ON TINY ZERO-SUM SEQUENCES OVER FINITE ABELIAN GROUPS

WEIDONG GAO, WANZHEN HUI*, XUE LI, XIAOER QIN, AND QIUYU YIN

ABSTRACT. Let G be an additive finite abelian group and $S = g_1 \cdot \ldots \cdot g_l$ be a sequence over G. Let $\mathsf{k}(S) = \text{ord}(g_1)^{-1} + \ldots + \text{ord}(g_l)^{-1}$ be its cross number. Let $\mathsf{t}(G)$ (resp. $\eta(G)$) be the smallest integer t such that every sequence of t elements (repetition allowed) from G has a non-empty zero-sum subsequence T with $\mathsf{k}(T) \le 1$ (resp. $|T| \le \exp(G)$). It is easy to see that $\mathsf{t}(G) \ge \eta(G)$. It is known that $\mathsf{t}(G) = \eta(G) = |G|$ when G is cyclic, and for any integer $r \ge 3$, there are infinitely many groups G of rank r such that $\mathsf{t}(G) > \eta(G)$. It is conjectured in 2012 [G12] that $\mathsf{t}(G) = \eta(G)$ for all finite abelian groups of rank two. This conjecture has been verified only for the groups $G \cong C_{p^\alpha} \oplus C_{p^\alpha}$, $G \cong C_2 \oplus C_{2p}$ and $G \cong C_3 \oplus C_{3p}$ with $p \ge 5$, where p is a prime. In this paper, among other results, we confirm this conjecture for more groups including the groups $G \cong C_n \oplus C_n$ with the smallest prime divisor of n not less than the number of the distinct prime divisors of n.

1. Introduction and main results

Let G be a finite abelian group, written additively. If G is cyclic of order n, it will be denoted by C_n. In the general case, we can decompose G as a direct sum of cyclic groups $C_{n_1} \oplus \ldots \oplus C_{n_r}$ such that $1 < n_1 \mid \ldots \mid n_r \in \mathbb{N}$ (if $n_1 = \ldots = n_r = n$, it will be abbreviated as C_n^r), where r and n_r are respectively called the *rank* and *exponent* of G. Usually, the exponent of G is simply denoted by $\exp(G)$. The order of an element g of G will be written $\text{ord}(g)$.

Given a sequence $S = g_1 \cdot \ldots \cdot g_l$ over G, we denote by $S_{(d)}$ the subsequence of S consisting of all terms of S of order d and S_H the subsequence of S consisting of all terms of S belonging to a subgroup H of G. And by $\mathsf{k}(S)$ the *cross number* of S, which is defined as follows:

$$\mathsf{k}(S) = \sum_{i=1}^{l} \frac{1}{\text{ord}(g_i)}.$$

The cross number is an important concept in factorization theory. For more information on the cross number we refer to ([GG09, GS94, G09, G12]).

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*Corresponding author.

Let $\mathsf{t}(G)$ denote the smallest integer $t \in \mathbb{N}$ such that every sequence S over G of length $|S| \geq t$ has a non-empty zero-sum subsequence S' with $\mathsf{k}(S') \leq 1$. Such a subsequence will be called a *tiny zero-sum* subsequence.

The study of $\mathsf{t}(G)$ goes back to the late 1980s, Lemke and Kleitman [LK89] proved that $\mathsf{t}(C_n) = n$, which confirmed a conjecture by Erdős and Lemke. More generally, Lemke and Kleitman [LK89] conjectured that $\mathsf{t}(G) \leq |G|$ holds for every finite abelian group G. This conjecture was proved by Geroldinger [G93] in 1993. Furthermore, Elledge and Hurlbert [EH05] gave a different proof in 2005.

In 2012, Girard [G12] proved that, by using a result of Alon and Dubiner [AD95], for finite abelian groups of fixed rank, $\mathsf{t}(G)$ grows linearly in the exponent of G, which gives the correct order of magnitude.

Let $\eta(G)$ denote the smallest integer $t \in \mathbb{N}$ such that every sequence S over G of length $|S| \geq t$ has a non-empty zero-sum subsequence S' with $|S'| \leq \exp(G)$. Such a subsequence is called a *short zero-sum* subsequence. The constant $\eta(G)$ is one of many classical invariants in so-called zero-sum theory. For zero-sum theory and its application, the interested reader is referred to [GG06] and [GH06].

Since $\mathsf{k}(T) \leq 1$ implies $|T| \leq \exp(G)$, we know that $\eta(G) \leq \mathsf{t}(G)$ always holds. Girard [G12] noticed that if $\mathsf{t}(G) = \eta(G)$ for some finite abelian group G, then $\eta(H) \leq \eta(G)$ for any subgroup H of G, and then he deduced that for any positive integer $r \geq 4$, there is a finite abelian group of rank r such that $\mathsf{t}(G) > \eta(G)$. Concerning groups of rank three, the first author with coauthors [FGPWZ13] found that $\mathsf{t}(G) > \eta(G)$ if $G \cong C_2 \oplus C_2 \oplus C_{2n}$, where $n > 1$ is a positive integer. Girard [G12] also proved that $\mathsf{t}(C_{p^\alpha}^2) = \eta(C_{p^\alpha}^2) = 3p^\alpha - 2$ for any prime p and conjectured that $\mathsf{t}(G) = \eta(G)$ for all finite abelian groups of rank two. Girard also [G12] noticed the easy fact that $\mathsf{t}(G) = \eta(G)$ for all elementary p-groups G, since all non-zero elements of G have same order in this case, and conjectured that $\mathsf{t}(G) = \eta(G)$ for $G \cong C_n^r$.

Conjecture 1.1. *(*[G12]*) For all positive integers m, n with $m \mid n$, we have*

$$\mathsf{t}(C_m \oplus C_n) = \eta(C_m \oplus C_n) = 2m + n - 2.$$

Conjecture 1.2. *(*[G12]*) For all positive integers r, n, we have*

$$\mathsf{t}(C_n^r) = \eta(C_n^r).$$

Conjectures 1.1 and 1.2 have been confirmed only for a few classes of groups.

Theorem 1.3. *(*[FGPWZ13, GHST07, G12, W20]*) Let G be a finite abelian group, and n, r, α, β be positive integers and p be a prime number. Then $\mathsf{t}(G) = \eta(G)$ for the following groups.*

(1) $G \cong C_n$,
(2) $G \cong C_{p^\alpha} \oplus C_{p^\alpha}$,
(3) $G \cong C_2 \oplus C_{2p}$,
(4) $G \cong C_3 \oplus C_{3p}$ *with* $p \geq 5$,
(5) $G \cong C_n^3$ *with* $n = 3^\alpha$ *or* $n = 5^\beta$,
(6) $G \cong C_n^r$ *with* $n = p$ *or* $n = 2^\alpha$.

In this paper, we will confirm both Conjecture 1.1 and Conjecture 1.2 for more groups. Now we state our main results.

Theorem 1.4. *Let n be a positive integer, and let $G \cong C_n \oplus C_n$. If $\sum_{p|n} \frac{1}{p} < 1$, where p runs over all distinct prime divisors of n, then*

$$\mathsf{t}(G) = \eta(G).$$

In particular, if $p(n) \geq \omega(n)$, then $\mathsf{t}(G) = \eta(G)$, where $p(n)$ denotes the smallest prime divisor of n and $\omega(n)$ denotes the number of distinct prime divisors of n.

Theorem 1.5. *Let α, β be positive integers and p be a prime number. Then $\mathsf{t}(G) = \eta(G)$ for the following groups.*

(a) $G \cong C_2 \oplus C_{2^\alpha}$,
(b) $G \cong C_2 \oplus C_{2p^\beta}$,
(c) $G \cong C_{3^\alpha 5^\beta}^3$.

The paper is organized as follows. Section 2 provides some notation and concepts which will be used in the sequel. In Section 3 we prove the main results.

2. Notation and preliminaries

Let $\mathbb{N}$ denote the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any two integers $a, b \in \mathbb{N}_0$, we set $[a, b] = \{x \in \mathbb{N}_0 \mid a \leq x \leq b\}$. Throughout this paper, all abelian groups will be written additively.

Let G be an additive finite abelian group with rank r. An r-tuple $(e_1, \ldots, e_r)$ in $G \setminus \{0\}$ is called a *basis* of G if $G \cong \langle e_1 \rangle \oplus \ldots \oplus \langle e_r \rangle$. We denote by $\mathcal{F}(G)$ the free (abelian, multiplicative) monoid with basis G. The elements of $\mathcal{F}(G)$ are called *sequences* over G. We write sequences $S \in \mathcal{F}(G)$ in the form

$$S = \prod_{g \in G} g^{\mathsf{v}_g(S)}, \text{ with } \mathsf{v}_g(S) \in \mathbb{N}_0 \text{ for all } g \in G.$$

We call $\mathsf{v}_g(S)$ the *multiplicity* of g in S, and we say that S contains g if $\mathsf{v}_g(S) > 0$. A sequence S' is called a *subsequence* of S if $\mathsf{v}_g(S') \le \mathsf{v}_g(S)$ for all $g \in G$, denote by $S' \mid S$, and SS'^{-1} denotes the subsequence obtained from S by deleting S', two subsequences S_1 and S_2 of S are called *disjoint* if $S_1 \mid SS_2^{-1}$. The unit element $1 \in \mathcal{F}(G)$ is called the *empty* sequence.

For a sequence

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G} g^{\mathsf{v}_g(S)} \in \mathcal{F}(G),$$

we call

- $|S| = l = \sum_{g \in G} \mathsf{v}_g(S) \in \mathbb{N}_0$ the *length* of S,
- $\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g \in G} \mathsf{v}_g(S) g \in G$ the *sum* of S,
- $\text{supp}(S) = \{g \in G \mid \mathsf{v}_g(S) > 0\} \subset G$ the *support* of S,
- S a *zero-sum sequence* if $\sigma(S) = 0 \in G$,
- S a *zero-sum free sequence* if there is no non-empty zero-sum subsequence of S,
- S a *minimal zero-sum sequence* if it is a non-empty zero-sum sequence and has no proper zero-sum subsequence,
- S a *short zero-sum sequence* if S is zero-sum and $1 \le |S| \le \exp(G)$,
- S a *tiny zero-sum sequence* if S is a non-empty zero-sum sequence and $\mathsf{k}(S) \le 1$.

Let $\mathsf{D}(G)$ denote the smallest integer $t \in \mathbb{N}$ such that every sequence S over G of length $|S| \ge t$ has a non-empty zero-sum subsequence. The invariant $\mathsf{D}(G)$ is called the *Davenport constant* of G.

Every map of abelian groups $\varphi : G \longrightarrow H$ extends to a homomorphism $\varphi : \mathcal{F}(G) \longrightarrow \mathcal{F}(H)$, where $\varphi(S) = \varphi(g_1) \cdot \ldots \cdot \varphi(g_l)$. If φ is a homomorphism then $\varphi(S)$ is a zero-sum sequence if and only if $\sigma(S) \in \ker(\varphi)$.

Given a positive integer n, let $p(n)$ denote the smallest prime divisor of n, by convention $p(1) = 1$, let $\omega(n)$ denote the number of distinct prime divisors of n.

We list some results on $\eta(G)$ which will be used frequently in the sequel.

Lemma 2.1. *(*[EEGKR07, GHST07]*) Let m, n be positive integers. Then*

(1) $\eta(C_m \oplus C_n) = 2m + n - 2$ for $m \mid n$,
(2) $\eta(C_n^3) = 8n - 7$ for $n = 3^\alpha 5^\beta$, where $\alpha, \beta \in \mathbb{N}_0$.

Lemma 2.2. *(*[E04]*) If n is an odd integer, then $\eta(C_n^3) \ge 8n - 7$.*

Lemma 2.3. *([GH06, Proposition 5.7.11]) Let G be a finite abelian group, and let H be a subgroup of G with $\exp(G) = \exp(H)\exp(G/H)$. Then*

$$\eta(G) \leq \exp(G/H)(\eta(H) - 1) + \eta(G/H).$$

Lemma 2.4. *Let m, n be odd integers. Suppose that $\eta(C_m^3) = 8m - 7$ and $\eta(C_n^3) = 8n - 7$, then $\eta(C_{mn}^3) = 8mn - 7$.*

Proof. By Lemma 2.2 we have $\eta(C_{mn}^3) \geq 8mn - 7$. Let $G \cong C_{mn}^3$ and $H \cong C_m^3$ be a subgroup of G, then $G/H \cong C_n^3$. It follows from Lemma 2.3 that

$$\eta(G) \leq \exp(G/H)(\eta(H) - 1) + \eta(G/H) = 8mn - 7.$$

Therefore, $\eta(C_{mn}^3) = 8mn - 7$. □

Lemma 2.5. *([S12, Corollary 3.2]) Let $H \cong C_m \oplus C_{mn}$ with integers $m \geq 2$ and $n \geq 1$. Every sequence S over H of length $|S| = \eta(H) - 1$ having not any short zero-sum subsequence has the following form*

$$S = b_1^{m-1} b_2^{sm-1} (-xb_1 + b_2)^{(n+1-s)m-1},$$

where $\{b_1, b_2\}$ is a generating set of H with $\mathrm{ord}(b_2) = mn$, $s \in [1, n]$, $x \in [1, m]$ with $\gcd(x, m) = 1$ and either

(1) $\{b_1, b_2\}$ is an independent generating set of H, or
(2) $s = n$ and $x = 1$.

Lemma 2.6. *([GH06, Theorem 5.4.5]) Let $n > 1$ be a positive integer, and let $S \in \mathcal{F}(C_n)$ be a sequence of length $n - 1$. If S is zero-sum free then $S = g^{n-1}$ for some generating element $g \in C_n$.*

Lemma 2.7. *([FGPWZ13, Lemma 2.3]) Let $n > 1$ be a positive integer, and let $S \in \mathcal{F}(C_n)$ be a sequence of length $2n - 1$. If S has no two disjoint non-empty zero-sum subsequences then $S = g^{2n-1}$ for some generating element $g \in C_n$.*

3. Proof of main results

In this section we shall prove Theorem 1.4 and Theorem 1.5, and we begin with some preliminary results.

Lemma 3.1. *Let G be a finite abelian group and H be a subgroup of G. Let S be a sequence over G. Suppose that SS_H^{-1} has a factorization $SS_H^{-1} = S_1 S_2 \cdot \ldots \cdot S_k S'$ such that $\sigma(S_i) \in H$ and $\mathsf{k}(S_i) \leq \mathsf{k}(\sigma(S_i))$ for every $i \in [1, k]$. If $k + |S_H| \geq \mathsf{t}(H)$, then S has a tiny zero-sum subsequence.*

Proof. By the hypothesis of this lemma, $\sigma(S_1)\sigma(S_2)\cdot\ldots\cdot\sigma(S_k)S_H$ is a sequence over H of length $k+|S_H|\geq \mathsf{t}(H)$. Therefore, it has a tiny zero-sum subsequence $T\prod_{i\in I}\sigma(S_i)$, where $T\mid S_H$ and $I\subset[1,k]$. Let $W=T\prod_{i\in I}S_i$. Then W is a zero-sum subsequence of S with $\mathsf{k}(W)=\mathsf{k}(T)+\sum_{i\in I}\mathsf{k}(S_i)\leq \mathsf{k}(T)+\sum_{i\in I}\mathsf{k}(\sigma(S_i))=\mathsf{k}(T\prod_{i\in I}\sigma(S_i))\leq 1$. □

Lemma 3.2. *Let G be a finite abelian group and H be a subgroup of G. Let S be a sequence over G. Suppose that SS_H^{-1} has a subsequence L such that for every $T\mid L$ with $|T|\leq \exp(G/H)$ we have $\mathsf{k}(T)\leq \frac{1}{\exp(H)}$. If $|S_H|+\lceil\frac{|L|-(\eta(G/H)-1)}{\exp(G/H)}\rceil\geq \mathsf{t}(H)$, then S has a tiny zero-sum subsequence.*

Proof. Let ϕ be the projection from G onto G/H with $\ker(\phi)=H$. By applying $\eta(\phi(G))=\eta(G/H)$ repeatedly on the sequence $\phi(L)$, we can get a factorization $L=S_1\cdot\ldots\cdot S_kS'$ such that $\phi(S_i)$ is a short zero-sum sequence over $\phi(G)=G/H$ for every $i\in[1,k]$, and such that $\phi(S')$ has no short zero-sum subsequence over $\phi(G)=G/H$. It follows that

$$|S'|=|\phi(S')|\leq \eta(G/H)-1.$$

Therefore,

$$k\geq\lceil\frac{|L|-(\eta(G/H)-1)}{\exp(G/H)}\rceil.$$

By the hypothesis, $\mathsf{k}(S_i)\leq\frac{1}{\exp(H)}\leq\frac{1}{\mathrm{ord}(\sigma(S_i))}=\mathsf{k}(\sigma(S_i))$ for every $i\in[1,k]$. Now the result follows from Lemma 3.1 since $k+|S_H|\geq\lceil\frac{|L|-(\eta(G/H)-1)}{\exp(G/H)}\rceil+|S_H|\geq\mathsf{t}(H)$. □

Proposition 3.3. *Let c,n,r be three positive integers such that for every positive divisor $m(>1)$ of n, we have $\eta(C_m^r)=c(m-1)+1$. If $\sum_{p\mid n}\frac{1}{p}<1$, where p runs over all distinct prime divisors of n, then*

$$\mathsf{t}(C_n^r)=\eta(C_n^r).$$

Proof. Let $G\cong C_n^r$. Let $p_1,\ldots,p_s$ be the all distinct prime divisors of n. By the hypothesis of this proposition,

$$\sum_{i=1}^{s}\frac{1}{p_i}<1.$$

For every positive integer $m\mid n$, let $G_m=\{x\in G\mid mx=0\}$. Clearly, G_m is a subgroup of G with $G_m\cong C_m^r$.

Let $d(n)$ denote the number of positive divisors (>1) of n. We proceed by induction on $d(n)$. If $d(n)=1$ then n is a prime, therefore $\mathsf{t}(G)=\eta(G)$ follows from Theorem 1.3(6) and we are done. Suppose that the proposition is true for $d(n)<k$ $(k\geq 2)$ and then we want to prove it is true also for $d(n)=k$.

As mentioned in the introduction we always have $\mathsf{t}(G) \ge \eta(G)$. So, it suffices to prove that

$$\mathsf{t}(G) \le \eta(G) = c(n-1)+1.$$

Let S be a sequence of length $|S| = c(n-1)+1$ over G. We want to show that S has a tiny zero-sum subsequence. If $0 \mid S$, then $S' = 0$ has the required property and we are done. Next we suppose that $0 \nmid S$. Assume to the contrary that S has no tiny zero-sum subsequence. Let

$$S = TW$$

such that $\mathrm{ord}(g) = n$ for all $g \in \mathrm{supp}(T)$, and $\mathrm{ord}(h) < n$ for all $h \in \mathrm{supp}(W)$. If $S = T$, then it is easy to see that S has a tiny zero-sum subsequence, a contradiction. Next we assume that T is a proper subsequence of S. For every $i \in [1, s]$, let W_i be the subsequence of W consisting of all terms of W in $G_{\frac{n}{p_i}}$. Then,

$$|W_1| + \ldots + |W_s| \ge |W|.$$

Since for every $T' \mid T$ with $|T'| \le \exp(G/G_{\frac{n}{p_i}})$ we have $\mathsf{k}(T') \le \frac{\exp(G/G_{\frac{n}{p_i}})}{\exp(G)} = \frac{1}{\exp(G_{\frac{n}{p_i}})}$, by Lemma 3.2 we obtain that

$$|W_i| + \lceil \frac{|T| - (\eta(G/G_{\frac{n}{p_i}}) - 1)}{\exp(G/G_{\frac{n}{p_i}})} \rceil \le \mathsf{t}(G_{\frac{n}{p_i}}) - 1.$$

Therefore, by induction we have

$$\frac{|T| - c(p_i - 1)}{p_i} + |W_i| \le c(\frac{n}{p_i} - 1)$$

for every $i \in [1, s]$, or equivalently,

$$\frac{|T|}{p_i} + |W_i| \le \frac{c(n-1)}{p_i}.$$

So,

$$|T| \sum_{i=1}^{s} \frac{1}{p_i} + |W_1| + \ldots + |W_s| \le c(n-1) \sum_{i=1}^{s} \frac{1}{p_i},$$

it follows that $|W_1| + \ldots + |W_s| \le (cn - c - |T|) \sum_{i=1}^{s} \frac{1}{p_i}$. Since $|W_1| + \ldots + |W_s| \ge |W|$, we deduce that

$$c(n-1)+1-|T| = |S|-|T| = |W| \le |W_1|+\ldots+|W_s| \le (cn-c-|T|) \sum_{i=1}^{s} \frac{1}{p_i}.$$

So we have $1 \le (cn - c - |T|)(\sum_{i=1}^{s} \frac{1}{p_i} - 1)$. It follows from $\sum_{i=1}^{s} \frac{1}{p_i} < 1$ and $|T| \le |S| - 1 = c(n-1)$ that

$$1 \le (cn - c - |T|)(\sum_{i=1}^{s} \frac{1}{p_i} - 1) \le 0,$$

a contradiction. □

Proof of Theorem 1.4. Since $\eta(C_m \oplus C_m) = 3m-2 = 3(m-1)+1$ for every positive integer m, the first part of this theorem follows from Proposition 3.3. If $p(n) \ge \omega(n)$, we clearly have

$$\sum_{p|n} \frac{1}{p} \le \frac{\omega(n)}{p(n)}$$

with equality holding if and only if $\omega(n) = 1$. Therefore, we have $\sum_{p|n} \frac{1}{p} < 1$ and the result follows from the first part of this theorem. □

Remark 3.4. Clearly, if $\omega(n) \le 2$ then $\sum_{p|n} \frac{1}{p} < 1$. If $\omega(n) = 3$ and $n \ne 2^\alpha 3^\beta 5^\gamma$ then we also have $\sum_{p|n} \frac{1}{p} < 1$. It would be interesting to prove $\mathsf{t}(C_n \oplus C_n) = \eta(C_n \oplus C_n)$ for $n = 2^\alpha 3^\beta 5^\gamma$.

Lemma 3.5. *Let n be a positive even integer and let $G \cong C_2 \oplus C_{2n}$. Let S be a sequence over G with $|S| = 2n+1$. If $\mathrm{ord}(x) = 2n$ for every $x \in \mathrm{supp}(S)$, then S has a tiny zero-sum subsequence.*

Proof. Let (e_1, e_2) be a basis of G. If S has a short zero-sum subsequence S', then $\mathsf{k}(S') = \frac{|S'|}{2n} \le 1$ and we are done. Next we assume that S has no short zero-sum subsequence. Since $|S| = 2n+1 = \eta(G) - 1$, then by Lemma 2.5 we have

$$S = b_1 b_2^{2s-1} (-b_1 + b_2)^{2(n+1-s)-1},$$

where $\{b_1, b_2\}$ is a generating set of G with $\mathrm{ord}(b_2) = 2n$, $s \in [1, n]$. Let $b_1 = x_1 e_1 + y_1 e_2$ and $b_2 = x_2 e_1 + y_2 e_2$, where $x_i \in [0,1]$, $y_i \in [0, 2n-1]$ for $i \in \{1,2\}$. Since $\mathrm{ord}(b_1) = \mathrm{ord}(b_2) = 2n$ and since n is assumed to be even, y_1, y_2 are odd. It follows that $-b_1 + b_2 = (-x_1 + x_2)e_1 + (-y_1 + y_2)e_2$, since $-y_1 + y_2$ is even, we have $\mathrm{ord}(-b_1 + b_2) \le n$, a contradiction. □

Lemma 3.6. *Let $G \cong C_n^3$ be a finite abelian group with $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$, where $p_1, \ldots, p_s$ are distinct odd prime numbers and $\alpha_1, \ldots, \alpha_s \in \mathbb{N}$. If $\sum_{i=1}^{s} \frac{1}{p_i} < 1$ and $\eta(C^3_{p_i^{\alpha_i}}) = 8p_i^{\alpha_i} - 7$, then*

$$\mathsf{t}(G) = \eta(G).$$

Proof. By Lemma 2.4 we have $\eta(C_n^3) = 8n - 7$. By Proposition 3.3 we have $\mathsf{t}(G) = \eta(G)$. □

Proof of Theorem 1.5. (a) Let $G \cong C_2 \oplus C_{2^\alpha}$ with $\alpha \in \mathbb{N}$ and (e_1, e_2) be a basis of G. The result follows from Theorem 1.3(3) for $\alpha \le 2$. Next we may assume that $\alpha \ge 3$.

We proceed by induction on α. Suppose that $\mathsf{t}(C_2 \oplus C_{2^l}) = \eta(C_2 \oplus C_{2^l})$ for $l \le \alpha - 1$. Next we need to prove it holds for $l = \alpha$.

As mentioned in the introduction we always have that $\mathsf{t}(G) \ge \eta(G)$. So, it suffices to prove that

$$\mathsf{t}(G) \le \eta(G) = 2^\alpha + 2.$$

Let S be a sequence of length $|S| = 2^\alpha + 2$ over G. We want to show that S has a tiny zero-sum subsequence. If $0 \mid S$, then $S' = 0$ has the required property and we are done. Next we suppose that $0 \nmid S$. Assume to the contrary that S has no tiny zero-sum subsequence.

Let us recall that we denote by $S_{(d)}$ the subsequence of S consisting of all terms of S of order d. Let H_1 be a subgroup of G isomorphic to $C_2 \oplus C_{2^{\alpha-1}}$ such that $H_2 = G/H_1$ is isomorphic to C_2. Then $S = S_{H_1} S_{(2^\alpha)}$ and

$$|S| = |S_{H_1}| + |S_{(2^\alpha)}| = 2^\alpha + 2. \tag{3.1}$$

Since for every $T \mid S_{(2^\alpha)}$ with $|T| \le \exp(G/H_1)$ we have $\mathsf{k}(T) \le \frac{\exp(G/H_1)}{\exp(G)} = \frac{1}{\exp(H_1)}$, by Lemma 3.2 we obtain that

$$|S_{H_1}| + \lceil \frac{|S_{(2^\alpha)}| - (\eta(G/H_1) - 1)}{\exp(G/H_1)} \rceil \le \mathsf{t}(H_1) - 1.$$

Therefore,

$$2|S_{H_1}| + |S_{(2^\alpha)}| \le 2^\alpha + 3.$$

Combining equality (3.1), we obtain that $|S_{H_1}| \le 1$. If $|S_{H_1}| = 0$, then $S = S_{(2^\alpha)}$. Hence S has a short zero-sum subsequence T' with $\mathsf{k}(T') \le 1$, a contradiction.

Next we assume that $|S_{H_1}| = 1$, by (3.1) we have $|S_{(2^\alpha)}| = 2^\alpha + 1$. By Lemma 3.5 we obtain that $S_{(2^\alpha)}$ has a tiny zero-sum subsequence, so S has a tiny zero-sum subsequence, a contradiction again.

(b) Let $G \cong C_2 \oplus C_{2p^\beta}$ with $\beta \in \mathbb{N}$ and p be a prime number and (e_1, e_2) be a basis of G. The results follow from Theorem 1.3 and (a) for $\beta = 1$ or $p = 2$. Next we may assume that $\beta \ge 2$ and $p \ge 3$.

We proceed by induction on β. Suppose that $\mathsf{t}(C_2 \oplus C_{2p^s}) = \eta(C_2 \oplus C_{2p^s})$ for $s \le \beta - 1$. Next we need to prove it holds for $s = \beta$.

As mentioned in the introduction we always have that $\mathsf{t}(G) \ge \eta(G)$. So, it suffices to prove that

$$\mathsf{t}(G) \le \eta(G) = 2p^\beta + 2.$$

Let S be a sequence of length $|S| = 2p^\beta + 2$ over G. We want to show that S has a tiny zero-sum subsequence. If $0 \mid S$, then $S' = 0$ has the required property and we are done. Next we suppose that $0 \nmid S$. Assume to the contrary that S has no tiny zero-sum subsequence.

Let H_1 be a subgroup of G isomorphic to C_{p^β} such that $G/H_1 \cong C_2 \oplus C_2$. Let also H_2 be a subgroup of G isomorphic to $C_2 \oplus C_{2p^{\beta-1}}$ such that $G/H_2 \cong C_p$. Let φ_1 and φ_2 be projections from G to G/H_1 and G/H_2, respectively, then $\ker(\varphi_1) = H_1 \cong C_{p^\beta}$ and $\ker(\varphi_2) = H_2 \cong C_2 \oplus C_{2p^{\beta-1}}$. Therefore,

$$S = S_{H_1} \cdot S_{(2)} \cdot S_{(2p)} \cdot \ldots \cdot S_{(2p^{\beta-1})} \cdot S_{(2p^\beta)} = S_{H_2} \cdot S_{(p^\beta)} \cdot S_{(2p^\beta)}$$

and

$$|S| = |S_{H_1}|+|S_{(2)}|+|S_{(2p)}|+\ldots+|S_{(2p^{\beta-1})}|+|S_{(2p^\beta)}| = |S_{H_2}|+|S_{(p^\beta)}|+|S_{(2p^\beta)}|. \tag{3.2}$$

Since for every $T \mid S_{(2p^\beta)}$ with $|T| \leq \exp(G/H_1)$ we have $\mathsf{k}(T) \leq \frac{\exp(G/H_1)}{\exp(G)} = \frac{1}{\exp(H_1)}$, by Lemma 3.2 we obtain that

$$|S_{H_1}| + \lceil \frac{|S_{(2p^\beta)}| - (\eta(G/H_1) - 1)}{\exp(G/H_1)} \rceil \leq \mathsf{t}(H_1) - 1.$$

Therefore,

$$2|S_{H_1}| + |S_{(2p^\beta)}| \leq 2p^\beta + 1.$$

Combining equality (3.2), we obtain that

$$|S_{H_1}| \leq 2p^\beta+1-(|S_{H_1}|+|S_{(2p^\beta)}|) = |S_{(2)}|+|S_{(2p)}|+\ldots+|S_{(2p^{\beta-1})}|-1 \leq |S_{H_2}|-1. \tag{3.3}$$

Since for every $T \mid S_{(2p^\beta)}$ with $|T| \leq \exp(G/H_2)$ we have $\mathsf{k}(T) \leq \frac{\exp(G/H_2)}{\exp(G)} = \frac{1}{\exp(H_2)}$, by Lemma 3.2 we obtain that

$$|S_{H_2}| + \lceil \frac{|S_{(2p^\beta)}| - (\eta(G/H_2) - 1)}{\exp(G/H_2)} \rceil \leq \mathsf{t}(H_2) - 1. \tag{3.4}$$

Therefore,

$$|S_{(2p^\beta)}| + p|S_{H_2}| \leq 2p^\beta + 2p - 1.$$

Combining equality (3.2) and inequality (3.3),

$$\begin{aligned} p|S_{H_2}| &\leq 2p^\beta + 2p - 1 - |S_{(2p^\beta)}| \\ &= 2p^\beta + 2p - 1 - (|S| - |S_{H_2}| - |S_{(p^\beta)}|) \\ &= 2p - 3 + |S_{H_2}| + |S_{(p^\beta)}| \\ &\leq 2p - 3 + |S_{H_2}| + |S_{H_1}| \\ &\leq 2p - 4 + 2|S_{H_2}|. \end{aligned}$$

Therefore, $|S_{H_2}| \leq 2$ and $|S_{(p^\beta)}| \leq |S_{H_1}| \leq |S_{H_2}| - 1 \leq 1$. Hence, we have the following possibilities:

$$|S_{H_2}| = 1 \text{ and } |S_{(p^\beta)}| = 0, |S_{H_2}| = 2 \text{ and } |S_{(p^\beta)}| = 1, |S_{H_2}| = 2 \text{ and } |S_{(p^\beta)}| = 0.$$

We proceed case by case.

Case 1. $|S_{H_2}| = 1$ and $|S_{(p^\beta)}| = 0$, then $|S_{(2p^\beta)}| = 2p^\beta + 1 = \mathsf{D}(G)$ and $S_{(2p^\beta)}$ is a minimal zero-sum subsequence.

It follows that we can decompose $S_{(2p^\beta)}$ into

$$S_{(2p^\beta)} = V_1 \cdot \ldots \cdot V_n$$

such that $\sigma(\varphi_2(V_i)) = 0$ and $|V_i| \le p$ for every $1 \le i \le n$, then $\sigma(V_i) \in \ker(\varphi_2) = H_2$ and $\mathsf{k}(V_i) = \frac{|V_i|}{\exp(G)} \le \frac{p}{\exp(G)} = \frac{1}{\exp(H_2)} \le \mathsf{k}(\sigma(V_i))$ for $1 \le i \le n$. So we have $n \ge \lceil \frac{|S_{(2p^\beta)}|}{p} \rceil = \lceil \frac{2p^\beta+1}{p} \rceil = 2p^{\beta-1} + 1$, then

$$n + |S_{H_2}| \ge 2p^{\beta-1} + 1 + 1 = 2p^{\beta-1} + 2 = \eta(H_2) = \mathsf{t}(H_2),$$

a contradiction with Lemma 3.1.

Case 2. $|S_{H_2}| = 2$ and $|S_{(p^\beta)}| = 1$. Recall that $|S_{H_1}| = 1$, then $|S_{(2p^\beta)}| = 2p^\beta - 1$. Let

$$S_{(2p^\beta)} = U_1 \cdot \ldots \cdot U_m U' = V_1 \cdot \ldots \cdot V_n V',$$

where $\sigma(\varphi_1(U_i)) = 0 \in G/H_1$ and $|U_i| = 2$ for $1 \le i \le m$ and $\varphi_1(U')$ has no short zero-sum subsequence over G/H_1, $\sigma(\varphi_2(V_j)) = 0 \in G/H_2$ and $|V_j| \le p$ for $1 \le j \le n$ and $\varphi_2(V')$ has no short zero-sum subsequence over G/H_2.

By Lemmas 3.1 and 3.2 we have

$$\lceil \frac{|S_{(2p^\beta)}| - (\eta(G/H_2) - 1)}{\exp(G/H_2)} \rceil + |S_{H_2}| \le n + |S_{H_2}| \le \mathsf{t}(H_2) - 1,$$

therefore $n = 2p^{\beta-1} - 1$, and every subsequence of $\varphi_2(S_{(2p^\beta)})$ of length $p-1$ is zero-sum free. Otherwise, suppose that there exists a subsequence $S'_{(2p^\beta)} \mid S_{(2p^\beta)}$ of length $|S'_{(2p^\beta)}| \le p-1$ such that $\varphi_2(S'_{(2p^\beta)})$ is zero-sum, then $|\varphi_2(S_{(2p^\beta)} S'^{-1}_{(2p^\beta)})| \ge 2p^\beta - p$, we can find at least $2p^{\beta-1} - 1$ disjoint zero-sum subsequences of length at most p of $\varphi_2(S_{(2p^\beta)} S'^{-1}_{(2p^\beta)})$ by Lemma 3.2, so we can find at least $2p^{\beta-1}$ disjoint zero-sum subsequences of length at most p of $\varphi_2(S_{(2p^\beta)})$, a contradiction with $n = 2p^{\beta-1} - 1$. Therefore,

$$\varphi_2(S_{(2p^\beta)}) = h^{2p^\beta - 1}$$

for some $h \in \varphi_2(G) = G/H_2$ by Lemma 2.7.

By Lemmas 3.1 and 3.2 we have

$$\lceil \frac{|S_{(2p^\beta)}| - (\eta(G/H_1) - 1)}{\exp(G/H_1)} \rceil + |S_{H_1}| \le m + |S_{H_1}| \le \mathsf{t}(H_1) - 1,$$

therefore $m = p^\beta - 2$.

Let $S_{(2p^\beta)} = U_1 \cdot \ldots \cdot U_{p^\beta - 2} \cdot U_0$, where $U_0 = S_{(2p^\beta)}(U_1 \cdot \ldots \cdot U_{p^\beta-2})^{-1}$. Since $\varphi_1(U_0)$ has no short zero-sum subsequence over G/H_1 and $|U_0| = 3 = \mathsf{D}(G/H_1)$, $\sigma(\varphi_1(U_0)) = 0 \in G/H_1$ and $\text{supp}(\varphi_1(U_0)) = G/H_1 \setminus \{0\} = \{h_1, h_2, h_3\}$. Since $|S_{(p^\beta)} \cdot \sigma(U_0) \cdot \sigma(U_1) \cdot \ldots \cdot \sigma(U_{p^\beta-2})| = p^\beta = \mathsf{t}(H_1)$, $S_{(p^\beta)} \cdot \sigma(U_0) \cdot \sigma(U_1) \cdot \ldots \cdot \sigma(U_{p^\beta-2})$ has a tiny zero-sum subsequence W_0. If $|W_0| \le$

$p^\beta-1$, suppose that $W_0=S'_{(p^\beta)}\Pi_{i\in I}\sigma(U_i)$, where $S'_{(p^\beta)}\mid S_{(p^\beta)}$, $I\in[0,p^\beta-2]$ and $|S'_{(p^\beta)}|+|I|\le p^\beta-1$, then $W'_0=S'_{(p^\beta)}\Pi_{i\in I}U_i$ is a tiny zero-sum subsequence of S, a contradiction. Therefore $S_{(p^\beta)}\cdot\sigma(U_0)\cdot\sigma(U_1)\cdot\ldots\cdot\sigma(U_{p^\beta-2})$ is a minimal zero-sum sequence over C_{p^β}. So we have $S_{(p^\beta)}=\sigma(U_0)=\sigma(U_1)=\ldots=\sigma(U_{p^\beta-2})$. Then

$$\varphi_1(S_{(2p^\beta)})=h_1^{1+2l_1}h_2^{1+2l_2}h_3^{1+2l_3},$$

where $l_i\in[0,p^\beta-2]$ and $l_1+l_2+l_3=p^\beta-2$.

Claim. Let $h_i\in\operatorname{supp}(\varphi_1(S_{(2p^\beta)}))$ with $\mathsf{v}_{h_i}(\varphi_1(S_{(2p^\beta)}))\ge3$ and let $g_1,g_2\in\operatorname{supp}(S_{(2p^\beta)})$. If $\varphi_1(g_1)=\varphi_1(g_2)=h_i$, then $g_1=g_2$.

Proof of the Claim. Assume to the contrary that $g_1\ne g_2$. Without loss of generality we may assume that $g_1\mid U_1$ and $g_2\mid U_0$. Let $U'_1=U_1g_1^{-1}g_2$. Thus, both $S_{(p^\beta)}\cdot\sigma(U_1)\cdot\ldots\cdot\sigma(U_{p^\beta-2})$ and $S_{(p^\beta)}\cdot\sigma(U'_1)\cdot\ldots\cdot\sigma(U_{p^\beta-2})$ are zero-sum free of length $p^\beta-1$. It follows from Lemma 2.6 that $\sigma(U_1)=\sigma(U'_1)$ and hence $g_1=g_2$, a contradiction. □

Since $\varphi_1(S_{(2p^\beta)})=h_1^{1+2l_1}h_2^{1+2l_2}h_3^{1+2l_3}$, where $l_i\in[0,p^\beta-2]$ and $l_1+l_2+l_3=p^\beta-2$. For $i\in[1,3]$, if $l_i\ge1$, then $\mathsf{v}_{h_i}(\varphi_1(S_{(2p^\beta)}))\ge3$, by the Claim we have that there exists a subsequence $g_i^{1+2l_i}\mid S_{(2p^\beta)}$ such that $\varphi_1(g_i^{1+2l_i})=h_i^{1+2l_i}$, if $l_i=0$, then there exists a subsequence $g_i\mid S_{(2p^\beta)}$ such that $\varphi_1(g_i)=h_i$.

Therefore, we have

$$S_{(2p^\beta)}=g_1^{1+2l_1}g_2^{1+2l_2}g_3^{1+2l_3},$$

where $\varphi_1(g_i)=h_i$ for $i\in[1,3]$. If there exist $i,j\in[1,3]$ and $i\ne j$ such that $l_i\ge1$ and $l_j\ge1$. Without loss of generality, we assume that $\{i,j\}=\{1,2\}$. Since $S_{(p^\beta)}=\sigma(U_0)=\sigma(U_1)=\ldots=\sigma(U_{p^\beta-2})$, we have $g_1+g_2+g_3=2g_1$ and $g_1+g_2+g_3=2g_2$, it deduces that $2g_3=0$, a contradiction. Therefore, there at least exist two zeros among l_1,l_2,l_3 and without loss of generality, we assume that $l_2=l_3=0$. Then

$$S_{(2p^\beta)}=g_1^{2p^\beta-3}g_2(-g_2+g_1),$$

and $S_{(p^\beta)}=2g_1$.

Let $h\mid S_{H_2}$, then $\operatorname{ord}(h)=2p^l$, $l\in[0,\beta-1]$. We write $h=a_1e_1+p^{\beta-l}y_1g_1$, where $a_1\in[0,1]$ and $(y_1,p)=1$. Let $U_1=\ldots=U_{p^\beta-2}=g_1^2$ and $U_{p^\beta-1}=S_{p^\beta}=2g_1$. Without loss of generality, we assume that $\varphi_1(h)=\varphi_1(g_1)$.

Let $T_1=hg_1$, $T_2=hg_2(-g_2+g_1)$, $T_3=g_1g_2(-g_2+g_1)$. Then $\sigma(T_i)\in\ker(\varphi_1)$ for $i\in[1,3]$. So, for every $i\in[1,3]$, the sequence $\sigma(U_1)\cdot\ldots\cdot\sigma(U_{p^\beta-1})\cdot\sigma(T_i)$ has a zero-sum subsequence X_i over $\ker(\varphi_1)$, i.e., there exists a subset $J_i\subset[1,p^\beta-1]$ such that $X_i=\sigma(T_i)\Pi_{j\in J_i}\sigma(U_j)$ for each

$i \in [1,3]$. Let $Y_i = T_i \Pi_{j\in J_i} U_j$ for each $i \in [1,3]$. Then Y_1, Y_2 and Y_3 are zero-sum subsequences of S. Let $t_i = |J_i|$ for $i \in [1,3]$. Then $X_1 = (2g_1)^{t_1}(h+g_1)$, $X_2 = (2g_1)^{t_2}(h+g_2+(-g_2+g_1))$, $X_3 = (2g_1)^{t_3}(g_1+g_2+(-g_2+g_1))$.

Since $\mathsf{k}(Y_i) > 1$ for every $i \in [1,3]$, we have

$$\mathsf{k}(Y_1) = \frac{1}{\mathrm{ord}(h)} + \frac{1}{\mathrm{ord}(g_1)} + \frac{2t_1}{\mathrm{ord}(g_1)} = \frac{p^{\beta-l}+2t_1+1}{2p^\beta} > 1,$$

$$\mathsf{k}(Y_2) = \frac{1}{\mathrm{ord}(h)} + \frac{1}{\mathrm{ord}(g_2)} + \frac{1}{\mathrm{ord}(-g_2+g_1)} + \frac{2t_2}{\mathrm{ord}(g_1)} = \frac{p^{\beta-l}+2t_2+2}{2p^\beta} > 1,$$

$$\mathsf{k}(Y_3) = \frac{1}{\mathrm{ord}(g_1)} + \frac{1}{\mathrm{ord}(g_2)} + \frac{1}{\mathrm{ord}(-g_2+g_1)} + \frac{2t_3}{\mathrm{ord}(g_1)} = \frac{2t_3+3}{2p^\beta} > 1.$$

Combining $t_i \le p^\beta - 1$, by a straightforward computation we obtain that

$$p^\beta - \frac{p^{\beta-l}-1}{2} \le t_1 \le p^\beta - 1, p^\beta - \frac{p^{\beta-l}+1}{2} \le t_2 \le p^\beta - 1, t_3 = p^\beta - 1.$$

From X_i is zero-sum over $\ker(\varphi_1)$ we infer that

$$2t_1g_1+h+g_1 = 2t_2g_1+h+g_2+(-g_2+g_1) = 2(p^\beta-1)g_1+g_1+g_2+(-g_2+g_1) = 0.$$

Therefore,

$$2t_1g_1+h+g_1+2t_2g_1+h+g_2+(-g_2+g_1)-2(p^\beta-1)g_1-g_1-g_2-(-g_2+g_1) = 0.$$

This deduces that $(2t_1+2t_2+2)g_1 + 2h = 0$. Therefore $(2t_1+2t_2+2)g_1 + 2p^{\beta-l}yg_1 = 0$, then $(t_1+t_2+1+p^{\beta-l}y) \equiv 0 \pmod{p^\beta}$, but $2p^\beta - p^{\beta-l} + 1 + p^{\beta-l}y \le t_1+t_2+1+p^{\beta-l}y \le 2p^\beta - 1 + p^{\beta-l}y$, a contradiction.

Case 3. $|S_{H_2}| = 2$ and $|S_{(p^\beta)}| = 0$, then $|S_{(2p^\beta)}| = 2p^\beta$. Therefore,

$$|S_{H_2}| + \lceil \frac{|S_{(2p^\beta)}| - (\eta(G/H_2)-1)}{\exp(G/H_2)} \rceil = 2 + 2p^{\beta-1} = \eta(H_2),$$

a contradiction with inequality (3.4).

(c) The result follows from Lemma 2.1 and Lemma 3.6. □

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Center for Combinatorics, LPMC-TJKLC Nankai University, Tianjin 300071, P.R. China
E-mail address: wdgao@nankai.edu.cn, weidong.gao@tju.edu.cn

Center for Combinatorics, LPMC-TJKLC Nankai University, Tianjin 300071, P.R. China
E-mail address: huiwanzhen@163.com

College of Science, Tianjin University of Commerce, Tianjin 300134, P.R. China
E-mail address: lixue931006@163.com

School of Mathematics and Statistics, Yangtze Normal University, Chongqing 408100, P.R. China
E-mail address: qincn328@sina.com

Center for Combinatorics, LPMC-TJKLC Nankai University, Tianjin 300071, P.R. China
E-mail address: yinqiuyu26@126.com