31}{:}\mathbb{Z}_{15}{\times}\mathrm{S}_3, 31)$,
$(\mathrm{M}, \mathbb{Z}_{41}{:}\mathbb{Z}_{40}, 41)$, $(\mathrm{M}, \mathbb{Z}_{47}{:}\mathbb{Z}_{23}{\times}2, 47)$.

Recall that p^f is a divisor of $|X|$ and r is a primitive prime divisor of $p^f - 1$. Searching all possible pairs (p^f, r), we get the following table:

X	J_1	J_2	J_4	Co_1	$\mathrm{O'N}.2$	He	B
$\lvert X_{\{u,v\}}\rvert$	$2{\cdot}3{\cdot}7$	$2^2{\cdot}3{\cdot}5^2$	$2{\cdot}7{\cdot}43$	$2^3{\cdot}3^2{\cdot}7^2$	$2{\cdot}3{\cdot}5{\cdot}31$	$2^4{\cdot}3{\cdot}5^2$	$2^5{\cdot}3^4{\cdot}13$
r	7	5	43	7	31	5	13
p^f	2^3	2^4	2^{14}	2^3, 3^6	2^5	2^4	$3^3, 5^4, 2^{12}$
$p^f - 1 \mid \lvert G_{uv}\rvert$	✓	✓	×	✓, ×	✓	✓	✓, ✓, ×
X	B	B	M	M	M	M	M
$\lvert X_{\{u,v\}}\rvert$	$2^2{\cdot}3^2{\cdot}19$	$2{\cdot}11{\cdot}23$	$2^3{\cdot}3{\cdot}11{\cdot}23$	$2^2{\cdot}3{\cdot}7{\cdot}29$	$2{\cdot}3^2{\cdot}5{\cdot}31$	$2^3{\cdot}5{\cdot}41$	$2{\cdot}23{\cdot}47$
r	19	23	23	29	31	41	47
p^f	2^{18}	$2^{11}, 3^{11}$	$2^{11}, 3^{11}$	2^{28}	$2^5, 5^3$	$2^{20}, 3^8$	2^{23}
$p^f - 1 \mid \lvert G_{uv}\rvert$	×	×, ×	×, ×	×	✓, ×	×, ×	×

Recalling that $G_{\{u,v\}} = X_{\{u,v\}}.(G/X)$, we have $2|G_{uv}| = |G_{\{u,v\}}| = |X_{\{u,v\}}||G{:}X| = 2|X_{uv}||G{:}X|$, and so $|G_{uv}| = |X_{uv}||G{:}X|$. Since G_v is 2-transitive on $\Gamma(v)$, we know that $(p^f - 1)$ is a divisor of $|G_{uv}| = |X_{uv}||G{:}X|$. It follows that $(X, X_{\{u,v\}}, r, p^f)$ is one of the following quadruples:

$(\mathrm{J}_1, \mathbb{Z}_7{:}\mathbb{Z}_6, 7, 2^3)$, $(\mathrm{J}_2, \mathbb{Z}_5^2{:}\mathrm{D}_{12}, 5, 2^4)$, $(\mathrm{Co}_1, \mathbb{Z}_7^2{:}(3{\times}2\mathrm{A}_4), 7, 2^3)$,
$(\mathrm{O'N}.2, \mathbb{Z}_{31}{:}\mathbb{Z}_{30}, 31, 2^5)$, $(\mathrm{He}, \mathbb{Z}_5^2{:}4\mathrm{A}_4, 5, 2^4)$, $(\mathrm{B}, \mathbb{Z}_{13}{:}\mathbb{Z}_{12}{\times}\mathrm{S}_4, 13, 3^3)$,
$(\mathrm{B}, \mathbb{Z}_{13}{:}\mathbb{Z}_{12}{\times}\mathrm{S}_4, 13, 5^4)$, $(\mathrm{M}, \mathbb{Z}_{31}{:}\mathbb{Z}_{15}{\times}\mathrm{S}_3, 31, 2^5)$.

For $(\mathrm{Co}_1, \mathbb{Z}_7^2{:}(3{\times}2\mathrm{A}_4), 7, 2^3)$, we have $X_{uv} \lesssim \Gamma\mathrm{L}_1(2^3){\times}\Gamma\mathrm{L}_1(2^3)$, yielding that $|X_{uv}|$ is odd, a contradiction. Similarly, for $(\mathrm{B}, \mathbb{Z}_{13}{:}\mathbb{Z}_{12}{\times}\mathrm{S}_4, 13, 3^3)$, the order of X_{uv} is indivisible by 2^4, a contradiction; for $(\mathrm{M}, \mathbb{Z}_{31}{:}\mathbb{Z}_{15}{\times}\mathrm{S}_3, 31, 2^5)$, the order of X_{uv} is indivisible by 3, a contradiction. For $(\mathrm{He}, \mathbb{Z}_5^2{:}4\mathrm{A}_4, 5, 2^4)$, the order of X_{uv} is divisible by $2^3 \cdot 3 \cdot 5^2$ and, since $p^f = 2^4$, the order of X_u is divisible by $2^7 \cdot 3 \cdot 5^2$; however, He has no soluble subgroup of order divisible by $2^7 \cdot 3 \cdot 5^2$, a contradiction. Similarly, $(\mathrm{O'N}.2, \mathbb{Z}_{31}{:}\mathbb{Z}_{30}, 31, 2^5)$ is excluded as O′N.2 has no soluble subgroup with order divisible by $2^5 \cdot 31$. (Note that G_v is soluble.) By the Altas [3], J_2 has no subgroup with order divisible by $2^4 \cdot 5^2$, and then $(\mathrm{J}_2, \mathbb{Z}_5^2{:}\mathrm{D}_{12}, 5, 2^4)$ is excluded. By the Altas [3] and [35, Theorem 2.1], B has no subgroup with order divisible by $3^2 \cdot 5^4 \cdot 13$, and so $(\mathrm{B}, \mathbb{Z}_{13}{:}\mathbb{Z}_{12}\times\mathrm{S}_4, 13, 5^4)$ is excluded. Then only $(\mathrm{J}_1, \mathbb{Z}_7{:}\mathbb{Z}_6, 7, 2^3)$ is left, which gives $X_v \cong \mathbb{Z}_2^3{:}\mathbb{Z}_7{:}\mathbb{Z}_3$, $d = p^f = 8$ and $G = X$. This completes the proof. □

Finally, we summarize the argument for proving Theorem 1.1 as follows.

Proof of Theorem 1.1. Clearly, each $(G, G_v, G_{\{u,v\}})$ in Table 1 gives a G-edge-primitive graph $\mathsf{Cos}(G, G_v, G_{\{u,v\}})$. It is not difficult to check the 2-arc-transitivity of G acting on $\mathsf{Cos}(G, G_v, G_{\{u,v\}})$, we omit the details.

Now let G and $\Gamma = (V, E)$ satisfy the assumptions in Theorem 1.1. Let $T = \mathrm{soc}(G)$ and $\{u, v\} \in E$. Choose a minimal X among the normal subgroups of G which act primitively on E. Then $\mathrm{soc}(X) = T$. Since $G_{\{u,v\}}$ is soluble, $X_{\{u,v\}}$ is soluble. Then $(X, X_{\{u,v\}})$ is one of the pairs (G_0, H_0) listed in [18, Tables 14-20]. Thus Γ, G, $G_{\{u,v\}}$, X and $X_{\{u,v\}}$ satisfy Hypothesis 3.1, and then Lemmas 3.3 and 3.5 work here. If $G_v^{\Gamma(v)}$ is an almost simple 2-transitive group then, by Lemma 3.3 and Lemmas 4.1-4.4, the triple $(G, G_v, G_{\{u,v\}})$ is listed in Table 1. Assume next that $G_v^{\Gamma(v)}$ is a soluble 2-transitive group of degree $d = p^f$, where p is a prime.

If X_{uv} is abelian then the triple $(G, G_v, G_{\{u,v\}})$ is desired as in Table 1 by Lemma 5.2. Thus assume further that X_{uv} is nonabelian. Then G_{uv} is nonabelian. By Lemma 3.5, either $G_v^{\Gamma(v)} \not\leq \mathrm{GL}_1(p^f)$ and $G_v^{\Gamma(v)} \leq \Gamma\mathrm{L}_1(p^f)$, or $G_v^{\Gamma(v)}$ has a normal subgroup $\mathrm{SL}_2(3)$ or 2_+^{1+4}. For the latter case, the triple $(G, G_v, G_{\{u,v\}})$ is known by Lemma 5.3. Let $G_v^{\Gamma(v)} \leq \Gamma\mathrm{L}_1(p^f)$ and consider the primitive prime divisors of $p^f - 1$. If $p^f - 1$ has no primitive prime divisor then, by Lemma 5.5 and Corollary 5.7, $(G, G_v, G_{\{u,v\}})$ is listed in Table 1. If $p^f - 1$ has primitive prime divisors, then $(G, G_v, G_{\{u,v\}})$ is known by Lemma 5.10. This completes the proof.

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