Digraphs with proper connection number two*

Luyi Li, Xueliang Li
Center for Combinatorics and LPMC
Nankai University, Tianjin 300071, China
Email: liluyi@mail.nankai.edu.cn, lxl@nankai.edu.cn

Abstract

A directed path in a digraph is proper if any two consecutive arcs on the path have distinct colors. An arc-colored digraph D is proper connected if for any two distinct vertices x and y of D, there are both proper (x, y)-directed paths and proper (y, x)-directed paths in D. The proper connection number $\overrightarrow{pc}(D)$ of a digraph D is the minimum number of colors that can be used to make D proper connected. Obviously, if a digraph has a proper connection number, it must be strongly connected, and $\overrightarrow{pc}(D) = 1$ if and only if D is complete. Magnant et al. showed that $\overrightarrow{pc}(D) \leq 3$ for all strong digraphs D, and Ducoffe et al. proved that deciding whether a given digraph has proper connection number at most two is NP-complete. In this paper, we give a few classes of strong digraphs with proper connection number two, and from our proofs one can construct an optimal arc-coloring for a digraph of order n in time $O(n^3)$.

Keywords: arc-colored (strong) digraph, proper connected, proper connection number, algorithmic complexity.

AMS subject classification 2020: 05C15, 05C40, 05C20, 68Q25, 68R10.

1 Introduction

Throughout this paper, we use standard terminology and notation in graph theory. For those not defined here, we refer to [3].

Let $G = (V, E)$ be an undirected graph with vertex-set V and edge-set E. An *edge-coloring* of G is a mapping $c : E \mapsto \mathbb{N}$, where $\mathbb{N}$ is the set of colors. We use (G, c) to denote an edge-colored graph with edge-coloring c of G. An edge-colored graph (G, c) is said to be *proper colored* if no two adjacent edges share the same color. We say that a path P in (G, c) is *proper* if any two adjacent edges of P receive different colors. A connected edge-colored graph (G, c) is *proper connected* if there exists at least one proper colored path between each pair of vertices in G. The *proper connection number* of a connected graph G is the minimum number of colors that are needed in order to make G proper connected.

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The concepts of proper connected graphs and proper connection numbers were introduced by Borozan et al. in [5] and have attracted much attention during the last decade. For more details, the reader can see surveys [10, 11] and paper [9]. Melville and Goddard introduced in [13, 14] the notions of *proper connected walk* and *proper connected trail*, i.e., a walk (trail) in an edge-colored graph G is said to be proper if and only if it does not use two consecutive edges of the same color. For a connected graph, the *proper-trail (proper-walk) connection number* is the minimum number of colors that one needs in order to get a proper colored trail (walk) between each pair of vertices in (G, c). Bang-Jansen et al. in [1] considered the proper-walk connection number of connected graphs. They established that the problem can always be solved in polynomial time in the size of the graph and provided a characterization of the graphs that can be proper-connected colored with k colors for every possible value of k.

In fact, the concepts of proper connection number, proper-trail connection number and proper-walk connection number for undirected graphs can be naturally generalized to directed graphs or digraphs. The directed versions of the proper connection and the proper-walk connection were introduced by Magnant et al. in [12] and Melville et al. in [13], respectively. In this paper, we study the proper connection numbers of some digraphs.

Let $D = (V, A)$ be a digraph with vertex-set V and arc-set A. In this paper, we only consider digraphs that do not contain any parallel arcs or loops. A digraph D is *strongly connected* (or *strong*) if for each pair of distinct vertices x, y of D, there exist both directed paths from x to y and directed paths from y to x in D. An *arc-coloring* of D is a mapping $c : A \mapsto \mathbb{N}$, where $\mathbb{N}$ is the set of colors. We use (D, c) to denote an arc-colored digraph with arc-coloring c of D. An arc-colored digraph (D, c) is said to be *proper colored* if no two adjacent arcs share the same color. An arc-colored directed path (walk, trail) is *proper* if it does not contain two consecutive arcs with the same color. An arc-colored digraph (D, c) is *proper connected* if, between each ordered pair of vertices, there is a proper directed path connecting them. In that case, we say that the corresponding arc-coloring is a *proper connection arc-coloring* of D. The *proper connection number* of a digraph D, denoted by $\overrightarrow{pc}(D)$, is the minimum number of colors that are needed to color the arcs of D so that D is proper connected. An arc-colored digraph (D, c) is *proper-trail (proper-walk) connected* if, between each ordered pair of vertices, there is a proper directed trail (proper directed walk) connecting them. Again, we say that the corresponding arc-coloring is a *proper-trail (proper-walk) connection arc-coloring* of D. Clearly, every proper connected digraph is also a proper-trail (proper-walk) connected and every proper-trail connected digraph is also proper-walk connected. The *proper-trail (proper-walk) connection number* of a digraph D, denoted by $\overrightarrow{tc}(D)$ ($\overrightarrow{wc}(D)$), is the minimum number of colors that are needed to color the arcs of D so that D is proper-trail (proper-walk) connected. Note that in order to admit an arc-coloring which makes it proper (proper-trail or proper-walk) connected, a digraph must be strongly connected, or it must be a strong digraph. We can obverse that $\overrightarrow{pc}(D) \geq \overrightarrow{tc}(D) \geq \overrightarrow{wc}(D)$ for any strong digraph. For an arc xy in an arc-colored digraph D, let $c(xy)$ denote the color of xy. For two vertex-disjoint subdigraphs F and H of D, we denote by $A(F, H)$ the set of arcs of D with the arcs from F to H. For convenience, let $c(F, H) = \{c(xy), xy \in A(F, H)\}$. If $F = \{v\}$, then we write $c(v, H)$ for $c(\{v\}, H)$.

A digraph D is *complete* if, for every pair x, y of distinct vertices of D, both arcs xy and yx are in D. A digraph D is *semicomplete* if there is an arc between every pair of vertices in D. A digraph D is *locally in-semicomplete* (*locally out-semicomplete*, respectively) if, for every vertex x of D, all in-neighbours (out-neighbours, respectively) of x induce a semicomplete digraph. A digraph D is *locally semicomplete* if it is both locally in- and locally out-semicomplete. Similarly, we can define the arc version of locally semicomplete. For two disjoint subsets X and Y of $V(D)$, $X \rightarrow Y$ means that some vertices of X dominate some vertices of Y and $X \nrightarrow Y$ means that $A(X, Y) = \emptyset$. $X \mapsto Y$ means that every vertex of X dominates every vertex of Y. Also, $X \Rightarrow Y$ stands for $X \mapsto Y$ and no vertex of Y dominates a vertex in X. When u, v are adjacent vertices of D, we will write $\overline{uv}$. A digraph D is called *quasi-transitive* if whenever $x \rightarrow y$ and $y \rightarrow z$ $(x \neq z)$ we have that $\overline{xz}$. It was a natural step to introduce a new class of digraphs. A digraph D is *k-quasi-transitive* if for every pair of vertices u, v of D, the existence of a (u, v)-path of length k in D implies that $\overline{uv}$. Clearly, a quasi-transitive digraph is a 2-quasi-transitive digraph.

We often use the following operation, called *composition*, to construct bigger digraphs from smaller ones. Let D be a digraph with vertex-set $\{v_i : i \in [n]\}$, and let $G_1, G_2, ..., G_n$ be digraphs which are pairwise vertex-disjoint. The c*omposition* $D[G_1, G_2, \cdots, G_n]$ is the digraph L with vertex-set $V(G_1) \cup V(G_2) \cup \cdots \cup V(G_n)$ and arc-set $(\cup_{i=1}^{n} A(G_i)) \cup \{g_i g_j : g_i \in V(G_i), g_j \in V(G_j), v_i v_j \in A(D)\}$. If $D = H[S_1, \cdots, S_h]$ and none of the digraphs $S_1, \cdots, S_h$ has an arc, then D is an *extension* of H. A digraph on n vertices is *round* if we can label its vertices $v_1, v_2, \cdots, v_n$ so that for each i, we have $N^+(v_i) = \{v_{i+1}, \cdots, v_{i+d^+}(v_i)\}$ and $N^-(v_i) = \{v_{i-d^-}(v_i), \cdots, v_{i-1}\}$ (all subscripts are taken modulo n). We will refer to the labeling $v_1, v_2, \cdots, v_n$ as a *round labeling* of D.

2 Preliminaries

To begin with, we introduce some useful definitions and basic properties.

Observation 2.1 *A digraph D is complete if and only if $\overrightarrow{pc}(D) = 1$ ($\overrightarrow{tc}(D) = 1, \overrightarrow{wc}(D) = 1$).*

So, we always suppose that D is a noncomplete digraph in the sequel.

Lemma 2.1 *(monotonicity) Let D be a strong digraph and H be a strong spanning subdigraph of D. Then $\overrightarrow{pc}(D) \leq \overrightarrow{pc}(H)$, $\overrightarrow{tc}(D) \leq \overrightarrow{tc}(H)$ and $\overrightarrow{wc}(D) \leq \overrightarrow{wc}(H)$.*

In fact, if a strong digraph D contains a strong spanning bipartite subdigraph $H = (X \cup Y, A')$, we only need to color all the arcs with tail in X with red and all the arcs with tail in Y with blue. Then we know that H is proper connected. Combining with Lemma 2.1, we have the following observation.

Observation 2.2 *If D contains a strong spanning bipartite subdigraph, then $\overrightarrow{pc}(D) = \overrightarrow{tc}(D) = \overrightarrow{wc}(D) = 2$.*

A digraph D is called *vertex-pancyclic* if each vertex of D is contained in a directed cycle of length k for every k with $3 \leq k \leq n$.

Lemma 2.2 *[15] Every strong semicomplete digraph is vertex-pancyclic.*

A locally semicomplete digraph D is *round decomposable* if there exists a round local tournament R on $r(\geq 2)$ vertices such that $D = R[S_1, \cdots, S_r]$, where each S_i is a strong semicomplete digraph. We call $R[S_1, \cdots, S_r]$ a *round decomposition* of D.

Lemma 2.3 *[2] Let D be a strong locally semicomplete digraph on n vertices which is not round decomposable. Then D is vertex-pancyclic.*

Bang-Jensen and Huang gave an excellent structure for quasi-transitive digraphs in [4].

Lemma 2.4 *[4] Let D be a quasi-transitive digraph.*

(1) If D is not strong, then there exists a transitive oriented graph T with vertices $\{u_1, u_2, \cdots, u_t\}$ and strong quasi-transitive digraphs $H_1, H_2, \cdots, H_t$ such that $D = T[H_1, H_2, \cdots, H_t]$, where H_i is substituted for u_i, $i \in \{1, 2, \cdots, t\}$.

(2) If D is strong, then there exists a strong semicomplete digraph S with vertices $\{v_1, v_2, \cdots, v_s\}$ and quasi-transitive digraphs $Q_1, Q_2, \cdots, Q_s$ such that Q_i is either a vertex or is non-strong and $D = S[Q_1, Q_2, \cdots, Q_s]$, where Q_i is substituted for v_i, $i \in \{1, 2, \cdots, s\}$.

Let F_n be the digraph on n vertices consisting of a directed 3-cycle $xyzx$, together with $n-3$ vertices $v_1, \cdots, v_{n-3}$, such that yv_jz is a directed path for each $1 \leq j \leq n-3$ (see Figure 1).

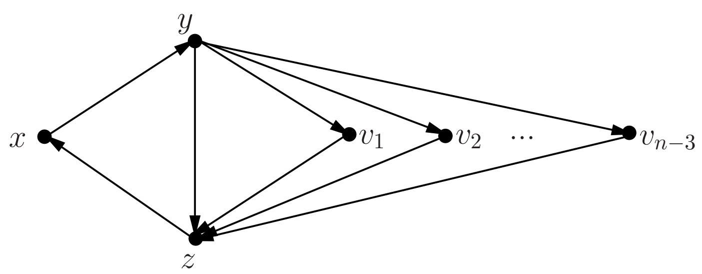

Figure 1: F_n

Lemma 2.5 *[7] Let D be a strong 3-quasi-transitive digraph. Then D is either semicomplete, semicomplete bipartite, or isomorphic to F_n for some $n \geq 4$.*

At the end of this section, we give a few lemmas for the structure of strong k-quasi-transitive digraphs.

Lemma 2.6 *[16] Let k be an integer with $k \geq 2$, and let D be a strong k-quasi-transitive digraph. Suppose that $C = v_0v_1 \cdots v_{r-1}v_0$ is a cycle of length r in D with $r \geq k$. Then, for any $v \in V(D) \setminus V(C)$, v and C are adjacent.*

Lemma 2.7 *[16] Let k be an integer with $k \geq 2$, and D be a strong k-quasi-transitive digraph, and let $C = v_0v_1 \cdots v_{r-1}v_0$ be a cycle of length r in D with $r \geq k$. Suppose that r and $k-1$ are coprime. For any $v \in V(D) \setminus V(C)$, if $(V(C), v) = \emptyset$, then $v \Rightarrow V(C)$; if $(v, V(C)) = \emptyset$, then $V(C) \Rightarrow v$.*

Lemma 2.8 *[8] Let k be an integer with $k \geq 2$, D be a k-quasi-transitive digraph and $u, v \in V(D)$ such that $d(u, v) = k+2$. Suppose that $P = x_0x_1 \cdots x_{k+2}$ is a shortest (u, v)-path, where $u = x_0$, and $v = x_{k+2}$. Then each of the following statements holds:*

(1) $x_{k+2}x_{k-i} \in A(D)$, for every odd i such that $1 \leq i \leq k$;

(2) $x_{k+1}x_{k-i} \in A(D)$, for every even i such that $1 \leq i \leq k$.

Lemma 2.9 *[16] Let k be an even integer with $k \geq 4$ and D be a strong k-quasi-transitive digraph. Suppose that $P = x_0x_1 \cdots x_{k+2}$ is a shortest (x_0, x_{k+2})-path in D. For any $x \in V(D) \setminus P$, if $(x, P) \neq \emptyset$ and $(P, x) \neq \emptyset$, then either x is adjacent to every vertex of $V(P)$ or $\{x_{k+2}, x_{k+1}, x_k, x_{k-1}\} \Rightarrow x \Rightarrow \{x_0, x_1, x_2, x_3\}$. In particular, if $k = 4$, then x is adjacent to every vertex of $V(P)$.*

3 Digraphs with proper connection number two

From Observation 2.1, we know that D is complete if and only if $\overrightarrow{pc}(D) = 1$. Magnant et al. showed that the proper connection number of every strong digraph is at most three in [12] and Ducoffe et al. proved that deciding whether a given digraph has proper connection number at most two is NP-complete in [6]. Then it makes sense to find some sufficient conditions for a digraph with $\overrightarrow{pc}(D) \leq 2$. In this section, we show a few classes of digraphs with proper connection number two.

Theorem 3.1 *[12] If D is a strong digraph, then $\overrightarrow{pc}(D) \leq 3$.*

A *partial arc-coloring* of $D = (V, A)$ is a mapping $c : A' \mapsto \mathbb{N}$, where $\mathbb{N}$ is set of colors and $A' \subseteq A$. Note that if a partial arc-coloring c of D with k colors can make (D, c) proper connected, then $\overrightarrow{pc}(D) \leq k$. Let $C = v_1v_2 \cdots v_rv_1$ be a directed cycle of a strong digraph D. We use v_iCv_j to denote the directed path $v_iv_{i+1} \cdots v_{j-1}v_j$ on C.

Lemma 3.1 *Let D be a strong digraph of order n and C be an even directed cycle in D. If for any vertex $x \in V(D) \setminus V(C)$ we have $N^+(x) \cap V(C) \neq \emptyset$ and $N^-(x) \cap V(C) \neq \emptyset$, then $\overrightarrow{pc}(D) = 2$ and one can construct an optimal arc-coloring c of D in time $O(n^2)$.*

Proof. Suppose that $C = v_1v_2 \cdots v_rv_1$ is an even directed cycle of a strong digraph D. We define a partial arc-coloring c of D using two colors in the following:

(1) $c(v_iv_{i+1}) = c_1$ if i is odd and $c(v_iv_{i+1}) = c_2$ if i is even;

(2) $c(vv_i) = c(v_{i-1}v_i)$ and $c(v_iv) = c(v_iv_{i+1})$ for any vertex v of $V(D) \setminus V(C)$, where all subscripts are taken modulo r.

Note that we can construct the above arc-coloring c of D in time $O(n^2)$ to guarantee that any two vertices are proper connected in C. Next, we assert that (D, c) is proper connected. For any two distinct vertices $x_1, x_2 \in V(D) \setminus V(C)$, we suppose that v_i is an in-neighbor of x_1 and v_j is an out-neighbor of x_2 in C, respectively. Then $x_2 v_j C v_i x_1$ is a proper directed path in D. We suppose that v_a is an in-neighbor of x_2 and v_b is an out-neighbor of x_1 in C, respectively. Then $x_1 v_b C v_a x_2$ is a proper directed path in D. Hence, x_1 and x_2 are proper connected. For any two distinct vertices $x_1 \in V(D) \setminus V(C)$ and $x_2 \in V(C)$, we suppose that v_i is an in-neighbor of x_1 and v_j is an out-neighbor of x_1 in C. Then $x_1 v_j C x_2$ and $x_2 C v_i x_1$ are two proper directed paths in D. Hence, x_1 and x_2 are proper connected. Consequently, (D, c) is proper connected and $\overrightarrow{pc}(D) = 2$. □

From the above lemma, we thus obtain the following corollary.

Corollary 3.1 *If D is vertex-pancyclic, then $\overrightarrow{pc}(D) = 2$.*

Theorem 3.2 *Let D be a strong locally semicomplete digraph. Then $\overrightarrow{pc}(D) = 2$ or D is an odd directed cycle.*

Proof. Suppose that D is a strong locally semicomplete digraph. If D is a strong semicomplete digraph, then D is vertex-pancyclic by Lemma 2.2. If D is not round decomposable, then D is vertex-pancyclic by Lemma 2.3. In such two cases, we can easily show that $\overrightarrow{pc}(D) = 2$ by Corollary 3.1. Now we only need to consider the case that D is not a semicomplete digraph and has a round decomposition $D = R[S_1, S_2, \cdots, S_r]$. From the definition of round decomposition, we know that R is a round local tournament and S_i is a strong semicomplete digraph.

Claim 3.1 *R is Hamiltonian.*

Proof. To prove Claim 3.1, we first show that R is strongly connected. In fact, for every nonempty proper subset $X = \{S_{i_1}, S_{i_2}, \cdots, S_{i_a}\}$ of $V(R)$, we know that $X' = V(S_{i_1}) \cup V(S_{i_2}) \cup \cdots \cup V(S_{i_a})$ is a nonempty proper subset of $V(D)$, where $1 \leq a < r$. Because D is strongly connected, we have $\partial_D^+(X') \neq \emptyset$ and $\partial_D^-(X') \neq \emptyset$, which means that $\partial_R^+(X) \neq \emptyset$ and $\partial_R^-(X) \neq \emptyset$. We have $\partial_R^+(X) = \partial_D^+(\{V(S_{i_1}) \cup V(S_{i_2}) \cup \cdots \cup V(S_{i_a})\}) \neq \emptyset$, where $1 \leq a < r$. Consequently, R is strongly connected. We can obverse that $d_R^+(S_i) \neq 0$ and $d_R^-(S_i) \neq 0$ for all $1 \leq i \leq r$. Since R is a round digraph, without loss of generality, we suppose that $S_1, S_2, \cdots, S_r$ is a round labeling of R. Then $S_1 S_2 \cdots S_r S_1$ is a Hamiltonian cycle in R. The claim thus follows. □

From Claim 3.1, we suppose that $C = S_1 S_2 \cdots S_r S_1$ is a Hamiltonian cycle of R. If r is even, then we can color the edges of C with two colors red and blue alternately. We denote by c the above coloring of C. Now we define a partial arc-coloring c of D: Color every arc of $A(S_i, S_{i+1})$ in D with the color of $S_i S_{i+1}$ in R for all $1 \leq i \leq r$, where the index i is taken module r. We can easily prove that (D, c) is proper connected and $\overrightarrow{pc}(D) = 2$.

If r is odd and $|S_i| = 1$ for all $1 \leq i \leq r$, then D is an odd directed cycle, and the result follows. If r is odd and $|S_i| \geq 2$ for some $1 \leq i \leq r$, without loss of generality, we suppose that $|S_1| \geq 2$. Since D is a locally semicomplete digraph, we know that $D[S_1]$ is a semicomplete digraph. Then we choose an arc $s_1s'_1 \in D[S_1]$. We can obverse that $C = s'_1s_1s_2 \cdots s_rs'_1$ is an even directed cycle, where $s_i \in S_i$ for all $2 \leq i \leq r$. Note that for each vertex $v \in V(D) \setminus V(C)$, we always have $N^+(v) \cap V(C) \neq \emptyset$ and $N^-(v) \cap V(C) \neq \emptyset$. Using Lemma 3.1, we know that $\overrightarrow{pc}(D) = 2$.

In conclusion, if D is a strong locally semicomplete digraph, then $\overrightarrow{pc}(D) = 2$ or D is an odd directed cycle. □

The *underlying multigraph* $UMG(D)$ of D is an undirected multigraph obtained from D by replacing every arc (x, y) with the edge xy. The *underlying graph* $UG(D)$ of D is obtained from $UMG(D)$ by deleting all multiple edges between every pair of vertices apart from one. The complement $\overline{G}$ of an undirected graph G is the undirected graph with vertex set $V(G)$ in which two vertices are adjacent if and only if they are not adjacent in G.

Lemma 3.2 *[4] Let D be a strong quasi-transitive digraph on at least two vertices. Then the following holds:*

(a) $\overline{UG(D)}$ is disconnected;

(b) If S and S' are two subdigraphs of D such that $\overline{UG(S)}$ and $\overline{UG(S')}$ are distinct connected components of $UG(D)$, then either $S \Rightarrow S'$ or $S' \Rightarrow S$, or both $S \mapsto S'$ and $S' \mapsto S$, in which case $|V(S)| = |V(S')| = 1$.

Theorem 3.3 *Let D be a strong quasi-transitive digraph of order n. Then $\overrightarrow{pc}(D) = 2$ and one can construct an optimal arc-coloring c of D in time $O(n^2)$.*

Proof. Let $Q_1, \cdots, Q_s$ be the subdigraphs of D such that each $\overline{UG(Q_i)}$ is a connected component of $\overline{UG(D)}$. According to Lemma 3.2 (a), each Q_i is either non-strong or just a single vertex. By Lemma 3.2 (b), we obtain a strong semicomplete digraph S if each Q_i is contracted to a vertex. Hence, we can find $s + 1$ digraphs: $S, Q_1, Q_2, \cdots, Q_s$ in time $O(n^2)$. By Lemma 2.4, we know that S is a strong semicomplete digraph with s vertices and $Q_1, Q_2, \cdots, Q_s$ are quasi-transitive digraphs. Suppose that $V(S) = \{v_1, v_2, \cdots, v_s\}$ and $D = S[Q_1, Q_2, \cdots, Q_s]$, where Q_i is substituted for v_i, $i \in \{1, 2, \cdots, s\}$. Then S is vertex-pancyclic from Lemma 2.2. Without loss of generality, we suppose that $C_1 = v_1v_2 \cdots v_sv_1$ is a directed Hamiltonian cycle of S. If s is even, then there is an even directed cycle $C'_1 = q_1q_2 \cdots q_sq_1$ in D, where $q_i \in Q_i$ for all $1 \leq i \leq s$. From the definition of composition, we know that for any vertex $v \in V(D) \setminus V(C'_1)$, we always have $N^+(v) \cap V(C'_1) \neq \emptyset$ and $N^-(v) \cap V(C'_1) \neq \emptyset$. From Lemma 3.1, we know that $\overrightarrow{pc}(D) = 2$ one can construct an optimal arc-coloring c of D in time $O(n^2)$.

If s is odd, by Lemma 2.2 we know that D must contain a directed $(s-1)$-cycle C_2. Without loss of generality, suppose that $C_2 = v_1v_2 \cdots v_{s-1}v_1$. Then there is an even directed cycle $C'_2 = q_1q_2 \cdots q_{s-1}q_1$ in D, where $q_i \in Q_i$ for all $1 \leq i \leq s-1$. From the definition of composition, we know that for any vertex $v \in V(D) \setminus V(C'_2)$, we have $q_{i+1} \in N^+_D(v)$ and $q_{i-1} \in N^-_D(v)$ for all $1 \leq i \leq s$, where all subscripts are taken modulo s. From Lemma 3.1, we know that $\overrightarrow{pc}(D) = 2$ and one can construct an optimal arc-coloring c of D in time $O(n^2)$. □

Theorem 3.4 *Let D be a strong 3-quasi-transitive digraph of order n. Then $\overrightarrow{pc}(D) = 2$ and one can construct an optimal arc-coloring c of D in time $O(n^3)$.*

Proof. Suppose that D is a strong 3-quasi-transitive digraph of order n. According to Lemma 2.5, we know that D is either semicomplete, semicomplete bipartite or isomorphic to F_n for some $n \geq 4$. We can check whether D is semicomplete in time $O(n^2)$. If D is not semicomplete, then we can check whether D contains a directed triangle in time $O(n^3)$. If D contains a directed triangle, then D must be isomorphic to F_n. Otherwise, D is semicomplete bipartite. If D is a strong semicomplete or strong semicomplete bipartite, then it is clear that $\overrightarrow{pc}(D) = 2$. If D is a copy of F_n, then we can color yv_i and zx with red for all $1 \leq i \leq n-3$ and the other arcs with blue. Hence, we can construct an arc-coloring c of D in time $O(n^3)$. We can obverse that D is proper connected and $\overrightarrow{pc}(D) = 2$. This completes the proof. □

The *distance* $dist(x, y)$ from a vertex x to a vertex y is the length of a shortest (x, y)-directed path in a digraph D. The *distance* $dist(X, Y)$ from a vertex set X to another vertex set Y is the length of a shortest (x, y)-directed path for any pair of vertices $x \in X$ and $y \in Y$ in a digraph D. This means that $dist(X, Y) = \min\{dist(x, y) : x \in X \text{ and } y \in Y\}$. If there is no a directed path from x to y, then we have $dist(x, y) = \infty$; otherwise, $dist(x, y) < \infty$. The *diameter* of D is the maximum of the distances $dist(x, y)$ over all pairs of vertices x and y in D. Let DFS denote the *depth-first search* on a digraph. A digraph T_s is an *out-tree (in-tree)* if T_s is an oriented tree with just one vertex s of in-degree zero (out-degree zero). The vertex s is the root of T_s. If an out-tree (in-tree) T_s is a spanning subdigraph of D, T_s is called an *out-branching (in-branching).*

Inspired by Theorems 3.3 and 2.5, we thus want to determine the proper connection number of strong k-quasi-transitive digraphs. However, all digraphs D with $diam(D) \leq k-1$ must be k-quasi-transitive digraph. Then, in the next section we shall study the proper connection number of strong k-quasi-transitive digraphs with $diam(D) \geq k$. We will consider k-quasi-transitive digraphs by the parity of k. Then we give the following theorem.

Theorem 3.5 *Let D be a strong k-quasi-transitive digraph of order n with $diam(D) \geq k+2$. Then $\overrightarrow{pc}(D) = 2$ and one can construct an optimal arc-coloring c of D in time $O(n^3)$.*

We will prove Theorem 3.5 in two parts. To begin with, we consider the case that k is even.

Theorem 3.6 *Let k be an even integer with $k \geq 4$, D be a strong k-quasi-transitive digraph of order n with $diam(D) \geq k+2$. Then $\overrightarrow{pc}(D) = 2$ and one can construct an optimal arc-coloring c of D in time $O(n^3)$.*

Proof. Since $diam(D) \geq k+2$, there exist two vertices $t, t' \in V(D)$ such that $d(t, t') = k+2$ in D. Using DFS for every vertex $v \in D$, we can find a shortest (t, t')-path P of D in time $O(n^3)$. Without loss of generality, we suppose that $P = t_0t_1 \cdots t_{k+2}$, where $t = t_0$ and $t' = t_{k+2}$. Because k is even, we know that $k-3$ is odd. From Lemma 2.8, we have $t_{k+2}t_3 \in A(D)$. Thus, $C = t_3t_4 \cdots t_{k+2}t_3$ is a directed cycle of length k. For the sake of simplicity, let $C = s_1s_2 \cdots s_ks_1$. Choosing any vertex $v \in V(D) \setminus V(C)$, one can check whether v is adjacent to every vertex of C in time $O(n)$. Then we can get the following three vertex sets in time $O(n^2)$:

$$X = \{v \in V(D) \setminus V(C) : v \rightarrow V(C) \text{ and } V(C) \nrightarrow v\},$$

$$Y = \{v \in V(D) \setminus V(C) : v \nrightarrow V(C) \text{ and } V(C) \rightarrow v\},$$

$$Z = \{v \in V(D) \setminus V(C) : v \rightarrow V(C) \text{ and } V(C) \rightarrow v\}.$$

Since $k-2$ is even, from Lemma 2.8 we know that $t_{k+1} \rightarrow t_2 \rightarrow t_3$. This means $s_{k-1} \rightarrow t_2 \rightarrow s_1$. Then $t_2 \in Z$. It is clear that $t_0, t_1 \notin Z$. From Lemma 2.6, we know that for any $v \in V(D) \setminus V(C)$, v and C are adjacent. Hence, $(X, Y, Z, V(C))$ is a vertex partition of D. We define a partial arc-coloring c of D using two colors in the following:

(1) $c(s_is_{i+1}) = c_1$ if i is odd and $c(s_is_{i+1}) = c_2$ if i is even;

(2) $c(vs_i) = c(s_{i-1}s_i)$ and $c(s_iv) = c(s_is_{i+1})$ for any vertex v of $V(D) \setminus V(C)$, where all subscripts are taken modulo k.

It is clear that c is also a partial arc-coloring of $H_1 = D[V(C) \cup Z]$. By Lemma 3.1, we can conclude that (H_1, c) is proper connected and $\overrightarrow{pc}(H_1) = 2$. Since k and $k-1$ are coprime, we can get that $V(C) \Rightarrow x$ for any vertex $x \in X$ and $y \Rightarrow V(C)$ for any vertex $y \in Y$ by Lemma 2.7. Then we can obverse that $c(O(C), x) = c(y, E(C)) = \{c_1\}$ and $c(E(C), x) = c(y, O(C)) = \{c_2\}$, where $O(C) = \{s_1, s_2, \cdots, s_{k-1}\}$ and $E(C) = \{s_2, s_4, \cdots, s_k\}$. Hence, we have the following claim.

Claim 3.2 *(a) For any two vertices $x \in X$ and $v \in V(D) \setminus X$, there exists a proper directed (x, v)-path in (D, c);*

(b) For any two vertices $y \in Y$ and $v \in V(D) \setminus Y$, there exists a proper directed (v, y)-path in (D, c).

Proof. Choose an arbitrary vertex $x \in X$, if $v \in V(C)$, then xv is a proper directed (x, v)-path. If $v \in Z$, then xs_iv is a proper directed (x, v)-path for any vertex $s_i \in V(C)$ and $s_i \rightarrow v$. If $v \in Y$, then xs_iv is a proper directed (x, v)-path for any vertex $s_i \in V(C)$ and $s_i \rightarrow v$. The statement of (a) is right. By a similar argument, we can prove (b). □

Next, we consider the vertices of X and Y in more detail and give the following partitions of X and Y in time $O(n^2)$ (see Figure 2).

$$X_1 = \{v \in X : v \rightarrow Y\},$$

$$Y_1 = \{v \in Y : X \rightarrow v\},$$

$$X_2 = \{v \in X \setminus X_1 : v \rightarrow Z\},$$

$$Y_2 = \{v \in Y \setminus Y_1 : Z \rightarrow v\},$$

$$X_3 = X \setminus (X_1 \cup X_2) \text{ and } Y_3 = Y \setminus (Y_1 \cup Y_2).$$

For any vertex $z \in Z$, there may be many out-neighbors and in-neighbors of z in C. If $z \in Z \setminus t_2$, from Lemma 2.9, we know that either z adjacent to every vertex of C or $\{s_k, s_{k-1}, s_{k-2}, s_{k-3}\} \Rightarrow z \Rightarrow s_1$. If $z = t_2$, then $s_{k-1} \to t_2 \to s_1$. Consequently, we always can find two vertices s_{z_+} and s_{z_-} in C such that $s_{z_-} \to z \to s_{z_+}$ and $c(s_{z_-}z) \neq c(zs_{z_+})$ for every vertex $z \in Z$. Now we extend the partial arc-coloring c of D in the following method:

(1) $c(xy) = c_1$ for every arc $xy \in A(X_1, Y_1)$;

(2) $c(xz) \neq c(zs_{z_+})$ for any vertex $x \in X_2$, where $x \to z \in Z$;

(3) $c(zy) \neq c(s_{z_-}z)$ for any vertex $y \in Y_2$, where $z \to y$ and $z \in Z$.

This vertex partition and arc-coloring of D is illustrated in Figure 2. It is clear that c is also a partial arc-coloring of $H_2 = D[V(D) \setminus (X_3 \cup Y_3)]$.

Claim 3.3 (H_2, c) *is proper connected.*

Proof. Choose any two vertices $x \in X_1$ and $w \in V(H_2) \setminus x$, we suppose that y is an out-neighbor of x in Y_1. If $w \in X_1 \setminus x$, then xys_1w is a proper (x, w)-path in D. If $w \in V(C)$, then xys_1Cw is a proper (x, w)-path in D. If $w \in Y_1$ and $x \nrightarrow w$, then $xys_1s_2x'w$ is a proper (x, w)-path in D, where $x' \in X$ and $x' \to w$. If $w \in X_2$, then xys_1w is a proper (x, w)-path in D. If $w \in Z$, then $xys_1Cs_{w_-}w$ is a proper (x, w)-path in D. If $w \in Y_2$, then $xys_1Cs_{z_-}zw$ is a proper (x, w)-path in D, where $z \in W$ and $z \to w$. Similarly, for any two vertices $y \in Y_1$ and $w \in V(H_2) \setminus y$, we can also find a proper (w, y)-path in D.

Choose any two vertices $x \in X_2$ and $w \in V(H_2) \setminus x$, we suppose that z is an out-neighbor of x in Z. If $w \in X_2 \setminus x$, then $xzs_{z_+}Cs_1w$ is a proper (x, w)-path in D. If $w \in V(C)$, then $xzs_{z_+}Cw$ is a proper (x, w)-path in D. If $w \in Y_2$ and $z \to w$, then xzw is a proper (x, w)-path in D, If $w \in Y_2$ and $z \nrightarrow w$, then $xzs_{z_+}Cs_{z'_-}z'w$ is a proper (x, w)-path in D, where $z' \in Z$ and $z' \to w$. If $w \in X_1$, then $xzs_{z_+}w$ is a proper (x, w)-path in D. If $w \in Y_1$, then $xzs_{z_+}Cx_kx'w$ is a proper (x, w)-path in D, where $x' \in X_1$ and $x' \to w$. If $w \in Z$ and $x \nrightarrow w$, then $xzs_{z_+}s_{w_-}w$ is a proper (x, w)-path in D. Similarly, for any two vertices $y \in Y_2$ and $w \in V(H_2) \setminus y$, we can also find a proper (w, y)-path in D. Combining with Claim 3.2, the claim follows. □

From the definition of X_3 and the fact that D is strongly connected, we have that $N^+(x') \subseteq X$ for any vertex $x' \in X_3$. Then, for every vertex $x' \in X_3$, we know that $dist(x', X_1) < \infty$ or $dist(x', X_2) < \infty$. Set

$$X_3^1 = \{x' \in X_3 : dist(x', X_1) < \infty\}$$

and

$$X_3^2 = \{x' \in X_3 : dist(x', X_1) = \infty \text{ and } dist(x', X_2) < \infty\}.$$

The vertex partition (X_3^1, X_3^2) of X_3 is illustrated in Figure 2. We can obtain a digraph D_1 from $D[X_1 \cup X_3^1]$ by shrinking X_1 to a vertex e and a digraph D_2 from $D[X_2 \cup X_3^2]$ by shrinking X_2 to a vertex f. Using DFS for the vertex e in D_1 and the vertex f in D_2, we can find an in-branching T_e of D_1 and an in-branching T_f of D_2 in time $O(n^2)$, respectively.

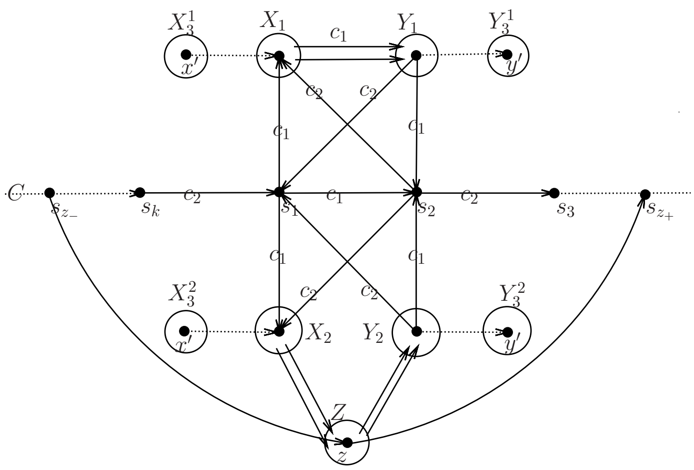

Figure 2: Vertex partition and arc-coloring of D.

We can obverse that there is a unique (x', e)-path in T_e for every vertex $x' \in X_3^1$ and a unique (x', f)-path in T_f for every vertex $x' \in X_3^2$, which means that we can find a shortest directed (x', x)-path $P_1(x', x)$ such that $x \in X_1$ in $D[X_1 \cup X_3^1]$ corresponding to every (x', e)-path of T_e and a shortest directed (x', x)-path $P_2(x', x)$ such that $x \in X_2$ in $D[X_2 \cup X_3^2]$ corresponding to every (x', f)-path of T_f. If $x \in X_1$ (X_2), then we choose a vertex $y \in Y$ $(z \in Z)$ such that $x \to y$ $(x \to z)$. We suppose that $P_1(x') = P(x', x)y$ for every vertex $x' \in X_3^1$ and $P_2(x') = P_2(x', x)z$ for every vertex $x' \in X_3^2$. Note that xy and xz have been colored in the previous step. Now we extend the partial arc-coloring c of D in the following method again:

(1) Color the arcs of $P_i(x')$ for every vertex $x' \in X_3^i$ with $\{c_1, c_2\}$ such that $P_i(x')$ is proper, where $i = 1, 2$;

(2) Color the uncolored arcs of $A(X)$ with either c_1 or c_2.

It is clear that c is also a partial arc-coloring of $H_3 = D[V(D) \setminus Y_3]$.

Claim 3.4 (H_3, c) *is proper connected.*

Proof. Choose two vertices $x' \in X_3^1$ and $u \in V(H_3) \setminus x'$, if $u \in X$ and $x \nrightarrow u$, then $P_1(x')s_1u$ is a proper (x', u)-path in D. If $u \in V(C)$, then $P_1(x')s_1Cu$ is a proper (x', u)-path in D. If $u \in Z$, then $P_1(x')s_1Cs_{u_-}u$ is a proper (x', u)-path in D. If $u \in Y_1 \setminus \{y_{x_+}\}$ and $x \nrightarrow u$, then $P_1(x')s_1Cs_kx_{u_-}u$ is a proper (x', u)-path in D. If $u \in Y_1 \setminus \{y_{x_+}\}$ and $x \to u$, then $P_1(x', x)u$ is a proper (x', u)-path in D. If $u \in Y_2$, then $P_1(x')s_1Cs_{z_-}zu$ is a proper (x', u)-path in D, where $z \to u$.

Choose two vertices $x' \in X_3^2$ and $u \in V(H_3) \setminus x'$, if $u \in X \setminus \{x\}$, then $P_2(x')s_{z_+}u$ is a proper (x', u)-path in D. If $u \in V(C)$, then $P_2(x')s_{z_+}Cu$ is a proper (x', u)-path in D. If $u \in Y_1$, then $P_2(x')s_{z+}x_1u$ is a proper (x', u)-path in D, where $x_1 \to u$. If $u \in Y_2$ and $z_{x_+} \to u$, then $P_2(x')u$ is a proper (x', u)-path in D. If $u \in Y_2$ and $z \nrightarrow u$, then $P_2(x')s_{z_+}Cs_{z'_-}z'u$ is a proper (x', u)-path in D, where $z' \in Z$ and $z' \to u$. If $u \in Z \setminus \{z_{x_+}\}$, then $P_2(x')s_{z_+}Cs_{u_-}u$ is a proper (x', u)-path in D.

In conclusion, there exists a proper directed (x', u)-path in D for any two vertices $x' \in X_3$ and $u \in V(H_3) \setminus x'$. Combining with Claim 3.3, the claim follows. $\square$

From the definition of Y_3 and the fact that D is strongly connected, we know that $N^-(y') \subseteq Y$ for any vertex $y' \in Y_3$. Then for every vertex $y' \in Y_3$, we have $dist(Y_1, y') < \infty$ or $dist(Y_2, y') < \infty$. Set

$$Y_3^1 = \{y' \in Y_3 : dist(Y_1, y') < \infty\}$$

and

$$Y_3^2 = \{y' \in Y_3 : dist(Y_1, y') = \infty \text{ and } dist(Y_2, y') < \infty\}.$$

The vertex partition (Y_3^1, Y_3^2) of Y_3 is illustrated in Figure 2.

We can obtain a digraph F_1 from $D[Y_1 \cup Y_3^1]$ by shrinking Y_1 to a vertex g and a digraph F_2 from $D[Y_2 \cup Y_3^2]$ by shrinking Y_2 to a vertex h. Using DFS for the vertex g in D_1 and the vertex h in D_2, we can find an out-branching T_g of D_1 and an out-branching T_h of D_2 in time $O(n^2)$, respectively. We can obverse that there is a unique (g, y')-path in T_g for every vertex $y' \in Y_3^1$ and a unique (h, y')-path in T_h for every vertex $y' \in Y_3^2$. This means that we can find a shortest directed (y, y')-path $Q_1(y, y')$ such that $y \in Y_1$ in $D[Y_1 \cup Y_3^1]$ corresponding to every (g, y')-path of T_g and a shortest directed (y, y')-path $Q_2(y, y')$ such that $y \in Y_2$ in $D[Y_2 \cup Y_3^2]$ corresponding to every (h, y')-path of T_h. If $y \in Y_1$ (Y_2), then we choose a vertex $x \in X_1$ $(z \in Z)$ such that $x \to y$ $(z \to y)$. Let $Q_1(y') = xQ_1(y, y')$ for every vertex $y' \in Y_3^1$ and $Q_2(y') = zQ_2(y, y')$ for every vertex $y' \in Y_3^2$. Note that xy and zy have been colored in the previous step. Now we extend the partial arc-coloring c of D in the following method again:

(1) Color the arcs of $Q_i(y')$ for every vertex $y' \in Y_3^i$ with $\{c_1, c_2\}$ such that $Q_i(y')$ is proper, where $i = 1, 2$;

(2) Color the uncolored arcs of $A(Y)$ with either c_1 or c_2.

It is clear that c is also a partial arc-coloring of $H_4 = D[V(D) \setminus X_3]$. Then we give the following claim.

Claim 3.5 (H_4, c) *is proper connected.*

Proof. In fact, the proof of Claim 3.5 is similar to Claim 3.4, and so we omit it. $\square$

We extend c by coloring the uncolored arcs of $A(D)$ with either c_1 or c_2. So, c is an arc-coloring of D. Note that we can construct such an arc-coloring c of D in time $O(n^2)$. Finally, we prove that (D, c) is proper connected. From Claim 3.2 to Claim 3.5, we only need to show that there is a proper (x', y')-path for any two vertices $x' \in X_3$ and $y' \in Y_3$ in D.

Choose any two vertices $x' \in X_3^1$ and $y' \in Y_3^1$, there exist two proper directed paths $P_1(x', x)$ and $Q_1(y, y')$. If $x \to y$, then $P_1(x', x)Q_1(y, y')$ is a proper directed path in D. If $x \not\to y$, then $P_1(x')s_1s_2Q_1(y')$ is a proper directed path in D. For any two vertices $x' \in X_3^1$ and $y' \in Y_3^2$, there exist two proper directed paths $P_1(x')$ and $Q_2(y')$. Hence, $P_1(x')s_1Cs_{z_-}Q_2(y')$ is a proper directed path in D. For any two vertices $x' \in X_3^2$ and $y' \in$

Y_3^1, there exist two proper directed paths $P_2(x')$ and $Q_1(y')$. Hence, $P_2(x')s_{z_+}Cs_2Q_1(y')$ is a proper directed path in D. For any two vertices $x' \in X_3^2$ and $y' \in Y_3^2$, there exist two proper directed paths $P_2(x') = P_2(x', x)z$ and $Q_2(y') = Q_2(y, y')$. If $z = z_{x_+} = z_{y_-} = z'$, then $P_2(x', x)zQ_2(y, y')$ is a proper directed path in D. If $z = z_{x_+} \neq z_{y_-} = z'$, then $P_2(x')s_{z_+}Cs_{z'_-}Q_2(y')$ is a proper directed path in D. Consequently, we find a proper directed path for any two vertices $x' \in X_3$ and $y' \in Y_3$. Then (D, c) is proper connected and $\overrightarrow{pc}(D) \doteq 2$, the result follows. □

To study the case that k is odd, we need some more lemmas and notations below. Now let k be an odd integer with $k \geq 5$, D be a strong k-quasi-transitive digraph of order n with $diam(D) \geq k$. Because $diam(D) \geq k$, there exist two vertices s_0 and s_k such that $dist(s_0, s_k) = k$ in D. Using DFS for every vertex $v \in D$, we can find a shortest (s_0, s_k)-path $P = s_0s_1 \cdots s_k$ of D in time $O(n^3)$. Because D is a k-quasi-transitive digraph, we know that $C = s_0s_1 \cdots s_ks_0$ is a $(k+1)$-cycle in D. By the parity of the subscripts, we divide $\{s_0, s_1, \cdots, s_k\}$ into two vertex sets: $E(C) = \{s_0, s_2, \ldots, s_{k-1}\}$ and $O(C) = \{s_1, s_3 \ldots, s_k\}$. Choosing any vertex $v \in V(D) \setminus V(C)$, one can check whether v is adjacent to every vertex of C in time $O(n)$. Then, combining with Lemma 2.6, we can get the following three vertex sets in time $O(n^2)$:

$$X = \{v \in V(D) \setminus V(C) : v \to V(C)) \ and \ V(C) \nrightarrow v\};$$

$$Y = \{v \in V(D) \setminus V(C) : v \nrightarrow V(C)) \ and \ V(C) \to v\};$$

$$Z = \{v \in V(D) \setminus V(C) : v \to V(C)) \ and \ V(C) \to v\}.$$

Lemma 3.3 *For every vertex $v \in V(D) \setminus V(C)$, we have the following properties:*

If $v \in X$, then either $v \mapsto O(C)$ or $v \mapsto E(C)$;

If $v \in Y$, then either $E(C) \mapsto v$ or $O(C) \mapsto v$;

If $v \in Z$, then there exist two vertices s_i and s_{i+2} such that $s_i \to z \to s_{i+2}$ or $E(C) \mapsto v \mapsto O(C)$ or $O(C) \mapsto v \mapsto E(C)$, where the subscripts are taken modulo n.

Proof. If $v \in X$, then $(v, V(C)) \neq \emptyset$. Hence, there exists one vertex $s_i \in V(C)$ such that $v \to s_i$. Since D is a k-quasi-transitive digraph, we have $\overline{vs_{i-2}}$. Note that vs_iCs_{i-2} is a k-path in D. Since $(V(C), v) = \emptyset$, we can get $v \to s_{i-2}$. It is clear that $vs_{i-2}Cs_{i-4}$ is also a k-path in D. Note that C is an even cycle. Repeating the above discussions, we know that if $s_i \in O(C)$, then $v \mapsto O(C)$. If $s_i \in E(C)$, then $v \mapsto E(C)$. Similarly, we can show that either $E(C) \mapsto v$ or $O(C) \mapsto v$ if $v \in Y$.

If $v \in Z$ and there exists two vertices s_i and s_{i+2} such that $s_i \to z \to s_{i+2}$ in $V(C)$, then the assertion holds. Next, we suppose that there do not exist such two vertices in $V(C)$. Since $(v, V(C)) \neq \emptyset$, without loss of generality, suppose that there exists one vertex $s_i \in E(C)$ such that $v \to s_i$. Note that vs_iCs_{i-2} is a k-path in D. If $s_{i-2} \to v$, which contradicts the hypothesis. If $v \to s_{i-2}$, then $vs_{i-2}Cs_{i-4}$ is a k-path in D. Repeating the above discussions, we know that $v \mapsto E(C)$. Since $(V(C), v) \neq \emptyset$, there is one vertex $s_j \in O(C)$ such that $s_j \to v$. Then $s_{j+2}Cs_jv$ is a k-path in D. If $v \to v_{j+2}$, which contradicts the hypothesis. If $v_{j+2} \to v$, then $s_{j+4}Cs_{j+2}v$ is a k-path in D. Repeating the above discussions, we know that $O(C) \mapsto v$. By symmetry, we can conclude that $E(C) \mapsto v \mapsto O(C)$ when $s_i \in O(C)$. $\square$

From Lemma 3.3, as shown in Figure 3, we can give the following partitions of X and Y in time $O(n^2)$.

$$X_1 = \{v \in X : v \mapsto O(C) \text{ and } v \not\mapsto E(C)\};$$

$$X_2 = \{v \in X : v \not\mapsto O(C) \text{ and } v \mapsto E(C)\};$$

$$X_3 = \{v \in X : v \mapsto V(C)\};$$

$$Y_1 = \{v \in Y : O(C) \mapsto v \text{ and } E(C) \not\mapsto v\};$$

$$Y_2 = \{v \in Y : O(C) \not\mapsto v \text{ and } E(C) \mapsto v\};$$

$$Y_3 = \{v \in Y : V(C) \mapsto v\};$$

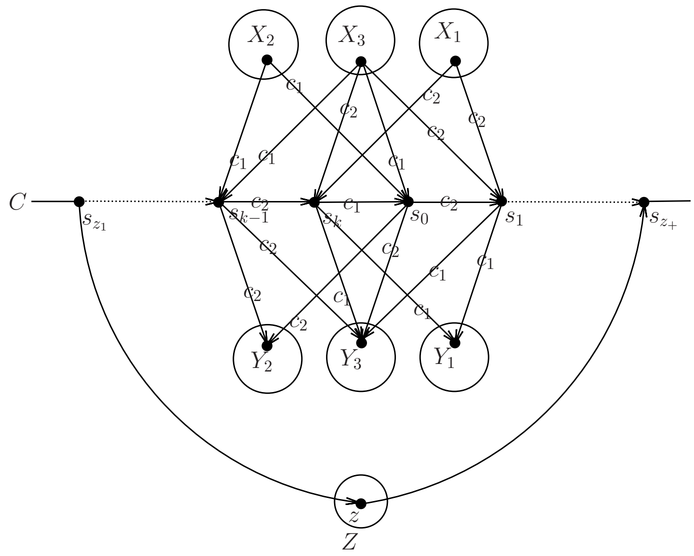

Figure 3: Vertex partitions of X and Y.

Lemma 3.4 (1) $Y_2 \not\mapsto X_2 \cup X_3$ *and* $Y_1 \not\mapsto X_1 \cup X_3$*;*

(2) $X_1 \cup X_2 \not\mapsto X_3$ *and* $Y_3 \not\mapsto Y_1 \cup Y_2$*;*

(3) $A(X_1) = A(X_2) = A(Y_1) = A(Y_2) = \emptyset$.

Proof. Suppose to the contrary that there exist two vertices $x \in X_2 \cup X_3$ and $y \in Y_2$ such that $y \to x$. Then yxs_0Cs_{k-2} is k-path in D. Hence, y and s_{k-2} are adjacent. If $s_{k-2} \to y$, from Lemma 3.3 we know that $O(C) \mapsto y$, which contradicts the definition of Y_2. If $y \to s_{k-2}$, then $y \in Z$, a contradiction. Then $y \not\mapsto x$, which means that $Y_2 \not\mapsto X_2 \cup X_3$. Similarly, we can conclude that $Y_1 \not\mapsto X_1 \cup X_3$. Then statement (1) is right. The proof of statement (2) is similar to statement (1), and so we omit it here.

We show statement (3) by contradiction. If $A(X_1) \neq \emptyset$, then there exists at least one arc $x_1x_2 \in A(X_1)$. Then $x_1x_2s_1Cs_{k-1}$ is k-path in D. Hence, x_1 and s_{k-1} are adjacent. If $x_1 \to s_{k-1}$, from Lemma 3.3 we know that $x_1 \mapsto E(C)$, which contradicts the definition of X_1. If $s_{k-2} \to x_1$, then $x_1 \in Z$, a contradiction. Similarly, we can conclude that $A(X_2) = A(Y_1) = A(Y_2) = \emptyset$. Then statement (3) is right. □

From Lemma 3.4, we can get the following corollary.

Corollary 3.2 *(1) For any two vertices $y \in Y$ and $z \in Z$, if $y \in Y_1$ and $y \to z$, then $z \nrightarrow O(C)$. If $y \in Y_2$ and $y \to z$, then $z \nrightarrow E(C)$.*

(2) For any two vertices $x \in X$ and $z \in Z$, if $x \in X_1$ and $z \to x$, then $O(C) \nrightarrow z$. If $x \in X_2$ and $z \to x$, then $E(C) \nrightarrow z$.

Theorem 3.7 *Let k be an odd integer with $k \geq 5$, D be a strong k-quasi-transitive digraph of order n with $diam(D) \geq k$. Then $\overrightarrow{pc}(D) = 2$ and one can construct an optimal arc-coloring c of D in time $O(n^3)$.*

Proof. We define a partial arc-coloring c of D using two colors in the following:

(1) $c(s_is_{i+1}) = c_1$ if i is odd and $c(s_is_{i+1}) = c_2$ if i is even;

(2) $c(vs_i) = c(s_{i-1}s_i)$ and $c(s_iv) = c(s_is_{i+1})$ for any vertex v of $V(D) \setminus V(C)$, where all subscripts are taken modulo k.

This partial arc-coloring of D is illustrated in Figure 3. It is clear that c is also a partial arc-coloring of $H_1 = D[V(C) \cup Z]$. By Lemma 3.1, we can conclude that (H_1, c) is proper connected and $\overrightarrow{pc}(H_1) = 2$.

Claim 3.6 *(a) For any two vertices $x \in X$ and $v \in V(D) \setminus X$, there exists a proper directed (x, v)-path in (D, c);*

(b) For any two vertices $y \in Y$ and $v \in V(D) \setminus Y$, there exists a proper directed (v, y)-path in (D, c);

Proof. The proof is similar to that of Claim 3.2, and so we omit it here. □

We can obverse that $c(X, E(C)) = c(O(C), Y) = c_1$ and $c(X, O(C)) = c(E(C), Y) = c_2$. For every vertex $z \in Z$, we fix an out-neighbor s_{z^+} and an in-neighbor s_{z^-} in C. Now we extend the partial arc-coloring c of D in the following method:

(1) $c(yv) = c_1$ for any arc $yv \in A(Y_2, X_1 \cup Z)$ and $c(yv) = c_2$ for any arc $yv \in A(Y_1, X_2 \cup Z)$;

(2) $c(yx) = c_1$ for any arc $yx \in A(Y_3, X_1)$ and $c(yx) = c_2$ for any arc $yx \in A(Y_3, X_2 \cup X_3)$;

(3) $c(yz) \neq s(zs_{z^+})$ for any arc $yz \in A(Y_3, Z)$ and $c(zx) \neq c(s_{z^-}z)$ for any arc $zx \in A(Z, X)$.

Let X^1 be a subset of X such that for any vertex $x \in X^1$, all the in-arcs of x are uncolored, and let Y^1 be a subset of Y such that for any vertex $y \in Y^1$, all the out-arcs of y are uncolored. Let $X_i^1 = X_i \setminus X^1$ and $Y_i^1 = Y_i \setminus Y^1$ for every $i = 1, 2, 3$ (see Figure 4). It is obvious that c is also a partial arc-coloring of $H_2 = D[V \setminus (X^1 \cup Y^1)]$.

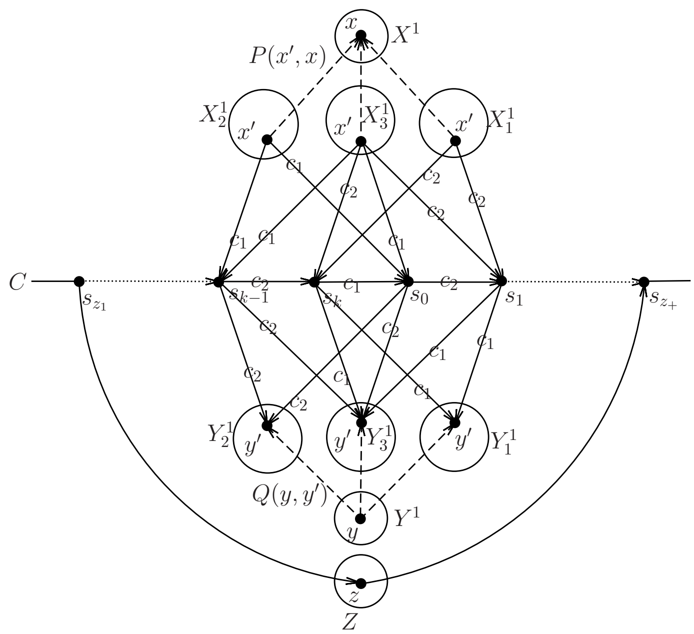

Figure 4: Vertex partition and arc-coloring of D.

Claim 3.7 (H_2, c) *is proper connected.*

Proof. To begin with, using Corollary 3.2, we will find a proper 2-path for all $u \in X \cup Y \setminus (X^1 \cup Y^1)$ in H_2. If $u \in X_1^1$, then we have $Y_2 \to u$ or $Z \to u$ or $Y_3 \to u$. We can obverse that there exists a vertex $s_i \in E(C)$ such that $P_{X_1}(u) = s_i v u$ is a proper path in H_2, where $v \in Y_2$ or $v \in Z$ or $v \in Y_3$. If $u \in X_2^1$, then we have $Y_1 \to u$ or $Z \to u$ or $Y_3 \to u$. We can obverse that there exists a vertex $s_i \in O(C)$ such that $P_{X_2}(u) = s_i v u$ is a proper path in H_2, where $v \in Y_1$ or $v \in Z$ or $v \in Y_3$. If $u \in X_3^1$, then we have $Z \to u$ or $Y_3 \to u$. We can obverse that there exists a vertex $s_i \in V(C)$ such that $P_{X_3}(u) = s_i v u$ is a proper path in H_2, where $v \in Z$ or $v \in Y_3$. If $u \in Y_1^1$, then we have $Y_1 \to X_2$ or $Y_1 \to Z$. We can obverse that there exists a vertex $s_i \in E(C)$ such that $P_{Y_1}(u) = u v s_i$ is a proper path in H_2, where $v \in X_2$ or $v \in Z$. If $u \in Y_2^1$, then we have $Y_1 \to X_1$ or $Y_1 \to Z$. We can obverse that there exists a vertex $s_i \in O(C)$ such that $P_{Y_2}(u) = u v s_i$ is a proper path in H_2, where $v \in X_1$ or $v \in Z$. If $u \in Y_3^1$, then we have $Y_3 \to X \setminus X^1$ or $Y_3 \to Z$. We can obverse that there exists a vertex $s_i \in V(C)$ such that $P_{Y_3}(u) = u v s_i$ is a proper path in H_2, where $v \in X \setminus X^1$ or $v \in Z$.

Choose any two distinct vertices $u, v \in H_2$, if $u, v \in X_1^1$, then $u s_0 C s_i P_{X_1}(v)$ and $v s_0 C s_i P_{X_1}(u)$ are two proper paths in H_2. If $u \in X_1^1$ and $v \in X_2^1$, then $u s_0 C s_i P_{X_2}(v)$ and $v s_0 C s_i P_{X_1}(u)$ are two proper paths in H_2. If $u \in X_1^1$ and $v \in X_3^1$, then $u s_0 C s_i P_{X_3}(v)$ and $v s_0 C s_i P_{X_1}(u)$ are two proper paths in H_2. If $u \in X_1 \setminus X^1$ and $v \in Y_1^1$, then $P_{Y_1}(v) C P_{X_1}(u)$ is a proper path in H_2. If $u \in X_1^1$ and $v \in Y_2^1$, then $P_{Y_2}(v) C P_{X_1}(u)$ is a proper path in H_2. If $u \in X_1^1$ and $v \in Y_3^1$, then $P_{Y_3}(v) C P_{X_1}(u)$ is a proper path in H_2. If $u \in X_1^1$ and $v \in V(C)$, then $v C P_{X_1}(u)$ is a proper path in H_2. If $u \in X_1^1$ and $v \in Z$, then $v s_j C P_{X_1}(u)$ is a proper path in H_2. By a similar discussion and combining with Claim 3.6, we can show that (H_2, c) is proper connected. □

Let D_1 be a digraph by shrinking the subset $X \setminus X^1$ of $D[X]$ to a vertex e, and let D_2 be a digraph by shrinking the subset $Y \setminus Y^1$ of $D[Y]$ to a vertex f. Using DFS for the vertex e in D_1 and the vertex f in D_2, we can find an in-branching T_e of D_1 and an out-branching T_f of D_2 in time $O(n^2)$, respectively. We can obverse that there is a unique (x', e)-path in T_e for every vertex $x' \in X^1$ and a unique (f, y')-path in T_f for every vertex $y' \in Y^1$. These mean that for every vertex $x' \in X^1$, we can find a shortest directed (x', x)-path $P(x', x)$ such that $x' \in X \setminus X^1$ in $D[X]$ corresponding to every (x', e)-path of T_e, and for every vertex $y' \in Y^1$, we can find a shortest directed (y, y')-path $Q(y, y')$ such that $y' \in Y \setminus Y^1$ in $D[Y]$ corresponding to every (f, y')-path of T_f, see Figure 4.

Then we define another path $P(x)$ for every vertex $x \in X^1$ and another path $Q(y)$ for every vertex $y \in Y^1$:

$$P(x) = \begin{cases} P_{X_1}(x')P(x', x), x' \in X_1; \\ P_{X_2}(x')P(x', x), x' \in X_2; \\ P_{X_3}(x')P(x', x), x' \in X_3. \end{cases} \tag{1}$$

and

$$Q(y) = \begin{cases} Q(y, y')P_{Y_1}(y'), x' \in Y_1; \\ Q(y, y')P_{Y_2}(y'), x' \in Y_2; \\ Q(y, y')P_{Y_3}(y'), x' \in Y_3. \end{cases} \tag{2}$$

Note that $P_{X_i}(x')$ and $P_{Y_i}(y')$ have been colored in the previous step for all $i = 1, 2, 3$. Now we extend the partial arc-coloring c of D in the following method again:

(1) Color the arcs of $P(x)$ for all vertex $x \in X^1$ with $\{c_1, c_2\}$ such that $P(x)$ is proper;

(2) Color the arcs of $Q(y)$ for all vertex $y \in Y^1$ with $\{c_1, c_2\}$ such that $Q(y)$ is proper;

(3) Color the uncolored arcs of $A(D)$ with either c_1 or c_2.

Note that we can construct such an arc-coloring c of D in time $O(n^2)$. Then we assert that (D, c) is proper connected. In fact, for any two vertices u and v in D, if $u, v \notin X^1 \cup Y^1$, then u and v are proper connected by Claim 3.7. If $u, v \in X^1$, then $us_0Cs_iP(v)$ and $vs_0Cs_iP(u)$ are two proper paths in D. If $u \in X^1$ and $v \in X_1^1$, then $us_0Cs_iP_{X_2}(v)$ and $vs_0Cs_iP(u)$ are two proper paths in D. If $u \in X^1$ and $v \in X_3^1$, then $us_0Cs_iP_{X_3}(v)$ and $vs_0Cs_iP(u)$ are two proper paths in D. If $u \in X^1$ and $v \in Y_1^1$, then $P_{Y_1}(v)CP(u)$ is a proper path in D. If $u \in X^1$ and $v \in Y_2^1$, then $P_{Y_2}(v)CP(u)$ is a proper path in D. If $u \in X^1$ and $v \in Y_3^1$, then $P_{Y_3}(v)CP(u)$ is a proper path in D. If $u \in X^1$ and $v \in Y^1$, then $Q(v)CP(u)$ is a proper path in D. If $u \in X^1$ and $v \in V(C)$, then $vCP(u)$ is a proper path in D. If $u \in X^1$ and $v \in Z$, then $vs_jCP(u)$ is a proper path in D. By a similar discussion and combining with Claim 3.6, we can show that (D, c) is proper connected and $\overrightarrow{pc}(D) = 2$, the result thus follows. □

Combining Theorem 3.3, Theorem 3.4, Theorem 3.6 and Theorem 3.7, we can easily show Theorem 3.5.

Remark: Let $D_1 = C_{k+1}$ and let D_2 be a digraph on $k+2$ vertices consisting of a directed $(k+1)$-cycle $C = v_0v_1 \cdots v_kv_0$, together with one vertex v_{k+1}, such that $v_k \to v_{k+1} \to v_1$. Then we can obverse that D_1 is a k-quasi-transitive digraph with $diam(D_1) = k$ and D_2 is a k-quasi-transitive digraph with $diam(D_2) = k+1$. If $k \geq 4$ and k is even, then we can easily conclude that $\overrightarrow{pc}(D_i) = 3$, where $i = 1, 2$. Thus, the bound of the condition $diam(D) \geq k+2$ in Theorem 3.5 is sharp.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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