Algebra Colloquium **28** : 4 (2021) 645–654
DOI: 10.1142/S100538672100050X

Algebra Colloquium

Arc-Transitive Graphs of Square-free Order with Valency 11 *

Guang Li
School of Mathematics and Statistics, Shandong University of Technology
Zibo, Shandong 255049, China
E-mail: lig@sdut.edu.cn

Bo Ling
School of Mathematics and Computer Science, Yunnan Minzu University
Kunming 650000, China
E-mail: bolinggxu@163.com

Zaiping Lu
Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, China
E-mail: lu@nankai.edu.cn

Received 24 September 2019
Revised 15 July 2020

Communicated by Genghua Fan

Abstract. In this paper, we present a complete list of connected arc-transitive graphs of square-free order with valency 11. The list includes the complete bipartite graph $\mathsf{K}_{11,11}$, the normal Cayley graphs of dihedral groups and the graphs associated with the simple group J_1 and $\mathrm{PSL}(2,p)$, where p is a prime.

2010 Mathematics Subject Classification: 05C25, 20B25

Keywords: arc-transitive graph, coset graph, normal quotient, vertex-stabilizer, (almost) simple group

1 Introduction

All graphs considered in this paper are assumed to be finite and simple. Any unexplained notation and terminology for graphs and permutation groups is as in [2] and [10].

Let $\Gamma = (V, E)$ be a connected graph with vertex set V and edge set E. The number of vertices $|V|$ is called the *order* of Γ. Let $\mathrm{Aut}\,\Gamma$ be the automorphism group of Γ, and G be a subgroup of $\mathrm{Aut}\,\Gamma$, written as $G \leq \mathrm{Aut}\,\Gamma$. The graph Γ is said to be *G-vertex-transitive* (resp., *G-edge-transitive*) if G acts transitively on V (resp., E). Recall that an *arc* in Γ is an ordered pair of adjacent vertices. Then Γ is

*Supported by the National Natural Science Foundation of China (11731002, 11701503) and the Fundamental Research Funds for the Central Universities.

said to be *G-arc-transitive* if G acts transitively on the set of all arcs in Γ. (Note that the arc-transitivity here yields the vertex-transitivity and edge-transitivity.) For $u \in V$, we denote the stabilizer of u in G and the neighborhood of u in Γ by G_u and $\Gamma(u)$, respectively; that is, $G_u = \{g \in G \mid u^g = u\}$, $\Gamma(u) = \{v \in V \mid \{u,v\} \in E\}$. Then G_u induces a permutation group $G_u^{\Gamma(u)}$ (on $\Gamma(u)$). We call Γ a *G-locally primitive* graph if $G_u^{\Gamma(u)}$ is a primitive group for any vertex $u \in V$. It is well known that Γ is G-edge-transitive if it is G-locally primitive, and Γ is G-arc-transitive if it is both G-vertex-transitive and G-locally primitive.

This paper is one of a series of papers devoted to characterizing edge-transitive graphs of square-free order. The classisfication of edge-transitive graphs of square-free order has received considerable attention in the literature. A lot of interesting results have been given, especially for those of order being a prime or a product of two primes (see [1, 7, 23–27] for example). In [18], Li et al. gave a reduction theorem for locally primitive arc-transitive graphs of square-free order. Let Γ be a connected locally primitive arc-transitive graph of square-free order. It was proved that, besides the complete bipartite graphs, either $\mathrm{Aut}\,\Gamma$ is soluble, or Γ is a cover of some 'basic' graph arising from $\mathrm{PSL}(2,p)$, $\mathrm{PGL}(2,p)$ or a finite number (depending only on the valency of Γ) of other almost simple groups. Such a reduction makes it possible to classify arc-transitive graphs of square-free order and special valencies. For example, the reader may find some classification results on graphs of valency less than 8 and valency 10 in [17, 19–21]. In this paper, we deal with such graphs of valency 11. The main result is stated as follows.

Theorem 1.1. *Let $\Gamma = (V,E)$ be a connected graph of square-free order with valency 11, and $G \le \mathrm{Aut}\,\Gamma$. Assume that G acts transitively on the arc set of Γ. Then either $\Gamma \cong \mathsf{K}_{11,11}$ or one of the following statements holds:*

(1) $G = \mathrm{D}_{2n} : \mathrm{Z}_{11}$ *and Γ is isomorphic to a graph $\mathrm{D}_n(r,s)$ given in Example* 2.4;
(2) $G = \mathrm{J}_1$ *and Γ is isomorphic to the graph in Example* 2.4;
(3) $G = \mathrm{PSL}(2,p)$ *or* $\mathrm{PGL}(2,p)$ *for prime p with $p \equiv \pm 1 \pmod{11}$, and for an edge $\{u,v\} \in E$ the triple $(G_u, G_{uv}, \mathbf{N}_G(G_{uv}))$ is listed in Table* 1, *where $G_{uv} = G_u \cap G_v$ and $\mathbf{N}_G(G_{uv})$ is the normalizer of G_{uv} in G.*

Table 1 The $\mathrm{PSL}(2,p)$-graphs of valency 11

G	G_u	G_{uv}	$\mathbf{N}_G(G_{uv})$	Remark	Bipartite?
$\mathrm{PSL}(2,p)$	D_{22}	$\mathbb{Z}_2$	$\mathrm{D}_{p\pm1}$	$p \equiv \pm 3 \pmod 8$	No
$\mathrm{PGL}(2,p)$	D_{44}	$\mathbb{Z}_2^2$	S_4	$p \equiv \pm 3 \pmod 8$	No
$\mathrm{PSL}(2,p)$	D_{44}	$\mathbb{Z}_2^2$	S_4	$p \equiv \pm 7 \pmod{16}$	No
$\mathrm{PGL}(2,p)$	D_{44}	$\mathbb{Z}_2^2$	S_4	$p \equiv \pm 3 \pmod 8$	Yes

2 Preliminaries

Let V be a nonempty set and G be a transitive permutation group on V. For a subset B of V, denote by G_B the set-wise stabilizer of B in G, and by G_B^B the permutation group induced by G_B on B. For a G-invariant partition $\mathcal{B}$ of V, denote by $G^{\mathcal{B}}$ the permutation group induced by G on $\mathcal{B}$.

• **Normal quotient.** Let $\Gamma = (V,E)$ be a connected G-arc-transitive graph, $G \le \mathrm{Aut}\,\Gamma$ and N be a normal subgroup of G, written as $N \trianglelefteq G$.

Assume that N has at least three orbits on V, and let $\mathcal{B}$ be the set of N-orbits. Then $\mathcal{B}$ is a G-invariant partition of V. The *normal quotient*, denoted by Γ_N, is defined on $\mathcal{B}$ such that $B_1, B_2 \in \mathcal{B}$ are adjacent if and only if there are some $u \in B_1$ and $v \in B_2$ adjacent in Γ. The graph Γ is called a *normal cover* of Γ_N if, for every edge $\{B_1, B_2\}$ of Γ_N, the induced subgraph of Γ by $B_1 \cup B_2$ is a matching. The next lemma collect some well-known facts about normal quotient and normal cover of arc-transitive graphs; refer to [20, Lemma 2.6].

Lemma 2.1. *Let $\Gamma = (V, E)$ be a connected G-arc-transitive graph, and $N \trianglelefteq G$. Assume that N has at least three orbits on V, and let $\mathcal{B}$ be the set of N-orbits. Then Γ_N is a $G^{\mathcal{B}}$-arc-transitive graph. Moreover, the following statements hold:*

(1) *If Γ is a normal cover of Γ_N, then Γ and Γ_N have the same valency, N is semiregular on V and N itself is the kernel of G acting on $\mathcal{B}$; in particular, $G^{\mathcal{B}} \cong G/N$ and $|\mathcal{B}| = |V|/|N|$.*

(2) *If further Γ is G-locally primitive, then Γ is a normal cover of Γ_N.*

For the case where N is not semiregular on V, we have the following lemma (see [20, Lemma 2.5] for example).

Lemma 2.2. *Let $\Gamma = (V, E)$ be a connected G-locally primitive arc-transitive graph, and $N \trianglelefteq G \leq \mathrm{Aut}\,\Gamma$. If N is not semiregular on V, then for $u \in V$ the stabilizer N_u is transitive on $\Gamma(u)$; in particular, Γ is N-edge-transitive and N has at most two orbits on V.*

• **Coset graph and Cayley graph.** Let G be a finite group and H a core-free subgroup of G. For a 2-element $x \in G \backslash H$ with $x^2 \in H$, the *coset graph* $\mathrm{Cos}(G, H, x)$ is defined on the set $[G : H]$ such that $\{Hg_1, Hg_2\}$ is an edge if and only if $g_2 g_1^{-1} \in HxH$. Note that $\mathrm{Cos}(G, H, x)$ is a well-defined (undirected) graph. View G as a subgroup of $\mathrm{Aut}\,\mathrm{Cos}(G, H, x)$, where G acts on $[G : H]$ by the right multiplication. Then we have the following basic fact.

Lemma 2.3. *Let $\Gamma = \mathrm{Cos}(G, H, x)$ be defined as above. Then Γ is a G-arc-transitive graph with valency $|H : (H \cap H^x)|$, and Γ is connected if and only if $G = \langle x, H \rangle$.*

Example 2.4. By the Atlas [9], all the subgroups isomorphic to $\mathrm{PSL}(2, 11)$ of the first Janko group J_1 are maximal and conjugate. Let $\mathrm{PSL}(2, 11) \cong H < \mathrm{J}_1$. Then H has exactly two conjugation classes of subgroups isomorphic A_5. Checking by GAP [11], we find that one of these classes consists of the subgroups having normalizer isomorphic to $\mathbb{Z}_2 \times \mathrm{A}_5$ in J_1, while the other contains only self-normalized subgroups. Take a subgroup K of H with $K \cong \mathrm{A}_5$ and $\mathbf{N}_{\mathrm{J}_1}(K) \cong \mathbb{Z}_2 \times \mathrm{A}_5$. Let o be the involution in the center of $\mathbf{N}_{\mathrm{J}_1}(K)$. Then $o \notin H$, and so $\langle H, o \rangle = G$. Thus, $\mathrm{Cos}(\mathrm{J}_1, H, o)$ is a connected arc-transitive graph of order $2 \cdot 7 \cdot 19$ with valency 11. By a direct computation by MAGMA [3], it is easy to see that $\mathrm{Cos}(\mathrm{J}_1, H, o)$ is the unique connected arc-transitive graph of square-free order with valency 11.

Now let $\Gamma = (V, E)$ be a G-arc-transitive graph, $G \leq \mathrm{Aut}\,\Gamma$ and $\{u, v\} \in E$. Then there exists $x \in G$ with $(u, v)^x = (v, u)$. Replacing x by its odd power, we may choose x as a 2-element. Clearly, $x \notin G_u$. By $(u, v)^x = (v, u)$, we have

$G_{uv}^x = G_u^x \cap G_v^x = G_v \cap G_u = G_{uv}$, so $x \in \mathbf{N}_G(G_{uv})$, and $|\mathbf{N}_G(G_{uv}) : G_{uv}|$ is even. Since G is transitive on V, we have a bijection $V \to [G : G_u]$ given by $u^g \mapsto G_u g$, which is in fact an isomorphism from Γ to $\mathrm{Cos}(G, G_u, x)$.

Lemma 2.5. *Let $\Gamma = (V, E)$ be a graph, $E \neq \emptyset$, and $G \leq \mathrm{Aut}\,\Gamma$. If Γ is G-arc-transitive, for $\{u, v\} \in E$ there is a 2-element $x \in \mathbf{N}_G(G_{uv})$ such that $(u, v)^x = (v, u)$ and $\Gamma \cong \mathrm{Cos}(G, G_u, x)$; moreover, Γ is connected if and only if $\langle x, G_u\rangle = G$.*

Let R be a finite group, and $1 \notin S = S^{-1} \subset R$. The *Cayley graph* $\mathrm{Cay}(R, S)$ is defined on R such that $x, y \in R$ are adjacent if and only if $yx^{-1} \in S$. The underlying group R may be viewed as a regular subgroup of $\mathrm{Aut}\,\mathrm{Cay}(R, S)$, where R acts on the vertices by the right multiplication. The following result is well known.

Lemma 2.6. *Let $\Gamma = (V, E)$ be a G-vertex-transitive graph and $G \leq \mathrm{Aut}\,\Gamma$. If G contains a regular subgroup R, then $\Gamma \cong \mathrm{Cay}(R, S)$ for some S with $1 \notin S = S^{-1} \subset R$; in this case, Γ is connected if and only if $G = \langle S\rangle$.*

Using the Cayley graph, one can easily construct arc-transitive graphs. For example, taking a finite group R, a subgroup $H \leq \mathrm{Aut}(R)$ and some involution $x \in R$, we get a G-arc-transitive graph $\mathrm{Cay}(R, \{x^\sigma \mid \sigma \in H\})$, where $G = R{:}H$ is the semidirect product of R and H.

Example 2.7. Consider the dihedral group $\mathrm{D}_{2n} = \langle a, b \mid a^n = 1 = b^2,\ bab = a^{n-1}\rangle$, where $n = p_1 p_2 \cdots p_l$ for distinct primes p_i no less than 13. Suppose that there are $0 \leq s < n$ and $2 \leq r < n$ with $(n, r + s - 1) = 1$ and $1 + r + \cdots + r^{10} \equiv 0 \pmod n$. Set $S_{n,r,s} = \{(ab)^{\sigma^i} \mid 0 \leq i \leq 10\} = \{ab, a^{(1+r+\cdots+r^{i-1})s+r^i} b \mid 1 \leq i \leq 10\}$ and $\mathrm{D}_n(r, s) = \mathrm{Cay}(\mathrm{D}_{2n}, S_{n,r,s})$. Note that $\langle S_{n,r,s}\rangle \geq \langle ab, a^{r+s}b\rangle = \langle ab, a^{r+s}bab\rangle = \langle ab, a^{r+s-1}\rangle = \mathrm{D}_{2n}$. Then $\mathrm{D}_n(r, s)$ is connected. By [16, Lemma 2.2], there is $\sigma \in \mathrm{Aut}(\mathrm{D}_{2n})$ such that $a^\sigma = a^r$ and $b^\sigma = a^s b$. It is easily shown that σ has order 11 and $S_{n,r,s} = \{(ab)^{\sigma^i} \mid 0 \leq i \leq 10\}$. Thus, $\mathrm{D}_{2n} : \langle\sigma\rangle$ acts transitively on the arc set of $\mathrm{D}_n(r, s)$. For example, taking $n = 2047$, $r = 2$ and $s = 1$, we get an arc-transitive graph $\mathrm{D}_{2047}(2, 1)$ of order 4094 with valency 11.

Remark. The reader can find out the enumeration of these Cayley graphs from [22, Theorem 1.2].

3 The Vertex-Stabilizers

In this section, we assume that $\Gamma = (V, E)$ is a connected G-arc-transitive graph with valency 11, where $G \leq \mathrm{Aut}\,\Gamma$. Let $u \in V$. Then $G_u^{\Gamma(u)}$ is a transitive permutation group of degree 11. By [4], $G_u^{\Gamma(u)}$ is either soluble or 2-transitive. Then $G_u^{\Gamma(u)}$ is known by checking the list of 2-transitive permutation groups; refer to [5, Theorem 5.3].

Lemma 3.1. *Let X be a transitive permutation group on a set Ω with $|\Omega| = 11$, and let $\alpha \in \Omega$. Then, up to isomorphism, X and X_α are listed as follows:*

X	$\mathrm{PSL}(2, 11)$	M_{11}	A_{11}	S_{11}	$\mathbb{Z}_{11}{:}\mathbb{Z}_{10}$	$\mathbb{Z}_{11}{:}\mathbb{Z}_l$
X_α	A_5	M_{10}	A_{10}	S_{10}	$\mathbb{Z}_{10}$	$\mathbb{Z}_l$
Remark	2-trans.	2-trans.	2-trans.	2-trans.	2-trans.	$l \in \{1, 2, 5\}$

Let $G_u^{[1]}$ be the kernel of G_u acting on $\Gamma(u)$. Then $G_u^{\Gamma(u)} \cong G_u/G_u^{[1]}$. Take $v \in \Gamma(u)$ and set $G_{uv}^{[1]} := G_u^{[1]} \cap G_v^{[1]}$. By [12, 2.3], $G_{uv}^{[1]}$ is an r-group for some prime r. Then the next result follows from Lemma 3.1 and [28].

Lemma 3.2. *One of the following statements holds:*

(1) $G_u \cong \mathrm{PSL}(2,11)$ *or* $\mathrm{A}_5 \times \mathrm{PSL}(2,11)$, *and* $|G_u| = 2^2 \cdot 3 \cdot 5 \cdot 11$ *or* $2^4 \cdot 3^2 \cdot 5^2 \cdot 11$, *respectively*;

(2) $G_u \cong \mathrm{M}_{11}$, $\mathrm{A}_6 \times \mathrm{M}_{11}$ *or* $\mathrm{M}_{10} \times \mathrm{M}_{11}$, *and* $|G_u| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$, $2^7 \cdot 3^4 \cdot 5^2 \cdot 11$ *or* $2^8 \cdot 3^4 \cdot 5^2 \cdot 11$, *respectively*;

(3) $G_u \cong \mathrm{A}_{11}$, $\mathrm{A}_{10}\times\mathrm{A}_{11}$, S_{11}, $(\mathrm{A}_{10}\times\mathrm{A}_{11}).\mathbb{Z}_2$ *or* $\mathrm{S}_{10}\times\mathrm{S}_{11}$, *and* $|G_u| = 2^7{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}11$, $2^{14} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11$, $2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$, $2^{15} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11$ *or* $2^{16} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11$, *respectively*;

(4) $G_u \cong (\mathbb{Z}_{l'} \times \mathbb{Z}_{11}).\mathbb{Z}_l$, $|G_u| = 11l'l$, *where* $l \in \{1,2,5,10\}$ *and* l' *is a divisor of* l;

(5) $G_u \cong (G_u^{[1]}{:}\mathbb{Z}_{11}).\mathbb{Z}_r$, $|G_u^{[1]}| = 11r^{k+1}$ *and* $|G_u| = 11r^{k+2}$, *where* $r \in \{2,5\}$ *and* $k \geq 1$.

Proof. Let $\{u,v\} \in E$. Consider the action of $G_u^{[1]}$ on $\Gamma(v)$, and let $(G_u^{[1]})^{\Gamma(v)}$ be the resulting permutation group. Then $(G_u^{[1]})^{\Gamma(v)} \cong G_u^{[1]}G_v^{[1]}/G_v^{[1]} \cong G_u^{[1]}/G_{uv}^{[1]}$. Since Γ is G-arc-transitive, there exists some $x \in G$ interchanging u and v. Thus x interchanges $G_u^{[1]}$ and $G_v^{[1]}$ by conjugation, and hence $G_u^{[1]}/G_{uv}^{[1]} \cong G_v^{[1]}/G_{uv}^{[1]}$. Therefore, $(G_u^{[1]})^{\Gamma(v)} \cong G_v^{[1]}/G_{uv}^{[1]} \cong (G_v^{[1]})^{\Gamma(u)}$, and we may write

$$G_u \cong G_{uv}^{[1]}.(G_v^{[1]})^{\Gamma(u)}.G_u^{\Gamma(u)}.$$

Moreover, since $G_v^{[1]} \trianglelefteq G_{uv}$, we have $(G_v^{[1]})^{\Gamma(u)} \trianglelefteq G_{uv}^{\Gamma(u)} = (G_u^{\Gamma(u)})_v$.

Assume first that $G_u^{\Gamma(u)}$ is 2-transitive. By Lemma 3.1 and [28], we have $G_{uv}^{[1]} = 1$. Then $G_u \cong (G_v^{[1]})^{\Gamma(u)}.G_u^{\Gamma(u)}$. Check $G_u^{\Gamma(u)}$ and $(G_u^{\Gamma(u)})_v$ (see Lemma 3.1). Recall that $(G_v^{[1]})^{\Gamma(u)} \trianglelefteq (G_u^{\Gamma(u)})_v$, and then one of (1)–(4) of this lemma follows.

Assume that $G_u^{\Gamma(u)}$ is not 2-transitive. Then $G_u^{\Gamma(u)} \cong \mathbb{Z}_{11} : \mathbb{Z}_l$ with $l \in \{1,2,5\}$. If $G_{uv}^{[1]} = 1$, then a similar argument to that above yields (4). Thus, we let $G_{uv}^{[1]}$ be a nontrivial r-group for some prime r. It is easily shown that r is a divisor of l; see [8, Lemma 1.1] for example. Then $l = r$. Suppose that $G_u^{[1]} = G_v^{[1]}$. Then $G_{uv}^{[1]} = G_u^{[1]} \trianglelefteq \langle G_u, x\rangle = G$ by Lemma 2.5, where $x \in G$ with $(u,v)^x = (v,u)$. This implies that $G_{uv}^{[1]}$ fixes every vertex of Γ, and so $G_{uv}^{[1]} = 1$, a contradiction. Thus $G_u^{[1]} \neq G_v^{[1]}$, and hence $(G_u^{[1]})^{\Gamma(v)} \cong G_u^{[1]}/G_{uv}^{[1]} \neq 1$. Then $(G_u^{[1]})^{\Gamma(v)} \cong \mathbb{Z}_r$, $G_u^{[1]} = G_{uv}^{[1]}.\mathbb{Z}_r$ is an r-group of order divisible by r^2, and so item (5) follows. □

4 Graphs Arising from Almost Simple Groups

Let $\Gamma = (V,E)$ be a connected G-arc-transitive graph of square-free order with valency 11, where $G \leq \mathrm{Aut}\,\Gamma$. In what follows, we assume that G is an almost simple group with socle $\mathrm{soc}(G) = T$.

By Lemma 2.2, Γ is T-edge-transitive, and T has at most two orbits on V. Take a T-orbit U and let $u \in U$. Then $|T : T_u| = |U| = |V|$ or $|V|/2$; in particular, $|U|$ is square-free. Since T_u is transitive on $\Gamma(u)$, the order of T_u is divisible by 11.

Lemma 4.1. $\Gamma \ncong \mathsf{K}_{11,11}$.

Proof. Suppose that $\Gamma \cong \mathsf{K}_{11,11}$. Then T has two orbits on V, and hence we obtain $11 = |U| = |T : T_u|$. By Lemma 2.2, Γ is T-edge-transitive, and then T_u is transitive on $\Gamma(u)$. In particular, $|T_u|$ is divisible by 11. It follows that $|T|$ is divisible by 11^2. Let K be the kernel of the action of T on U. It is easily shown that K fixes every vertex of $\mathsf{K}_{11,11}$, and we have $K = 1$. Then T is a transitive permutation group of degree 11 (on U). By Lemma 3.1, $T \cong \mathrm{PSL}(2, 11)$, M_{11} or A_{11}. Thus, $|T|$ is not divisible by 11^2, a contradiction. □

By Lemma 4.1 and [18, Theorem 4], T is isomorphic to one of the following simple groups:

(1) $\mathrm{M}_{11}, \mathrm{M}_{12}, \mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{M}_{24}$, J_1, A_n with $11 \le n < 33$, $\mathrm{PSL}(2, p)$, $\mathrm{PSL}(2, 11^2)$;

(2) $\mathrm{PSL}(d, p^f)$ with $d \ge 3$, $\mathrm{PSU}(d, p^f)$ with $d \ge 3$, $\mathrm{PSp}(d, p^f)$ with even $d \ge 4$, $\Omega(d, p^f)$ with odd $d \ge 7$, $\mathrm{P}\Omega^{\pm}(d, p^f)$ with even $d \ge 6$, where $p \le 11$;

(3) $\mathrm{G}_2(p^f)$, ${}^3\mathrm{D}_4(p^f)$, $\mathrm{F}_4(p^f)$, ${}^2\mathrm{E}_6(p^f)$ and $\mathrm{E}_7(p^f)$ with $p \le 11$;

here p is a prime. Note that $|T| = |T_u||T : T_u|$ and that $|U| = |T : T_u|$ is square-free. Since T_u is a normal subgroup of G_u, the order of T_u is a divisor of G_u. In particular, $|T_u|$ is not divisible by 11^2 or s, and so $|T|$ is not divisible by 11^3 or s^2, where s is a prime no less than 13.

Checking the orders of simple groups in (1)–(3) (refer to [15, Tables 5.1.A–C]), we conclude that T is one of the simple groups listed in the following lemma.

Lemma 4.2. *T is one of* J_1, $\mathrm{PSL}(2, 11^2)$ *and* $\mathrm{PSL}(2, p)$, *where p is a prime with* $p^2 \equiv 1 \pmod{11}$.

Proof. If $T = \mathrm{PSL}(2, p)$, then since $|T|$ is divisible by 11, either $p = 11$ or $p^2 \equiv 1 \pmod{11}$. In the following, we prove this lemma by excluding the simple groups not involved in this lemma. First, the groups $\mathrm{PSL}(2, 11)$ and M_{11} are excluded, as they have no subgroup of square-free index and of order divisible by 11. Next we lay out the argument in three cases.

Case 1. Suppose that T is a simple group listed in (1) other than M_{11}, J_1, $\mathrm{PSL}(2, p)$ and $\mathrm{PSL}(2, 11^2)$. For $T = \mathrm{M}_n$ with $n \in \{12, 22, 23, 24\}$, the order of T is divisible by $2^6 \cdot 3^2$, and hence the order of T_u is divisible by $2^5 \cdot 3$ since $|T : T_u|$ is square-free. Then the order of G_u is divisible by $2^5 \cdot 3$, and by Lemma 3.2, $|G_u|$, and hence $|G|$, is divisible by 3^4, which is impossible when $T = \mathrm{M}_n$, where $n \in \{12, 22, 23, 24\}$.

Now let $T = \mathrm{A}_n$ with $n \ge 11$. Note that $|T|$ is divisible by $2^7 \cdot 3^4$, and it follows that $|T_u|$, and hence $|G_u|$, has order divisible by $2^6 \cdot 3^3$. By Lemma 3.2, $|G_u|$, and hence $|T_u|$, is indivisible by 2^{17}, and G_u has a subgroup isomorphic to A_{11}. Then $|T|$ is indivisible by 2^{18} and T has a proper subgroup A_{11}, and thus $12 \le n \le 21$. In particular, $|T_u|$, and hence $|G_u|$, has order divisible by $2^8 \cdot 3^4$. Again by Lemma 3.2, G_u, and hence G, has a subgroup isomorphic to $\mathrm{A}_{10} \times \mathrm{A}_{11}$. It follows that $n = 21$ and $\mathrm{A}_{10} \times \mathrm{A}_{11} \le G_u \le \mathrm{S}_{10} \times \mathrm{S}_{11}$. Thus, 4 is a divisor of $|V| = |G : G_u|$, a contradiction.

Case 2. Suppose that T is listed in (2). Then $|T|$ has a divisor p^{df}, and $|T_u|$, and hence $|G_u|$, is divisible by p^{df-1}. Check the order of G_u. Since $df - 1 \ge 2$, by Lemma 3.2, $|G : G_u|$ is divisible by p^2 when $p \ge 11$, and hence we have $p < 11$.

For $p = 7$, by Lemma 3.2, 7^3 is not a divisor of $|G_u|$, and so 7^4 is not a divisor of $|T|$. This implies $f = 1$. Checking the orders of simple classical groups, we conclude that $T = \mathrm{PSL}(3,7)$ or $\mathrm{PSU}(3,7)$. Thus, $|T|$ is indivisible by 11, a contradiction.

Let $p = 5$. Then by Lemma 3.2, either 3 is not a divisor of $|G_u|$ or 5^5 is not a divisor of $|G_u|$. The latter case yields $T = \mathrm{PSL}(3,5)$, $\mathrm{PSU}(3,5)$ or $\mathrm{PSp}(4,5)$, and hence $|T|$ is indivisible by 11, a contradiction. Thus 3 is not a divisor of $|G_u|$, and $|T|$ is not divisible by 3^2. Then one of (4) and (5) in Lemma 3.2 occurs; thus, $|T_u|$ has no divisor being 2^3, and $|T|$ is indivisible by 2^4, which is impossible as $d \geq 3$.

Let $p = 3$. Recalling that $|T|$ has a divisor 3^{df}, we know that T_u has order divisible by 3^2. By Lemma 3.2, $|T_u|$ is indivisible by 3^9, and so 3^{10} is not a divisor of $|T|$. It follows that T is isomorphic to one of $\mathrm{PSL}(3,3)$, $\mathrm{PSL}(3,9)$, $\mathrm{PSL}(3,27)$, $\mathrm{PSL}(4,3)$, $\mathrm{PSU}(3,3)$, $\mathrm{PSU}(3,9)$, $\mathrm{PSU}(3,27)$, $\mathrm{PSU}(4,3)$, $\mathrm{PSp}(4,3)$, $\mathrm{PSp}(4,9)$, $\mathrm{PSp}(6,3)$. However, none of these simple groups has order divisible by 11, a contradiction.

Let $p = 2$. Suppose that $|T_u|$ is indivisible by 2^5. Checking the order of T, we get $T = \mathrm{PSL}(3,2)$, and then $|T|$ is indivisible by 11, a contradiction. Suppose that $|T_u|$ is indivisible by 3. Then one of (4) and (5) in Lemma 3.2 occurs, and so $|T_u|$ is not divisible by 2^3. Thus, $|T|$ is indivisible by 2^4, a contradiction. Accordingly, $|T_u|$ is divisible by $3 \cdot 2^5$. By Lemma 3.2, $|T_u|$ is indivisible by 2^{17}, and G_u has a subgroup isomorphic to one of $\mathrm{A}_6 \times \mathrm{M}_{11}$ and A_{11}. Noting that G/T is soluble, we see that T has a subgroup isomorphic to $\mathrm{A}_6 \times \mathrm{M}_{11}$ or A_{11}. By [15, Propositions 5.3.7, 5.3.8 and 5.5.7], we conclude that $d \geq 8$. Checking the orders of simple classical groups, we have $T = \mathrm{PSp}(8,2)$, $\mathrm{P}\Omega^+(8,2)$ or $\mathrm{P}\Omega^-(8,2)$. Then $|T|$ is indivisible by 11, a contradiction.

Case 3. Suppose that T is listed in (3). Then $|T|$ has a divisor p^{6f}, and $|T_u|$, and hence $|G_u|$, is divisible by p^{6f-1}. Check the order of G_u. Since $6f - 1 \geq 5$, by Lemma 3.2, $|G : G_u|$ is divisible by p^2 when $p \geq 7$, and then we have $p \in \{2,3,5\}$.

Assume that $p = 5$. Then 5^6 is a divisor of $|T|$, and hence $|G|$. We conclude that $|G_u|$ is divisible by 5^5. Thus, by Lemma 3.2, the only possibility of G_u is the case (5) of this lemma, and hence $|G_u| = 11 \cdot 5^{k+2}$ for $k \geq 3$. Therefore, 3 is not a divisor of $|T_u|$, and $|T|$ is not divisible by 3^2 since $|T : T_u|$ is square-free, which is impossible.

Let $p = 3$. Then by Lemma 3.2, $|G_u|$ is not divisible by 3^9, and so T is not divisible by 3^{10}. This forces $T = \mathrm{G}_2(3)$, which has order indivisible by 11, a contradiction.

Now let $p = 2$. Since $|T|$ has a divisor $(p^{2f} - 1)^2$, we know that $|T|$ is divisible by 3^2, and hence T_u has order divisible by 3. By Lemma 3.2, we conclude that 2^{17} is not a divisor of $|G_u|$. Thus, 2^{18} is not a divisor of $|T|$. Checking the order of T, we get $T = \mathrm{G}_2(4)$ or ${}^3\mathrm{D}_4(2)$; then $|T|$ has no divisor being 11, a contradiction. □

Theorem 4.3. *Let $\Gamma = (V,E)$ be a connected G-arc-transitive graph of square-free order with valency* 11. *Assume that G is almost simple with socle T. Then either*

(1) $T = \mathrm{J}_1$ *and Γ is isomorphic to the graph given in Example* 2.4, *or*

(2) $T = \mathrm{PSL}(2,p)$ *for a prime p with* $p^2 \equiv 1 \pmod{11}$, *and for* $\{u,v\} \in E$, G_u, G_{uv} *and* $\mathbf{N}_G(G_{uv})$ *are listed in Table* 1.

Proof. Suppose that $T = \mathrm{PSL}(2,11^2)$. Check the subgroups of T with index square-free and order divisible by 11 (refer to [14, II.8.27]). Then $T_u \cong \mathrm{PSL}(2,11)$

or $\mathrm{PGL}(2,11)$. Checking the maximal subgroups of $\mathrm{PGL}(2,11)$ by the Atlas [9], we deduce that $\mathrm{PGL}(2,11)$ has no transitive permutation representation of degree 11, and we have $T_u \cong \mathrm{PSL}(2,11)$, so $T_{uv} \cong \mathrm{A}_5$. In this case, $|T:T_u|$ is even, and $|V|$ is divisible by 4 when T has two orbits on V, a contradiction. Thus, T is transitive on V, and Γ is T-arc-transitive. However, it is easy to check that T_{uv} is self-normalized in T, which contradicts Lemma 2.5.

By Lemma 3.2, we only need to deal with two cases: $T = \mathrm{J}_1$ and $T = \mathrm{PSL}(2,p)$.

Case 1. Let $T = \mathrm{J}_1$. Then $G = T$, and T_u has order divisible by 44, where $u \in V$. Let M be a maximal subgroup of T with $T_u \leq M$. Checking the maximal subgroups of J_1 in [9], we conclude that $M \cong \mathrm{PSL}(2,11)$ and $|T:M| = 266$. Note that $|V| = |T:T_u| = |T:M||M:T_u|$. We know that $(|T:M|, |M:T_u|) = 1$ and $|M:T_u|$ is square-free. Checking the (maximal) subgroups of $\mathrm{PSL}(2,11)$, we conclude that M has no subgroup with square-free index and order divisible by 11. It follows that $T_u = M \cong \mathrm{PSL}(2,11)$, and so $T_{uv} \cong \mathrm{A}_5$ for $v \in \Gamma(u)$. By Lemma 2.5, Γ is isomorphic to the graph given in Example 2.4.

Case 2. Let $T = \mathrm{PSL}(2,p)$ for prime p with $p^2 \equiv 1 \pmod{11}$. Then $G = T$ or $\mathrm{PGL}(2,p)$. Let $\{u,v\} \in E$. Checking the subgroups of G (refer to [14, II.8.27] and [6, Theorem 2]). Since $|G_u|$ is divisible by 11, we have $G_u \cong \mathrm{D}_{22m}$ for some integer $m \geq 1$, and hence $G_{uv} \cong \mathrm{D}_{2m}$. Suppose that $m > 2$. Then both D_{22m} and D_{2m} have a unique cyclic subgroup of order m. Let Z be the cyclic subgroup of G_{uv} of order m. Then Z is a characteristic subgroup of G_u and of G_{uv}. Thus, $Z \trianglelefteq \langle G_u, x\rangle$ for every $x \in \mathbf{N}_G(G_{uv})$. Since Γ is connected, we may choose $x \in \mathbf{N}_G(G_{uv})$ with $G = \langle G_u, x\rangle$. Then Z is normal in G, and then Z fixes every vertex of Γ, a contradiction. Therefore, $m \leq 2$, and $G_u \cong \mathrm{D}_{22}$ or D_{44}.

Let $G_u \cong \mathrm{D}_{44}$. Then $G_{uv} \cong \mathbb{Z}_2^2$. If $G = \mathrm{PGL}(2,p)$, since $|G:G_u|$ is even and square-free, we have $p \equiv \pm 3 \pmod 8$, and $\mathrm{S}_4 \cong \mathbf{N}_G(G_{uv}) \not\leq T$; in this case, T is transitive on V since $|V|$ is divisible by 4 when T has two orbits on V. Thus, Γ is T-arc-transitive. For $G = T = \mathrm{PSL}(2,p)$, we get $p \equiv \pm 7 \pmod{16}$, $\mathbf{N}_G(G_{uv}) \cong \mathrm{S}_4$.

Let $G_u \cong \mathrm{D}_{22}$. Then $G = T$, and since $|G:G_u|$ is square-free, we obtain $p \equiv \pm 3 \pmod 8$. In this case, G_{uv} has order 2, and $\mathbf{N}_G(G_{uv}) = \mathbf{C}_G(G_{uv}) \cong \mathrm{D}_{p\pm 1}$. □

5 The Proof of Theorem 1.1

Let $\Gamma = (V,E)$ be a connected G-arc-transitive graph of square-free order with valency 11. Assume that $\Gamma \ncong \mathsf{K}_{11,11}$, and let $\{u,v\} \in E$.

Case 1. Assume first that G is soluble. By [18, Theorem 4], $G_u \cong \mathbb{Z}_{11}$ and G has a normal regular subgroup isomorphic to D_{2n}, where $n = p_1p_2\cdots p_l$ for distinct primes no less than 13. Then Γ is a Cayley graph of D_{2n}, and thus we write $\Gamma = \mathrm{Cay}(\mathrm{D}_{2n}, S)$, where $1 \notin S = S^{-1} \subset \mathrm{D}_{2n}$ (see Lemma 2.6). Let u be the vertex corresponding to the identity of D_{2n}. By [13, Lemma 2.1], $G_u = \langle\sigma\rangle$ for $\sigma \in \mathrm{Aut}(\mathrm{D}_{2n})$ with $S^\sigma = S$. Since Γ is G-arc-transitive, $\langle\sigma\rangle$ is transitive on S, and so all elements in S have the same order in D_{2n}. On the other hand, Γ has odd valency and $S^{-1} = S$, and it follows that S consists of 11 involutions of D_{2n}. Write $\mathrm{D}_{2n} = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle$. Since n is odd, all involutions in D_{2n} are conjugate under $\langle a\rangle$. Without loss of generality, we let $ab \in S$.

Noting that $\langle a\rangle$ is a characteristic subgroup of D_{2n}, we may let $a^\sigma = a^r$ and

$b^\sigma = a^s b$ for some integers $1 \le r \le n-1$ and $0 \le s \le n-1$ with $(n,r)=1$. Then $S = \{(ab)^{\sigma^i} \mid 0 \le i \le 10\} = \{ab,\, a^{(1+r+\cdots+r^{i-1})s+r^i}b \mid 1 \le i \le 10\}$. Since σ has order 11, we have $a = a^{\sigma^{11}} = a^{r^{11}}$ and $b = b^{\sigma^{11}} = a^{(1+r+\cdots+r^{10})s}b$. This yields $r^{11} \equiv 1 \pmod n$ and $(1+r+\cdots+r^{10})s \equiv 0 \pmod n$. If $r=1$, then s is divisible by n as $(11,n)=1$, and so σ is an identity map, a contradiction. Thus, we have $r \ge 2$ and $1+r+\cdots+r^{10} \equiv 0 \pmod n$. Since Γ is connected,

$$\begin{aligned} G = \langle S\rangle &= \langle ab,\, a^{(1+r+\cdots+r^{i-1})s+r^i}b \mid 1 \le i \le 10\rangle \\ &= \langle ab,\, a^{(1+r+\cdots+r^{i-1})s+r^i}bab \mid 1 \le i \le 10\rangle \\ &= \langle ab,\, a^{(1+r+\cdots+r^{i-1})s+r^i-1} \mid 1 \le i \le 10\rangle \\ &= \langle a^{(1+r+\cdots+r^{i-1})s+r^i-1} \mid 1 \le i \le 10\rangle\langle ab\rangle. \end{aligned}$$

This implies

$$\begin{aligned} \langle a\rangle &= \langle a^{(1+r+\cdots+r^{i-1})s+r^i-1} \mid 1 \le i \le 10\rangle \\ &= \langle a^{(1+r+\cdots+r^{i-1})(r-1+s)} \mid 1 \le i \le 10\rangle = \langle a^{r-1+s}\rangle. \end{aligned}$$

Then a^{r-1+s} generates $\langle a\rangle$, so $(n, r-1+s)=1$. Thus, (1) of Theorem 1.1 follows.

Case 2. Let G be insoluble, and M be the maximal soluble normal subgroup of G. By [18, Theorem 4], $|M|$ is square-free, Γ is a normal cover of Γ_M (see also Lemma 2.1), and G has an almost simple subgroup X such that $G = M:X$. Moreover, let $T = \mathrm{soc}(X)$, and thus we have $MT = M \times T$.

Let $\mathcal{B}$ the the set of M-orbits on V. Then $|\mathcal{B}| = |V|/|M|$ is square-free. Since Γ_M has valency 11, we know that $|\mathcal{B}|$ is even, and so $|M|$ is odd as $|V|$ is square-free.

Take a T-orbit U on V. Since $MT = M \times T$, we have a T-orbit U^x for each $x \in M$. By Lemma 2.2, T has at most two orbits on V. It follows that $|M : M_U| \le 2$, and so $M = M_U$ as $|M|$ is odd. Let $u \in U$ and $B \in \mathcal{B}$ with $u \in B$. Then we have $B \subseteq U$. It follows that T_B is transitive on B. Consider the action of MT_B on B. Since M is regular on B and M centralizes T_B, by [10, Theorem 4.2A], T_B induces a (semi)regular permutation group on B. This implies that T_u is normal in T_B, and thus T_B has a normal subgroup of odd and square-free index $|M|$.

It is easily shown that $G^{\mathcal{B}} = X^{\mathcal{B}} \cong X$, and thus Γ_M is $X^{\mathcal{B}}$-arc-transitive and of square-free order $|\mathcal{B}|$. Note that $X^{\mathcal{B}}$ is an almost simple group with socle $T^{\mathcal{B}}$. Thus, up to isomorphism, the graph Γ_M is known by Theorem 4.3. In particular, $T_B \cong (T^{\mathcal{B}})_B \cong \mathrm{D}_{22}$, D_{44} or $\mathrm{PSL}(2,11)$. Then the only normal subgroup of T_B with odd index is T_B itself. It follows that $M=1$. Hence, G is almost simple, and then (2) or (3) of Theorem 1.1 holds by Theorem 4.3. This completes the proof.

References

[1] B. Alspach, M.Y. Xu, 1/2-transitive graphs of order $3p$, *J. Algebraic Combin.* **3** (4) (1994) 347–355.

[2] N.L. Biggs, *Algebraic Graph Theory*, 2nd edition, Cambridge University Press, New York, 1992.

[3] W. Bosma, C. Cannon, C. Playoust, The MAGMA algebra system I: the user language, *J. Symbolic Comput.* **24** (3-4) (1997) 235–265.

[4] W. Burnside, On some properties of groups of odd order, *Proc. Lond. Math. Soc.*

33 (1901) 257–268.

[5] P.J. Cameron, Finite permutation groups and finite simple groups, *Bull. London Math. Soc.* **13** (1) (1981) 1–22.

[6] P.J. Cameron, G.R. Omidi, B. Tayfeh-Rezaie, 3-Designs from $\mathrm{PGL}(2,p)$, *Electron. J. Combin.* **13** (1) (2006) #R50, 11 pp.

[7] Y. Cheng, J. Oxley, On weakly symmetric graphs of order twice a prime, *J. Combin. Theory B* **42** (1987) 196–211.

[8] M.D. Conder, C.H. Li, C.E. Praeger, On the Weiss conjecture for finite locally primitive graphs, *Proc. Edinburgh Math. Soc.* **43** (1) (2000) 129–138.

[9] J.H. Conway, R.T. Curtis, S.P. Noton, R.A. Parker, R.A. Wilson, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.

[10] J.D. Dixon, B. Mortimer, *Permutation Groups*, Springer, New York, 1996.

[11] The GAP Group, GAP — groups, algorithms, and programming, Version 4.8.6, 2016 (`http://www.gapsystem.org`).

[12] A. Gardiner, Arc transitivity in graphs, *Quart. J. Math. Oxford* **24** (1973) 399–407.

[13] C.D. Godsil, On the full automorphism group of a graph, *Combinatorica* **1** (1981) 243–256.

[14] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin-New York, 1967.

[15] P.B. Kleidman, M.W. Liebeck, *The Subgroup Structure of the Finite Classical Groups*, Cambridge University Press, Cambridge, 1990.

[16] C.H. Li, Z. Liu, Z.P. Lu, Tetravalent edge-transitive Cayley graphs of square free order, *Discrete Math.* **312** (2012) 1952–1967.

[17] C.H. Li, Z.P. Lu, G.X. Wang, Vertex-transitive cubic graphs of square-free order, *J. Graph Theory* **75** (2014) 1–19.

[18] C.H. Li, Z.P. Lu, G.X. Wang, On edge-transitive graphs of square-free order, *Electron. J. Combin.* **22** (3) (2015) Paper 3.25, 22 pp.

[19] C.H. Li, Z.P. Lu, G.X. Wang, On vertex-transitive and edge-transitive tetravalent graphs of square-free order, *J. Algebra Combin.* **42** (2015) 25–50.

[20] C.H. Li, Z.P. Lu, G.X. Wang, Arc-transitive graphs of square-free order and small valency, *Discrete Math.* **339** (2016) 2907–2918.

[21] G. Li, Z.P. Lu, X.Y. Zhang, Locally-primitive arc-transitive 10-valent graphs of square-free order, *Algebra Colloq.* **25** (2) (2018) 243–264.

[22] J.M. Pan, B. Ling, S.Y. Ding, On prime-valent symmetric graphs of square-free order, *Ars Math. Contemp.* **15** (2018) 53–65.

[23] C.E. Praeger, R.J. Wang, M.Y. Xu, Symmetric graphs of order a product of two distinct primes, *J. Combin. Theory* (Ser. B) **58** (1993) 299–318.

[24] C.E. Praeger, M.Y. Xu, Vertex-primitive graphs of order a product of two distinct primes, *J. Combin. Theory* (Ser. B) **59** (1993) 245–266.

[25] J.M. Turner, Point-symmetric graphs with a prime number of points, *J. Combin. Theory* **3** (1967) 136–145.

[26] R.J. Wang, Half-transitive graphs of order a product of two distinct primes, *Comm. Algebra* **22** (1994) 915–927.

[27] R.J. Wang, M.Y. Xu, A classification of symmetric graph of order $3p$, *J. Combin. Theory* (Ser. B) **58** (1993) 197–216.

[28] R. Weiss, s-Transitive graphs, in: *Algebraic Methods in Graph Theory, Vol. I, II* (Szeged, 1978), North-Holland, Amsterdam-New York, 1981, pp. 827–847.