Strong rainbow disconnection in graphs

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Abstract. Let G be a nontrivial edge-colored connected graph. A *rainbow edge-cut* is an edge-cut R of G, and all edges of R have different colors in G. For two different vertices u and v of G, a *u-v-edge-cut* is an edge-cut separating them. An edge-colored graph G is called *strong rainbow disconnected* if for every two distinct vertices u and v of G, there exists a both rainbow and minimum u-v-edge-cut in G, and such an edge-coloring is called a *strong rainbow disconnection coloring* (srd-coloring for short) of G. For a connected graph G, the *strong rainbow disconnection number* (srd-number for short) of G, denoted by $\mathrm{srd}(G)$, is the minimum number of colors required to make G strong rainbow disconnected. In this paper, we first characterize the graphs with m edges satisfing $\mathrm{srd}(G) = k$ for each $k \in \{1, 2, m\}$, respectively, and we also show that the srd-number of a nontrivial connected graph G is equal to the maximum srd-number in the blocks of G. Secondly, we study the srd-numbers for the complete k-partite graphs, k-edge-connected k-regular graphs and grid graphs. Finally, we prove that for a connected graph G, computing $\mathrm{srd}(G)$ is NP-hard. In particular, we prove that it is NP-complete to decide if $\mathrm{srd}(G) = 3$ for a connected cubic graph. We also show that the following problem is NP-complete: given an edge-colored (with an unbounded number of colors) connected graph G, check whether the given coloring makes G strong rainbow disconnected.

1. Introduction

All graphs considered in this paper are simple, nontrivial, finite and undirected. Let G be a nontrivial connected graph with vertex-set $V(G)$ and edge-set $E(G)$. For $v \in V(G)$, let $d_G(v)$ and $N_G(v)$ denote the *degree* and the *neighborhood* of v in G (or simply $d(v)$ and $N(v)$, respectively, when it is clear which G it refers to). We follow the notations and terminology of Bondy and Murty [1].

An edge-coloring of G is *proper* if adjacent edges receive different colors. The *chromatic index* of G, denoted by $\chi'(G)$, is the minimum number of colors required in a proper edge-coloring of G. An important theorem due to Vizing [2], asserts that for any simple graph G, either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$. If $\chi'(G) = \Delta(G)$, then G is said to belong to *Class* 1; and the others to *Class* 2.

We know that there are two ways to study the connectivity of graphs, one is to use paths, and the other is to use cuts. The rainbow connection using paths has been studied extensively; see for examples, papers [3, 4, 5] and book [6] and the references therein. So, it is natural to consider the rainbow edge-cuts for the colored connectivity in edged-colored graphs. In [7], Chartrand et al. first discussed the rainbow edge-cut by introducing the concept of rainbow disconnection of graphs. In [8] we call all of them global colorings of graphs since they relate global structural property: connectivity of graphs.

An *edge-cut* of a connected graph G is a subset of edges which separates some pair of vertices. For distinct vertices u and v of G, we denote by $\lambda_G(u,v)$ (or simply $\lambda(u,v)$ when it is clear which G it refers to) the minimum cardinality in an edge-cut S such that u and v are in different components of $G-S$, and this kind of edge-cut S is called a *minimum u-v-edge-cut*. The minimum cardinality of an edge cut of G is the *edge-connectivity* of G, denoted by $\lambda(G)$ (i.e., $\lambda(G)$ is the minimum value of $\lambda_G(u,v)$ taken over all pairs of distinct vertices u, v); whereas the maximum value of $\lambda_G(u,v)$ taken over all pairs of distinct vertices u, v is the *upper edge-connectivity* of G, denoted by $\lambda^+(G)$, which was introduced and studied in [9, 10]. A *u-v-path* is a path with ends u and v. The following proposition presents an alternate interpretation of $\lambda(u,v)$ (see [11, 12]).

Proposition 1.1 For every two distinct vertices u and v of a graph G, $\lambda(u,v)$ is equal to the maximum number of pairwise edge-disjoint u-v-paths in G.

A *rainbow edge-cut* is an edge-cut R of an edge-colored connected graph G, and all edges of R have different colors in G. Let u and v be two vertices of G. A rainbow edge-cut R of G is called a *rainbow u-v-edge-cut* if u and v belong to different components of $G-R$. An edge-colored graph G is called *rainbow disconnected* if for each pair of different vertices u and v of G, there exists a rainbow u-v-edge-cut in G. Such an edge-coloring is called a *rainbow disconnection coloring* (abbreviated as rd-*coloring*) of G. The *rainbow disconnection number* (abbreviated as rd-*number*) of G, denoted by $\mathrm{rd}(G)$, is the minimum number of colors required to make G rainbow disconnected. An *optimal* rd-*coloring* of G is an rd-coloring with $\mathrm{rd}(G)$ colors.
Remember that in the above Menger's famous result of Proposition 1.1, only minimum edge-cuts play a role, however, in the definition of rd-colorings we only requested the existence of a u-v-edge-cut between vertices u and v, which could be any edge-cut (large or small are both OK). This may cause the failure of a colored version of such a nice Min-Max result of Proposition 1.1. In order to overcome this problem, we will introduce the concept of strong rainbow disconnection in graphs, with a hope to set up the colored version of the so-called Max-Flow Min-Cut Theorem.
An edge-colored graph G is called *strong rainbow disconnected* if for every two distinct vertices u and v of G, there exists a both rainbow and minimum u-v-edge-cut (*rainbow minimum u-v-edge-cut* for short) in G. Such an edge-coloring is called a *strong rainbow disconnection coloring* (abbreviated as srd-*coloring*) of G. For a connected graph G, similarly, the *strong rainbow disconnection number*(abbreviated as srd-*number*) of G, denoted by $\mathrm{srd}(G)$, is the minimum number of colors required to make G strong rainbow disconnected. An $\mathrm{srd}(G)$-coloring with $\mathrm{srd}(G)$ colors is called an *optimal* srd-*coloring* of G.
In the remainder of this paper, we first present some basic results for the srd-numbers of graphs in Section 2. Then we study the srd-numbers for some special graphs in Section 3. In the last section, we show that for a connected graph G, computing $\mathrm{srd}(G)$ is NP-hard. In particular, we prove that it is NP-complete to decide if $\mathrm{srd}(G)=3$ for a connected cubic graph. We also show that the following problem is NP-complete: given an edge-colored (with an unbounded number of colors) connected graph G, check whether the given coloring makes G strong rainbow disconnected.

2. Some basic results

Let G be a connected graph. Suppose that X is a vertex subset of G, and let $\overline{X}=V(G)\setminus X$. Recall that for a pair of distinct vertices x and y of G, we say that an edge-cut $\partial(X)$ *separates* x and y if $x\in X$ and $y\in V\setminus X$. We denote by $C_G(x,y)$ the minimum size of such an edge-cut in G. The graph G/X is obtained from G by *shrinking* X to a single vertex. A *trivial edge-cut* is an edge-cut associated with a single vertex. We denote by E_v the set of edges incident with v in G. A *block of a graph* is a subgraph which contains no cut-vertices and is maximal with respect to this property. From definitions, the following inequalities are obvious.

Proposition 2.1 *If G is a connected graph with edge-connectivity $\lambda(G)$, upper edge-connectivity $\lambda^+(G)$ and number $e(G)$ of edges, then*

$$\lambda(G) \leq \lambda^+(G) \leq \mathrm{rd}(G) \leq \mathrm{srd}(G) \leq e(G). \tag{1}$$

Our first question is that the new parameter srd-number is really something new, different from rd-number ? However, we have not found any connected graph G with $\mathrm{srd}(G) \neq \mathrm{rd}(G)$. So, we pose the following conjecture.

Conjecture 2.2 *For any connected graph G,* $\mathrm{srd}(G) = \mathrm{rd}(G)$.

In the rest of the paper we will show that for many classes of graphs the conjecture is true.
In this section, we characterize all nontrivial connected graphs of m edges such that $\mathrm{srd}(G) = k$ for each $k \in \{1, 2, m\}$, respectively. We first characterize the graphs with $\mathrm{srd}(G) = m$. The following are two lemmas which we will be used.

Lemma 2.3 [13] *Let $\partial(X)$ be a minimum edge-cut in a graph G separating two vertices x and y, where $x \in X$, and let $\partial(Y)$ be a minimum edge-cut in G separating two vertices u and v of X ($\overline{X}$), where $y \in Y$. Then every minimum u-v-edge-cut in $G/\overline{X}$ (G/X) is a minimum u-v-edge-cut in G.*

It follows from Lemma 2.3 that we get the following result.

Lemma 2.4 *Let G be a connected graph of order at least 3. Then* $\mathrm{srd}(G) \leq e(G) - 1$.

Proof. We distinguish the following two cases.
Case 1. There is at least one pair of vertices having nontrivial minimum edge-cut.
Let $C_G(x, y)$ be a nontrivial minimum u-v-edge-cut of G, where $x, y \in V(G)$, and let $\partial(X) = \min\{C_G(x, y)|x, y \in V(G)\}$. Suppose that $\partial(X)$ is a nontrivial minimum x_0-y_0-edge-cut in graph G, where $x_0 \in X$, and let $\partial(Y)$ be a minimum u-v-edge-cut in G, where $u, v \in X$ and $y_0 \in Y$. By Lemma 2.3, we get that every minimum u-v-edge-cut in $G/\overline{X}$ is a minimum u-v-edge-cut in G. Now we give an edge-coloring c for G by assigning different colors for each edge of $G[X]$ using colors from $[e(G[X])]$ and assigning different colors for each edge of $G[\overline{X}]$ using colors from $[e(G[\overline{X}])]$, respectively, and assigning $|\partial(X)|$ new colors for $\partial(X)$. Note that the E_w is rainbow for each vertex w of G, and $|c| = \max\{e(G[X]), e(G[\overline{X}])\} + |\partial(X)| \leq e(G) - 1$ since $e(G[X]), e(G[\overline{X}]) \geq 1$.
We can verify that the coloring c is an srd-coloring of G. Let p and q be two vertices of G. If p and q have a nontrivial minimum edge-cut $C_G(p, q)$ in G, then $|C_G(p, q)| \geq |\partial(X)|$. Suppose that $p \in X$ and $q \in \overline{X}$. Without loss of generality, let $d(p) \leq d(q)$. If $d(p) < |\partial(X)|$, then the E_p is a rainbow minimum p-q-edge-cut in G under the coloring c; if $|\partial(X)| \leq d(p) \leq d(q)$, then the $\partial(X)$ is a rainbow minimum p-q-edge-cut in G under the coloring c. If $p, q \in X$ ($\overline{X}$), then the minimum p-q-edge-cut in $G/\overline{X}$ (G/X) is a rainbow minimum p-q-edge-cut in G since the colors of the edges in graph $G/\overline{X}$ (G/X) are different from each other under the restriction of coloring c.
Case 2. For any two vertices of G, there are only trivial minimum edge-cut.
If G is a tree, then $\mathrm{srd}(G) = 1$. Obviously, $\mathrm{srd}(G) \leq e(G) - 1$ since G is a connected graph with $n \geq 3$. Otherwise, we give a proper edge-coloring for G using $n - 1$ colors. Since G is not a tree, we have $n - 1 \leq e(G) - 1$. For any two vertices p, q of G, without loss of generality, let $d(p) \leq d(q)$, the E_p is a rainbow minimum p-q-edge-cut in G. $\square$

By Lemma 2.4, we immediately obtain the following result.

Corollary 2.5 *Let G be a connected graph. Then $\mathrm{srd}(G) = e(G)$ if and only if $G = P_2$.*

Next, we further characterize the graphs G with $\mathrm{srd}(G) = 1$ and 2, respectively. We first restate two results as lemmas.

Lemma 2.6 [7] *Let G be a nontrivial connected graph. Then $\mathrm{rd}(G) = 1$ if and only if G is a tree.*

Lemma 2.7 [7] *Let G be a nontrivial connected graph. Then $\mathrm{rd}(G) = 2$ if and only if each block of G is either K_2 or a cycle and at least one block of G is a cycle.*

Furthermore, we obtain the following two results.

Theorem 2.8 *Let G be a nontrivial connected graph. Then $\mathrm{srd}(G) = 1$ if and only if $\mathrm{rd}(G) = 1$.*

Proof. First, if $\mathrm{srd}(G) = 1$, then we have $1 \leq \mathrm{rd}(G) \leq \mathrm{srd}(G)$ by Proposition 2.1. Next, if $\mathrm{rd}(G) = 1$, then the graph G has no cycle, namely, the G is a tree. We give one color for all edges of G. Obviously, the coloring is an optimal srd-coloring of G, and so $\mathrm{srd}(G) = 1$ by Proposition 2.1. $\square$

Theorem 2.9 *Let G be a nontrivial connected graph. Then $\mathrm{srd}(G) = 2$ if and only if $\mathrm{rd}(G) = 2$.*

Proof. First, if $\mathrm{srd}(G) = 2$, then G has no cycle with a chord by Proposition 2.1. Furthermore, we know $\mathrm{srd}(G) = 1$ if G is a tree. Therefore, each block of G can only be K_2 or cycle, and at least one of the blocks of G is a cycle. By Lemma 2.7, we get $\mathrm{rd}(G) = 2$.
Conversely, suppose $\mathrm{rd}(G) = 2$. Then each block of G can only be K_2 or cycle, and at least one of the blocks of G is a cycle. We can give a 2-edge-coloring c for G as follows. Choose one edge from each cycle to give color 1. The remaining edges are assigned color 2. We can verify that the c is strong rainbow disconnected. Combined with Proposition 2.1, we have $\mathrm{srd}(G) = 2$. $\square$

By Lemmas 2.6 and 2.7, and Theorems 2.8 and 2.9, we immediately get Corollary 2.10.

Corollary 2.10 *Let G be a connected graph. Then*
(i) $\mathrm{srd}(G) = 1$ *if and only if G is a tree.*
(ii) $\mathrm{srd}(G) = 2$ *if and only if each block of G is either a K_2 or a cycle and at least one block of G is a cycle.*

Furthermore, we get $\mathrm{srd}(G)$ is equal to the maximum srd-number among all blocks of G. It implies that we only need to study the srd-numbers of 2-connected graphs.

Lemma 2.11 *If H is a block of a graph G, then $\mathrm{srd}(H) \leq \mathrm{srd}(G)$.*

Proof. Let c be an optimal srd-coloring of G, and let u, v be two vertices of H. Suppose R is a rainbow minimum u-v-edge-cut in G. Then $R \cap E(H)$ is a rainbow minimum u-v-edge-cut in H. Assume that there exists a smaller u-v-edge-cut R' in H. Then there is no u-v-path in $G \setminus R'$. This contradicts to the definition of R since $|R'| < |R|$. Denote by c_H the coloring c restricted to H. So, c_H is an srd-coloring of H. Therefore, $\mathrm{srd}(H) \leq \mathrm{srd}(G)$. $\square$

Theorem 2.12 *Let G be a connected graph with the set of blocks $\{B_1, B_2, \ldots, B_t\}$, where t is a positive integer. Then $\mathrm{srd}(G) = \max\{\mathrm{srd}(B_i) : i \in [t]\}$.*

Proof. Let $k = \max\{\mathrm{srd}(B_i) : i \in [t]\}$. If G is 2-connected, then $G = B_1$ and the result is obviously true. Thus, suppose that G has at least one cut-vertex. By Lemma 2.11, we have $k \leq \mathrm{srd}(G)$.
Let c_i be an optimal srd-coloring of B_i. We define the edge-coloring c of G by $c(e) = c_i(e)$ if $e \in E(B_i)$ using colors from [k]. Let u and v be two vertices of G. If $u, v \in B_i$ $(i \in [t])$, let $C_G(u, v) = C^r_{B_i}(u, v)$, where $C^r_{B_i}(u, v)$ is the rainbow minimum u-v-edge-cut in B_i. Obviously,

$C_G(u,v)$ is rainbow under the coloring c_i. Moreover, it is minimum u-v-edge-cut in G. Otherwise, assume that R is a smaller u-v-edge-cut in G. Then $R \cap E(B_i)$ is also a u-v-edge-cut in B_i, which contradicts to the definition of $C^r_{B_i}(u,v)$ since $|R \cap E(B_i)| < |C_{B_i}(u,v)|$. Hence, the $C_G(u,v)$ is a rainbow minimum u-v-edge-cut in G. Suppose that $u \in B_i$ and $v \in B_j$, where $i < j$ and $i,j \in [t]$. Let $B_i x_i B_{i+1} x_{i+1} \dots x_{j-1} B_j$ be a unique B_i-B_j-path in the block-tree of G, and let x_i be the cut-vertex between blocks B_i and B_{i+1}. If $u = x_i$ and $v = x_{j-1}$, let $C_G(u,v) = \min\{C^r_{B_{i+1}}(x_i,x_{i+1}),\dots,C^r_{B_{j-1}}(x_{j-2},x_{j-1})\}$. If $u = x_i$ and $v \neq x_{j-1}$, let $C_G(u,v) = \min\{C^r_{B_{i+1}}(x_i,x_{i+1}),\dots,C^r_{B_{j-1}}(x_{j-2},x_{j-1}),C^r_{B_j}(x_{j-1},v)\}$. If $u \neq x_i$ and $v = x_{j-1}$, let $C_G(u,v) = \min\{C^r_{B_i}(u,x_i),C^r_{B_{i+1}}(x_i,x_{i+1}),\dots,C^r_{B_{j-1}}(x_{j-2},x_{j-1})\}$. If $u \neq x_i$ and $v \neq x_{j-1}$, let $C_G(u,v) = \min\{C^r_{B_i}(u,x_i),C^r_{B_{i+1}}(x_i,x_{i+1}),\dots,C^r_{B_j}(x_{j-1},v)\}$. By the connectivity of G, we know that $\lambda_G(u,v) = |C_G(u,v)|$, and $C_G(u,v)$ is rainbow. Then $C_G(u,v)$ is a rainbow minimum u-v-edge-cut in G. Hence, $\mathrm{srd}(G) \le k$, and so $\mathrm{srd}(G) = k$. □

Remark 2.13 *As one has seen that all the above results for the* srd*-number behave the same as for the* rd*-number. This supports Conjecture 2.2.*

3. The srd-numbers of some classes of graphs

In this section, we investigate the srd-numbers of complete graphs, complete multipartite graphs, regular graphs and grid graphs. Again, we will see that the results for srd-number behave the same as for the rd-number. At first, we restate several results as lemmas which will be used in the sequel.

Lemma 3.1 [14] *Let G be a connected graph. If every connected component of G_Δ is a unicyclic graph or a tree, and G_Δ is not a disjoint union of cycles, then G is in Class 1.*

Lemma 3.2 [7] *For each integer $n \ge 4$, $\mathrm{rd}(K_n) = n-1$.*

Lemma 3.3 [15] *If $G = K_{n_1,n_2,\dots,n_k}$ is a complete k-partite graph with order n, where $k \ge 2$ and $n_1 \le n_2 \le \cdots \le n_k$, then*

$$\mathrm{rd}(K_{n_1,n_2,\dots,n_k}) = \begin{cases} n-n_2, & \text{if } n_1 = 1, \\ n-n_1, & \text{if } n_1 \ge 2. \end{cases} \tag{2}$$

Lemma 3.4 [15] *If G is a connected k-regular graph, then $k \le \mathrm{rd}(G) \le k+1$.*

Lemma 3.5 [14] *The* rd*-number of the grid graph $G_{m,n}$ is as follows.*
(i) *For all $n \ge 2$, $\mathrm{rd}(G_{1,n}) = \mathrm{rd}(P_n) = 1$.*
(ii) *For all $n \ge 3$, $\mathrm{rd}(G_{2,n}) = 2$.*
(iii) *For all $n \ge 4$, $\mathrm{rd}(G_{3,n}) = 3$.*
(iv) *For all $4 \ge m \ge n$, $\mathrm{rd}(G_{m,n}) = 4$.*

First, we get the srd-number for complete graphs.

Theorem 3.6 *For each integer $n \ge 2$, $\mathrm{srd}(K_n) = n-1$.*

Proof. By Proposition 2.1 and Lemma 3.2, $n-1 \le \mathrm{rd}(K_n) \le \mathrm{srd}(K_n)$. It remains to show that there exists an srd-coloring for K_n using $n-1$ colors. Suppose first that $n \ge 2$ is even. Let u and v be two vertices of K_n, and let c be a proper edge-coloring of K_n using $n-1$ colors. Since $\lambda(K_n) = n-1$, the E_u is a rainbow minimum u-v-edge-cut in G. Next suppose $n \ge 3$ is odd. We give the same edge-coloring for graph G as the coloring in Lemma 3.2. We now restate it as follows. Let x be a vertex of K_n and $K_{n-1} = K_n - x$. Then K_{n-1} has a proper edge-coloring c using $n-2$ colors since $n-1$ is even. Now we extend an edge-coloring c of K_{n-1} to K_n by assigning color $n-1$ for each edge incident with vertex x. Let u and v be two vertices of K_n, say $u \neq x$. Then the E_u is a rainbow minimum u-v-edge-cut in G since $\lambda(K_n) = n-1$. $\square$

Then, we give the srd-number for complete multipartite graphs.

Theorem 3.7 *If $G = K_{n_1,n_2,\ldots,n_k}$ is a complete k-partite graph with order n, where $k \geq 2$ and $n_1 \leq n_2 \leq \cdots \leq n_k$, then*

$$\operatorname{srd}(K_{n_1,n_2,\ldots,n_k}) = \begin{cases} n-n_2, & \text{if } n_1 = 1, \\ n-n_1, & \text{if } n_1 \geq 2. \end{cases} \tag{3}$$

Proof. It remains to prove that $\operatorname{srd}(G) \leq n-n_2$ for $n_1 = 1$, and $\operatorname{srd}(G) \leq n-n_1$ for $n_1 \geq 2$ by Proposition 2.1 and Lemma 3.3. Let $V_1, V_2, \ldots V_k$ be the k-partition of the vertices of G, and $V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}\}$ for each $i \in [k]$. We consider two cases.

Case 1. $n_1 = 1$.

First, we have $V_1 = \{v_{1,1}\}$ and $d(v_{1,1}) = n-1$. Let $H = G - \{v_{1,1}\}$. Then $\Delta(H) = n-n_2-1$. Then, we construct a proper edge-coloring c_0 of H using colors from $[\Delta(H)+1]$. For each vertex $x \in V(H)$, since $d_H(x) \leq \Delta(H)$, there is an $a_x \in [\Delta(H)+1]$ which does not appear on any edge incident with x in H. Since $E(G) = E(H) \cup \{v_{1,1}x \mid x \in N_G(v_{1,1})\}$, we now extend the edge-coloring c_0 of H to an edge-coloring c of G by assigning $c(v_{1,1}x) = a_x$ for every vertex $x \in N_G(v_{1,1})$. Note that the E_x is a rainbow set for each vertex $x \in V(G) \setminus v_{1,1}$ in G. Suppose p and q are two vertices of G. If $p \in V_i$ and $q \in V_j$ $(1 \leq i < j \leq t)$, then the E_q is a rainbow minimum p-q-edge-cut in G since $\lambda_G(p,q) = n-n_j$. If $p, q \in V_i$, then the E_q is a rainbow minimum p-q-edge-cut in G since $\lambda_G(p,q) = n-n_i$. Hence, we obtain $\operatorname{srd}(G) \leq \Delta(H)+1 = n-n_2$.

Case 2. $n_1 \geq 2$.

Pick a vertex u of V_1 and let $F = G-u$. Then $\Delta(F) = n-n_1$ since $n_1 \geq 2$ and $F_\Delta = G[V_1-u]$. By Lemma 3.1, F belongs to Class 1, and so $\chi'(F) = n-n_1$. Furthermore, for each vertex $x \in N_G(u)$, we know $d_F(x) \leq \Delta(F)-1 = n-n_1-1$. Similar to the argument of Case 1, we can construct an edge-coloring c for G such that the E_x is a rainbow set for each vertex $x \in V(G) \setminus u$ using $n-n_1$ colors. Suppose p and q are two vertices of G. If $p \in V_i$ and $q \in V_j$ $(1 \leq i < j \leq t)$, then the E_q is a rainbow minimum p-q-edge-cut in G since $\lambda_G(p,q) = n-n_j$. If $p, q \in V_i$ $(i \in [t])$, say $q \neq u$, then the E_q is a rainbow minimum p-q-edge-cut in G since $\lambda_G(p,q) = n-n_i$. Hence, $\operatorname{srd}(G) \leq n-n_1$. $\square$

For regular graphs, we only study the srd-number of k-edge-connected k-regular graphs. Moreover, for a k-edge-connected k-regular graph G, where k is odd, we obtain that $\operatorname{srd}(G) = k$ if and only if $\chi'(G) = k$.

Lemma 3.8 [16] *Let k be an odd integer, and G a k-edge-connected k-regular graph of order n. Then $\chi'(G) = k$ if and only if $\operatorname{rd}(G) = k$.*

Theorem 3.9 *Let G be a k-edge-connected k-regular graph. Then $k \leq \operatorname{srd}(G) \leq \chi'(G)$.*

Proof. It follows from Proposition 2.1 that $\operatorname{srd}(G) \geq k$. Let u, v be two vertices of G. Using the fact that G is a k-edge-connected k-regular graph, one may verify that the E_v is a rainbow minimum u-v-edge-cut under a proper edge-coloring of G. $\square$

Theorem 3.10 *Let k be an odd integer, G a k-edge-connected k-regular graph. Then $\operatorname{srd}(G) = k$ if and only if $\operatorname{rd}(G) = k$.*

Proof. First, assume that $\mathrm{srd}(G) = k$. Since $\lambda(G) = k$, we have $\mathrm{rd}(G) = k$ by Proposition 2.1. Conversely, if $\mathrm{rd}(G) = k$, then we have $\mathrm{srd}(G) = k$ by Proposition 2.1 and Lemma 3.8 and Theorem 3.9. $\square$

By Lemma 3.8 and Theorem 3.10, we immediately get Corollary 3.11.

Corollary 3.11 *Let k be an odd integer, G a k-edge-connected k-regular graph. Then* $\mathrm{srd}(G) = k$ *if and only if* $\chi'(G) = k$.

The cartesian product of graphs G and H is the graph $G\Box H$ whose vertex-set is $V(G) \times V(H)$ and whose edge set is the set of all pairs $(u_1, v_1)(u_2, v_2)$ such that either $u_1u_2 \in E(G)$ and $v_1 = v_2$, or $v_1v_2 \in E(H)$ and $u_1 = u_2$. We consider the $m \times n$ grid graph $G_{m,n} = P_m\Box P_n$. Now we determine the srd-number for grid graphs.

Theorem 3.12 *The* srd*-number of the grid graph $G_{m,n}$ is as follows.*
(i) *For $n \geq 2$,* $\mathrm{srd}(G_{1,n}) = \mathrm{srd}(P_n) = 1$.
(ii) *For $n \geq 3$,* $\mathrm{srd}(G_{2,n}) = 2$.
(iii) *For $n \geq 4$,* $\mathrm{srd}(G_{3,n}) = 3$.
(iv) *For $4 \geq m \geq n$,* $\mathrm{srd}(G_{m,n}) = 4$.

Proof. First, it follows from Proposition 2.1 and Lemma 3.5 that the lower bounds on $\mathrm{srd}(G_{m,n})$ in (i)-(iv) hold. It remains to show that the upper bound on $\mathrm{srd}(G_{m,n})$ in each of (i)-(iv) also holds.
(i) We get $\mathrm{srd}(G_{1,n}) = \mathrm{srd}(P_n) = 1$ by Corollary 2.10.
Similar to the proof of Lemma 3.5, we regard the vertices of $G_{m,n}$ as a matrix. Let $x_{i,j}$ be the vertex of the i row and j column, where $1 \leq i \leq m$ and $1 \leq j \leq n$.
(ii) We give the same edge-coloring c for $G_{2,n}$ ($n \geq 3$) using colors from the elements of Z_3 of the integer modulo 3 as in Lemma 3.5 (ii). We now restate it as follows.
$\star$ $c(x_{i,j}x_{i,j+1}) = i + j + 1$ for $1 \leq i \leq 2$ and $1 \leq j \leq n - 1$;
$\star$ $c(x_{1,j}x_{2,j}) = j$ for $1 \leq j \leq n - 1$.
One can verify that the c is an srd-coloring for $G_{2,n}$. Let u and v be two vertices of $G_{2,n}$. If u and v are not in the same column, then two parallel edges between u and v which join vertices in the same two columns form a rainbow minimum u-v-edge-cut in $G_{2,n}$ since $\lambda(u, v) = 2$. Suppose u and v are in the same column. Because the E_u is rainbow and $\lambda(u, v) = d(u) = d(v)$, the E_u is a rainbow minimum u-v-edge-cut in $G_{2,n}$.
(iii) Give the same edge-coloring c as for $G_{3,n}$ ($n \geq 3$) in Lemma 3.5 (iii). Again we use the elements of Z_3 as the colors here. It can be restated as follows.
$\star$ $c(x_{i,j}x_{i,j+1}) = i + j + 1$ for $1 \leq i \leq 3$ and $1 \leq j \leq n - 1$;
$\star$ $c(x_{1,j}x_{2,j}) = j$ for $1 \leq j \leq n - 1$;
$\star$ $c(x_{2,j}x_{3,j}) = j + 2$ for $1 \leq j \leq n - 1$.
Now we show that the coloring c is an srd-coloring of $G_{3,n}$. Observe that the E_x is rainbow for each vertex x with $d(x) \leq 3$ in $G_{3,n}$ under the coloring c. Let u and v be two vertices of $G_{3,n}$. When u and v have at most one vertex with degree 4, without loss of generality, $2 \leq d(u) \leq d(v) \leq 4$, the E_u is a rainbow minimum u-v-edge-cut in $G_{3,n}$ since $\lambda(u, v) = d(u)$. If $d(u) = d(v) = 4$, then three parallel edges between u and v which join vertices in the same two columns form a rainbow minimum u-v-edge-cut in $G_{3,n}$ since $\lambda(u, v) = 3$.
(iv) For the graph $G_{m,n}$ ($4 \leq m \leq n$), because $G_{m,n}$ is bipartite and $\Delta(G_{m,n}) = 4$, there exists a proper edge-coloring c using 4 colors. Now we prove that the c is an srd-coloring of $G_{m,n}$. Let u and v be two vertices of $G_{m,n}$. Suppose $d(u) \leq d(v)$. Then the E_u is a rainbow minimum u-v-edge-cut in $G_{m,n}$ ($4 \leq m \leq n$) since $\lambda(u, v) = d(u)$. $\square$

4. Hardness results

First, we show that our problem is in NP for any fixed integer k.

Lemma 4.1 *For a fixed positive integer k, given a k-edge-colored graph G, deciding whether G is a strong rainbow disconnected under the coloring is in P.*

Proof. Let n, m be the number of the vertices and edges of G, respectively. Let u, v be two vertices of G. Because G has at most k colors, we have at most $\sum_{l=1}^{k}\binom{m}{l}$ rainbow edge subsets in G, denoted the set of the subsets by $\mathcal{S}$. One can see that this number is upper bounded by a polynomial in m when k is a fixed integer (say km^k, roughly speaking). Given a rainbow subset of edges $S \in \mathcal{S}$, it is checkable in polynomial time to decide whether S is a u-v-edge-cut of G, just to see whether u and v are not in the same component of $G \setminus S$, and the number of components is a polynomial in n. If each rainbow subset in $\mathcal{S}$ is not a u-v-edge-cut in G, then the coloring is not an srd-coloring of G, which can be checked in polynomial time since the number of such subsets is polynomial many in m. Otherwise, let the integer $l_0(\leq k)$ be the minimum size of a u-v-edge-cut in G, and this l_0 can be computed in polynomial time. Then, if one of the rainbow subsets of $\mathcal{S}$ is a u-v-edge-cut of G with size l_0, then it is a rainbow minimum u-v-edge-cut of G, which can be done in polynomial time since the number of such subsets is polynomial many in m. Otherwise, the coloring is not an srd-coloring. Moreover, there are at most $\binom{n}{2}$ pairs of vertices in G. Since k is an integer, we can deduce that deciding wether a k-edge-colored graph G is strong rainbow disconnected can be checked in polynomial time. □

In particular, it is NP-complete to determine whether $\mathrm{srd}(G) = 3$ for a cubic graph. We first restate the following result as a lemma.

Lemma 4.2 [15] *It is NP-complete to determine whether the* rd*-number of a cubic is* 3 *or* 4.

Theorem 4.3 *It is NP-complete to determine whether the* srd*-number of a cubic is* 3 *or* 4.

Proof. The problem is in NP from Lemma 4.1. Furthermore, we get that it is NP-hard to determine whether the srd-number of a 3-edge-connected cubic is 3 or 4 by Theorem 3.10 and the proof of Lemma 4.2. □

Lemma 4.1 tells us that deciding whether a given k-edge-colored graph G is strong rainbow disconnected for a fixed integer k is in P. However, the following problem is NP-complete: given an edge-colored connected graph G, check whether the given coloring makes G strong rainbow disconnected.

Theorem 4.4 *Given an edge-colored graph G and two vertices s, t of G, deciding whether there is a rainbow minimum s-t-edge-cut is NP-complete.*

Proof. We know the problem is in NP, since for a graph G checking whether a given set of edges is a rainbow minimum s-t-edge-cut in G can be done in polynomial time, just to see whether it is an s-t-edge-cut and it has the minimum size $\lambda_G(s,t)$ by solving the maximum flow problem. We exhibit a polynomial reduction from the problem 3SAT. Given a 3CNF for $\phi = \wedge_{i=1}^{m} c_i$ over variables $x_1, x_2, \ldots, x_n$, we construct a graph G_ϕ with two vertices s, t and give an edge-coloring f such that G_ϕ has a rainbow minimum s-t-edge-cut if and only if ϕ is satisfiable.
The G_ϕ is defined as follows:

$$\begin{aligned} V(G_\phi) = \quad & \{s,t\} \cup \{x_{i,0}, x_{i,1} | i \in [n]\} \cup \{c_{i,j} | i \in [m], j \in \{0,1,2,3\}\} \\ \cup \quad & \{p_{i,j}, q_{i,j} | i \in [n], j \in [\ell_i]\} \cup \{y_i | i \in [5m+1]\}, \end{aligned}$$

where ℓ_i is the number of times of each variable x_i appearing among the clauses of ϕ.

$$
\begin{aligned}
E(G_\phi) = \quad & \big\{sp_{i,l}, sq_{i,l} \mid i \in [n], l \in [\ell_i]\big\} \\
\cup \quad & \big\{p_{i,l}x_{i,0}, q_{i,l}x_{i,1} \mid i \in [n], l \in [\ell_i]\big\} \\
\cup \quad & \big\{x_{j,0}c_{i,0}, c_{i,0}c_{i,k}, c_{i,k}x_{j,1} \mid \\
& \text{if variable } x_j \text{ is positive in the } k\text{-th literal of clause } c_i, \\
& i \in [m], j \in [n], k \in \{1,2,3\}\big\}. \\
\cup \quad & \big\{x_{j,1}c_{i,0}, c_{i,0}c_{i,k}, c_{i,k}x_{j,0} \mid \\
& \text{if variable } x_j \text{ is negative in the } k\text{-th literal of clause } c_i, \\
& i \in [m], j \in [n], k \in \{1,2,3\}\big\}. \\
\cup \quad & \big\{E(K_{6m+2}) \mid V(K_{6m+2}) = \{c_{1,0}, \ldots, c_{m,0}, y_1, \ldots, y_{5m+1}, t\}\big\}.
\end{aligned}
$$

The edge-coloring f is defined as follows (see Figure 1):

- The edges $\big\{sp_{i,l}, p_{i,l}x_{i,0}, sq_{i,l}, q_{i,l}x_{i,1} \mid i \in [n], l \in [\ell_i]\big\}$ are colored with a special color $r^0_{i,l}$.
- The edge $x_{j,0}c_{i,0}$ or $x_{j,1}c_{i,0}$ is colored with a special color $r_{i,k}$ when x_j is the k-th literal of clause c_i, $i \in [m], j \in [n], k \in \{1,2,3\}$.
- The edge $c_{i,k}x_{j,0}$ or $c_{i,k}x_{j,1}$ is colored with a special color $r_{i,4}$, $i \in [m], j \in [n], k \in \{1,2,3\}$.
- The edge $c_{i,k}c_{i,0}$ is colored with a special color $r_{i,5}$, $i \in [m], k \in \{1,2,3\}$.
- The remaining edges are colored with a special color r_0.

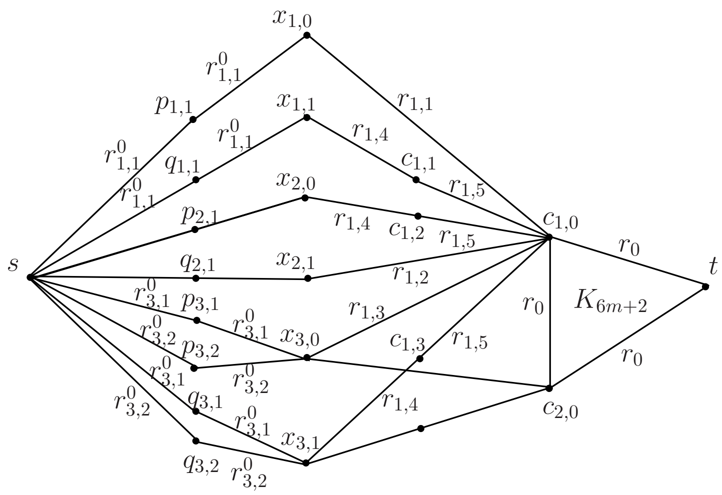

Figure 1: The clause $c_1 = (x_1, \overline{x_2}, x_3)$ and the variable x_3 is in clause c_1 and c_2.

Now we verify that G_ϕ has a rainbow minimum s-t-edge-cut under the f if and only if ϕ is satisfiable.

Assume that there exists a rainbow minimum s-t-edge-cut S in G_ϕ under the coloring f, and let us show that ϕ is satisfiable. Note that for each $j \in [n]$, $l \in l_j$, if S has an edge in $\{sp_{j,l}, p_{j,l}x_{j,0}\}$ (or $\{sq_{j,l}, q_{j,l}x_{j,0}\}$), then a rainbow s-$x_{j,0}$(or s-$x_{j,1}$)-edge-cut in $G[s \cup x_{j,0} \cup \{p_{j,l} | l \in l_j\}]$ is in S, and no edge of $\{sq_{j,l}, q_{j,l}x_{j,1} | l \in [l_j]\}$ (or $\{sp_{j,l}, p_{j,l}x_{j,0} | l \in [l_j]\}$) is in S. Otherwise, it contradicts to the assumption that S is a rainbow minimum s-t-edge-cut in G_ϕ. For each $j \in [n]$, if a rainbow s-$x_{j,0}$-edge-cut in $G[s \cup x_{j,0} \cup \{p_{j,l} | l \in l_j\}]$ is in S under the coloring f, then set $x_j = 0$; if a rainbow s-$x_{j,1}$-edge-cut in $G[s \cup x_{j,1} \cup \{q_{j,l} | l \in l_j\}]$ is in S under the coloring f, then set

$x_j = 1$. First, we have $|S| = 6m$ and $S \subseteq G[V(G_\phi) \setminus \{y_1, \dots, y_{5m+1}, t\}]$. Moreover, for given $c_{i,0}$ ($i \in [m]$), we know that S has at most two edges from three paths of length two between $c_{i,0}$ and $\{x_{j,0}, x_{j,1} | x_j$ in c_i and $j \in [n]\}$ under the coloring f of G_ϕ. Suppose, without loss of generality, that the path of length two between $x_{j,0}$ (or $x_{j,1}$) and $c_{i,0}$ has no edge belonging to S for some $j \in [n]$. If x_j in c_i is positive, then there exists a rainbow s-$x_{j,1}$-edge-cut with size ℓ_j in $G[s \cup x_{j,1} \cup \{q_{j,l} | l \in l_j\}]$ belonging to S, where $i \in [m]$, $j \in [n]$. Then $x_j = 1$ and c_i is satisfiable. If x_j in c_i is negative, then there exists a rainbow s-$x_{j,0}$-edge-cut with size ℓ_j in $G[s \cup x_{j,1} \cup \{p_{j,l} | l \in l_j\}]$ belonging to S, where $i \in [m]$, $j \in [n]$. Then $x_j = 0$ and c_i is satisfiable. Since this is true for each c_i ($i \in [m]$), we imply that ϕ is a YES instance of the 3-SAT.
Now suppose ϕ is a YES instance of the 3-SAT, and let us construct a rainbow minimum s-t-edge-cut in G_ϕ under the coloring f. First, there exists a satisfiable assignment of ϕ. If $x_j = 0$, we put the rainbow s-$x_{j,0}$-edge-cut in $G[s \cup x_{j,0} \cup \{p_{j,l} | l \in l_j\}]$ into S for each $j \in [n]$. If the vertex $x_{j,0}$ is adjacent to $c_{i,0}$, then let one edge of $c_{i,k}x_{j,1}, c_{i,k}c_{i,0}$ be in S for each $i \in [m], j \in [n], k \in \{1, 2, 3\}$. If the vertex $x_{j,0}$ is adjacent to $c_{i,k}$, then let the edge $x_{j,1}c_{i,0}$ be in S for each $i \in [m], j \in [n], k \in \{1, 2, 3\}$. If $x_j = 1$, we put the rainbow s-$x_{j,1}$-edge-cut in $G[s \cup x_{j,1} \cup \{q_{j,l} | l \in l_j\}]$ into S for each $j \in [n]$. If the vertex $x_{j,1}$ is adjacent to $c_{i,0}$, then let one edge of $c_{i,k}x_{j,0}, c_{i,k}c_{i,0}$ be in S for each $i \in [m], j \in [n], k \in \{1, 2, 3\}$. If the vertex $x_{j,1}$ is adjacent to $c_{i,k}$, then let the edge $x_{j,0}c_{i,0}$ be in S for each $i \in [m], j \in [n], k \in \{1, 2, 3\}$. Now we verify that S is indeed a rainbow minimum s-t-edge-cut. First, we can verify that $|S| = 6m$ and it is a minimum s-t-edge-cut. In fact, if a literal of c_i is false, then one edge colored with r_i^4 or r_i^5 is in S. Since the three literals of c_i cannot be false at the same time, we can find a rainbow minimum s-t-edge-cut in G_ϕ under the coloring f. $\square$

5. Concluding remarks

In this paper we defined a new colored connection parameter srd-number for connected graphs. We hope that with this new parameter, avoiding the drawback of the parameter rd-number, one could get a colored version of the famous Menger's Min-Max Theorem. We do not know if this srd-number is actually equal to the rd-number for every connected graph, and then posed a conjecture to further study on the two parameters. The results in the last sections fully support the conjecture.

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