Proper disconnection of graphs*

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Abstract

For an edge-colored graph G, a set F of edges of G is called a *proper edge-cut* if F is an edge-cut of G and any pair of adjacent edges in F are assigned different colors. An edge-colored graph is *proper disconnected* if for each pair of distinct vertices of G there exists a proper edge-cut separating them. For a connected graph G, the *proper disconnection number* of G, denoted by $pd(G)$, is the minimum number of colors that are needed in order to make G proper disconnected. In this paper, we first give the exact values of the proper disconnection numbers for some special families of graphs. Next, we obtain a sharp upper bound of $pd(G)$ for a connected graph G of order n, i.e, $pd(G) \leq \min\{\chi'(G) - 1, \lceil \frac{n}{2} \rceil\}$. Finally, we show that for given integers k and n, the minimum size of a connected graph G of order n with $pd(G) = k$ is $n-1$ for $k = 1$ and $n + 2k - 4$ for $2 \leq k \leq \lceil \frac{n}{2} \rceil$.

Keywords: edge-coloring; proper edge-cut; proper disconnection number; outerplanar graph

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1 Introduction

All graphs considered in this paper are simple, nontrivial, finite and undirected. Let $G = (V(G), E(G))$ be a connected graph with vertex set $V(G)$ and edge set

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$E(G)$. For $v \in V(G)$, let $d(v)$ denote the *degree* of v, $N(v)$ denote the *neighborhood* of v, and $N[v]$ denote the closed neighborhood of v in G. For a subset S of $V(G)$, denote by $N(S)$ the set of neighbors of S in G. Denote the diameter of G by $D(G)$. For any notation or terminology not defined here, we follow those used in [2].

Throughout this paper, we use P_n, C_n, K_n to denote the path, the cycle and the complete graph of order n, respectively. Given two disjoint graphs G and H, the *join* of G and H, denoted by $G \vee H$, is obtained from the vertex-disjoint copies of G and H by adding all edges between $V(G)$ and $V(H)$.

For a graph G, let $c : E(G) \to [k] = \{1, 2, ..., k\}$, $k \in \mathbb{N}$, be an edge-coloring of G. For an edge e of G, we denote the color of e by $c(e)$. When adjacent edges of G receive different colors by c, the edge-coloring c is called *proper*. The *chromatic index* of G, denoted by $\chi'(G)$, is the minimum number of colors needed in a proper edge-coloring of G.

Chartrand et al. in [3] introduced the concept of rainbow disconnection of graphs. An *edge-cut* of a graph G is a set R of edges such that $G - R$ is disconnected. An edge-coloring is called a *rainbow disconnection coloring* of G if for every two vertices of G, there exists a rainbow cut in G separating them. For a connected graph G, the *rainbow disconnection number* of G, denoted by $rd(G)$, is the smallest number of colors required for a rainbow disconnection coloring of G. A rainbow disconnection coloring with $rd(G)$ colors is called an rd-*coloring* of G. In [1] the authors have obtained many results.

Inspired by the concept of rainbow disconnection, we naturally put forward a concept of proper disconnection. For an edge-colored graph G, a set F of edges of G is a *proper edge-cut* if F is an edge-cut of G and any pair of adjacent edges in F are assigned different colors. An edge-colored graph is called *proper disconnected* if for each pair of distinct vertices of G, there exists a proper edge-cut separating them. For a connected graph G, the *proper disconnection number* of G, denoted by $pd(G)$, is defined as the minimum number of colors that are needed in order to make G proper disconnected. A proper disconnection coloring with $pd(G)$ colors is called an pd-*coloring* of G. Clearly, for any pair of vertices of a graph, a rainbow cut is definitely a proper edge-cut. In [3], we know that if G is a nontrivial connected graph, then $\lambda(G) \leq \lambda^+(G) \leq rd(G) \leq \chi'(G) \leq \Delta(G) + 1$. Hence, we immediately have the following observation.

Observation 1.1 *If G is a nontrivial connected graph, then* $1 \leq pd(G) \leq rd(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Some complexity results on the proper disconnection of graphs are obtained in our recent paper [4].

The concept of monochromatic disconnection of graphs is discussed in [5, 6].

2 Preliminaries

At the very beginning, we state some fundamental results on the proper disconnection numbers of graphs, which will be used in this paper. Let G be a connected graph. An edge-cut of G is a *matching cut* if the edge-cut is a matching of G. For vertices u and v of G, a matching cut F is called a u-v matching cut if F separates u and v in G. For a vertex $v \in V(G)$, let E_v be the set of all the edges incident with v in G.

Theorem 2.1 *Let G be a nontrivial connected graph. Then $pd(G) = 1$ if and only if for any two vertices of G, there exists a matching cut separating them.*

Proof. Let $pd(G) = 1$ and c be a pd-coloring of G with one color. Assume, to the contrary, that there exist two vertices x and y which have no matching cut, i.e. each x-y proper cut has two adjacent edges. Obviously, the two adjacent edges are colored differently. That is, $pd(G) \geq 2$. This is a contradiction.

For the converse, define an edge-coloring c such that $c(e) = 1$ for every $e \in E(G)$. For any two vertices x and y in G, there is a matching cut which is an x-y proper edge-cut. Thus, c is a proper disconnection coloring of G and so $pd(G) = 1$. □

For trees and cycles, we get the following results immediately by Theorem 2.1.

Proposition 2.2 *If G is a tree, then $pd(G) = 1$.*

Proposition 2.3 *If C_n is a cycle, then*

$$\mathrm{pd}(C_n) = \begin{cases} 2, & if\ n = 3, \\ 1, & if\ n \geq 4. \end{cases}$$

Lemma 2.4 *If H is a connected subgraph of a connected graph G, then $pd(H) \leq pd(G)$.*

Proof. Let c be a pd-coloring of G and c_H be a coloring of H by restricting c to H. Let x and y be two vertices of H. Suppose that F is an x-y proper edge-cut in G. Then $F \cap E(H)$ is an x-y proper edge-cut in H. Hence, the coloring c restricted to H is a proper disconnection coloring of H. Thus, $pd(H) \leq pd(G)$. □

A *block* of a graph G is a maximal connected subgraph of G that has no cut vertex. Then the block is either a cut edge, say trivial block, or a maximal 2-connected subgraph. Let $\{B_1, B_2, ..., B_t\}$ be the block set of G.

Lemma 2.5 *Let G be a nontrivial connected graph with blocks $B_1, B_2, \cdots, B_t$. Then $pd(G) = \max\{pd(B_i)|\ i \in [t]\}$.*

Proof. Let $k = \max\{pd(B_i) \mid 1 \leq i \leq t\}$. If G has no cut vertex, then $G = B_1$ and the result follows. Next, we assume that G has at least one cut vertex. Since each block is a subgraph of G, $pd(G) \geq k$ by Lemma 2.4.

For each $i \in [t]$, let c_i be a pd-coloring of B_i. We define the edge-coloring $c : E(G) \rightarrow [k]$ of G by $c(e) = c_i(e)$ if $e \in E(B_i)$.

Let $x, y \in V(G)$. If there exists a block, say B_i, that contains both x and y, then any x-y proper edge-cut in B_i is an x-y proper edge-cut in G. Next, we consider that no block of G contains both x and y. Assume that $x \in B_i$ and $y \in B_j$, where $i \neq j$. Now every x-y path contains a cut vertex, say v, of G in B_i and a cut vertex, say w, of G in B_j. Note that v could equal w. If $x \neq v$, then any x-v proper cut of B_i is an x-y proper edge-cut in G. Similarly, if $y \neq w$, then any y-w proper cut of B_j is an x-y proper edge-cut in G. Thus, we may assume that $x = v$ and $y = w$. It follows that $v \neq w$. Consider the x-y path $P = (x = v_1, v_2, \ldots, v_p = y)$. Since x and y are cut vertices in different blocks and no block contains both x and y, we can select the first cut vertex z of G on P except x, that is, $z = v_k$ for some k $(2 \leq k \leq p-1)$. Then x and z belong to the same block, say B_s $(s \in \{1, 2, \cdots, t\} \backslash \{i, j\})$. Then any x-z proper edge-cut of B_s is an x-y proper edge-cut of G. Hence, $pd(G) \leq k$, and so $pd(G) = k$. □

We next present a useful structural property.

Lemma 2.6 *Let G be a nontrivial connected graph. If there exist two nonadjacent vertices u and v sharing t $(t \geq 1)$ common neighbors in G, then $pd(G) \geq \lceil \frac{t}{2} \rceil$. Furthermore, if $uv \in E(G)$, then $pd(G) \geq \lceil \frac{t}{2} \rceil + 1$.*

Proof. Let c be a pd-coloring of G. Let u, v be two vertices of G and $F(u, v)$ be a u-v proper edge-cut in G. If $uv \notin E(G)$, let $W = N(u) \cap N(v) = \{w_1, w_2, \cdots, w_t\}$. Then there are t internally disjoint paths of length two. Let $E_1 = \{uw_i|1 \leq i \leq t\}$ and $E_2 = \{vw_i|1 \leq i \leq t\}$. Then $|F(u, v) \cap E_1| \geq \lceil \frac{t}{2} \rceil$ or $|F(u, v) \cap E_2| \geq \lceil \frac{t}{2} \rceil$. Otherwise there exists at least one u-v path of length two in $G \backslash F(u, v)$, which is a contradiction. Since $E_1 \subseteq E_u$ and $E_2 \subseteq E_v$, $pd(G) \geq \lceil \frac{t}{2} \rceil$. Moreover, if $uv \in E(G)$, then $uv \in F(u, v)$.

So $|F(u,v) \cap E_u| \geq |F(u,v) \cap E_1| + 1$ or $|F(u,v) \cap E_v| \geq |F(u,v) \cap E_2| + 1$. Hence, $pd(G) \geq \lceil \frac{t}{2} \rceil + 1$. $\square$

3 Main results

In this section, we give the exact values of the proper disconnection numbers for the wheel graphs, the complete graphs, the complete bipartite graphs and the outerplanar graphs. Furthermore, we obtain a sharp upper bound of $pd(G)$, and derive the minimum size of a graph G of order n with $pd(G) = k$, where $1 \leq k \leq \lceil \frac{n}{2} \rceil$.

3.1 Wheel graphs

Lemma 3.1 *Let $G = K_4 - \{e\}$. Then $pd(G) = 2$. Furthermore, the colors of matching edges in G are the same for any pd-coloring.*

Proof. Let $V(G) = \{v_1, v_2, v_3, v_4\}$ and $E(G) = \{v_iv_j\}|1 \leq i < j \leq 4\} \setminus \{v_2v_4\}$. First, we have $pd(G) \geq 2$ since K_3 is a subgraph of G. Define an edge-coloring c: $E(G) \to [2]$ of G as follows. Let $c(v_1v_2) = c(v_1v_3) = c(v_3v_4) = 1$, $c(v_1v_4) = c(v_2v_3) = 2$. Let u and v be two vertices of G. If $d(u) = 2$ or $d(v) = 2$, then E_u (or E_v) is a u-v proper edge-cut. Otherwise, If $d(u) = d(v) = 3$, then the edge set $\{v_1v_4, v_1v_3, v_2v_3\}$ is a proper edge-cut of u, v. Thus, $pd(G) \leq 2$.

For any pd-coloring c: $E(G) \to [2]$ of G, assume that $c(v_1v_3) = 1$. Since any edge-cut of v_1 and v_3 has at least three edges, there exist two matching edges with color 2 incident with v_1, v_3 respectively. We claim that the remaining two matching edges must have the same color. Otherwise, there exists a vertex, say v_2, such that all edges of E_{v_2} have color 2. Then E_{v_1} or E_{v_3} has two edges with color 2. Therefore, there is no v_1-v_2 or v_2-v_3 proper edge-cut. $\square$

Theorem 3.2 *If $W_n = C_n \vee K_1$ is the wheel of order $n + 1 \geq 4$, then*

$$\mathrm{pd}(W_n) = \begin{cases} 2, & \text{if } n = 3k \ (k \in \mathbb{Z}), \\ 3, & \text{otherwise.} \end{cases}$$

Proof. Let $V(W_n) = \{v_0, v_1, \ldots, v_n\}$ and $E(W_n) = \{v_0v_i, v_0v_n, v_iv_{i+1}, v_1v_n|1 \leq i \leq n-1\}$. For convenience, each subscript of vertices is expressed as an integer $0, 1, 2, \cdots, n-1$ modular n. First, $\mathrm{pd}(W_n) \geq 2$, since K_3 is a subgraph of W_n.

Case 1. $n = 3k$.

Define an edge-coloring c: $E(W_n) \to [2]$ of W_n. Let $c(v_0v_{3i}) = c(v_{1+3j}v_{2+3j}) = 2$ where $1 \le i \le k$ and $0 \le j \le k-1$ and assign color 1 to the remaining edges. Let v_i be a vertex of W_n, where $i = 3t$ $(1 \le t \le k)$. Then the edge set $\{v_{i-1}v_i, v_0v_i, v_0v_{i+1}, v_{i+1}v_{i+2}\}$ is a proper edge-cut between $\{v_i, v_{i+1}\}$ and $V(W_n) \setminus \{v_i, v_{i+1}\}$ and the edge set $\{v_iv_{i+1}, v_0v_i, v_0v_{i-1}, v_{i-2}v_{i-1}\}$ is also a proper edge-cut between $\{v_{i-1}, v_i\}$ and $V(W_n) \setminus \{v_{i-1}, v_i\}$.

Let v_k, v_l be any two vertices of W_n where k, l are integers. If v_k is nonadjacent to v_l, then there exists an edge v_kv_p such that $p = 3t$ or $k = 3t$. By above argument, we have a proper edge-cut between $\{v_k, v_p\}$ and $V(W_n) \setminus \{v_k, v_p\}$, which is a v_k-v_l proper edge-cut. Assume v_k is adjacent to v_l with $k \le l$. If k or l is a multiple of 3, without loss of generality, $k = 3t$, then there exists a proper cut between $\{v_k, v_p\}$ and $V(W_n) \setminus \{v_k, v_p\}$ where $v_p \in N(v_k) \setminus \{v_0, v_l\}$, which is a v_k-v_l proper edge-cut. If neither k nor l is a multiple of 3, then there exits a proper edge-cut between $\{v_k, v_s\}$ and $V(W_n) \setminus \{v_k, v_s\}$ where $v_s \in N(v_k) \setminus \{v_0, v_l\}$, which is a v_k-v_l proper edge-cut. Thus, $\mathrm{pd}(W_n) = 2$.

Case 2. $n \neq 3k$.

Assume that $\mathrm{pd}(W_n) = 2$. Let $c(v_0v_1) = 1$. Then for matching edges v_0v_1 and v_2v_3 in induced graph $G[\{v_0, v_1, v_2, v_3\}]$, we get $c(v_0v_1) = c(v_2v_3) = 1$ by Lemma 3.1. Using Lemma 3.1 repeatedly, we get $c(e) = 1$ for any edge e of W_n (i.e. $c(v_0v_1) = c(v_2v_3) = c(v_0v_4) = \cdots = 1$). This is a contradiction with $\mathrm{pd}(W_n) \ge 2$. Thus, $\mathrm{pd}(W_n) \ge 3$.

Now we define an edge-coloring $c : E(W_n) \to [3]$ of W_n. First, let c be a proper edge-coloring of C_n using the colors $1, 2, 3$. For each integer i with $1 \le i \le n$, let $a_i \in \{1, 2, 3\} \setminus \{c(v_{i-1}v_i), c(v_iv_{i+1})\}$, and let $c(v_0v_i) = a_i$. Thus, E_{v_i} is a proper set for $1 \le i \le n$. Let x, y be two distinct vertices of W_n. Then at least one of x and y belongs to C_n, say $x \in V(C_n)$. Since E_x separates x and y, it follows that c is a proper disconnection coloring of W_n using three colors. Therefore, $pd(W_n) = 3$ for $n \neq 3k$. □

3.2 Complete bipartite graphs and complete graphs

Now we introduce some notations. Let X and Y be sets of vertices of a graph G, we denote by $E[X, Y]$ the set of all the edges of G with one end in X and the other end in Y. We write $G[X, Y]$ for $G[E[X, Y]]$. For an edge-coloring c of $G[X, Y]$, if c is a proper coloring, then $E[X, Y]$ is called a *proper set.*

Theorem 3.3 *Let $K_{n,n}$ be a complete bipartite graph. Then $pd(K_{n,n}) = \lceil \frac{n}{2} \rceil$.*

Proof. Let $G = K_{n,n}$ and suppose that X and Y are two partite vertex sets of G, where $X = \{x_1, x_2, \cdots, x_n\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$. For any two vertices $x_i, x_j \in X$, there are n common neighbors in Y. Then $pd(G) \geq \lceil \frac{n}{2} \rceil$ by Lemma 2.6.

Now, for the upper bound, we define an edge-coloring $c : E(G) \to \{0, 1, \cdots, \lceil \frac{n}{2} \rceil - 1\}$ of G by assigning each edge $x_i y_j$ with $c(x_i y_j) \equiv i + j - 1 \pmod{\lceil \frac{n}{2} \rceil}$ for $1 \leq i, j \leq n$. Let $X = X_1 \cup X_2$, where $X_1 = \{x_1, x_2, \cdots, x_{\lceil \frac{n}{2} \rceil}\}$ and $X_2 = \{x_{\lceil \frac{n}{2} \rceil + 1}, x_{\lceil \frac{n}{2} \rceil + 2}, \cdots, x_n\}$. Let $Y = Y_1 \cup Y_2$, where $Y_1 = \{y_1, y_2, \cdots, y_{\lceil \frac{n}{2} \rceil}\}$ and $Y_2 = \{y_{\lceil \frac{n}{2} \rceil + 1}, y_{\lceil \frac{n}{2} \rceil + 2}, \cdots, y_n\}$.

Claim 3.4 *For each pair of vertex sets X_k and Y_l $(k, l \in [2])$, $E[X_k, Y_l]$ is a proper set.*

Proof. By symmetry, we only consider the vertex sets X_1 and Y_1. For each vertex x_i of X_1, since $c(x_i y_j) \equiv i + j - 1 \pmod{\lceil \frac{n}{2} \rceil}$, it follows that $c(x_i y_1) \neq c(x_i y_2) \neq \cdots \neq c(x_i y_{\lceil \frac{n}{2} \rceil})$. Therefore, the edges of $G[X_1, Y_1]$ incident with the same vertex are colored by different colors. Thus, $E[X_1, Y_1]$ is a proper set. □

We now show that for each pair of vertices u and w of G, there is a proper edge-cut separating them. Two cases are needed to be discussed.

Case 1. $u \in X$, $w \in Y$.

Suppose that $u = x_i$ and $w = y_j$. Let $x_i \in X_1$, $y_j \in Y_1$ or $x_i \in X_2$, $y_j \in Y_2$ and let $F(x_i, y_j)$= $E[X_1, Y_1] \cup E[X_2, Y_2]$. Consider the subgraph H of G obtained by deleting $F(x_i, y_j)$ from G, then H has two components $G[X_1, Y_2]$ and $G[X_2, Y_1]$ (See Figure 1). Since $x_i \in G[X_1, Y_2]$ and $y_j \in G[X_2, Y_1]$, we can know that $F(x_i, y_j)$ is an edge cut separating x_i and y_j. Moreover, $E[X_1, Y_1]$ and $E[X_2, Y_2]$ are proper sets by the claim and $E[X_1, Y_1] \cap E[X_2, Y_2] = \phi$, so $F(x_i, y_j)$ is an x_i-y_j proper edge-cut. If $x_i \in X_1$, $y_j \in Y_2$ or $x_i \in X_2$, $y_j \in Y_1$, we can similarly show that $E[X_1, Y_2] \cup E[X_2, Y_1]$ is an x_i-y_j proper edge-cut.

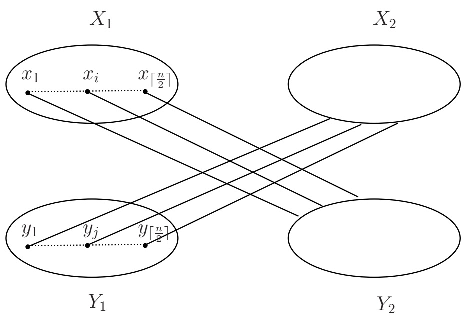

Figure 1: A graph H

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Case 2. $u, w \in X$ *or* $u, w \in Y$.

By symmetry, suppose without loss of generality that $u = x_i$ and $w = x_j$. If $x_i, x_j \in X_1$, where $1 \le i < j \le \lceil \frac{n}{2} \rceil$. Let $X_1' = X_1 - x_i + x_{i+\lceil \frac{n}{2} \rceil}$ and $X_2' = X_2 + x_i - x_{i+\lceil \frac{n}{2} \rceil}$. Then $x_i \in X_2'$ and $x_j \in X_1'$. For every vertex y_t of Y, we know that $c(x_i y_t) = c(x_{i+\lceil \frac{n}{2} \rceil} y_t) \equiv i + t - 1 \pmod{\lceil \frac{n}{2} \rceil}$. By the same method used in the claim, for vertex sets X_k' and Y_l ($k, l \in [2]$), $E[X_k', Y_l]$ is a proper set. Let $F'(x_i, x_j) = E[X_1', Y_1] \cup E[X_2', Y_2]$. Let $H' = G[X_1', Y_2] \cup G[X_2', Y_1]$ be a graph obtained by deleting $F'(x_i, x_j)$ from G (See Figure 2). According to Case 1, $F'(x_i, x_j)$ is an x_i-x_j proper edge-cut. If $x_i \in X_1$ and $x_j \in X_2$, similarly, we can get $E[X_1, Y_1] \cup E[X_2, Y_2]$ is an x_i-x_j proper edge-cut. □

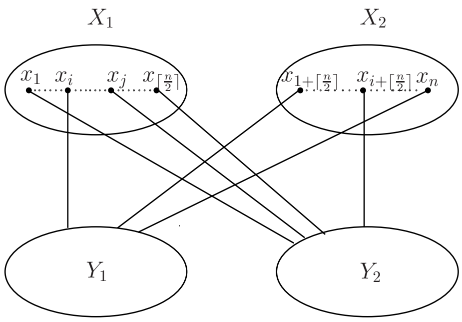

Figure 2: A graph H'

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Theorem 3.5 *Let $K_{m,n}$ be a complete bipartite graph with $2 \le m \le n$. Then $pd(K_{m,n}) = \lceil \frac{n}{2} \rceil$.*

Proof. Let $G = K_{m,n}$ and $V(G) = X \cup Y$, where $X = \{x_1, \cdots, x_m\}$ and $Y = \{y_1, \cdots, y_n\}$. For any two vertices $x_i, x_j \in X$, there are n common neighbors in Y. From Lemma 2.6, it follows that $pd(G) \ge \lceil \frac{n}{2} \rceil$. Also, G is a subgraph of $K_{n,n}$. By Theorem 3.3 and Lemma 2.4, we have that $pd(G) \le pd(K_{n,n}) = \lceil \frac{n}{2} \rceil$. Hence, it follows that $pd(G) = \lceil \frac{n}{2} \rceil$. □

Theorem 3.6 *For each integer $n \ge 2$, $pd(K_n) = \lceil \frac{n}{2} \rceil$.*

Proof. Let $a = \lceil \frac{n}{2} \rceil$ and $V(K_n) = X \cup Y$, where $X = \{v_1, v_2, \cdots, v_a\}$ and $Y = \{v_{a+1}, v_{a+2}, \cdots, v_n\}$. For any two vertices $v_i, v_j \in V(K_n)$, there are $n-2$ common neighbors in K_n and $v_i v_j \in E(K_n)$. Then $pd(K_n) \ge \lceil \frac{n}{2} \rceil$ by Lemma 2.6. Define an edge-coloring $c : E(G) \to \{0, 1, \cdots, a-1\}$ such that $c(v_i v_j) \equiv i + j - 1 \pmod{a}$. For any two vertices u, $w \in V(G)$, assume $u \in X$ and $w \in Y$. Since $E[X, Y]$ separates u and w, and $E[X, Y]$ is a proper set in K_n by a similar proof method as

Claim 3.4, $E[X,Y]$ is a u-w proper edge-cut. If $u, w \in X$, assume that $u = v_i$ and $w = v_j$, where $i < j$. Let $i_0 = i + a$. Then $v_{i_0} \in Y$. For $v_r \in V(K_n)$ $(r \neq i, i_0)$, we have $c(v_i v_r) = c(v_{i_0} v_r)$. Let $X' = X \setminus \{v_i\}$ and $Y' = Y \setminus \{v_{i_0}\}$. Similarly, $F(v_i, v_j) = E[X' \cup \{v_{i_0}\}, Y' \cup \{v_i\}]$ is a proper set. Since $F(v_i, v_j)$ separates u and w, $F(v_i, v_j)$ is a u-w proper edge-cut in K_n. If $u, w \in Y$, we can obtain a u-w proper edge-cut in the similar way. Thus, c is a proper disconnection coloring of K_n and so $pd(K_n) \leq \lceil \frac{n}{2} \rceil$. □

3.3 Outerplanar graphs

An *outerplanar graph* is a graph that can be drawn in the plane without crossings in such a way that all of the vertices belong to the unbounded face of the drawing. A *minor* of a graph G is any graph obtained from G by means of a sequence of vertex and edge deletions and edge contractions. A *chord* of a cycle C in a graph G is an edge in $E(G) \setminus E(C)$ both of whose ends lie on C. There is a characterization of outerplanar graphs as follows.

Theorem 3.7 [2] A graph is outerplanar if and only if it does not contain K_4 or $K_{2,3}$ as a minor.

Let P_n be a path of order n. We denote $K_1 \vee P_n$ by $F_{1,n}$, which is called a *fan graph*. We first show that the proper disconnection number of fan graph is 2, which will be used to characterize the outerplanar graphs with diameter 2.

Lemma 3.8 *Let $F_{1,n} = K_1 \vee P_n$ be a fan graph, where $n \geq 2$. Then $pd(F_{1,n}) = 2$.*

Proof. Firstly, we have that $pd(F_{1,n}) \geq pd(K_3) = 2$ by Lemma 2.4. Clearly, $F_{1,n}$ is a subgraph of W_n. If $n = 3k$ $(k \geq 1)$, then $pd(F_{1,n}) \leq pd(W_n) = 2$ by Lemma 2.4 and Theorem 3.2. Thus, $F_{1,3k} = 2$. If $n = 3k-2$ or $3k-1$ $(k \geq 1)$, then $pd(F_{1,3k-2}) \leq pd(F_{1,3k-1}) \leq pd(F_{1,3k}) = 2$ since both $F_{1,3k-2}$ and $F_{1,3k-1}$ are subgraphs of $F_{1,3k}$. Hence, $pd(F_{1,n}) = 2$. □

Theorem 3.9 *Let G be an outerplanar graph. Then $pd(G) = 1$ if and only if G is a triangle-free graph.*

Proof. Let $pd(G) = 1$. Assume, to the contrary, that G contains a copy of K_3. Then $pd(G) \geq pd(K_3) = 2$ by Theorem 2.4, which is a contradiction. Then G is a triangle-free outerplanar graph.

For the converse, let B be a block of G with t vertices such that $pd(B)$ is maximum among all proper disconnection numbers of the other blocks. Then B is a triangle-free outerplanar graph. By Lemma 2.5, it suffices to show that $pd(B) = 1$. If B is trivial, then $pd(B) = 1$. If B is not trivial, then $t \geq 4$. Suppose that B is a cycle. Then $pd(B) = 1$ by Proposition 2.3.

It remains to consider that B is not a cycle. Since B is a triangle-free and outerplanar graph, we only consider $t \geq 6$. We proceed by induction on t. When $t = 6$, the block B has only one chord since B is a triangle-free and outerplanar graph. Clearly, $pd(B) = 1$ and the result is true. When $t \geq 7$, let $C = v_1e_1v_2e_2 \cdots e_{t-1}v_te_tv_1$ be the boundary of the outer face in B. Choose a chord v_iv_j such that the internal vertices of $P = v_ie_iv_{i+1}e_{i+1} \cdots e_{j-1}v_j (j \geq i + 3)$ have degree 2 in B. Let B' be a graph by removing internal vertices of P from B. Then $pd(B') = 1$ by induction hypothesis. By Theorem 2.1, for any two vertices x and y of B' there is an x-y matching cut, denoted by $F_{B'}(x, y)$. For any two vertices u, v in B, if u, $v \in B'$, then $F_{B'}(u, v) \cup \{e_{i+1}\}$ is a u-v matching cut in B. If u, $v \in B \setminus B'$, then $F_{B'}(v_i, v_j) \cup \{e\}$ is a u-v matching cut in B, where e is an edge cut separating u and v in P. If $u \in B'$ and $v \in B \setminus B'$, then $\{e_i, e_{j-1}\}$ is a u-v matching cut in B. Therefore, $pd(B) = 1$. □

Now we characterize the outerplanar graphs G with $D(G) = 2$. We first construct some graph classes. Let $\mathcal{D}$ be a family of graphs obtained from $W_n = K_1 \vee C_n$ by deleting t $(1 \leq t \leq n-1)$ edges from C_n. Let z be an isolated vertex and $v_1v_2v_3v_4v_5$ be a path of length 4. Then join z with v_1, v_2, v_4 and v_5. We denote the resulting graph by $F^-_{1,5}$. Let y be an isolated vertex and $v_1v_2v_3v_4$ be a path of length 3. Then join y with v_1, v_3 and v_4. We denote the resulting graph by $F^-_{1,4}$. Let $C_6 = v_1v_2v_3v_4v_5v_6v_1$ Then let $F' = C_6 \cup \{v_1v_3, v_3v_5, v_1v_5\}$.

Theorem 3.10 *Let G be an outerplanar graph with $D(G) = 2$. Then $pd(G) = 2$ if and only if $G \in \mathcal{D}$ or $G \cong F^-_{1,5}$ or $F^-_{1,4}$ or F'.*

Proof. Sufficiency. Since there is at least one triangle K_3 for every $G \in \mathcal{D}$, it is clear to see that $pd(G) \geq pd(K_3) = 2$ by Theorem 2.4 and Theorem 3.6. Meanwhile, G is a subgraph of a fan graph, therefore, $pd(G) \leq 2$ by Lemma 3.8 and Lemma 2.4. Hence, $pd(G) = 2$ for $G \in \mathcal{D}$. Similarly, $pd(F^-_{1,5}) = pd(F^-_{1,4}) = 2$. For the graph F', $pd(F') \geq 2$ by Lemma 2.4 and Theorem 3.6. We now assign a 2-edge-coloring $c : E(F') \to \{1, 2\}$ for F'.

Let $c(v_1v_5) = c(v_3v_4) = c(v_2v_3) = 2$ and the remaining edges are colored by 1. Thus, for every pair of vertices in F', there exists a proper edge-cut $F[V_1, V(F') \setminus V_1]$, where $V_1 = \{v_1, v_2, v_3\}$ and $V(F') \setminus V_1 = \{v_4, v_5, v_6\}$ or $V_1 = \{v_1, v_2, v_6\}$ and

$V(F') \setminus V_1 = \{v_4, v_5, v_3\}$ or $V_1 = \{v_1, v_2, v_3, v_5, v_6\}$ and $V(F') \setminus V_1 = \{v_4\}$ or $V_1 = \{v_1, v_3, v_4, v_5, v_6\}$ and $V(F') \setminus V_1 = \{v_2\}$. Hence, $pd(F') = 2$.

Necessity. Suppose that $pd(G) = 2$. Clearly, there is at most one cut vertex since $D(G) = 2$. Otherwise $D(G) \geq 3$. We now discuss it by two cases.

Case 1. Suppose that there exists exactly one cut vertex. Then the remaining vertices are adjacent to the cut vertex. Since the star $pd(K_{1,t}) = 1$, G is not a star. And since the wheel graph is not the outerplanar graph, G is not the graph containing the wheel graph. Thus, we get $G \in \mathcal{D}$ because G is a outerplanar graph.

Case 2. Suppose that there is not a cut vertex. Then $\delta(G) \geq 2$. Let r be a vertex with maximum degree and $N(r) = \{x_1, x_2, \cdots, x_\Delta\}$.

Subcase 2.1. $d(r) = n-1$. Since there is no cut vertex in G, the induced subgraph $G[N(r)]$ is connected. We claim that $G[N(r)]$ is a path. Otherwise, it is a tree with a vertex v of degree at least three, or it contains a cycle. Thus, G contains a minor of $K_{2,3}$ or K_4. By Theorem 3.7, we have a contradiction. Clearly, G is the graph from $\mathcal{D}$ with $t = 1$.

Subcase 2.2. $d(r) = n - 2$. Let x be a vertex which is nonadjacent to r. Suppose that $|N(x)| \geq 3$. Then, there is a minor of $K_{2,3}$ in G, which is a contradiction. Thus, $|N(x)| = 2$. We now illustrate our claim that $G \cong F_{1,5}^-$ or $F_{1,4}^-$ or $G \cong F'$. Without loss of generality, let x_1x, x_2x be two edges of G. Suppose that $n \geq 7$. Since $D(G) = 2$, the vertices x_3, x_4, x_5 are adjacent to x_1 or x_2. Then there are at least two vertices adjacent to the same vertex x_1 (or x_2). Then we obtain a minor of $K_{2,3}$ in G. Suppose that $n = 5$. Since $d(x_3) \geq 2$, we have exactly one edge $x_2x_3 \in E(G)$ or $x_1x_3 \in E(G)$. Otherwise, there is a minor of K_4. So $G \cong F_{1,4}^-$. Suppose that $n = 6$. The vertices x_3, x_4 have no common neighbor other than r. Otherwise, there is a minor of $K_{2,3}$ in G, and it is the same for x_1, x_2. Since $D(G) = 2$, there exist edges x_2x_3, x_1x_4 (or x_1x_3, x_2x_4). Hence, we have $G \cong F_{1,5}^-$ if $x_1x_2 \notin E(G)$ and $G \cong F'$ if $x_1x_2 \in E(G)$.

Subcase 2.3. $d(r) \leq n-3$. Let y_1 and y_2 be two vertices which are nonadjacent to r. Assume that $y_1y_2 \in E(G)$. If $|N(r) \cap N(\{y_1, y_2\})| \geq 3$, then there is a minor of $K_{2,3}$ in G. Thus, $|N(r) \cap N(\{y_1, y_2\})| \leq 2$. If $N(r) \cap N(\{y_1, y_2\}) = \emptyset$, then $d(r, y_1) \geq 3$, which is a contradiction. If $|N(r) \cap N(\{y_1, y_2\})| = 1$, then when there is exactly one vertex of $\{y_1, y_2\}$ which is adjacent to one vertex of $N(r)$, it contradicts with $D(G) = 2$. When both y_1 and y_2 are adjacent to one common vertex of $N(r) \cap N(\{y_1, y_2\})$, without loss of generality, let $y_1x_1, y_2x_1 \in E(G)$. Then there exists x_i $(i \neq 1)$ which is nonadjacent to $x_1 \in N(r) \cap N(\{y_1, y_2\})$ since r is a vertex with maximum degree. Then there exists two vertices $y_3, y_4 \in V(G) \setminus (N[r] \cup \{y_1, y_2\})$ such that, with loss of generality,

$x_iy_3, y_3y_1, x_iy_4, y_4y_2 \in E(G)$ since $D(G) = 2$, where y_3 may be equal to y_4. Then $|N(r) \cap N(\{y_1, y_3\})| = 2$. Namely, there are always two vertices, with loss of generality, $y_1, y_2 \in V(G) \setminus N[r]$, which satisfy $y_1y_2 \in E(G)$ and $|N(r) \cap N(\{y_1, y_2\})| = 2$. Therefore, we only consider the case that $|N(r) \cap N(\{y_1, y_2\})| = 2$. Without loss of generality, let $x_1y_1, x_2y_2 \in E(G)$.

When $|N(r)| = 2$, clearly, $G \cong C_5$, contradicting that $pd(G) = 2$. If $|N(r)| \geq 5$, then x_3, x_4 and x_5 belong to $N(r)$ but not $N(\{y_1, y_2\})$. So, there are at least two vertices of x_3, x_4 and x_5 adjacent to one vertex of x_1 and x_2, which induces a minor of $K_{2,3}$. Thus, $|N(r)| = 3$ or 4. If there exists a vertex $y \notin N(r)$ $(y \neq y_1, y_2)$ such that $x_3y, yy_1 \in E(G)$ (or $x_3y, yy_2 \in E(G)$), then it contains a minor of $K_{2,3}$. If $|N(r)| = 3$, then $N(x_3) \cap N(\{y_1, y_2\}) \neq \emptyset$ since $D(G) = 2$. However, there is at most one vertex in $N(r) \cap N(\{y_1, y_2\})$ adjacent to x_3. Otherwise, there is a minor of K_4 in G. Suppose that x_3 is adjacent to one vertex of $\{x_1, x_2\}$, say x_1, then $D(G) = 3$ since r has the maximum degree. This is a contradiction. If $|N(r)| = 4$, then x_3 and x_4 are not adjacent to one common vertex of x_1 and x_2. Otherwise, it produces a minor of $K_{2,3}$. Then let $x_2x_3, x_1x_4 \in E(G)$. In the sake of $D(G) = 2$, the pair of vertices x_3 and y_1 has at least one common neighbor and so does the pair of x_4 and y_2. However, it produces a minor of K_4.

Assume that $V(G) \setminus N[r]$ is an independent set. Clearly, y_1 and y_2 have at most two neighbors in $\{x_i | 1 \leq i \leq n-3\}$, respectively. Since $D(G) = 2$, y_1 and y_2 have at least one common neighbor in $\{x_i | 1 \leq i \leq n-3\}$. If they have at least two common neighbors, then G contains a minor of $K_{2,3}$, which is a contradiction. Thus, y_1 and y_2 have exactly one common neighbor in $\{x_i | 1 \leq i \leq n-3\}$. Without loss of generality, let $x_1, x_2 \in N(y_1)$ and $x_1 \in N(y_2)$. Then y_2 has a neighbor x where $x \neq x_1, x_2$. Without loss of generality, let $x = x_3$. Then there is an edge x_1x_3 or x_2x_3. Otherwise, it is a contradiction to $D(G) = 2$. If $x_2x_3 \in E(G)$, then there is a minor of K_4. If $x_1x_3 \in E(G)$, then $x_1x_2 \in E(G)$ since $D(G) = 2$. Since $d(r) \geq d(x_1) \geq 4$, there exists another neighbor x_4 of vertex r, which is nonadjacent to x_1. Otherwise, it produces a minor of $K_{2,3}$. In view of $D(G) = 2$, then $\{x_2x_4, x_3x_4\} \subseteq E(G)$. There is a minor of K_4, which is a contradiction. □

3.4 An upper bound and an extremal problem

We first consider the upper bound of the proper disconnection number for a graph of order n and chromatic index $\chi'(G)$.

Theorem 3.11 *If G is a nontrivial connected graph, then $pd(G) \leq \chi'(G) - 1$.*

Proof. Let c: $E(G) \rightarrow [\chi'(G)]$ be a proper edge-coloring of G. Then we define an edge-coloring c' of G as follows: for any edge e of G, if $c(e) = \chi'(G)$, then $c'(e) = 1$; otherwise, $c'(e) = c(e)$. Let x, y be two vertices of G. Assume $N(x) = \{v_1, v_2, \cdots, v_{d(x)}\}$. Obviously, at most two incident edges of x are assigned the color 1. If there exists at most one incident edge of x with color 1, then E_x is an x-y proper edge-cut. If there exist two edges with color 1, then we may assume $c'(xv_i) = c'(xv_j) = 1$ $(1 \le i < j \le d(x))$. If $y \in \{v_i, v_j\}$, then let t be the vertex in $\{v_i, v_j\}$ that is not y. Then $(E_x \cup E_t) \setminus \{xt\}$ is an x-y proper edge-cut. Otherwise, $(E_x \cup E_{v_i}) \setminus \{xv_i\}$ $(i \in [d(x)])$ is an x-y proper edge-cut. Thus, c' is a proper disconnection coloring of G and so $pd(G) \le \chi'(G) - 1$. □

According to Theorem 3.6 and Theorem 3.11, we get the following result.

Theorem 3.12 *Let G be a nontrivial connected graph of order n. Then $pd(G) \le min\{\chi'(G) - 1, \lceil \frac{n}{2} \rceil\}$, and the bound is sharp.*

Proof. By Theorem 3.11, $pd(G) \le \chi'(G) - 1$. Since G is a connected subgraph of K_n, $pd(G) \le \lceil \frac{n}{2} \rceil$ by Theorem 3.6. For the sharpness, $\lceil \frac{n}{2} \rceil$ can be reached by complete graphs, and $\chi'(G) - 1$ can be reached by even cycles and paths with at least 3 vertices. □

Now we investigate the following extremal problem: For given positive integers k and n with $1 \le k \le \lceil \frac{n}{2} \rceil$, what is the minimum possible size of a connected graph G of order n such that the proper disconnection number of G is k?

Lemma 3.13 *Let G be a connected graph of order n. Let M be a matching of G. Then $pd(G) \le \max\{pd(G_i) | 1 \le i \le t\} + 1$, where G_i is a connected component of $G - M$ and t is the number of components of $G - M$.*

Proof. Denote the components of $G - M$ by $G_1, G_2, \ldots, G_t$. Let $\ell = \max\{\mathrm{pd}(G_i) | 1 \le i \le t\}$. Let c_i be a pd-coloring of G_i and $F_{G_i}(u, v)$ be a u-v proper edge-cut in G_i for $i \in [t]$. We define an edge-coloring c: $E(G) \rightarrow [\ell + 1]$ of G by $c(e) = c_i(e)$ if $e \in E(G_i)$ and $c(e) = \ell + 1$ if $e \in M$. Let x, y be any two vertices of G. If $x, y \in G_i$, then $F_{G_i}(x, y) \cup M$ is an x-y proper edge-cut in G. If $x \in G_i$ and $y \in G_j$ where $i \ne j$, then M is an x-y proper edge-cut in G. Hence, $pd(G) \le \max\{pd(G_i) | 1 \le i \le t\} + 1$. □

Theorem 3.14 *For integers k and n with $1 \le k \le \lceil \frac{n}{2} \rceil$, the minimum size of a* *connected graph G of order n with* $\mathrm{pd}(G) = k$ *is*

$$|E(G)|_{min} = \begin{cases} n-1, & \text{if } k = 1, \\ n+2k-4, & \text{if } k \geq 2. \end{cases}$$

Proof. Since G is a connected graph, $|E(G)|_{min} = n-1$ for $k = 1$ by Proposition 2.2. For $k \geq 2$, we first show that if the size of a connected graph G of order n is at most $n + 2k - 5$, then $pd(G) \leq k - 1$. We proceed by induction on k. The result holds for $k = 2$ by Proposition 2.2. Suppose that G is a graph with $|E(G)| \leq n + 2k - 5$. If G is a graph with at most one block which is a cycle and other blocks are trivial, the result is true for G by Proposition 2.3 and Lemma 2.5. Otherwise, we claim that there exist two matching edges, say e_1, e_2, of G such that $G - \{e_1, e_2\}$ is connected graph of order n. Now, there are two cases as follows:

(i) G has exactly one nontrivial block which is not a cycle, and the other blocks are trivial;

(ii) G has at least two nontrivial blocks.

For (i), let B be a nontrivial block which is not a cycle. Then B contains two vertices x and y such that they are connected by at least three internally disjoint x-y paths. We can respectively pick one edge from two x-y paths as matching edges. For (ii), we can respectively pick one edge from two nontrivial blocks as matching edges. The edges from (i) and (ii) can insure that $H = G - \{e_1, e_2\}$ is a connected graph of order n. Since $|E(H)| \leq n + 2(k-1) - 5$, we have $pd(H) \leq k - 2$ by induction hypothesis. Then $pd(G) = pd(H + \{e_1, e_2\}) \leq pd(H) + 1 \leq k - 1$ by Lemma 3.13. Hence, we obtain that if $\mathrm{pd}(G) = k$, then $|E(G)| \geq n + 2k - 4$.

Next we show that for each pair integers k and n with $2 \leq k \leq \lceil \frac{n}{2} \rceil$, there is a connected graph G of order n and size $n + 2k - 4$ such that $pd(G) = k$. Let $H = K_{2,2k-3}$ with partite vertex sets $A = \{a_1, a_2\}$ and $B = \{b_1, b_2, \ldots, b_{2k-3}\}$. Let G be the graph of order n and size $n + 2k - 4$ obtained from H by adding an edge a_1a_2 and adding $n - 2k + 1$ pendent edges to a vertex of H. Now, define a coloring $c \to [k]$ for graph $H + a_1a_2$. Let c_H be a pd-coloring using colors $[k-1]$ by Theorem 3.5 and $c(a_1a_2) = k$. We can verify that the coloring c is a pd-coloring using colors $[k]$, so $pd(H + a_1a_2) \leq k$. Moreover, $pd(H + a_1a_2) \geq k$ by Lemma 2.6. Thus, We obtain that $pd(H + a_1a_2) = k$. Furthermore, since $H + a_1a_2$ is a block of G, we get $pd(G) = pd(H + a_1a_2) = k$ by Lemma 2.5. □

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