Discussiones Mathematicae
Graph Theory xx (xxxx) 1–18
doi:10.7151/dmgt.2206

THE VERTEX-RAINBOW CONNECTION NUMBER OF SOME GRAPH OPERATIONS

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Abstract

A path in an edge-colored (respectively vertex-colored) graph G is *rainbow* (respectively *vertex-rainbow*) if no two edges (respectively internal vertices) of the path are colored the same. An edge-colored (respectively vertex-colored) graph G is *rainbow connected* (respectively *vertex-rainbow connected*) if every two distinct vertices are connected by a rainbow (respectively vertex-rainbow) path. The *rainbow connection number* $rc(G)$ (respectively *vertex-rainbow connection number* $rvc(G)$) of G is the smallest number of colors that are needed in order to make G rainbow connected (respectively vertex-rainbow connected). In this paper, we show that for a connected graph G and any edge $e = xy \in E(G)$, $rvc(G) \leq rvc(G-e) \leq rvc(G) + d_{G-e}(x,y) - 1$ if $G - e$ is connected. For any two connected, non-trivial graphs G and H, $rad(G\Box H) - 1 \leq rvc(G\Box H) \leq 2rad(G\Box H)$, where $G\Box H$ is the Cartesian product of G and H. For any two non-trivial graphs G and H such that G is connected, $rvc(G \circ H) = 1$ if $diam(G \circ H) \leq 2$, $rad(G) - 1 \leq rvc(G \circ H) \leq 2rad(G)$ if $diam(G) > 2$, where $G \circ H$ is the lexicographic product of G and H. For the line graph $L(G)$ of a graph G we show that $rvc(L(G)) \leq rc(G)$, which is the first known nontrivial inequality between the rainbow connection number and vertex-rainbow connection number. Moreover, the bounds reported are tight or tight up to additive constants.

Keywords: rainbow connection number, vertex-rainbow connection number, Cartesian product, lexicographic product, line graph.

2010 Mathematics Subject Classification: 05C15, 05C40, 05C76.

1. Introduction

All graphs in this paper are undirected, finite, and simple. We refer to the book [4] for notation and terminology not described here. The *distance* between two vertices x and y in G, denoted by $d_G(x, y)$, is the number of edges of a shortest path between them. The *eccentricity* of a vertex x, denoted by $ecc_G(x)$, is $\max_{y\in V(G)} d_G(x, y)$. The *radius* and *diameter* of G, denoted by $rad(G)$ and $diam(G)$, are $\min_{x\in V(G)} ecc_G(x)$ and $\max_{x\in V(G)} ecc_G(x)$, respectively. A vertex u is a *center* if $ecc_G(u) = rad(G)$.

A path in an edge-colored (respectively vertex-colored) graph G is *rainbow* (respectively *vertex-rainbow*) if no two edges (respectively internal vertices) of the path are colored the same. An edge-colored (respectively vertex-colored) graph G is *rainbow connected* (respectively *vertex-rainbow connected*) if every two distinct vertices are connected by a rainbow (respectively vertex-rainbow) path. Such an edge-coloring (respectively a vertex-coloring) is a *rainbow coloring* (respectively *vertex-rainbow coloring*). A rainbow coloring (respectively *vertex-rainbow coloring*) using k colors is a *rainbow k-edge-coloring* (respectively *rainbow k-vertex-coloring*). The *rainbow connection number* $rc(G)$ (respectively *vertex-rainbow connection number* $rvc(G)$) of G is the smallest number of colors that are needed in order to make G rainbow connected (respectively vertex-rainbow connected).

The rainbow connection number was introduced by Chartrand, Johns, McKeon, and Zhang [8]. It has an application in transferring information of high security in multicomputer networks. We refer the readers to [6, 28] for details. Since then, the rainbow connection number has gained much attention. Chakraborty, Fischer, Matsliah, and Yuster [6] investigated the hardness and algorithms for the rainbow connection number, and showed that given a graph G, deciding if $rc(G) = 2$ is NP-complete. Bounds of the rainbow connection number of a graph have also been studied in terms of other graph parameters, for example, minimum degree and connectivity [5, 7, 9, 12, 26, 27], radius and diameter, etc. [1, 14, 15, 20]. Extremal problems have been studied in [3, 21, 30, 33]. Vertex-rainbow connection number was introduced by Krivelevich and Yuster [19]. Sequentially, this parameter was further studied in [10, 11, 24, 29, 31].

The rainbow connection number of some graph products has got recent attention [2, 13, 17, 23]. In [32], Mao, Yanling, Wang and Ye study the vertex-rainbow connection number on the lexicographical, strong, Cartesian and direct product of graphs G and H and present several upper bounds in terms of $rvc(G)$ and $rvc(H)$. In this paper, we continue to study the vertex-rainbow connection number of some graph operations and show several upper and lower bounds in terms of radius.

This paper is organized as follows. In Section 2, we summarize some notations and known results. In Section 3, we study how the rainbow connection number of a graph behaves under edge deletion. In Section 4, we study the vertex-rainbow connection number of the Cartesian product of two connected non-trivial graphs. In Section 5, we study the vertex-rainbow connection number of the lexicographic product of two non-trivial graphs. In Section 6, we study the relation between $rvc(L(G))$ and $rc(G)$, and prove that $rvc(L(G)) \leq rc(G)$, which is the first known nontrivial inequality between the rainbow connection number and vertex-rainbow connection number. We further prove that if a graph G has a k-edge-coloring such that every two vertices are connected by ℓ edge-disjoint rainbow paths, then $L(G)$ has a k-vertex-coloring such that every two vertices are connected by ℓ internally vertex-disjoint vertex-rainbow paths. In Section 7, we show several applications of our results.

2. Preliminaries

In this section, we summarize some notations and known facts that will be used for the proofs of our results.

We use P_n to denote a path with n vertices. A path P is called a *u-v path*, denoted by P_{uv}, if u and v are the end vertices of P. For simplicity, we use (G, c) to denote a graph with edge-coloring (respectively vertex-coloring) c, and we say that (G, c) is *rainbow connected* (respectively *vertex-rainbow connected*) if G is rainbow connected (respectively vertex-rainbow connected) under this edge-coloring (respectively vertex-coloring) c.

Let G and H be two graphs. The *Cartesian product* of two graphs G and H, denoted by $G\Box H$, is the graph with vertex set $V(G) \times V(H)$, in which two vertices (u, v) and (u', v') are adjacent if and only if $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$. Clearly, the Cartesian product is commutative, that is, $G\Box H \cong H\Box G$. It is easy to check that $diam(G\Box H) = diam(G) + diam(H)$ and $rad(G\Box H) = rad(G) + rad(H)$.

Let G and H be two graphs. The *lexicographic product* of two graphs G and H, written as $G \circ H$, is the graph with vertex set $V(G) \times V(H)$, in which two vertices (u, v) and (u', v') are adjacent if and only if $uu' \in E(G)$, or $u = u'$ and $vv' \in E(H)$.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *line graph* of the graph G is the graph $L(G)$ with $E(G)$ as vertex set, and where two vertices are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G. If $L(H) = G$, then H is called the *underlying graph* of G. The *k-iterated line graph* $L^k(G)$ is defined as $L^k(G) = L(L^{k-1}(G))$.

The next theorems will be useful.

Theorem 1 [8]**.** (1) *Let T be a tree of order $n \geq 2$. Then*

$$rc(T) = n - 1.$$

(2) *Let C_n be a cycle of order $n \geq 4$. Then*

$$rc(C_n) = \left\lceil \frac{n}{2} \right\rceil.$$

Theorem 2 [29]**.** *Let C_n be a cycle of order $n \geq 16$. Then*

$$rvc(C_n) = \left\lceil \frac{n}{2} \right\rceil.$$

3. Edge Deletion

In this section, we study how the vertex-rainbow connection number of a graph behaves under edge deletion.

For the vertex-rainbow connection number, the following observation holds.

Observation 3. *If H is a spanning connected subgraph of a connected graph G, then*

$$rvc(G) \leq rvc(H).$$

Theorem 4. *Let G be a connected graph. If $xy \in E(G)$ such that $G - xy$ is connected, then*

$$rvc(G) \leq rvc(G - xy) \leq rvc(G) + d_{G-xy}(x, y) - 1.$$

Proof. Since $G - xy$ is a subgraph of G, it follows from Observation 3 that $rvc(G) \leq rvc(G - xy)$. It suffices to show that $rvc(G - xy) \leq rvc(G) + d_{G-xy}(x, y) - 1$.

Let $rvc(G) = \ell$ and $d_{G-xy}(x, y) = k$ for simplicity. Without loss of generality, assume that c is a vertex-rainbow coloring of G using ℓ colors, and $P_{xy} = x_0x_1 \cdots x_k$ is a shortest path between x and y in $G - xy$, where $x_0 = x$ and $x_k = y$. Pick $k-1$ new colors, say $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$. We define an $(\ell+k-1)$-vertex-coloring of $G - xy$ as follows

$$c'(v) = \begin{cases} c(v), & \text{if } v \in V(G)\backslash\{x_i : 1 \leq i \leq k-1\}, \\ \alpha_i, & \text{if } v = x_i, 1 \leq i \leq k-1. \end{cases}$$

Now, we check that $(G - xy, c')$ is vertex-rainbow connected. Let $u, v \in V(G-xy)$. Since (G, c) is vertex-rainbow connected, there exists a vertex-rainbow path Q_{uv} in (G, c). If $V(Q_{uv}) \cap V(P_{xy}) = \emptyset$, then Q_{uv} is also a vertex-rainbow u-v

path in $(G - xy, c')$ by the definition of c'. Thus, we can suppose that $|V(Q_{uv}) \cap V(P_{xy})| \geq 1$. Let u' be the first vertex on Q_{uv} such that $u' \in V(P_{xy}) \cap V(Q_{uv})$ and let v' be the last vertex on Q_{uv} such that $v' \in V(P_{xy}) \cap V(Q_{uv})$. Let $Q_{uu'}$ be the u-u' subpath of Q_{uv} and $Q_{v'v}$ the v'-v subpath of Q_{uv}. Let $P_{u'v'}$ be the subpath of P_{xy} joining u' and v'. Then $Q_{uu'} \cup P_{u'v'} \cup Q_{v'v}$ is a vertex-rainbow path in $(G - xy, c')$. Thus $rvc(G - xy) \leq rvc(G) + d_{G-xy}(x, y) - 1$. ■

Remark 1. Let G be a graph with diameter two. Let $xy \in E(G)$ be an edge in G such that $G - xy$ has diameter two. It is easy to see that $rvc(G) = rvc(G - xy)$ and there are many such graphs. Thus the first inequality in Theorem 4 is sharp.

Remark 2. Let G be the graph in Figure 1a, and let $G - e$ be the graph in Figure 1b. It is easy to check that this graph is a sharp example for the second inequality in Theorem 4.

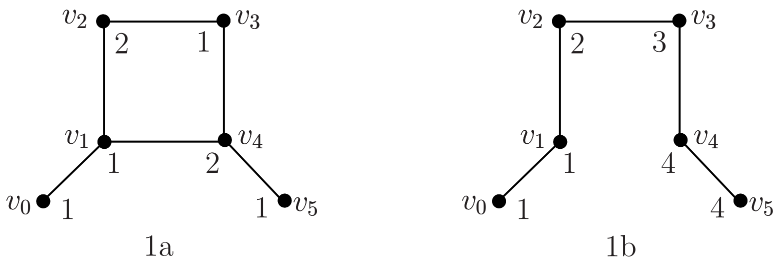

Figure 1. A graph G with $rvc(G - v_1v_4) = rvc(G) + d_{G-v_1v_4}(v_1, v_4) - 1$.

Corollary 5. *Let G be a connected graph, and let $xy \in E(G)$ be such that $G - xy$ is connected. Then*

$$rvc(G) \leq rvc(G - xy) \leq rvc(G) + diam(G - xy) - 1.$$

4. CARTESIAN PRODUCT

Let G and H be two graphs with $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$, respectively. For any subgraph $G_1 \subseteq G$, we use $G_1^{v_j}$ to denote the subgraph of $G \square H$ induced by the set $\{(u_i, v_j) : u_i \in V(G_1)\}$. Similarly, for any subgraph $H_1 \subseteq H$, we use $H_1^{u_i}$ to denote the subgraph of $G \square H$ induced by the set $\{(u_i, v_j) : v_j \in V(H_1)\}$.

For two vertices x, y in a tree T, we use xTy to denote the only x-y path in T. Recall that an *r-tree* is a tree with root r. Let T be an r-tree. The *level* of a vertex v in T, denoted by $\ell_T(v)$, is the length of the path rTv. The *depth* of an r-tree, denoted by $dep(T)$, is $\max\{\ell_T(v) : v \in V(T)\}$. Each vertex on the path

rTv, including the vertex v itself, is called an *ancestor* of v, and each vertex of which v is an ancestor is a *descendant* of v.

Given an r-tree T and a set of colors $c = \{c_i : 0 \leq i \leq dep(T)\}$, we define a *layer-wise vertex-coloring* of T as follows.

$$\text{For any } v \in V(T),\ c(v) = c_{\ell_T(v)}.$$

We are ready to prove the following theorem.

Theorem 6. *If G and H are two connected, non-trivial graphs, then*

$$rad(G\Box H) - 1 \leq rvc(G\Box H) \leq 2rad(G\Box H).$$

Proof. It follows from $rvc(G\Box H) \geq diam(G\Box H) - 1 \geq rad(G\Box H) - 1$ that the first inequality holds.

Next, we show that the second inequality holds. Let T be a breadth-first search tree (or BFS-tree) of G rooted at some center u_0, and let F be a BFS-tree of H rooted at some center v_0. We have $rad(T) = rad(G)$ and $rad(F) = rad(H)$ because we start a BFS-tree T and F in a center u_0 and v_0, respectively. In order to prove that $rvc(G\Box H) \leq 2rad(G\Box H)$, it suffices to prove that $rvc(T\Box F) \leq 2rad(G\Box H)$ by Observation 3.

Assume that $V(G) = V(T) = \{u_0, u_1, \dots, u_n\}$, $V(H) = V(F) = \{v_0, v_1, \dots, v_m\}$, $dep(T) = a$ and $dep(F) = b$. Clearly, $dep(T) = rad(T) = rad(G) = a$ and $dep(F) = rad(F) = rad(H) = b$. Let $\alpha = \{\alpha_0, \alpha_1, \dots, \alpha_a\}$, $\alpha' = \{\alpha'_0, \alpha'_1, \dots, \alpha'_a\}$, $\beta = \{\beta_0, \beta_1, \dots, \beta_b\}$, and $\beta' = \{\beta'_0, \beta'_1, \dots, \beta'_b\}$ be four sets of colors such that they are pairwise disjoint. We color the vertices in $T\Box F$ by the following two steps.

Step 1. Color T^{v_0} by a layer-wise vertex-coloring α, and color T^{v_i} by a layer-wise vertex-coloring α', where $i \geq 1$.

Step 2. For some vertices, we need modify their colors in this step. For F^{u_0}, recolor it by a layer-wise vertex-coloring β, and for F^{u_j} satisfying that u_j is a leaf in T, recolor it by a layer-wise vertex-coloring β'. Denoted by c this modified vertex-coloring of $T\Box F$. See Figure 2 for an illustration.

See Figure 2 for an example of our coloring process. In Figure 2, for a vertex (u, v) with $\alpha_i \to \beta_j$, it means that the vertex (u, v) is colored by α_i in Step 1, and the color α_i is modified by β_j in Step 2. For a vertex (u, v) with α_i, it means that the vertex (u, v) is colored by α_i in step 1, and the color α_i is not modified in Step 2.

Note that colors $\alpha_0, \alpha'_0, \alpha_a$, and α'_a do not appear in $(T\Box F, c)$. Thus we use $2(a+1)+2(b+1)-4 = 2a+2b$ colors in $(T\Box F, c)$. Now, we prove that $(T\Box F, c)$ is vertex-rainbow connected. First, the following two claims hold for the above vertex-coloring.

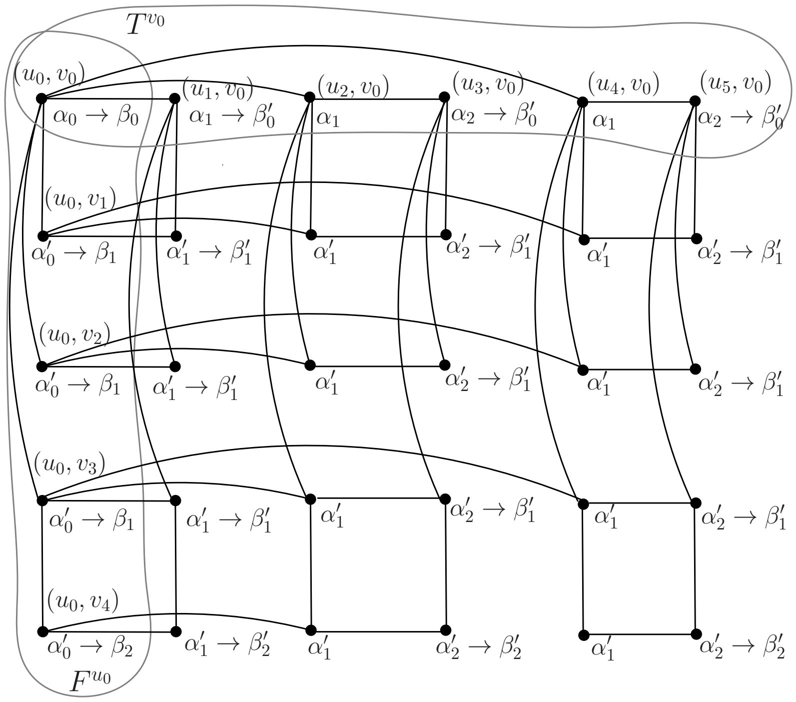

Figure 2. An example of our coloring process.

Claim 1. *For each v_i $(0 \le i \le m)$ in F, if x is a descendant of y in T^{v_i}, then the path $xT^{v_i}y$ is a vertex-rainbow path.*

Proof. Since x is a descendant of y in T^{v_i}, different vertices on $xT^{v_i}y$ have different levels in T^{v_i}, and obtain different colors in Step 1. So $xT^{v_i}y$ is vertex-rainbow in Step 1. Moreover, since every internal vertex of $xT^{v_i}y$ is not a leaf in T^{v_i}, its color is not modified in Step 2. Thus $xT^{v_i}y$ is also vertex-rainbow in (T^{v_i}, c), and the proof of Claim 1 is completed. $\square$

Claim 2. *Let u_i be a vertex in T such that u_i is a leaf or the root of T. If x is a descendant of y in F^{u_i}, then the path $xF^{u_i}y$ is a vertex-rainbow path.*

Proof. Since x is a descendant of y in F^{u_i}, different vertices on $xF^{u_i}y$ have different levels in F^{u_i}, and obtain different colors in Step 2. Thus $xF^{u_i}y$ is vertex-rainbow in $(T\square F, c)$. $\square$

Let $x = (u_i, v_j)$ and $y = (u_s, v_t)$ be any two vertices in $T\square F$. It suffices to show that there exists a vertex-rainbow x-y path in $T\square F$. Without loss of generality, assume that $\ell_T(u_i) \le \ell_T(u_s)$. We consider the following four cases.

Case 1. $v_j \ne v_0$ and $v_t \ne v_0$. Pick a leaf u_k in T such that u_s is an ancestor of u_k. We can easily check that $xT^{v_j}(u_0, v_j) + (u_0, v_j)F^{u_0}(u_0, v_0) +$

$(u_0, v_0)T^{v_0}(u_k, v_0)+(u_k, v_0)F^{u_k}(u_k, v_t)+(u_k, v_t)T^{v_t}y$ is our desired vertex-rainbow x-y path in $T\Box F$.

Case 2. $v_j = v_t = v_0$. Pick a leaf u_k in T such that u_s is an ancestor of u_k, and pick a leaf v_r in F. We can easily check that $xT^{v_0}(u_0, v_0) + (u_0, v_0)F^{u_0}(u_0, v_r) + (u_0, v_r)T^{v_r}(u_k, v_r) + (u_k, v_r)F^{u_k}(u_k, v_0) + (u_k, v_0)T^{v_0}y$ is our desired vertex-rainbow x-y path in $T\Box F$.

Case 3. $v_j = v_0$ and $v_t \neq v_0$. If $u_s = u_0$, then $u_i = u_0$ by our assumption that $\ell_T(u_i) \leq \ell_T(u_s)$. So (u_0, v_0) is an ancestor of (u_0, v_t) in F^{u_0}, and it follows from Claim 2 that they are connected by the vertex-rainbow path $(u_0, v_0)F^{u_0}(u_0, v_t)$. Otherwise, $u_s \neq u_0$, then the path $xT^{v_0}(u_0, v_0)+(u_0, v_0)F^{u_0}(u_0, v_t)+(u_0, v_t)T^{v_t}y$ is our desired vertex-rainbow x-y path in $T\Box F$.

Case 4. $v_j \neq v_0$ and $v_t = v_0$. In this case, $xT^{v_j}(u_0, v_j) + (u_0, v_j)F^{u_0}(u_0, v_0) + (u_0, v_0)T^{v_0}y$ is our desired vertex-rainbow x-y path in $T\Box F$.

Combining the above four cases, $(T\Box F, c)$ is vertex-rainbow connected, and we complete the proof of this theorem. ■

Remark 4. It is easy to check that the Cartesian product of two complete graphs of order 2 is a sharp example for the lower bound of Theorem 6. Let G and H be two graphs such that $diam(G) = 2rad(G)$ and $diam(H) = 2rad(H)$. Then $rvc(G\Box H) \geq diam(G\Box H)-1 = diam(G)+diam(H)-1 = 2rad(G)+2rad(H)-1$. Thus the upper bound of Theorem 6 is sharp up to an additive constant 1.

5. Lexicographic Product

Theorem 7. *Let G and H be two non-trivial graphs such that G is connected. The following assertions hold.*

(1) *If $diam(G \circ H) \leq 2$, then*

$$rvc(G \circ H) = 1.$$

(2) *If $diam(G \circ H) > 2$, then*

$$rad(G) - 1 \leq rvc(G \circ H) \leq 2rad(G) - 1.$$

Proof. (1) If $diam(G \circ H) \leq 2$, then we color each vertex in $G \circ H$ by 1. It is easy to see that this vertex-coloring is a vertex-rainbow coloring of $G \circ H$. Thus $rvc(G \circ H) = 1$.

(2) Suppose that $diam(G \circ H) > 2$. In this case, it is easy to check that $rad(G \circ H) = rad(G)$. It follows from $rvc(G \circ H) \geq diam(G \circ H) - 1 \geq rad(G \circ H) - 1 = rad(G) - 1$ that the first inequaltiy holds in (2).

Next, we prove that the second inequality holds in (2). Let T be a BFS-tree of G rooted at some center u_0, and let $v_0 \in V(H)$. In order to prove that $rvc(G \circ H) \leq 2rad(G) - 1$, it suffices to prove that $rvc(T \circ H) \leq 2rad(G) - 1$ by Observation 3. Assume that $V(G) = V(T) = \{u_0, u_1, \ldots, u_n\}$, $V(H) = \{v_0, v_1, \ldots, v_m\}$, and $dep(T) = a$. Clearly $dep(T) = rad(T) = rad(G) = a$. Let $\alpha = \{\alpha_0, \alpha_1, \ldots, \alpha_{a-1}\}$ and $\alpha' = \{\alpha'_1, \ldots, \alpha'_{a-1}\}$ be two sets of colors such that $\alpha \cap \alpha' = \emptyset$. Now we define a $(2a-1)$-vertex-coloring c of $T \circ H$ as follows

$$c((u,v)) = \begin{cases} \alpha_i, & \text{if } \ell_T(u) = i \text{ and } v = v_0, \text{ where } 1 \leq i \leq a-1, \\ \alpha'_i, & \text{if } \ell_T(u) = i \text{ and } v \neq v_0, \text{ where } 1 \leq i \leq a-1, \\ \alpha_0, & \text{if } \ell_T(u) = 0 \text{ or } \ell_T(u) = a. \end{cases}$$

Now, it suffices to prove that $T \circ H$ is vertex-rainbow connected. Let $x = (u_i, v_j)$ and $y = (u_s, v_t)$ be any two vertices in $T \circ H$. Assume that $u_0 T u_i = z_0 z_1 \cdots z_k$ and $u_0 T u_s = w_0 w_1 \cdots w_r$, where $z_0 = w_0 = u_0$, $z_k = u_i$, and $w_r = u_s$.

If $z_0 \neq z_k$ and $w_r \neq w_0$, then $(z_k, v_j)(z_{k-1}, v_0)(z_{k-2}, v_0) \cdots (z_0, v_0)(w_1, v_1)$ $(w_2, v_1) \cdots (w_{r-1}, v_1)(w_r, v_t)$ is a vertex-rainbow x-y path in $T \circ H$. If $z_0 = z_k$ and $w_r = w_0$, then every path connecting x and y with length 2 is a vertex-rainbow x-y path. If $z_0 = z_k$ and $w_r \neq w_0$, then $(z_0, v_j)(w_1, v_1)(w_2, v_1) \cdots (w_{r-1}, v_1)(w_r, v_t)$ is a vertex-rainbow x-y path in $T \circ H$. ■

Remark 5. Let C_{2k} $(k \geq 3)$ be a cycle of order $2k$, and G be a nontrivial graph. On one hand, we have that $diam(C_{2k} \circ G) = rad(C_{2k} \circ G) = k \geq 3$, and $rvc(C_{2k} \circ G) \geq rad(C_{2k}) - 1 = k - 1$ by Theorem 7. On the other hand, it is easy to check that $rvc(C_{2k} \circ G) = k$. Thus the first inequality in (2) of Theorem 7 is sharp up to an additive constant 1.

Remark 6. Let G and H be two graphs such that $diam(G) = 2rad(G) > 2$. Then $rvc(G \circ H) \geq diam(G \circ H) - 1 = diam(G) - 1 = 2rad(G) - 1$. Thus, the second inequality in (2) of Theorem 7 is sharp.

6. Line Graphs

It is very interesting to study the relation between the rainbow connection number and vertex-rainbow connection number. It is easy to see that $rc(G)$ and $rvc(G)$ can be seen as a kind of connectivity with more reinforced requirements. For connectedness, it is well-known that $\delta(G) \leq \lambda(G) \leq \kappa(G)$. But for rainbow connectedness, $rc(G)$ and $rvc(G)$ are not comparable. For connectedness, connectivity and edge-connectivity have another well-known relation, that is, $\lambda(G) \leq \kappa(L(G))$ for each graph, where $\lambda(G)$, $\kappa(G)$ and $L(G)$ are edge-connectivity, connectivity and the line graph of a graph G, respectively.

The concept of thorn graphs was proposed by Gutman [16] and different applications have been studied by many others. Let G be a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$, and let $(p_1, p_2, \ldots, p_n)$ be an n-tuple of non-negative integers. The *thorn graph* $G^*(p_1, p_2, \ldots, p_n)$ of the graph G is formed by attaching p_i new vertices of degree 1 to a vertex v_i of G for every $i \in \{1, \ldots, n\}$. We simply write G^* for $G^*(1, 1, \ldots, 1)$.

Lemma 8. *For any connected graph G, $rvc(L(G^*)) \geq rc(G)$.*

Proof. Recall that for a connected graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$, G^* is the thorn graph obtained from G by attaching a new vertex u_i to v_i, where $1 \leq i \leq n$. For simplicity, let $k = rvc(L(G^*))$, and $e_i = v_i u_i$ for each $1 \leq i \leq n$.

Assume that c^* is a vertex-rainbow coloring of $L(G^*)$ using k colors. We prove that $rvc(L(G^*)) \geq rc(G)$ by constructing a rainbow coloring of G using k colors as follows. For each edge e in G,

$$c(e) = c^*(e).$$

It suffices to show that (G, c) is rainbow connected. For any $v_s, v_t \in V(G)$, consider the vertices $e_s, e_t \in V(L(G^*))$, where $e_s = v_s u_s$ and $e_t = v_t u_t$. Pick a vertex-rainbow e_s-e_t path P^* in $(L(G^*), c^*)$, and let $P = P^* \setminus \{e_i : 1 \leq i \leq n\}$.

Claim 3. $P = P^* \setminus \{e_i : 1 \leq i \leq n\}$ *is a vertex-rainbow path in* $(L(G^*), c^*)$.

Proof. Delete e_s and e_t from P^*, if every e_i, $1 \leq i \leq n$, is no internal vertex of P^*. Then $P = P^*$ is our desired vertex-rainbow path in $(L(G^*), c^*)$. Otherwise, let e_i be such vertex, and let f and g be two neighbors of e_i on P^*. Since $e_i = v_i u_i$, $d_{G^*}(u_i) = 1$ and f and g are adjacent to e_i in $L(G^*)$, the vertex v_i is the common endvertex of e_i, f and g in G. So the vertices f and g are also adjacent in $L(G^*)$, and the path $P \setminus \{e_i\}$ is a vertex-rainbow path in $(L(G^*), c^*)$. We can repeat deleting similar vertices until we obtain our desired vertex-rainbow path. □

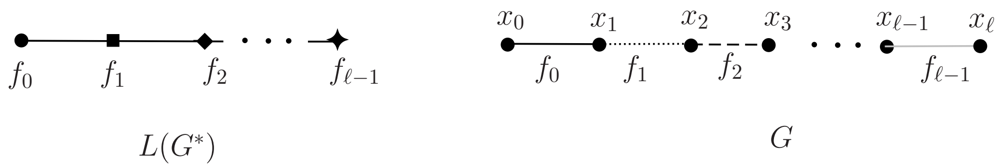

Figure 3. A vertex-rainbow path in $L(G^*)$ and its corresponding rainbow path in G.

Without loss of generality, assume that $P = f_0 f_1 \cdots f_{\ell-1}$ and $f_i = x_i x_{i+1}$, where $0 \leq i \leq \ell - 1$. It is easy to see that v_s and v_t are endvertices of f_0 and $f_{\ell-1}$, respectively, say $v_s = x_0$ and $v_t = x_\ell$. Then the path $Q = x_0 x_1 \cdots x_\ell$ is a rainbow v_s-v_t path. See Figure 3 for an illustration. Thus (G, c) is rainbow connected, and so $rvc(L(G^*)) \geq rc(G)$. ■

Lemma 9. *For any connected graph G, $rvc(L(G^*)) \leq rc(G)$.*

Proof. Recall that for a connected graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$, G^* is the thorn graph obtained from G by attaching a new vertex u_i to v_i, where $1 \leq i \leq n$. For simplicity, let $k = rc(G)$, and $e_i = v_i u_i$ for each $1 \leq i \leq n$.

Assume that c is a rainbow edge-coloring of G using k colors. We prove that $rvc(L(G^*)) \leq rc(G)$ by constructing a rainbow vertex-coloring c^* of $L(G^*)$ using k colors as follows. For each edge e in G,

$$c^*(e) = \begin{cases} c(e), & e \in E(G), \\ 1, & \text{otherwise}. \end{cases}$$

It suffices to show that $(L(G^*), c^*)$ is vertex-rainbow connected. For any $f, g \in V(L(G^*))$, let v_s and v_t be the endvertices of f and g in G^*, respectively, such that $v_s, v_t \in V(G)$. In (G, c), pick a rainbow v_s-v_t path $P = x_0 x_1 \cdots x_\ell$, where $x_0 = v_s$, $x_\ell = v_t$. Let $f_i = x_{i-1} x_i$, and $1 \leq i \leq \ell$. Then the path $Q \cup \{f, g\}$ is a vertex-rainbow f-g path in $(L(G^*), c^*)$, where $Q = f_1 f_2 \cdots f_\ell$. Thus $(L(G^*), c^*)$ is vertex-rainbow connected, and so $rvc(L(G^*)) \leq rc(G)$. ■

Combining Lemmas 8 and 9, the following theorem holds.

Theorem 10. *For any connected graph G, $rvc(L(G^*)) = rc(G)$.*

Remark 7. From the arguments of proofs of Lemmas 8 and 9, we can see that for any connected graph G of order n, $rvc(L(G^*(p_1, p_2, \ldots, p_n))) = rc(G)$ if $p_i \geq 1$ for each $1 \leq i \leq n$.

Lemma 11. *For any connected graph G, $rvc(L(G)) \leq rvc(L(G^*))$.*

Proof. Recall that for a connected graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$, G^* is the thorn graph obtained from G by attaching a new vertex u_i to v_i, where $1 \leq i \leq n$. For simplicity, let $k = rvc(L(G^*))$, and $e_i = v_i u_i$ for each $1 \leq i \leq n$.

Assume that c^* is a vertex-rainbow coloring of $L(G^*)$ using k colors. We prove that $rvc(L(G)) \leq rc(L(G^*))$ by constructing a vertex-rainbow coloring of $L(G)$ using k colors as follows. For each vertex e in $L(G)$,

$$c(e) = c^*(e).$$

It suffices to show that $(L(G), c)$ is vertex-rainbow connected. For any $g, g' \in V(L(G))$, since $(L(G^*), c^*)$ is vertex-rainbow connected, we can pick a vertex-rainbow g-g' path P^* in $(L(G^*), c^*)$.

Similar to Claim 3 of Lemma 8, we have that $P = P^* \setminus \{e_i : 1 \leq i \leq n\}$ is a vertex-rainbow path connecting g and g' in $L(G)$. Thus $(L(G), c)$ is vertex-rainbow connected, and so $rvc(L(G)) \leq rvc(L(G^*))$. ■

Combining Theorem 10 and Lemma 11, the following theorem holds.

Theorem 12. *The vertex-rainbow connection number of the line graph of a connected graph G is no more than the rainbow connection number of the graph G, that is, $rvc(L(G)) \leq rc(G)$.*

By Theorem 12, $rvc(L(G)) \leq rc(G)$. But there are other two questions, that is, firstly is this bound sharp and secondly could the difference $rc(G)-rvc(L(G))$ be any large? The following theorem show affirmative answers for these questions.

Theorem 13. *Let n and m be two integers. If $n = m \geq 16$, then there exists a connected graph G such that $rc(G) = rvc(L(G)) = n$. If $3 \leq n < m$, then there exists a connected graph G such that $rc(G) = m$ and $rvc(L(G)) = n - 1$.*

Proof. If $n = m \geq 16$, then it follows from Theorems 1 and 2 that C_n is our desired graph.

Suppose $n < m$ in the following arguments. Let $P_n = v_1v_2\cdots v_n$ be a path of order n, and let $(k_1, k_2, \ldots, k_n)$ be an n-tuple on non-negative integers such that $k_1, k_n \geq 1$ and $\sum_{i=1}^{n} k_i = m - n + 1$. Recall that $G = P_n^*(k_1, k_2, \ldots, k_n)$ is the thorn graph of P_n by attaching k_i new vertex $v_{i,1}, v_{i,2}, \ldots, v_{i,k_i}$ to v_i, where $1 \leq i \leq n$.

Now we show that $rc(G) = m$ and $rvc(L(G)) = n - 1$. Since G is a tree, if follows from Lemma 1 that $rc(G) = |V(G)| - 1 = m$. For $L(G)$, since $diam(L(G)) = n$, $rvc(L(G)) \geq n - 1$. Define a vertex-coloring c of $L(G)$ using $(n-1)$ colors as follows. For each vertex $e \in V(L(G))$,

$$c(e) = \begin{cases} i, & \text{if } \ e = v_iv_{i+1} \text{ or } e = v_iv_{i,j}, \text{ where } 1 \leq i \leq n-1,\ 1 \leq j \leq k_i, \\ 1, & \text{if } \ e = v_nv_{n,j}, \text{ where } 1 \leq n \leq k_n. \end{cases}$$

It is easy to check that $(L(G), c)$ is vertex-rainbow connected. Thus $rvc(L(G)) = n - 1$. That is, the thorn graph $G = P_n^*(k_1, k_2, \ldots, k_n)$ is our desired graph when $3 \leq n < m$. ■

If a graph should have more than one rainbow path, then we have the following further result.

Theorem 14. *If a graph G has an edge-coloring using k colors such that every two vertices are connected by ℓ edge-disjoint rainbow paths, then $L(G)$ has a vertex-coloring using k colors such that every two vertices are connected by ℓ internally disjoint vertex-rainbow paths.*

We show that Theorem 14 holds by the following three lemmas.

Lemma 15. *For any connected graph G, if $L(G^*)$ has a vertex-coloring using k colors such that every two vertices are connected by ℓ internally disjoint vertex-rainbow paths, then G has an edge-coloring using k colors such that every two vertices are connected by ℓ edge-disjoint rainbow paths.*

Proof. Given graphs G, G^* and $L(G^*)$ are as in the proof of Lemma 8. Let c^* be a vertex-coloring of $L(G^*)$ using k colors such that every two vertices are connected by ℓ internally disjoint vertex-rainbow paths. We define an edge-coloring c of G as in the proof of Lemma 8.

It suffices to show that every two vertices are connected by ℓ edge-disjoint rainbow paths in (G, c). For any $v_s, v_t \in V(G)$, consider the vertices $e_s, e_t \in V(L(G^*))$. Recall that $e_s = v_s u_s$ and $e_t = v_t u_t$. In $(L(G^*), c^*)$, pick ℓ internally vertex-disjoint vertex-rainbow e_s-e_t paths $P_1^*, P_2^*, \ldots, P_\ell^*$. Similar to Lemma 8, we can construct a rainbow v_s-v_t path P_i in (G, c) from each vertex-rainbow path P_i^* in $(L(G^*), c^*)$, where $1 \le i \le \ell$. It is easy to check that $P_1, P_2, \ldots, P_\ell$ are edge-disjoint rainbow paths. ∎

Lemma 16. *For any connected graph G, if G has an edge-coloring using k colors such that every two vertices are connected by ℓ edge-disjoint rainbow paths, then $L(G^*)$ has a vertex-coloring using k colors such that every two vertices are connected by ℓ internally disjoint vertex-rainbow paths.*

Proof. Given graphs G, G^* and $L(G^*)$ are as in Lemma 9. Let c be an edge-coloring of G using k colors such that every two vertices are connected by ℓ edge-disjoint rainbow paths. We define a vertex-coloring c^* of $L(G^*)$ as in the proof of Lemma 9.

It suffices to show that every two vertices are connected by ℓ internally disjoint vertex-rainbow paths in $(L(G^*), c^*)$. For any $f, g \in V(L(G^*))$, let v_s and v_t be the endvertices of f and g in G^*, respectively, such that $v_s, v_t \in V(G)$.

Pick ℓ edge-disjoint rainbow v_s-v_t paths $P_1, P_2, \ldots, P_\ell$ in (G, c). Similar to Lemma 9, we can construct a vertex-rainbow f-g path Q_i $(L(G^*), c^*)$ from each rainbow path P_i in (G, c), where $1 \le i \le \ell$. It is easy to check that $Q_1, Q_2, \ldots, Q_\ell$ are internally vertex-disjoint vertex-rainbow paths. ∎

Combining Lemmas 15 and 16, the following theorem holds.

Theorem 17. *For any connected graph G, $L(G^*)$ has a vertex-coloring using k colors such that every two vertices are connected by ℓ internally disjoint vertex-rainbow paths if and only if G has an edge-coloring using k colors such that every two vertices are connected by ℓ edge-disjoint rainbow paths.*

Lemma 18. *For any connected graph G, if $L(G^*)$ has a vertex-coloring using k colors such that every two vertices are connected by ℓ internally disjoint vertex-rainbow paths, then $L(G)$ also has a vertex-coloring using k colors such that every two vertices are connected by ℓ internally disjoint vertex-rainbow paths.*

Proof. Given graphs G, G^* and $L(G^*)$ are as in the proof of Lemma 11. Let c^* be a vertex-coloring of $L(G^*)$ using k colors such that every two vertices are connected by ℓ internally disjoint vertex-rainbow paths. We define an edge-coloring c of $L(G)$ as in the proof of Lemma 11.

For any $g, g' \in V(L(G))$, pick k internally vertex-disjoint vertex-rainbow g-g' paths $P_1^*, P_2^*, \ldots, P_\ell^*$ in $(L(G^*), c)$. Similar to Theorem 11, we can construct a vertex-rainbow g-g' path P_i in $(L(G), c)$ from a vertex-rainbow path P_i^* in $(L(G^*), c^*)$, where $1 \leq i \leq \ell$. It is easy to see that $P_1, P_2, \ldots, P_\ell$ are internally vertex-disjoint vertex-rainbow paths. ■

Combining Theorem 17 and Lemma 18, Theorem 14 holds.

7. Several Applications

In this section, we first present some known results on rainbow connection number, and secondly show some new results on vertex-rainbow connection number by combining these results and Theorem 12.

Ekstein, Holub, Kaiser, Koch, Camacho, Ryjáček and Schiermeyer [12] and Li, Liu, Chandran, Mathew and Rajendraprasad [25] showed the following bounds of rainbow connection number of a graph in connection with connectivity.

Theorem 19 [12, 25]**.** *Let G be a 2-connected graph of order n ($n \geq 3$). Then*

$$rc(G) \leq \left\lceil \frac{n}{2} \right\rceil.$$

Moreover, the upper bound is tight for $n \geq 4$.

Theorem 20 [25]**.** *For every $k \geq 1$, if G is a k-connected graph of order n, then for every $\varepsilon \in (0, 1)$,*

$$rc(G) \leq \left(\frac{2+\varepsilon}{k}\right) n + \frac{23}{\epsilon^2}.$$

Huang, Li, Li and Sun [14], Li, Li and Liu [20] and Li, Li and Sun [22] showed the following bounds of rainbow connection number of a graph in connection with diameter and radius.

Theorem 21 [14]**.** *For every bridgeless graph G,*

$$rc(G) \leq \sum_{i=1}^{rad(G)} \min\{2i+1, \eta(G)\} \leq rad(G)\eta(G),$$

where $\eta(G)$ is the smallest integer such that every edge of G belongs to a cycle of length at most $\eta(G)$.

Theorem 22 [20]**.** *For every bridgeless graph G with diameter 2,*

$$rc(G) \leq 5,$$

Moreover, the upper bound is sharp.

Theorem 23 [22]**.** *For every bridgeless graph G with diameter 3,*

$$rc(G) \leq 9.$$

Theorem 24. *For every $k \geq 2$, if G is the line graph of a k-connected graph, then $rvc(G) \leq \left\lceil \frac{|V(G)|}{k} \right\rceil$.*

Proof. Assume that $G = L(H)$. From Theorems 19 and 12, it follows that $rvc(G) = rvc(L(H)) \leq rc(H) \leq \left\lceil \frac{|V(H)|}{2} \right\rceil$. Since $\delta(H) \geq \kappa(H) \geq k$, we have that $|V(G)| = |E(H)| \geq \frac{\delta(H)|V(H)|}{2} \geq \frac{\kappa(H)|V(H)|}{2} \geq \frac{k|V(H)|}{2}$. So $|V(H)| \leq \frac{2|V(G)|}{k}$. Thus $rvc(G) \leq \left\lceil \frac{|V(G)|}{k} \right\rceil$. ■

Theorem 25. *For every $k \geq 1$, if G is the line graph of a k-connected graph, then for every $\varepsilon \in (0, 1)$,*

$$rvc(G) \leq \left(\frac{4 + 2\varepsilon}{k^2} \right) |V(G)| + \frac{23}{\epsilon^2}.$$

Proof. Assume that $G = L(H)$. From Theorems 20 and 12, it follows that $rvc(G) = rvc(L(H)) \leq rc(H) \leq \left(\frac{2+\varepsilon}{k}\right) |V(H)| + \frac{23}{\epsilon^2}$. Since $\delta(H) \geq \kappa(H) \geq k$, we have that $|V(G)| = |E(H)| \geq \frac{\delta(H)|V(H)|}{2} \geq \frac{\kappa(H)|V(H)|}{2} \geq \frac{k|V(H)|}{2}$. So $|V(H)| \leq \frac{2|V(G)|}{k}$. Thus $rvc(G) \leq \left(\frac{4+2\varepsilon}{k^2}\right) |V(G)| + \frac{23}{\epsilon^2}$. ■

Combining Theorems 22, 23 and 12, the following result holds.

Theorem 26. *Let G be the line graph of a bridgeless graph H.*

(1) *If $diam(H) = 2$, then $rvc(G) \leq 5$.*

(2) *If $diam(H) = 3$, then $rvc(G) \leq 9$.*

In [18], Knor, Niepel, and Šoltés obtained the following inequality.

Theorem 27 [18]**.** *For any connected graph G, $rad(G) - 1 \leq rad(L(G)) \leq rad(G) + 1$.*

Theorem 28. *Let G be the line graph of a bridgeless graph H. If every edge of H belongs to a cycle of length at most η, then*

$$rvc(G) \leq rad(H)\eta(H) \leq (rad(G) + 1)\eta(H).$$

Proof. By Theorems 12 and 21, $rvc(G) \leq rc(H) \leq rad(H)\eta(H)$. Moreover, it follows from Theorem 27 that $rad(H) \leq rad(L(H)) + 1 = rad(G) + 1$. Thus $rvc(G) \leq (rad(G) + 1)\eta(H)$. ■

Theorem 29. *Let G be a connected graph. If $\delta(G) \geq 3$, then*

$$rvc(L^2(G)) \leq 3(rad(G) + 1).$$

Proof. By Theorem 12, $rvc(L^2(G)) \leq rc(L(G))$. Since $\delta(G) \geq 3$, each edge of $L(G)$ belongs a cycle of length 3. By Theorem 21, we have that $rc(L(G)) \leq 3rad(L(G))$. Moreover, it follows from Theorem 27 that $rad(L(G)) \leq rad(G)+1$. Thus $rvc(L^2(G)) \leq 3(rad(G) + 1)$. ■

Acknowledgements

The authors are grateful to the referees for a very careful reading of the manuscript and very helpful comments which led to the improvement of the presentation of this paper. The paper was supported by NSFC (Nos. 11401181, 11531011 and 11701157).

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Received 4 July 2018
Revised 22 January 2019
Accepted 22 January 2019