Proper vertex-pancyclicity of edge-colored complete graphs without joint monochromatic triangles[1]

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Abstract

In an edge-colored graph (G, c), let $d^c(v)$ denote the number of colors on the edges incident with a vertex v of G and $\delta^c(G)$ denote the minimum value of $d^c(v)$ over all vertices $v \in V(G)$. A cycle of (G, c) is called proper if any two adjacent edges of the cycle have distinct colors. An edge-colored graph (G, c) on $n \geq 3$ vertices is called properly vertex-pancyclic if each vertex of (G, c) is contained in a proper cycle of length ℓ for every ℓ with $3 \leq \ell \leq n$. Fujita and Magnant conjectured that every edge-colored complete graph on $n \geq 3$ vertices with $\delta^c(G) \geq \frac{n+1}{2}$ is properly vertex-pancyclic. Chen, Huang and Yuan partially solve this conjecture by adding an extra condition that (G, c) does not contain any monochromatic triangle. In this paper, we show that this conjecture is true if the edge-colored complete graph contain no joint monochromatic triangles.

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1 Introduction

Let G be a simple graph and let c be an edge-coloring of G. We call (G, c) an *edge-colored graph*. G is called a *c-edge-colored graph* if its edges are colored in c colors. A subgraph in an edge-colored graph is called *proper* if any two adjacent edges in the subgraph are colored by distinct colors. Rainbow subgraphs and monochromatic subgraphs are two popular concepts related to proper subgraphs. A subgraph in an edge-colored graph is called *rainbow* if all the edges in the subgraph are colored by distinct colors, and *monochromatic* if all the edges in the subgraph are colored by the same color.

Edge-colored graphs contribute more to model certain real life problems. Some of them concerning about genetic and molecular biology, such as determining the spatial order of chromosomes; see [10, 11, 12]. Recently, the reconstruction of RNA molecule structure has been obtained by Nuclear Magnetic Resonance (NMR). This model has been extended to the more complex 3D case in [21]. 3D NMR maps display the results of NMR experiments, that allow to determine the shape of a biological molecular. Then in [21], the problem has been formalized as OCLP (Orderly Colored Longest Path Problem), and the authors in [13, 20] proposed different optimization models on OCLP, based on search of the longest path on certain expanded graphs. The other power of edge-colored graphs in modeling different types of problems, including Chinese Postman Problem, has been extensively discussed in [7, 15, 18].

Actually, we care more about the Hamiltonian properly colored paths and cycles. The problems of determining the existence of alternating paths, trails and cycles in 2-edge-colored multigraphs were suggested in [4]. In recent work (Guo et al. [16, 17]]), sufficient conditions for the existence of more general compatible spanning circuits (a closed trail that contains each vertex of G) in specific edge-colored graphs have been established. The authors in [5] found a sufficient condition for a complete graph to have a properly colored Hamiltonian path. In this paper, we mainly consider proper cycles in an edge-colored graph. A characterization of c-edge-colored graphs containing properly colored cycles was presented by Yeo [22] and generalized in [1] for properly colored closed trails. In both cases, the proposed results were used to construct polynomial time algorithms to check whether an edge-colored graph contains a properly colored cycle or a properly colored closed trail.

Once a cycle is found and denoted by the cyclic arrangement of its vertices such that two vertices are adjacent if they are consecutive in the sequence and nonadjacent otherwise in an edge-colored graph, one can easily checked whether it is proper or not. So, we often omit the checking process in the following.

In an edge-colored graph (G, c), let $d^c_G(v)$ denote the number of colors on the edges incident with a vertex v of G and let $\delta^c(G)$ denote the minimum value of $d^c_G(v)$ over all vertices $v \in V(G)$. When no confusion occurs, we use $d^c(v)$ instead of $d^c_G(v)$. The *length* of a path or a cycle is the number of its edges. Let $\Delta^{mon}(K^c_n)$ denote the maximum number of edges of the same color incident with a vertex of K^c_n. An edge-colored graph (G, c) is called *properly Hamiltonian* if it contains a properly colored Hamiltonian cycle. An edge-colored graph (G, c) is called *properly vertex-pancyclic* if every vertex of the graph is contained in a proper cycle of each length ℓ for every ℓ with $3 \leq \ell \leq n$.

In 1952, Dirac [9] obtained a classical theorem that if $\delta(G) \geq \frac{n}{2}$, then G is Hamiltonian. Inspired by this work, there have appeared lots of results and problems on the existence of proper cycles in different types of edge-colored graphs. In 1976, Bollobas and Erdos [6] conjectured that every K^c_n with $\Delta^{mon}(K^c_n) < \lfloor \frac{n}{2} \rfloor$ contains a properly colored Hamiltonian cycle. The author in [2] showed that for any $\varepsilon > 0$, there exists an integer n_0 such that every K^c_n with $\Delta^{mon}(K^c_n) < (\frac{n}{2} - \varepsilon)n$ and $n \geq n_0$ contains a properly colored Hamiltonian cycle, which implies a result obtained by Alon and Gutin [3] that for every $\varepsilon > 0$ and $n > n_0(\varepsilon)$, any complete graph K_n on n vertices whose edges are colored so that no vertex is incident with more than $(1-1/\sqrt{2}-\varepsilon)n$ edges of the same color, contains a Hamiltonian cycle in which adjacent edges have distinct colors. Moreover, for every k between 3 and n, any such K_n contains a cycle of length k in which adjacent edges have distinct colors.

2 Preliminaries

Fujita and Magnant in [14] posed the following conjecture.

Conjecture 1 ([14]). *Let (G, c) be an edge-colored graph on $n \geq 3$ vertices. If $\delta^c(G) \geq \frac{n+1}{2}$, then G is properly Hamiltonian.*

They showed there that the condition $\delta^c(G) \geq \frac{n+1}{2}$ in Conjecture 1 is sharp by constructing an example in [14]. Then, they further posed the following conjecture.

Conjecture 2 ([14]). *Let (G, c) be an edge-colored complete graph on $n \geq 3$ vertices. If $\delta^c(G) \geq \frac{n+1}{2}$, then G is properly vertex-pancyclic.*

Chen, Huang and Yuan partially solved the conjecture by adding a condition that (G, c) does not contain any monochromatic triangle.

Theorem 2.1. *[8] Let (G, c) be an edge-colored complete graph on $n \geq 3$ vertices such that $\delta^c(G) \geq \frac{n+1}{2}$. If (G, c) contains no monochromatic triangles, then (G, c) is properly vertex-pancyclic.*

They employed a term named as "follower vertex"; see the following definition.

Definition 2.1. *Let $C = v_1 v_2 \dots v_\ell v_1$ be a cycle in an edge-colored graph (G, c) and let $v_{\ell+1} = v_1$ and $v_0 = v_\ell$. We say that a vertex $x \in V(G) \setminus V(C)$ follows the colors of C increasingly if $c(xv_i) = c(v_i v_{i+1})$ for all $i = 1, 2, \dots, \ell$, and a vertex $x \in V(G) \setminus V(C)$ follows the colors of C decreasingly if $c(xv_i) = c(v_i v_{i-1})$ for all $i = 1, 2, \dots, \ell$. In either of these cases, we say that the vertex $x \in V(G) \setminus V(C)$ "follows" the colors of C and it is also called a follower vertex.*

In the proof of Theorem 2.1, they showed two claims which is stated as follows since we will use them later in our proof of Lemma 3.2.

Claim 1. *Suppose there is a cycle C of length ℓ containing v_1, but no proper cycle of length $\ell + 1$ containing v_1 in (G, c), and suppose there is no monochromatic triangle containing two vertices in $V(C)$ and a vertex in $V(G) \setminus V(C)$. If there are two vertices which follow the colors of C in different directions, then $c(v_i v_{i+1}) = c(v_{i+2} v_{i+3})$ for all indices i with $1 \leq i \leq \ell - 1$, which implies that C is an even cycle with two colors appearing alternatively on C.*

Claim 2. *Suppose there is a cycle C of length ℓ containing v_1, but no proper cycle of length $\ell + 1$ containing v_1 in (G, c), and suppose there is no monochromatic triangle contains two vertices in $V(C)$ and a vertex in $V(G) \setminus V(C)$. If the number of follower vertices is larger than 2, then*

(1) for every follower vertex w and every two distinct vertices v_i and v_j in C, we have $c(wv_i) \neq c(wv_j)$, and

(2) C has the DP_w for every $w \in W_2$, where DP_w is defined in Definition 2.4.

In this paper, we solve the conjecture by adding a looser condition that the edge-colored complete graph can have monochromatic triangles but not any two joint monochromatic triangles. Our main result is stated as follows.

Theorem 2.2. *Let (G, c) be an edge-colored complete graph on $n \geq 3$ vertices. If $\delta^c(G) \geq \frac{n+1}{2}$ and (G, c) contains no joint monochromatic triangles, then (G, c) is properly vertex-pancyclic.*

The following new definitions are needed in the sequel.

Definition 2.2. *Let v_i and v_j be two distinct vertices on a cycle C. The distance between v_i and v_j (denoted by d_{ij}) is the length of the shortest path of $v_i\overrightarrow{C}v_j$ and $v_i\overleftarrow{C}v_j$. Apparently, $d_{ij} = d_{ji} = min\{|i-j|, |j-i|, |i+l-j|, |j+l-i|\} \leq \frac{l}{2}$. Furthermore, we say that v_i is in front of v_j on C if $d_{ij} = |v_i\overrightarrow{C}v_j|$.*

The authors in [14] gave a property on set version.

Definition 2.3. (Set version) *In an edge-colored complete graph (G, c), a set A of vertices is said to have dependence property with respect to a vertex $v \notin A$ (denoted by DP_v) if $c(aa') \in \{c(va), c(va')\}$ for every two vertices $a, a' \in A$.*

Then, based on the definition on set version, we give a similar definition on vertex version.

Definition 2.4. (Vertex version) *In an edge-colored complete graph (G, c), a pair (u, w) of distinct vertices is said to have dependence (independence) property with respect to another vertex v (denoted by DP_v) if $c(uw) \in \{c(vu), c(vw)\}$ ($c(uw) \notin \{c(vu), c(vw)\}$) for every two vertices $u, w \in A$. The set of these vertices pairs is denoted by D_v (I_v).*

The following is an important fact appearing in [14], which will be used later.

Fact 1. *[14] If a set A of vertices in an edge-colored complete graph (G, c) has the DP_v for some vertex $v \notin A$, then there exists a vertex $a \in A$ such that*

a) $d^c_A(a) \leq \frac{|A|+1}{2}$, and

b) if $|A| \geq 2$, then at least one of the colors used at a in A is $c(va)$.

Theorem 2.3. *[14] Let (G, c) be an edge-colored completed graph on $n \geq 3$ vertices such that $\delta^c(G) \geq \frac{n+1}{2}$. Then every vertex of (G, c) is contained in a rainbow triangle.*

Theorem 2.4. *[14] Let (G, c) be an edge-colored completed graph on $n \geq 3$ vertices such that $\delta^c(G) \geq \frac{n+1}{2}$. If $n \geq 4$, then every vertex is contained in a proper cycle of length 4, and if $n \geq 13$, every vertex is contained in a proper cycle of length at least 5.*

In this paper, a subgraph induced by $V(C)$ union a vertex $w \in V(G) \setminus V(C)$ is denoted by G_C. Then, from Chen, Huang and Yuan [8] we can get the following properties about G_C.

Proposition 2.1. *Suppose there is no proper cycle of length $\ell+1$ containing v_1 in (G,c). Let P be a proper path on C. Let v_a and v_b be two distinct vertices in $V(P)$ such that $c(wv_a) = c(wv_b)$. If there are no monochromatic triangles containing w in (G_C, c), then $c(wv_{a-i}) = c(wv_{b-i})$ ($c(wv_{a+i}) = c(wv_{b+i})$) for $1 \le i \le \ell$ if w follows the colors of C increasingly (decreasingly), ($v_{a\pm i}$ and $v_{b\pm i}$ are in $V(P)$).*

Proposition 2.2. *Suppose there is no proper cycle of length $\ell+1$ containing v_1 in (G,c). Let P be a proper path on C. If w follows the colors of P increasingly (decreasingly) and there are two vertices $v_i, v_j \in V(P)$ such that v_j is in front of v_i and $c(wv_i) \neq c(wv_j)$, then we have $(v_{i+1}, v_{j+1}) \in D_w$ ($(v_{i-1}, v_{j-1}) \in D_w$).*

Proposition 2.3. *Suppose there is no proper cycle of length $\ell+1$ containing v_1 in (G,c). Let P be a proper path on C. If w follows the colors of P increasingly (decreasingly) and there are two vertices $v_i, v_j \in V(P)$ such that v_j is in front of v_i and $(v_i, v_j) \notin D_w$, then we have $c(wv_{i-1}) = c(wv_{j-1})$ ($c(wv_{i+1}) = c(wv_{j+1})$).*

3 Proof of Theorem 2.2

In this section we will use a few lemmas and propositions to prove our main result Theorem 2.2.

Let V_1 be a subset of $V(G)$ and w a vertex of G not in V_1. We give a vertex-induced subgraph $G[V_1]$ a coloring orientation. First, we orient the edges whose ends are the vertex pairs in D_w, that is, orient v_iv_j by $\overrightarrow{v_iv_j}$ if $c(v_iv_j) = c(wv_i)$, v_iv_j by $\overrightarrow{v_jv_i}$ if $c(v_iv_j) = c(wv_j)$, and arbitrarily orient v_iv_j if $c(wv_i) = c(wv_j)$. Next, orient v_iv_j if $(v_i, v_j) \in I_w$ by two inverse arcs $\overrightarrow{v_iv_j}$ and $\overleftarrow{v_iv_j}$. Thus, we get a digraph $D(G[V_1])$ of $(G[V_1], c)$, and $d^c_{G[V_1]}(v) \le d^-_{D(G[V_1])}(v) + 1$.

In the following, we always assume that (G,c) is an edge-colored complete graph on $n \ge 3$ vertices such that $\delta^c(G) \ge \frac{n+1}{2}$, and does not contain any joint monochromatic triangles. From Theorems 2.3 and 2.4, we know that every vertex v of (G,c) is contained in some proper cycles of lengths 3 and 4. To prove that (G,c) is properly vertex-pancyclic, it suffices to show that if a vertex is contained in a proper ℓ-cycle in (G,c) for some ℓ with $4 \le \ell \le n-1$, then it is also contained in a proper $(\ell+1)$-cycle in (G,c).

Suppose that (G,c) has a proper cycle $C = v_1v_2\cdots v_\ell v_1$ of length ℓ and let $v_1 = v$. Let $W_1(C)$ be the set of vertices in $V(G) \setminus V(C)$ such that for each vertex $w \in W_1$, the edges in $\partial(w, V(C))$ have just one color in (G,c). Let $W_2(C)$ be the set of vertices in $V(G) \setminus V(C)$ such that each vertex $w \in W_1$ follows the colors of C. Let $W_3(C) =$

$V(G)\setminus(V(C)\cup W_1(C)\cup W_2(C))$. Note that $V(C), W_1(C), W_2(C), W_3(C)$ form a partition of $V(G)$. For convenience, if $v_k \in V(C)$, we regard v_k and $v_{k+\ell}$ (or $v_{k-\ell}$) as the same vertex in the sequel. First, we analyze the coloring structure on $(G[V(C)\cup w], c)$ for $w \in W_3(C)$.

For each vertex $v \in V(G)$, we use $\partial(v)$ to denote the set of edges incident to v in G. Moreover, for two disjoint subsets $X, Y \subseteq V(G)$, we use $\partial(X,Y)$ to denote the set of edges between X and Y in G, i.e., $\partial(X,Y) = \{xy \in E(G) : x \in X, y \in Y\}$. $\partial(\{x\}, Y)$ will be simply written as $\partial(x, Y)$. The colors of $\partial(\{x\}, Y)$ will be simply written as $\mathcal{C}(x, Y)$, that is, $\mathcal{C}(x,Y) = \{c(xy) \mid y \in Y\}$. For a cycle $C = v_1v_2\dots v_\ell v_1$ and two vertices $v_i, v_j \in V(C)$ with $1 \le i \le j \le \ell$, we use $v_i\overrightarrow{C}v_j$ and $v_i\overleftarrow{C}v_j$ to denote the paths $v_iv_{i+1}\dots v_j$ and $v_iv_{i-1}\dots v_1v_\ell v_{\ell-1}\dots v_j$, respectively.

Lemma 3.1. *Suppose there is no proper cycle of length $\ell+1$ containing v_1 in (G,c). Then for any vertex w in $W_3(C)$, we can find three and only three vertices $v_{x(w)}$, $v_{y(w)}$ and $v_{z(w)}$ in $V(C)$ which can divide $V(C)$ into three subsets:*

$$\begin{cases} P_C^1(w) &= \{v_{x(w)}, v_{x(w)+1}, \dots, v_{y(w)}\}, \\ P_C^2(w) &= \{v_{y(w)+1}, v_{y(w)+2}, \dots, v_{z(w)}\}, \\ P_C^3(w) &= \{v_{z(w)+1}, v_{z(w)+2}, \dots, v_{x(w)-1}\}, \end{cases}$$

such that

$$(1)\ \begin{cases} c(wv_{x(w)}) = c(v_{x(w)}v_{x(w)+1}) \text{ while } c(wv_{x(w)-1}) \ne c(v_{x(w)}v_{x(w)-1}), \\ c(wv_{z(w)}) = c(v_{z(w)}v_{z(w)-1}) \text{ while } c(wv_{z(w)+1}) \ne c(v_{z(w)}v_{z(w)+1}), \\ c(wv_{y(w)}) = c(wv_{y(w)+1}) = c(v_{y(w)}v_{y(w)+1}); \end{cases}$$

$$(2)\ \begin{cases} c(wv_i) = c(v_iv_{i+1}) & \text{for } v_i \in P_C^1(w), \\ c(wv_i) = c(v_iv_{i-1}) & \text{for } v_i \in P_C^2(w), \\ c(wv_i) = c(wv_j) & \text{for } v_i, v_j \in P_C^3(w). \end{cases}$$

Proof. (1) Since w does not follow the colors of C, suppose, to the contrary, that $c(wv_i) \ne c(v_iv_{i+1})$ for all v_i in $V(C)$. Thus, (G_C, c) has no monochromatic triangles containing w. Then from Lemma, we know that w follows the colors of C decreasingly or w is a single color vertex of C, a contradiction. Thus, there exist two requested vertices $v_{x(w)}$ and $v_{z(w)}$.

As we know, $c(wv_{i+1}) \in \{c(wv_i),\ c(v_{i+1}v_{i+2})\}$ if $c(wv_i) = c(v_iv_{i+1})$ and $c(wv_{i-1}) \in \{c(wv_i),\ c(v_{i-1}v_{i-2})\}$ if $c(wv_i) = c(v_iv_{i-1})$. Since w dose not follow the colors of C, there exist two vertices $v_{y_1(w)}$ and $v_{y_2(w)}$ such that $c(wv_{y_1(w)}) = c(wv_{y_1(w)-1})$ and $c(wv_{y_2(w)}) =$

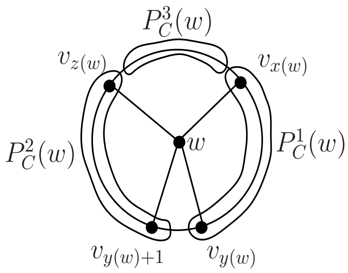

Figure 1: $G[V(C) \cup \{w\}]$ used in the proof of Lemma 3.1.

$c(wv_{y_2(w)+1})$. Thus, $wv_{y_1(w)}v_{y_1(w)-1}$ and $wv_{y_2(w)}v_{y_2(w)+1}$ are monochromatic. Then, we have $v_{y(w)} = v_{y_2(w)} = v_{y_1(w)-1}$. As (G, c) has no joint monochromatic triangles, $v_{x(w)}$, $v_{y(w)}$ and $v_{z(w)}$ are the only three requested vertices.

(2) Since $v_{x(w)}$, $v_{y(w)}$ and $v_{z(w)}$ are the only three vertices satisfying (1), for any vertex $v_i \in P_C^3(w)$, we have $c(wv_i) \notin \{c(v_iv_{i-1}), c(v_iv_{i+1})\}$. If there is a vertex v_j such that $c(wv_i) \neq c(wv_j)$, then there must exist two adjacent vertices $v_k, v_{k+1} \in P_C^3(w)$ on C such that $c(wv_k) \neq c(wv_{k+1})$. Thus, $v_kwv_{k+1}\overrightarrow{C}v_k$ is a requested cycle; see Figure 1, a contradiction. ∎

Note that if C is a proper cycle of length ℓ but does not contain v_1, then we can also divide $V(C)$ into three parts such that each part satisfying the results stated in Lemma 3.1. According to Lemma 3.1, for any $w \in W_3(C)$, $wv_{y(w)}v_{y(w)+1}$ is a monochromatic triangle. We denote the color of this monochromatic triangle by c_w. Since (G, c) has no joint monochromatic triangles, $|W_3(C)| \leq \frac{\ell}{2}$.

Lemma 3.2. *Suppose there is no proper cycle of length $\ell + 1$ containing v_1 in (G, c). If $|W_2(C)| \geq 2$, then*

(1) for every vertex $w \in W_2(C)$ and every two distinct vertices v_i and v_j in C, we have $c(wv_i) \neq c(wv_j)$, and

(2) C has the DP_w for every $w \in W_2(C)$.

Proof. First, we can assert that all vertices in W_2 following the colors of C in the same direction. Suppose the contrary holds. Then, from Claim 1 we have $c(w_1v_i) = c(w_1v_{i+2})$ for all indices i with $1 \leq i \leq \ell$. Since (G, c) has no joint monochromatic triangles, there exists at most one index k_i for $w_i \in W_2(C)$ such that $c(w_iv_{k_i}) = c(v_{k_i}v_{k_i+2})$ with $1 \leq k_i \leq \ell$. Hence, $v_\ell w_1v_1v_3w_2v_4\overrightarrow{C}v_\ell$, $v_1w_1v_2v_4w_5v_4\overrightarrow{C}v_1$ and $v_{\ell-2}w_1v_{\ell-1}v_1w_2v_2\overrightarrow{C}v_{\ell-2}$ are cycles of length $\ell + 1$ containing v_1. Then, we can easily verify that at least one of them is proper, a contradiction. The assertion follows. Thus, there are no monochromatic triangles containing a vertex in $W_2(C)$ and two vertices in $V(C)$. Then, from Claim 2 we can get the result. ∎

In the following, we consider the relation between $W_2(C)$ and $W_3(C)$. According to Lemma 3.1, we define some new vertex sets. Let $R_C(w) = V(v_{y(w)}\overleftarrow{C}v_{r(w)}) \subseteq P_C^1(w)$ such that $v_{y(w)+1}v_{y(w)}\ldots v_{r(w)}$ is a longest rainbow subpath of $v_{y(w)+1}v_{y(w)}\ldots v_{x(w)}$. In a similar way, we define $Q_C(w) = V(v_{y(w)+1}\overrightarrow{C}v_{q(w)}) \subseteq P_C^2(w)$. For convenience, we relabel the vertices of C depending on $w \in W_3(C)$ on a clockwise direction by $u_1u_2\cdots u_\ell$ such that $u_1 = v_{y(w)}$, $u_b = v_{q(w)}$, $u_t = v_{z(w)}$, $u_s = v_{x(w)}$ and $u_a = v_{r(w)}$.

Lemma 3.3. *Suppose there is no proper cycle of length $\ell+1$ containing v_1 in (G,c). If $W_2(C) \neq \emptyset$, then when $|W_3(C)| \geq 2$, C is an even cycle with two colors appearing alternatively on C. Furthermore, $|W_2(C)| \leq 1$.*

Proof. Since $|W_3(C)| \geq 2$, there is a vertex $w \in W_3$ such that $u_1 \neq v_1$. Without loss of generality, suppose $w' \in W_2(C)$ follows the colors of C increasingly. To avoid $u_{\ell-1}w'u_\ell wu_2\overrightarrow{C}u_{\ell-1}$ being a requested cycle, we have $c(wu_\ell) \in \{c(wu_2), c(w'u_\ell)\}$. If $c(wu_\ell) = c(w'u_\ell)$, then $c(ww') \notin \{c(wu_2), c(w'u_\ell)\}$. Hence, $u_\ell w'wu_2\overrightarrow{C}u_\ell$ is a requested cycle. Thus, $c(wu_\ell) = c(wu_2)$. Then, $u_\ell \notin P_C^1(w)$. Since $w \notin W_1$, we have $|P_C^2(w)| \geq 2$, that is, $u_3 \in P_C^2(w)$. To avoid $u_\ell w'u_2wu_3\overrightarrow{C}u_\ell$ being a requested cycle, we have $c(w'u_2) = c(w'u_\ell)$. Thus, C is an even cycle with two colors appearing alternatively on C. According to Lemma 3.2, we know that $|W_2(C)| \leq 1$. ∎

Proposition 3.1. *Let (G,c) be an edge-colored complete graph on $n \geq 3$ vertices such that $\delta^c(G) \geq \frac{n+1}{2}$, and not contain any joint monochromatic triangles.*

(1) If $W_1(C) \neq \emptyset$, then (G,c) is properly vertex-pancyclic.

(2) If $|W_2(C)| \geq 2$, then (G,c) is properly vertex-pancyclic.

(3) If $|W_2(C)| = 1$ and $W_3(C) = \emptyset$, then (G,c) is properly vertex-pancyclic.

Proof. Suppose, to the contrary, that there exists a vertex v which is contained in a proper ℓ-cycle C in (G,c) for some ℓ with $4 \leq \ell \leq n-1$, but no proper cycle of length $\ell+1$ in (G,c) contains vertex v.

(1) Let $w \in W_1(C)$ and $w' \in W_2(C)$. If $c(ww') \neq c(w, V(C))$, then at least one of $v_1w_2w_1v_3\overrightarrow{C}v_1$, $v_2w_2w_1v_4\overrightarrow{C}v_2$ and $v_3w_2w_1v_5\overrightarrow{C}v_3$ is a proper cycle of length $\ell+1$ containing v_1. Thus, $c(ww') = c(w, V(C))$, that is $c(w, W_2(C)) = c(w, V(C))$.

If $|W_1(C)| = 1$, then for $w \in W_1(C)$, we have $d^c(w) \leq 1 + |W_3| \leq 1 + \frac{\ell}{2} = \frac{\ell+2}{2} < \frac{n+1}{2}$, a contradiction. If $|W_1(C)| \geq 2$, then we can assert has $W_1(C)$ has the DP_{v_1}. Since

(G, c) has no joint monochromatic triangles, there is at most one edge on C colored by $c(w, V(C))$. Suppose, to the contrary, that $W_1(C)$ has no DP_{v_1}. Then, there are two vertices $w_1, w_2 \in W_1$ such that $c(w_1w_2) \notin \{c(v_1w_1), c(v_1w_2)\}$. Then, at least one of $v_1w_1w_2v_3\overrightarrow{C}v_1$ and $v_1w_2w_1v_3\overrightarrow{C}v_1$ is a requested cycle, a contradiction. According Fact 1, there exists a vertex $w \in W_1(C)$ such that $d^c_{W_1(C)}(w) \leq \frac{|W_1(C)|+1}{2}$, and at least one of the colors used in W_1 at w is $c(wv_1)$. Hence, $d^c(w) \leq \frac{|W_1(C)|+1}{2}+|W_3(C)| \leq \frac{|W_1(C)|+1}{2}+\frac{\ell}{2} < \frac{n+1}{2}$, a contradiction.

In the following, we might as well suppose $W_1(C) = \emptyset$.

(2) Since $|W_2(C)| \geq 2$, from Lemma 3.2, $V(C)$ has the DP_w for $w \in W_2(C)$. If $W_3(C) = \emptyset$, the result follows. If $W_3(C) \neq \emptyset$, then from Lemma 3.3 we have $|W_3(C)| \leq 1$. Thus, there is a vertex $v \in V(C)$ such that $d^c(v) \leq \frac{|W_2(C)+1|}{2} + 1 < \frac{n+1}{2}$.

(3) According to the proof of Case 3 of Theorem 2.1, (G, c) has a monochromatic triangle containing w. Otherwise, $V(C)$ has the DP_w. Then, there is a vertex $v \in V(C)$ such that $d^c(v) < \frac{n+1}{2}$. Let v_a and v_b be two distinct vertices on C such that $c(v_a, v_b) \in I_w$ (suppose may as well v_a is in front of v_b on C). If $wv_iv_{i+d_{a,b}}$ is not monochromatic for any $1 \leq i \leq \ell$, then $c(wv_{a-k}) = c(wv_{b-k})$ for $k = 0, 1, \ldots, \ell-1$. Consequently, $c(wv_x) = c(wv_{x+kd_{a,b}})$ for every vertex $v_x \in V(C)$ and for every positive integer k (except 1 when $v_x = v_a$). Furthermore, let (v_a, v_b) be such a vertex pair in I_w that $d_{a,b} = \min\{d_{i,j} \mid (d_i, d_j) \in I_w\}$, then $\ell \equiv 0 \pmod{d_{a,b}}$.

Let wv_xv_y be a unique monochromatic triangle containing w (we may as well suppose v_x is in front of v_y on C). Then, we assert $d_{x,y} = d_{a,b}$. Suppose, the contrary holds. Then, we can get that $c(wv_i) = c(wv_{i+kd_{a,b}})$ for every vertex $v_i \in V(C)$ and for every positive integer k (except 1 when $v_i = v_a$). Thus, $c(wv_x) = c(wv_{x+kd_{a,b}})$ for every positive integer k. Then, $d_{x,y} = qd_{a,b}$ where $q = 2, \cdots, \frac{n-1}{d_{a,b}}$. Otherwise, there is an integer p such that $d_{x+pd_{a,b},y} < d_{a,b}$. Since $c(wv_{x+pd_{a,b}}) = c(wv_x) = c(wv_y)$, we have $(v_{x+pd_{a,b}}, v_y) \in I_w$, which contradicts the choice of (v_a, v_b). Therefore, $\frac{n-1}{d_{a,b}} \geq 3$. Hence, $d^c(w) \leq 1+d_{a,b} \leq 1+\frac{n-1}{3} < \frac{n+1}{2}$, a contradiction.

For convenience, we relabel the vertices of C by $s_1s_2\cdots s_\ell$ on a clockwise direction such that $s_1 = s_x$ and $s_{1+d_{a,b}} = v_y$. Let s_q be such a vertex that $(s_q, s_{q+d_{a,b}}) \in I_w$ while $(s_{q+1}, s_{q+d_{a,b}+1}) \in D_w$. Then, from Proposition 2.3 we have $(s_i, s_{i+d_{a,b}}) \in D_w$ for $q+1 < i \leq \ell$ and $c(ws_i) = c(ws_{i+d_{a,b}})$ for $1 \leq i < q$. Since (G, c) has no joint monochromatic, $\{(s_i, s_{i+d_{a,b}}), 1 < i < q\} \subseteq I_w$. Suppose that there is such a vertex pair (s_f, s_g) in I_w while not in $\{(s_i, s_{i+d_{a,b}}), 1 < i < q\}$ (we may as well suppose s_f is in front of s_g). Then, according to the above content, we have $d_{g,f} > d_{a,b}$. Thus, $c(ws_i) = c(ws_{i+kd_{g,f}})$

for every vertex $s_i \in V(C)$ and for every positive integer k (except 1 when $s_i = s_f$). Then, $\mathcal{C}(w, C) = \{c(ws_i) \mid s_i \in V(s_1 \overrightarrow{C} s_{1+d_{f,g}})\}$. Since $c(ws_1) = c(ws_{1+d_{a,b}})$, $d^c(w) \leq 1 + d_{f,g} - 1 < \frac{n+1}{2}$. Thus, $\{(s_i, s_{i+d_{a,b}}), 1 < i < q\} = I_w$.

Let $V_1 = V(s_{q+d_{a,b}} \overrightarrow{C} s_\ell)$ and $V_2 = V(C) \setminus V_1$. Since $V_1 \times V_1 \cap I_w = \emptyset$, V_1 has the DP_w. Thus, there is a vertex $s \in V_1$ such that $d^c_{V_1}(s) \leq \frac{|V_1|+1}{2} = \frac{n-q-d_{a,b}+1}{2}$. Since $(s, V_2) \cap I_w = \emptyset$, $c(ss_i) \in \{c(ws), c(ws_i)\}$ for $s_i \in V_2$. Thus, $\mathcal{C}(s, V_2) \subseteq \mathcal{C}(w, V_2) \cup \{c(ws)\}$. Hence, $d^c(s) \leq \frac{n-q-d_{a,b}+1}{2} + d_{a,b}$. Apparently, $q < d_{a,b}$. Now we give a coloring orientation for $(G - w, c)$. Apparently, the edge set which is oriented arbitrarily is $\{s_i s_{i+d_{a,b}}, 2 \leq i \leq q\}$. Thus, $D(G - w)$ has at most $\frac{(n-1)(n-2)}{2} + q - 1$ arcs. Therefore,

$$\begin{aligned} d^c(G - w) &\leq d^-(D(G - w)) + 1 \\ &\leq \frac{n-2}{2} + \frac{q-1}{n-1} + 1 \\ &< \frac{n}{2} + \frac{d_{a,b} - 1}{n-1} \\ &< \frac{n+1}{2}. \end{aligned}$$

There exists a vertex with color degree less than $\frac{n+1}{2}$. ∎

Note that there is a special class of vertices w_i in $W_3(C)$ such that $|P^1_C(w_i)| = |P^2_C(w_i)| = 1$. By repeating the proof procedure of Proposition 3.1, we can get Proposition 3.2.

Proposition 3.2. *Let (G, c) be an edge-colored complete graph on $n \geq 3$ vertices and with no joint monochromatic triangles such that $\delta^c(G) \geq \frac{n+1}{2}$. If there exist such vertices $w_i \in W_3(C)$ that $|P^1_C(w_i)| = |P^2_C(w_i)| = 1$, then (G, c) is properly vertex-pancyclic.*

Hence, in the following we suppose that either $|P^1_C(w)|$ or $|P^2_C(w)|$ is larger than 1 for each $w \in W_3$.

Lemma 3.4. *Suppose there is no proper cycle of length $\ell + 1$ containing v_1 in (G, c). If $|W_2(C)| \leq 1$, then $|\mathcal{C}(w, C)| \geq 3$ for $w \in W_3(C)$.*

Proof. The result follows when $|W_3(C)| = 1$. Suppose now $|W_3(C)| \geq 2$. If $W_2 \neq \emptyset$, then from Lemma 3.3, we know that C is an even cycle with two colors appearing alternatively on C. Let $w_1 \in W_2$ and $w_2 \in W_3$. Without loss of generality, assume that w_1 follows the colors of C increasingly. Then, we assert $c(w_1w_2) \in \mathcal{C}(w_2, C)$. If $u_1 \neq v_1$, then to avoid $u_l w_1 w_2 u_2 \overrightarrow{C} u_1$ being a requested cycle, we have $c(w_1w_2) \in \{c(w_2u_2), c(w_1u_1)\} \subseteq \mathcal{C}(w_2, C)$. If $u_1 = v_1$, then to avoid $w_1 u_4 u_3 u_5 \overrightarrow{C} u_1 w_2 w_1$ being a requested cycle, we have $c(w_1w_2) \in \{c(w_2u_4), c(w_1u_1)\} \subseteq \mathcal{C}(w_2, C)$. Thus, $d^c(w) \leq |\mathcal{C}(w, C)| + |W_3(C)| - 1$. Then, $|\mathcal{C}(w, C)| \geq 3$. ∎

In the following we prove some lemmas to make the coloring structure of $(G[V(C) \cup \{w\}], c)$ clear for $w \in W_3(C)$.

Lemma 3.5. *Suppose there is no proper cycle of length $\ell+1$ containing v_1 in (G, c). For $w \in W_3(C)$, if $|P_C^3(w)| \geq 2$, then $R_C(w) = P_C^1(w)$ and $Q_C(w) = P_C^2(w)$.*

Proof. We prove Lemma 3.5 by contradiction. Assume that there is a vertex $u_i \in R_C(w)$ such that $c(wu_s) = c(wu_i)$. Then, $(u_s, u_i) \in I_w$. From Proposition 2.3, we have $c(wu_{s-1}) = c(wu_{i-1})$. Since $|P_C^3(w)| \geq 2$, $c(wu_{s-2}) = c(wu_{i-1}) \neq c(wu_{i-2})$. Thus, $c(u_{s-1}u_{i-1}) \in \{c(u_su_{s-1}), c(u_{i-1}u_i)\}$; otherwise, $wu_{i-2}\overleftarrow{C}u_{s-1}u_{i-1}\overrightarrow{C}u_{s-2}w$ is a requested cycle. Then, $c(u_{s-1}u_{i-1}) \neq c(u_{s-2}u_{s-1})$. Therefore, $wu_{i-2}\overleftarrow{C}u_su_i\overrightarrow{C}u_{s-1}u_{i-1}w$ is a requested cycle, a contradiction. ∎

Now we define a cycle $C_{u_i} = u_{i-1}wu_{i+1}\overrightarrow{C}u_{i+1}$ where $u_i \in \{u_1, u_2\}$. Taking an example of which C_{u_1} is proper, we can get a conclusion. Suppose there is no proper cycle of length $\ell+1$ containing v_1 in (G, c). If there is a vertex u_i such that $c(u_1u_i) \neq c(u_iu_{i+1})$, then $c(u_1u_{i-1}) \in \{c(u_1u_i), c(u_{i-1}u_{i-2})\}$. Otherwise, $wu_2\overrightarrow{C}u_{i-1}u_1u_i\overrightarrow{C}u_\ell w$ is a requested cycle. Thus, $c(u_1u_{i-1}) \neq c(u_iu_{i-1})$. By repeating this proof procedure, we can get that $c(u_1u_k) \in \{c(u_1u_{k+1}), c(u_{k-1}u_k)\}$ and $c(u_1u_k) \neq c(u_ku_{k+1})$ for $u_k \in V(u_{i-1}\overleftarrow{C}u_2)$. Notice that if $c(u_1u_j) = c(u_1u_i)$, then $c(u_1u_{j-1}) = c(u_1u_i)$, and once there is a vertex u_j such that $c(u_1u_j) = c(u_ju_{j-1})$, then $c(u_1u_k) = c(u_ku_{k-1})$ for $u_k \in V(u_j\overleftarrow{C}u_2)$. Consequently, $c(u_1u_k) \in \{c(u_1u_i), c(u_ku_{k-1})\}$ for $u_k \in V(u_{i-1}\overleftarrow{C}u_2)$. In a similar way, if there is a vertex u_i such that $c(u_1u_i) \neq c(u_iu_{i-1})$, then $c(u_1u_k) \in \{c(u_1u_i), c(u_ku_{k+1})\}$ for $u_k \in V(u_{i+1}\overrightarrow{C}u_\ell)$.

Lemma 3.6. *Suppose there is no proper cycle of length $\ell+1$ containing v_1 in (G, c). For $w \in W_3(C)$, we have*

(1) $R_C(w) \setminus \{u_a\}$ and $Q_C(w) \setminus \{u_b\}$ has the DP_w;

(2) $(R_C(w) \setminus \{u_a\}, Q_C(w) \setminus \{u_b\}) \subseteq D_w$;

(3) if neither $P_C^1(w) \setminus R_C(w)$ nor $P_C^2(w) \setminus Q_C(w)$ is empty, and then $(u_a, Q_C(w) \setminus \{u_2\}) \cup (R_C(w) \setminus \{u_1\}, u_b) \subseteq D_w$.

Proof. (1) Suppose, to the contrary, that there exist two vertices u_i and u_j in $R_C(w) \setminus \{u_a\}$ that have no DP_w (we might as well assume that u_i is in front of u_j). Then, according to Proposition 2.3, we have $c(wu_{i-1}) = c(wu_{j-1})$, a contradiction.

(2) Suppose, to the contrary, that $(u_i, u_j) \in I_w$, where $u_i \in R_C(w) \setminus \{u_a\}$ and $u_j \in Q_C(w) \setminus \{u_b\}$. Let $C' = wu_2\overrightarrow{C}u_ju_i\overrightarrow{C}u_1u_{i-1}\overleftarrow{C}u_{j+1}w$ be a cycle of length $\ell+1$ containing

v_1. If $u_{i-1} \neq u_a$, then according to (1) we have $c(u_1u_{i-1}) \in \{c(wu_1), c(wu_{i-1})\}$. Thus, $c(u_1u_{i-1}) \notin \{c(u_1u_2), c(u_{i-1}u_{i-2})\}$. Since $c(wu_1) \neq c(wu_{j+1})$, C' is proper, a contradiction. If $u_{i-1} = u_a$, then we can get that $c(u_1u_a) \in \{c(u_1u_l), c(u_au_{a-1})\}$. Otherwise, C' is proper. Thus, $c(u_1u_a) \neq c(v_av_{a+1})$. Hence, $c(u_1u_k) \in \{c(u_1u_{k+1}), c(u_ku_{k-1})\}$ for $u_k \in V(u_{a-1}\overrightarrow{C}u_2)$. Then, $c(u_1u_{j+1}) \neq c(u_{j+1}u_{j+2})$. Therefore, $wu_2\overrightarrow{C}u_ju_i\overrightarrow{C}u_1u_{j+1}\overrightarrow{C}u_{i-1}w$ or $wu_\ell\overleftarrow{C}u_iu_j\overleftarrow{C}u_1u_{j+1}\overrightarrow{C}u_{i-1}w$ is a requested cycle, a contradiction.

(3) We prove this statement by classified discussion. If $c(wu_{a-1}) = c_w$, then $c(u_1u_{a-1}) \neq c(u_{a-1}u_a)$. Since C_{u_1} is proper, $c(u_1u_k) \in \{c(u_1u_{a-1}), c(u_ku_{k-1})\}$ for $u_k \in V(u_{a-2}\overleftarrow{C}u_2)$. Therefore, $c(u_1u_k) \notin \{c_w, c(u_ku_{k+1})\}$ for $u_k \in P_C^2(w)$. Assume that there exist two vertices $u_i \in R_C(w) \setminus \{u_1\}$ and $u_j \in Q_C(w) \setminus \{u_2\}$ such that $(u_i, u_j) \in I_w$. Then, $c(u_iu_j) \notin \{c(u_iu_{i+1}), c(u_ju_{j-1})\}$. Since $c(u_1u_{j+1}) \notin \{c_w, c(u_{j+1}u_{j+2})\}$ and $c(wu_{i-1}) \neq c(wu_l)$, $wu_l\overleftarrow{C}u_iu_j\overleftarrow{C}u_1u_{j+1}\overrightarrow{C}u_{i-1}w$ is a requested cycle, a contradiction. Symmetrically, the result holds if $c(wu_{b+1}) = c_w$. Now suppose $c_w \notin \{c(wu_{a-1}), c(wu_{b+1})\}$. By repeating the proof procedure of (2), we know that $(u_a, Q_C(w) \setminus \{u_b\}) \cup (R_C(w) \setminus \{u_a\}, u_b) \subseteq D_w$. Hence, for $u_i, u_j \in R_C(w) \cup Q_C(w) \setminus \{u_1, u_a, u_b\}$ we have $c(wu_i) \neq c(wu_j)$. We prove $(u_a, u_b) \in D_w$ in the following cases by contradiction.

If $c(wu_{a-1}) = c(wu_\ell)$, then $c(u_1u_{a-1}) \in \{c(u_{a-1}u_{a-2}), c(u_1u_\ell)\}$; otherwise, $wu_2\overrightarrow{C}u_bu_a\overrightarrow{C}u_1u_{a-1}\overleftarrow{C}u_{b+1}w$ is a requested cycle. When $c(u_1u_{a-1}) = c(u_{a-1}u_{a-2})$, we have $c(u_1u_k) = c(u_ku_{k-1})$ for $u_k \in V(u_{a-1}\overleftarrow{C}u_2)$. Hence, $c(u_1u_{b+1}) = c(u_bu_{b+1}) \notin \{c(u_1u_\ell), c(u_{b+1}u_{b+2})\}$. Then, $wu_2\overrightarrow{C}u_bu_a\overrightarrow{C}u_1u_{b+1}\overrightarrow{C}u_{a-1}w$ is a requested cycle, a contradiction. When $c(u_1u_{a-1}) = c(u_1u_\ell)$, we have $c(u_1u_{a-1}) \notin \{c(u_{a-1}u_{a-2}), c_w\}$. Then, $wu_\ell\overleftarrow{C}u_au_b\overleftarrow{C}u_1u_{a-1}\overrightarrow{C}u_{b+1}w$ is a requested cycle, a contradiction. Symmetrically, the result holds if $c(wu_{b+1}) = c(wu_3)$.

The last case is that $c(wu_{a-1}) \neq c(wu_\ell)$. According to Proposition 2.1, we know that there is a vertex $u_i \in R_C(w)$ such that $c(wu_s) = c(wu_i)$. Since $c(wu_1) \notin \{c(wu_{b+2}), c(wu_{i-1})\}$, $c(u_1u_{b+1}) \in \{c(u_{b+1}u_b), c(u_1u_\ell)\}$; otherwise, one of $wu_2\overrightarrow{C}u_{b+1}u_1\overleftarrow{C}u_{b+2}w$ and $wu_2\overrightarrow{C}u_{b+1}u_1\overleftarrow{C}u_iu_s\overrightarrow{C}u_{i-1}w$ is a requested cycle. At the same time we have $c(u_1u_{b+1}) \in \{c(u_{b+1}u_{b+2}), c(u_1u_2)\}$; otherwise, $wu_\ell\overleftarrow{C}u_au_b\overleftarrow{C}u_1u_{b+1}\overrightarrow{C}u_{a-1}w$ is a requested cycle. Therefore, $c(u_1u_{b+1}) = c(u_1u_\ell) = c(u_{b+1}u_{b+2})$. Thus, we can get $c(u_1u_k) = c(u_ku_{k+1})$ for $u_k \in V(u_{b+1}\overrightarrow{C}u_\ell)$, Hence, $c(u_1u_{a-1}) = c(u_{a-1}u_a) \notin \{c(u_1u_\ell), c(u_{a-2}u_{a-1})\}$. Then, $wu_2\overrightarrow{C}u_bu_a\overrightarrow{C}u_1u_{a-1}\overleftarrow{C}u_{b+1}w$ is a requested cycle, a contradiction. So far, we have completed the proof of (3). ∎

Lemma 3.6 (1) and (2) claim that for $u_i, u_j \in R_C(w) \cup Q_C(w) \setminus \{u_1, u_a, u_b\}$, we have $c(wu_i) \neq c(wu_j)$. Lemma 3.6 (3) claims that if neither $P_C^1(w) \setminus R_C(w)$ nor $P_C^2(w) \setminus Q_C(w)$ is empty, then for $u_i, u_j \in R_C(w) \cup Q_C(w) \setminus \{u_1\}$, we have $c(wu_i) \neq c(wu_j)$. Then, from

Lemmas 3.5 and 3.6, we can get the following result.

Proposition 3.3. *Let (G, c) be an edge-colored complete graph on $n \geq 3$ vertices such that $\delta^c(G) \geq \frac{n+1}{2}$, and not contain any joint monochromatic triangles. For any $w \in W_3(C)$, if neither $P^1_C(w) \setminus R_C(w)$ nor $P^2_C(w) \setminus Q_C(w)$ is empty, then (G, c) is properly vertex-pancyclic.*

Proof. Suppose, to the contrary, that there exists a vertex v which is contained in a proper ℓ-cycle C in (G, c) for some ℓ with $4 \leq \ell \leq n-1$, but no proper cycle of length $\ell+1$ in (G, c) contains vertex v. According to Lemma 3.5, we can get $|P^3_C(w)| \leq 1$. If $P^3_C(w) = \emptyset$, then $c(wu_{a-1}) = c(wu_{b+1}) = c_w$; otherwise, (G, c) has a requested cycle. Thus, $c(u_2u_{b+2}) \neq c(u_{b+2}u_{b+3})$. Then, $wu_3\overrightarrow{C}u_{b+1}u_1u_2u_{b+2}\overrightarrow{C}u_\ell w$ or $wu_1u_{b+1}\overleftarrow{C}u_2u_{b+2}\overrightarrow{C}u_\ell w$ is a requested cycle, a contradiction. If $|P^3_C(w)| = 1$, then $c(w, P^3_C(w)) = c(wu_{a-1}) = c(wu_{b+1}) = c_w$. Thus, $c(u_1u_{t+1}) = c(u_{t+1}u_s)$. Since C_{u_1} is proper, we have $c(u_1u_k) = c(u_ku_{k+1})$ for $u_k \in P^1_C(w)$. Hence, wu_1u_{a-1} is monochromatic, a contradiction. ∎

Thus, if there is no proper cycle of length $\ell + 1$ containing v_1 in (G, c) and $W_3(C) \neq \emptyset$, then for any $w \in W_3(C)$, either $P^1_C(w) \setminus R_C(w)$ or $P^2_C(w) \setminus Q_C(w)$ is empty. Hence, we might as well suppose that $P^2_C(w) \setminus Q_C(w) = \emptyset$, that is $Q_C(w) = P^2_C(w)$.

Proposition 3.4. *Let (G, c) be an edge-colored complete graph on $n \geq 3$ vertices such that $\delta^c(G) \geq \frac{n+1}{2}$, and not contain any joint monochromatic triangles. If for any $w \in W_3$, $P^1_C(w) = V(u_3\overrightarrow{C}u_2)$ and $P^1_{C_{u_1}}(u_1) = V(u_3\overrightarrow{C_{u_1}}u_2)$, and $u_4\overrightarrow{C}u_2$ is a rainbow path, then (G, c) is properly vertex-pancyclic.*

Proof. Suppose, to the contrary, that there exists a vertex v which is contained in a proper ℓ-cycle C in (G, c) for some ℓ with $4 \leq \ell \leq n-1$, but no proper cycle of length $\ell + 1$ in (G, c) contains vertex v. Since $wu_{k-1}\overleftarrow{C}u_2u_k\overrightarrow{C}u_1w$ is of length $\ell + 1$ and contains v_1, we have $c(u_2u_k) \in \{c(u_2u_3), c(u_ku_{k+1})\}$ for $u_k \in V(u_3\overrightarrow{C}u_1)$. According to Lemma 3.6, we have $V(u_5\overrightarrow{C}u_\ell)$ has the DP_w. Then, there is a vertex u_p such that $d^c_{V(u_5\overrightarrow{C}u_\ell)}(u_p) \leq \frac{\ell-3}{2}$. If $c(u_2u_p) = c(u_pu_{p+1})$, then $d^c(u_p) \leq \frac{\ell+1}{2} < \frac{n+1}{2}$, a contradiction. Thus, $c(u_2u_p) = c(u_2u_3) \neq c(u_pu_{p+1})$. Then, to avoid $wu_2u_p\overrightarrow{C}u_1u_i\overleftarrow{C}u_3u_{i+1}\overrightarrow{C}u_{p-1}w$ for $u_i \in V(u_5\overrightarrow{C}u_{p-1})$ and $wu_{i-1}\overleftarrow{C}u_pu_2\overleftarrow{C}u_iu_3\overrightarrow{C}u_{p-1}w$ for $u_i \in V(u_{p+1}\overrightarrow{C}u_\ell)$ being requested cycles, we have $c(u_3u_i) \in \{c(wu_3), c(wu_i)\}$ for $u_i \in V(u_5\overrightarrow{C}u_1) \setminus \{u_p\}$. Note that if there is another vertex u such that $c(uu_2) = c(u_2u_3) \neq c(wu)$, we have $(u_3, V(u_4\overrightarrow{C}u_{p-1})) \subseteq DP_w$. Then, $u_3 \in R_C(w)$; otherwise, (G, c) has joint monochromatic triangles. Thus, there is a vertex in $V(u_3\overrightarrow{C}u_\ell)$ whose color degree is less than $\frac{n+1}{2}$, a contradiction. Hence, u_p is the unique vertex such that $c(u_2u_p) = c(u_2u_3) \neq c(u_pu_{p+1})$. If $n \leq 7$, we can easily find a vertex of $d^c(u) < \frac{n+1}{2}$. If $n > 8$, we give $(G[V(u_5\overrightarrow{C}u_\ell)], c)$ a coloring orientation. If there is a distinct vertex u such that $d^+_D(u) \geq \frac{\ell-5}{2}$, then $d^c(u) \leq \frac{\ell+1}{2} < \frac{n+1}{2}$, a contradiction. Thus, $d^+_D(u_p) \geq \frac{(\ell-4)(\ell-5)}{2} - \frac{(\ell-5)(\ell-6)}{2} = \ell - 5$. Then $d^c(u_p) \leq 4$, a contradiction. ∎

Lemma 3.7. *Suppose there is no proper cycle of length $\ell + 1$ containing v_1 in (G, c). Then for $w \in W_3(C)$ with $P^3_C(w) \neq \emptyset$, we have*

(1) $(R_C(w) \setminus \{u_1, u_a\}, u_b) \cup (u_a, Q_C(w) \setminus \{u_2, u_b\}) \subseteq D_w$.

(2) if all of $|P^1_C(w)|$, $|P^2_C(w)|$ and $|P^3_C(w)|$ are larger than 1, then $(u_a, u_b) \in D_w$.

Proof. (1) The proof method is the same as that of Lemma 3.6 (2) and (3).

(2) From (1) and Lemma 3.6 (2) , we can get $(R_C(w)\setminus\{u_1\}, Q_C(w)\setminus\{u_2\})\setminus\{(u_a, u_b)\} \subseteq D_w$. If $c(wu_l) = c(wu_3)$, we have $a = \ell$ and $b = 3$. Thus, $c(w, P^3_C(w)) \neq c(wu_3)$; otherwise, $d^c(w) = 2 < \frac{n+1}{2}$, a contradiction. Thus, $c(w, P^3_C(w)) \neq c(wu_\ell)$ or $c(w, P^3_C(w)) \neq c(wu_3)$ holds. Without loss of generality, suppose $c(w, P^3_C(w)) \neq c(wu_3)$.

Suppose, to the contrary, that $c(u_au_b) \notin \{c(u_au_{a+1}), c(u_bu_{b-1})\}$. Since $|P^3_C(w)| \geq 2$, we have $u_a = u_s$ and $u_b = u_t$. Since $c(wu_{t+2}) \neq c(wu_3)$, we can get $c(u_2u_{t+1}) \in \{c_w, c(u_{t+1}u_t)\}$. If $c(u_2u_{t+1}) = c_w$, then $c(w, P^3_C(w)) \neq c_w$. To avoid $wu_1\overleftarrow{C}u_su_t\overleftarrow{C}u_2u_{t+1}\overrightarrow{C}u_{s-1}w$ being a requested cycle, we have $c(u_2u_{t+1}) = c(u_{t+1}u_{t+2})$. Hence, $c(u_2u_{s-1}) = c(u_su_{s-1})$. Then, $wu_3\overrightarrow{C}u_tu_s\overrightarrow{C}u_2u_{s-1}\overleftarrow{C}u_{t+1}w$ or $wu_1\overleftarrow{C}u_su_t\overleftarrow{C}u_2u_{s-1}\overleftarrow{C}u_{t+1}w$ is a requested cycle, a contradiction. If $c(u_2u_{t+1}) = c(u_tu_{t+1}) \neq c_w$, then $c(u_2u_{t+1}) \neq c(u_{t+2}u_{t+1})$. Hence, $wu_3\overrightarrow{C}u_tu_s\overrightarrow{C}u_2u_{t+1}\overleftarrow{C}u_{s-1}w$ is a proper cycle of length ℓ containing v_1, a contradiction. ∎

Lemmas 3.6 (2) and 3.7 claim that if all of $|P^1_C(w)|$, $|P^2_C(w)|$ and $|P^3_C(w)|$ are larger than 1, then for $u_i, u_j \in R_C(w) \cup Q_C(w) \setminus \{u_1\}$, we have $c(wu_i) \neq c(wu_j)$.

Lemma 3.8. *Suppose there is no proper cycle of length $\ell + 1$ containing v_1 in (G, c). If $|W_3(C)| \geq 2$, then for each vertex $w \in W_3(C)$ such that neither wv_1v_l nor wv_1v_2 is monochromatic, we have that both C_{u_1} and C_{u_2} are proper cycles of length ℓ containing v_1.*

Proof. Let w be a vertex such that wv_1v_i is not monochromatic for $i = 2, \ell$, and w' be a distinct vertex. We again relabel the vertices of C depending on $w' \in W_3(C)$ by $u'_1u'_2\cdots u'_\ell$ in a clockwise direction such that $u_1 = v_{y(w')}$, $u_b = v_{q(w')}$, $u_t = v_{z(w')}$, $u_s = v_{x(w')}$ and $u_a = v_{r(w')}$. Without loss of generality, suppose $|P^1_C(w)| \geq 2$. Then, assume, to the contrary, that C_{u_2} is not proper, that is, $|P^2_C(w)| = 1$ and $c(w, P^3_C(w)) = c_w$ if $P^3_C(w) \neq \emptyset$. Then, $u_1, w' \in W_3(C_{u_1})$ and C_{u_1} is a proper cycle containing v_1.

In the first case we assume that $C_{u_2'}$ is proper. Then, the coloring of $\partial(u_2', C_{u_2'})$ follows the statement in Lemma 3.1. If $c(wu_2') = c_w$, then from Lemma 3.4 we have $|R_{C_{u_2'}}(w)| \geq 4$. Apparently, $u_1 \notin P^1_{C_{u_1'}}(u_2')$, and then $u_1, u_l \in P^2_{C_{u_1'}}(u_2') \cup P^3_{C_{u_1'}}(u_2')$. Thus, $u_{\ell-1}wu_2'u_1\overrightarrow{C}u_{\ell-1}$ or $u_{\ell-2}wu_2'u_\ell\overrightarrow{C}u_{\ell-2}$ is a requested cycle, a contradiction. If $c(wu_2') \neq c_w$, then $u_2' \in P^1_C(w)$ and $c(wu_2') = c(u_2'u_3')$. When $c(wu_2') = c(wu_\ell)$, we have $w \in P^2_{C_{u_1}}(u_2')$. Thus, $c(u_3'u_4') = c_w$. Then, $u_4'u_2'u_3'wu_1'\overleftarrow{C}u_4'$ is a requested cycle, a contradiction. When $c(wu_2') \neq c(wu_\ell)$, we have $w \in P^3_{C_{u_1}}(u_2')$. Note that $c(wu_\ell') \neq c(wu_2')$; otherwise, $wu_2'u_\ell'$ is monochromatic. Since $u_\ell'wu_2'u_1'\overrightarrow{C}u_\ell'$ is a proper cycle of length $\ell+1$, we have $u_1' = v_1$. Then, $u_1'ww'u_2'u_4'\overrightarrow{C}u_1'$ or $u_1'wu_3'u_2'u_4'\overrightarrow{C}u_1'$ is a requested cycle, a contradiction.

In the second case we assume that $C_{u_1'}$ is proper. Then, the coloring of $\partial(u_1', C_{u_1'})$ follows the statement in Lemma 3.1. If $c(wu_1') = c_w$, then it is easy to verify that there is a requested cycle in (G, c), a contradiction. If $c(wu_1') \neq c_w$, then $u_1' \in P^1_C(w)$ and $c(wu_1') = c(u_1'u_2')$. When $c(wu_1') = c(wu_\ell)$, we have $V(w\overleftarrow{C_{u_1}}u_2') \subseteq P^2_{C_{u_1}}(u_1')$. Since $w'wu_1'u_3'\overrightarrow{C}w'$ is a proper cycle of length ℓ, we have $u_2' = v_1$. To avoid $w'wu_2'u_1'u_4'\overrightarrow{C}w'$ being a requested cycle, we have $c(u_1'u_4') = c(u_1'u_2')$. Then, $|\mathcal{C}(w, C)| < 4$, a contradiction. When $c(wu_1') \neq c(wu_\ell)$, furthermore if $c(ww') = c(u_1'u_2')$, then $ww'u_1'$ is monochromatic. Thus, $c(ww') = c_w$, it is easy to verify that there is a requested cycle in (G, c), a contradiction. ∎

In the following we prove an important lemma which can transform a cycle C at $w \in W_3(C)$ with $|P^3_C(w)| \geq 3$ into a new cycle C_{u_i} at $u_i \in W_3(C_{u_i})$ with $|P^3_{C_{u_i}}(u_i)| \leq 1$, $i = 1, 2$ under the condition that there is no proper cycle of length $\ell + 1$ containing v_1 in (G, c).

Lemma 3.9. *Suppose there is no proper cycle of length $\ell+1$ containing v_1 in (G, c). Then for $w \in W_3(C)$, if $|P^3_C(w)| \geq 3$ then $u_1 \in W_3(C_{u_1})$ with $|P^3_{C_{u_1}}(u_1)| \leq 1$ or $u_2 \in W_3(C_{u_2})$ with $|P^3_{C_{u_2}}(u_2)| \leq 1$.*

Proof. In the first case we assume that both $|P^1_C(w)|$ and $|P^2_C(w)|$ are larger than 1. Then, according to Lemmas 3.6 (2) and 3.7, we have $c(wu_l) \neq c(wu_3)$. Apparently, $c_w \notin \{c(wu_\ell, c(wu_3))\}$, and then C_{u_1} and C_{u_2} are proper. Since $|P^3_C(w)| \geq 3$, there is a vertex $u_p \in P^3_C(w)$ such that $u_{p-1}, u_{p+1} \in P^3_C(w)$; see Figure 2.

If $c(w, P^3_C(w)) = c_w$, then $c(u_iu_k) \neq c_w$ for $u_k \in P^3_C(w)$ and $i = 1, 2$. It is simple to verify $c(u_1u_p) = c(u_{p+1}u_p)$ and $c(u_2u_p) = c(u_{p-1}u_p)$. Thus, $c(u_2u_p) \notin \{c(u_pu_{p+1}), c_w\}$ and $c(u_1u_p) \notin \{c(u_pu_{p-1}), c_w\}$. Then, to avoid $wu_3\overrightarrow{C}u_{p-1}u_1u_2u_p\overrightarrow{C}u_\ell w$ and $wu_3\overrightarrow{C}u_pu_1u_2u_{p+1}\overrightarrow{C}u_\ell w$ being requested cycles, we have $c(u_1u_{p-1}) = c(u_{p-1}u_{p-2})$ and $c(u_2u_{p+1}) = c(u_{p+1}u_{p+2})$. Hence, $u_1 \in W_3(C_{u_1})$ with $P^3_{C_{u_1}}(u_1) = \emptyset$ and $u_2 \in W_3(C_{u_2})$ with $P^3_{C_{u_2}}(u_2) = \emptyset$.

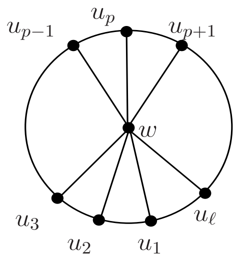

Figure 2: $G[V(C) \cup \{w\}]$ used in the proof of Lemma 3.9.

If $c(w, P_C^3(w)) = c(wu_3)$, then $c(w, P_C^3(w)) \neq c(wu_\ell)$. It is simple to verify $c(u_1u_p) \in \{c(u_{p-1}u_p), c(u_1u_\ell)\} \cap \{c(u_{p+1}u_p), c_w\}$ and $c(u_2u_p) \in \{c(u_{p+1}u_p), c(u_2u_3)\}$. If $c(u_1u_p) = c(u_pu_{p-1}) = c_w$, then $c(u_2u_p) \neq c_w$. Since C_{u_1} is proper, we have $c(u_1u_k) = c(u_ku_{k-1})$ for $u_k \in V(u_p\overleftarrow{C}u_2)$. Then, $c(u_1u_3) = c(u_3u_2) \notin \{c(u_3u_4), c(u_1u_\ell)\}$. Thus, $wu_2u_p\overleftarrow{C}u_3u_1 \overleftarrow{C}u_{p+1}w$ is a requested cycle, a contradiction. Hence, $c(u_1u_p) = c(u_1u_\ell) = c(u_pu_{p+1})$. Then, $c(u_1u_p) \notin \{c_w, c(u_pu_{p-1})\}$. Furthermore, if $c(u_2u_p) = c(u_pu_{p+1})$, then since C_{u_2} is proper, we have $c(u_2u_k) = c(u_ku_{k+1})$ for $u_k \in V(u_p\overrightarrow{C}u_\ell)$. Then, $c(u_2u_\ell) = c(u_\ell u_1) \notin \{c(u_2u_3), c(u_\ell u_{\ell-1})\}$. Thus, $wu_1u_p\overleftarrow{C}u_2u_\ell\overleftarrow{C}u_{p+1}w$ is a requested cycle. Hence, $c(u_2u_p) = c(u_2u_3) \neq c(u_pu_{p+1})$. Then, to avoid $wu_3\overrightarrow{C}u_{p-1}u_1u_2u_p\overrightarrow{C}u_\ell w$ being a requested cycle, we have $c(u_1u_{p-1}) \in \{c_w, c(u_{p-1}u_{p-2})\}$. Since $c(wu_p) \neq c_w$, it is simple to verify $c(u_1u_{p-1}) \in \{c(u_{p-1}u_{p-2}), c(u_\ell u_{\ell-1})\}$. Thus, $c(u_1u_{p-1}) = c(u_{p-1}u_{p-2})$. Therefore, $u_1 \in W_3(C_{u_1})$ with $P^3_{C_{u_1}}(u_1) = \emptyset$. Symmetrically, if $c(w, P_C^3(w)) = c(wu_\ell)$, we can get $u_2 \in W_3(C_{u_2})$ with $P^3_{C_{u_2}}(u_2) = \emptyset$.

If $c(w, P_C^3(w)) \notin \{c_w, c(wu_\ell), c(wu_3)\}$, then it is simple to verify $c(u_1u_p) \in \{c(u_{p-1}u_p), c(u_1u_\ell)\} \cap \{c_w, c(u_{p+1}u_p)\}$ and $c(u_2u_p) \in \{c(u_{p-1}u_p), c_w)\} \cap \{c(u_{p+1}u_p), c(u_2u_3)\}$. If $c(u_1u_p) = c(u_{p-1}u_p) = c_w$, then $c(u_2u_p) = c_w$. Hence, (G, c) has joint monochromatic triangles, a contradiction. Thus, $c(u_1u_p) = c(u_1u_\ell) = c(u_{p+1}u_p)$ and $c(u_2u_p) = c(u_{p-1}u_p) = c(u_2u_3)$. Then, $c(u_1u_p) \notin \{c(u_{p-1}u_p), c_w\}$. Thus, we have $c(u_2u_{p+1}) \in \{c_w, c(u_{p+1}u_{p+2})\}$; otherwise, $wu_3\overrightarrow{C}u_pu_1u_2u_{p+1}\overrightarrow{C}u_\ell w$ is a requested cycle. Since $c(wu_p) \neq c_w$, it is simple to verify $c(u_2u_{p+1}) \in \{c(u_{p+1}u_{p+2}), c(u_2u_3)\}$. Thus, $c(u_2u_{p+1}) = c(u_{p+1}u_{p+2})$. Symmetrically, we have $c(u_1u_{p-1}) = c(u_{p-1}u_{p-2})$. Since both C_{u_1} and C_{u_2} are proper, we have $u_1 \in W_3(C_{u_1})$ with $P^3_{C_{u_1}}(u_1) = \emptyset$ and $u_2 \in W_3(C_{u_2})$ with $P^3_{C_{u_2}}(u_2) = \emptyset$.

Thus the cycle C with a vertex $w \in W_3(C)$ which is of $|P_C^2(w)| \geq 2$ and $|P_C^3(w)| \geq 3$ can be changed into another cycle C_{u_i} with $u_i \in W_3(C_{u_i})$ and $P^3_{C_{u_i}}(u_i) = \emptyset$, $i = 1$ or 2; see Figures 2 and 3.

In the second case we assume that either $|P_C^1(w)|$ or $|P_C^2(w)|$ is 1. Without loss of generality, suppose $|P_C^2(w)| = 1$. Then, C_{u_1} is proper. Since $|P_C^3(w)| \geq 3$, $u_3, u_4, u_5 \in$

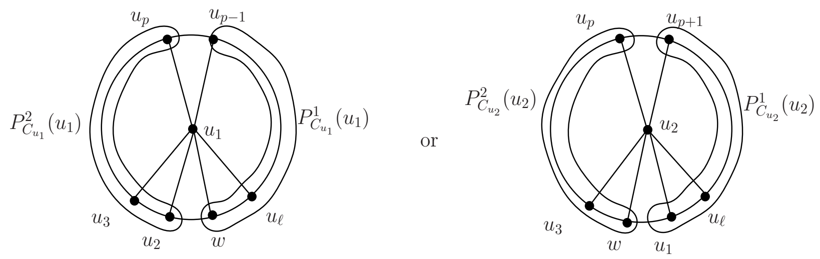

Figure 3: $G[V(C_{u_i}) \cup \{u_i\}]$ used in the proof of Lemma 3.9.

$P^3_C(w)$. If $c(wu_3) \notin \{c_w, c(wu_\ell)\}$, we can get $u_1 \in W_3(C_{u_1})$ with $P^3_{C_{u_1}}(u_1) = \emptyset$. If $c(wu_3) = c_w$, then it is simple to verify $c(u_1u_4) = c(u_4u_5)$. Thus, $u_1 \in W_3(u_1)$ with $P^3_{C_{u_1}} \subseteq \{u_3\}$. If $c(wu_3) = c(wu_\ell)$, then C_{u_2} is proper. To avoid $wu_3u_2u_4\overleftarrow{C}u_1w$ being a requested cycle, we have $c(u_2u_4) \in \{c(u_3u_2), c(u_4u_5)\}$. Once $c(u_2u_4) = c(u_4u_5)$, we get $u_2 \in W_3(C_{u_2})$ with $P^3_{C_{u_2}} \subseteq \{u_3\}$. Once $c(u_2u_4) = c(u_3u_2) \neq c(u_4u_5)$, then $u_3 \in P^3_{C_{u_2}}(u_2)$. Thus, $u_\ell \in P^1_{C_{u_2}}(u_2)$. It is easy to verify $c(u_2u_3) \notin \{c_w, c(u_2u_\ell)\}$. Then, we can get $u_1 \in W_3(C_{u_1})$ with $P^3_{C_{u_1}}(u_1) = \emptyset$.

Thus the cycle C with a vertex $w \in W_3(C)$ which is of $|P^2_C(w)| = 1$ and $|P^3_C(w)| \geq 3$ can be changed into another cycle C_{u_i} with $u_i \in W_3(C_{u_i})$ and $|P^3_{C_{u_i}}(u_i)| \leq 1$, $i = 1$ or 2; see Figures 4 and 5.

∎

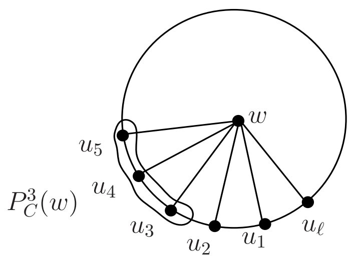

Figure 4: $G[V(C) \cup \{u\}]$ used in the proof of Lemma 3.9.

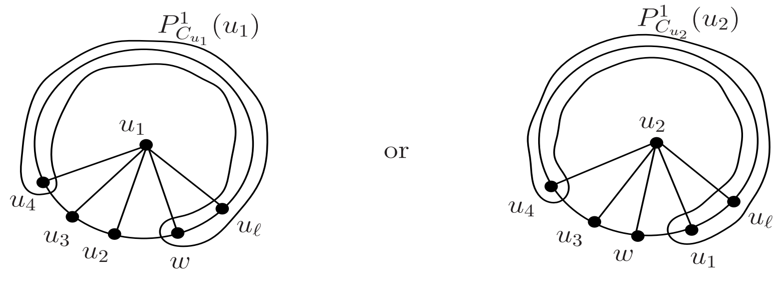

Figure 5: $G[V(C_{u_i}) \cup \{u_i\}]$ used in the proof of Lemma 3.9.

From Lemmas 3.8 and 3.9, we get the following important corollary.

Corollary 3.1. *Suppose there is no proper cycle of length $\ell+1$ containing v_1 in (G,c). Then, $|W_3(C)| \le 3$.*

Proof. First, we prove a claim: For a vertex w in $W_3(C)$ such that wv_1v_i is not monochromatic for $i = 2, \ell$, we have $u_1 \notin P^1_C(w')$ and $u_2 \notin P^2_C(w')$ for any distinct $w' \in W_3(C)$. According to Lemma 3.8, we know that C_{u_1} and C_{u_2} are proper cycles of length ℓ containing v_1. If $u_1 \in P^1_C(w')$, then $u_3 \in P^1_C(w')$. Thus, $c(ww') = c(wu_3)$. According to Lemma 3.8 again, we know that $c(u_1u_3) \notin \{c(wu_1), c(u_3u_4)\}$. Thus, $ww'u_1u_3\overrightarrow{C}w$ is proper cycle of length $\ell+1$ containing v_1, a contradiction. Thus, $u_1 \notin P^1_C(w')$. In a similar way, we can get $u_2 \notin P^2_C(w')$.

Suppose, to the contrary, that $|W_3(C)| \ge 4$. Since (G,c) has no joint monochromatic triangles, there exists a vertex $w \in W_3(C)$ with $|P^3_C(w)| \ge 4$ such that wv_1v_i, $i = 2, \ell$, is not monochromatic; see Figure 6.

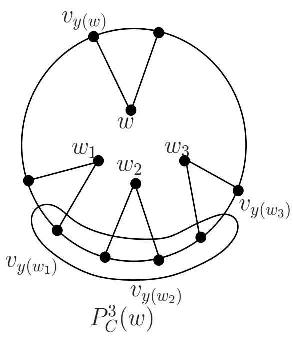

Figure 6: A cycle C with $|W_3(C)| \ge 4$ used in the proof of Corollary 3.1.

According to Lemma 3.9, without loss of generality, suppose $u_1 \in W_3(C_{u_1})$ with $|P^3_{C_{u_1}}(u_1)| \le 1$. Then, $|W_3(C_{u_1})| \le 3$. Since $V(C_{u_1}) \cap V(C) = V(u_2\overrightarrow{C}u_\ell)$, we have $W_3(C_{u_1}) = W_3(C) \cup \{u_1\} \setminus \{w\}$, that is, $|W_3(C_{u_1})| \ge 4$, a contradiction. ∎

Lemma 3.10. *Suppose there is no proper cycle of length $\ell+1$ containing v_1 in (G,c). Then for $w \in W_3(C)$ with $|P^3_C(w)| \le 2$, if both $|P^1_C(w)|$ and $|P^2_C(w)|$ are larger than 1, there is a vertex set V_1 of size $\ell - 4$ such that $c(wu) = c(uu_1) = c(uu_2)$ for $u \in V_1$.*

Proof. In the first case suppose $c(wu_\ell) = c(wu_3)$. When $\ell = 5$, the result follows apparently. In the following assume $\ell \ge 6$. From Lemmas 3.4 and 3.7, we get $P^3_C(w) = \emptyset$. Let $C_1 = wu_\ell u_3\overleftarrow{C}u_1u_4\overrightarrow{C}u_{\ell-1}w$, $C_2 = wu_2u_3u_\ell u_1u_4\overrightarrow{C}u_{\ell-1}w$, $C_3 = wu_3u_\ell\overleftarrow{C}u_2u_4\overrightarrow{C}u_{\ell-1}w$ and $C_4wu_1u_\ell u_3u_2u_4\overrightarrow{C}u_{\ell-1}w$ be cycles of length $\ell+1$ containing v_1. We might as well suppose $u_b = u_3$. (Note that when $R_C(w) \subsetneq P^1_C(w)$, we have $u_a = u_\ell$; otherwise, C_1

is proper.) Then, $|R_C(w)| \geq 3$. To avoid C_1 and C_2 being proper, we have $c(u_1u_4) \in \{c_w, c(u_4u_5)\} \cap \{c(u_1u_{\ell-1}), c(u_4u_5)\}$. Then, $c(u_1u_4) = c(u_4u_5)$. Since C_{u_1} is proper, we have $c(u_1u_k) = c(u_ku_{k+1})$ for $u_k \in V(u_4\overrightarrow{C}u_\ell)$. Then, $c(wu_k) \neq c(wu_1)$ for $u_k \in V(u_4\overrightarrow{C}u_\ell)$. To avoid C_3 being proper, we have $c(u_2u_4) \in \{c_w, c(u_4u_5)\}$. Thus, $c(u_2u_4) = c(u_4u_5)$; otherwise, C_4 is proper. Since C_{u_2} is proper, we have $c(u_2u_k) = c(u_ku_{k+1})$ for $u_k \in V(u_4\overrightarrow{C}u_\ell)$. The result follows as $V_1 = V(u_4\overrightarrow{C}u_\ell)$.

In the second case suppose $c(wu_\ell) \neq c(wu_3)$. First, we prove a claim: There is a vertex set V_1 of size $\ell - 4$ such that $c(u_iu_2) \neq c_w$ for $u_i \in P_C^1(w) \cap V_1$ and $c(u_ju_1) \neq c_w$ for $u_j \in P_C^2(w) \cap V_1$. Apparently, the claim holds for $u \in R_C(w) \cup Q_C(w) \setminus \{u_a, u_b, u_1, u_2\}$. Let $V_0 = R_C(w) \cup Q_C(w) \setminus \{u_1, u_2\}$. If $R_C(w) = P_C^1(w)$, then the claim follows apparently when $P_C^3(w) = \emptyset$ or $c(w, P_C^3(w)) \neq c(wu_1)$. While when $|P_C^3(w)| = 1$ and $c(w, P_C^3(w)) = c(wu_1)$, suppose, to the contrary, that $c(u_su_2) = c(u_tu_1) = c_w$. Then, $c(u_2u_{t+1}) \neq c_w$. Thus, $wu_{t+1}u_2u_s\overrightarrow{C}u_1u_t\overleftarrow{C}u_3w$ is a requested cycle, a contradiction. Hence, at least one of $c(u_2u_s)$ and $c(u_1u_t)$ is not c_w. Therefore, the claim follows as $V_1 = V_0 \setminus \{u_s\}$ or $V_1 = V_0 \setminus \{u_t\}$. When $|P_C^3(w)| = 2$ and $c(w, P_C^3(w)) = c_w$, since $c(wu_3) \neq c(wu_{t+2})$, we have $c(u_2u_{t+1}) \in \{c_2, c(u_tu_{t+1})\}$. Thus, $c(u_2u_{t+1}) \neq c(u_{t+1}u_{t+2})$. Then, $c(u_tu_1) \neq c_w$; otherwise, $wu_2u_{t+1}\overrightarrow{C}u_1u_t\overrightarrow{C}u_3w$ is a requested cycle. Therefore, the claim follows as $V_1 = V_0$. If $R_C(w) \subsetneq P_C^1(w)$, according to Lemma 3.5 we know $|P_C^3(w)| \leq 1$. When $c(wu_{a-1}) \neq c(wu_1)$, the claim follows as $V_1 = V_0 \setminus \{u_s, u_{s-1}\}$. When $c(wu_{a-1}) = c(wu_1)$, we can get an assertion that to avoid $wu_2u_{a-1}\overrightarrow{C}u_1u_{a-2}\overleftarrow{C}u_3w$ and $wu_l\overleftarrow{C}u_{a-1}u_2u_1u_{a-2}\overleftarrow{C}u_3w$ being requested cycles, we have $c(u_1u_{a-2}) \in \{c_w, c(u_{a-2}u_{a-3})\} \cap \{c(u_1u_l), c(u_{a-2}u_{a-3})\}$. In the following we prove the assertion that $|P_C^1(w) \cup P_C^3(w) \setminus R_C(w)| \leq 1$. If not, since $c(wu_{a-2}) = c(wu_\ell)$, we have $c(u_1u_\ell) \neq c(u_{a-2}u_{a-3})$. Thus, $c(u_1u_{a-2}) = c(u_{a-2}u_{a-3})$. Since C_{u_1} is proper, we have $c(u_1u_k) = c(u_ku_{k-1})$ for $u_k \in V(u_{a-2}\overleftarrow{C}u_2)$. Then, $c(u_1u_3) = c(u_2u_3) \notin \{c(u_3u_4), c(u_1u_\ell)\}$. Thus, $wu_\ell\overleftarrow{C}u_{a-1}u_2u_1u_3\overrightarrow{C}u_{a-2}w$ is a requested cycle, a contradiction. The assertion thus follows. Since $R_C(w) \subsetneq P_C^1(w)$, we have $P_C^3(w) = \emptyset$. Thus, $c(wu_s) = c_w$. Then, $c(u_1u_t) \neq c_w$; otherwise, $wu_\ell\overleftarrow{C}u_su_2u_1u_t\overleftarrow{C}u_3w$ is a requested cycle. Hence, the claim follows as $V_1 = V_0 \setminus \{u_s, u_t\}$.

Next suppose, to the contrary, that there exists a vertex $u_i \in P_C^1w \cap V_1$ such that $c(u_iu_2) \neq c(wu_i)$. Since $c(u_iu_2) \neq c_w$, to avoid $wu_2u_{a-1}\overrightarrow{C}u_1u_{a-2}\overleftarrow{C}u_3w$ and $wu_\ell\overleftarrow{C}u_{a-1}u_2u_1$ $u_{a-2}\overleftarrow{C}u_3w$ being requested cycles, we have $c(u_1u_{i-1}) = c(u_{i-1}u_{i-2})$. Since C_{u_1} is proper, we have $c(u_1u_k) = c(u_ku_{k-1})$ for $u_k \in V(u_{i-1}\overleftarrow{C}u_2)$. In a similar way, we can get $c(u_1u_j) = c(wu_j)$ for $u_j \in P_C^2(w) \cap V_1$. Then, $u_2u_1u_3\overrightarrow{C}u_lu_2$ is a proper cycle of length ℓ containing v_1. Then, by repeating this as above, we can get that $c(u_iu_1) = c(wu_i)$ for $u_i \in P_C^1(w) \cap V_1$ and $c(u_2u_j) = c(wu_j)$ for $u_j \in P_C^2(w) \cap V_1$. The result thus follows. ∎

Lemma 3.11. *Suppose there is no proper cycle of length $\ell+1$ containing v_1 in (G,c). For $w \in W_3(C)$ with $|P_C^2(w)| = 1$ and $|P_C^3(w)| \geq 3$, if both $c(w, P_C^3(w)) = c_w$ and $c(u_1u_3) \neq c(u_2u_3)$, then $c(u_2u_k) = c(u_3u_k) = c(u_ku_{k+1})$ for $u_k \in V(u_5\overrightarrow{C}u_\ell)$.*

Proof. Since $c(wu_3) \neq c(wu_\ell)$, we have $c(u_1u_4) = c(u_4u_5)$. From 3.4 we know that C_{u_1} is proper. Thus, $c(u_1u_k) = c(u_ku_{k+1})$ for $u_k \in V(u_4\overrightarrow{C}u_\ell)$. Hence, $c(u_1u_k) \notin \{c(u_ku_{k-1}), c_w\}$ for $u_k \in V(u_4\overrightarrow{C}u_\ell)$. Since for $u_k \in V(u_5\overrightarrow{C}u_\ell)$, $wu_1u_{k-1}\overleftarrow{C}u_2u_k\overrightarrow{C}u_\ell w$, $wu_3\overrightarrow{C}u_{k-1}u_1u_2u_k\overrightarrow{C}u_\ell w$ and $wu_4\overrightarrow{C}u_{k-1}u_1\overrightarrow{C}u_3u_k\overrightarrow{C}u_\ell w$ are of length $\ell+1$ and contain v_1, we have $c(u_2u_k) \in \{c(u_2u_3), c(u_ku_{k+1})\} \cap \{c_w, c(u_ku_{k+1})\}$ and $c(u_3u_k) \in \{c(u_2u_3), c(u_ku_{k+1})\}$. Thus, apparently we have $c(u_2u_k) = c(u_ku_{k+1})$ for $u_k \in V(u_5\overrightarrow{C}u_\ell)$. While if $c(u_3u_k) = c(u_2u_3) \neq c(u_ku_{k+1})$, then $c(u_1u_3) \notin \{c_w, c(u_3u_k)\}$. Thus, $wu_\ell\overleftarrow{C}u_ku_3u_1u_2u_{k-1}\overleftarrow{C}u_4w$ is a requested cycle, a contradiction. The result thus follows. ∎

Lemma 3.12. *Suppose there is no proper cycle of length $\ell+1$ containing v_1 in (G,c). Then for $w \in W_3(C)$ with $|P_C^2(w)| = 1$ and $|P_C^3(w)| = 2$, we have $c(u_1u_k) = c(u_2u_k) = c(u_ku_{k+1})$ for $u_k \in V(u_5\overrightarrow{C}u_\ell)$.*

Proof. In the first case suppose $c(w, P_C^3(w)) = c_w$. Then, to avoid $wu_3\overleftarrow{C}u_1u_4\overrightarrow{C}u_\ell w$ being a requested cycle, we have $c(u_1u_4) = c(u_4u_5)$. Thus, $c(u_1u_k) = c(u_ku_{k+1})$ for $u_k \in V(u_4\overrightarrow{C}u_1)$. Apparently, $c(u_2u_k) \in \{c_w, c(u_ku_{k+1})\}$ for $u_k \in V(u_5\overrightarrow{C}u_1)$; otherwise, $wu_1u_{k-1}\overleftarrow{C}u_2u_k\overrightarrow{C}u_\ell w$ is a requested cycle. If $c(u_2u_k) = c(u_2u_3) \neq c(u_ku_{k+1})$, then $c(u_2u_k) \neq c(u_1u_2)$. Thus, $wu_3\overrightarrow{C}u_{k-1}u_1u_2u_k\overleftarrow{C}u_\ell w$ is a requested cycle, a contradiction. Hence, $c(u_2u_k) = c(u_ku_{k+1})$ for $u_k \in V(u_5\overrightarrow{C}u_1)$.

In the second case suppose $c(w, P_C^3(w)) \neq c_w$. Then, to avoid $wu_4\overrightarrow{C}u_1u_3u_2w$ being a requested cycle, we have $c(u_1u_3) \in \{c(u_1u_\ell), c(u_2u_3)\}$. Hence, $c(u_1u_3) \neq c_w$. If $c(w, P_C^3(w)) \neq c(wu_\ell)$, then $c(u_1u_3) \neq c(wu_3)$. Thus, $c(u_2u_4) = c(u_4u_5)$; otherwise, $wu_3u_1u_2u_4\overrightarrow{C}u_\ell w$ is a requested cycle. Therefore, $c(u_2u_k) = c(u_ku_{k+1})$ for $u_k \in V(u_5\overrightarrow{C}u_\ell)$. If $c(w, P_C^3(w)) = c(wu_\ell)$, then to avoid $wu_3u_\ell\overleftarrow{C}u_2u_4\overrightarrow{C}u_{\ell-1}w$ and $wu_3u_2u_4\overrightarrow{C}u_1w$ being requested cycles, we have $c(u_2u_4) \in \{c_w, c(u_4u_5)\} \cap \{c(u_2u_3), c(u_4u_5)\}$. Thus, $c(u_2u_4) = c(u_4u_5)$. Since C_{u_2} is proper, we have $c(u_2u_k) = c(u_ku_{k+1})$ for $u_k \in V(u_4\overrightarrow{C}u_\ell)$. Furthermore, if $c(u_1u_3) = c(u_3u_4)$, then since C_{u_1} is proper, we can get $c(u_1u_k) = c(u_ku_{k+1})$ for $u_k \in V(u_5\overrightarrow{C}u_\ell)$; if $c(u_1u_3) \neq c(u_3u_4)$, then $C' = u_1u_3\overrightarrow{C}u_\ell u_2u_1$ is proper such that both $|P_{C'}^1(w)|$ and $|P_{C'}^2(w)|$ are larger than 1 or $|P_{C'}^3(w)| = 2$. Thus, according to Lemma 3.10 or the proof above, we can get $c(u_1u_k) = c(u_ku_{k+1})$ for $v_k \in V(u_5\overrightarrow{C}u_\ell)$. ∎

Lemma 3.13. *Suppose there is no proper cycle of length $\ell+1$ containing v_1 in (G,c). Then for $w \in W_3(C)$ with $|P_C^2(w)| = 1$ and $|P_C^3(w)| \leq 1$, we have $c(u_1u_k) = c(wu_k)$ and $c(wu_k) \in \{c(u_2u_k), c(u_3u_k)\}$ for $u_k \in V(u_5\overrightarrow{C}u_\ell)$.*

Proof. First, we prove the assertion that $|P_C^1(w) \cup P_C^3(w) \setminus R_C(w)| \leq 2$. Suppose, the contrary holds and let $c(wu_{a-1}) = c(wu_p)$, $u_p \in R_C(w)$. If $c(wu_{a-3}) = c(wu_\ell)$, then $c(wu_{a-1}) = c(wu_\ell)$ and $|R_C(w)| = 3$. Apparently, $d^c(w) < \frac{n+1}{2}$ if $|W_2(C)| + |W_3(C)| \leq 3$. Thus, $|W_2(C)| + |W_3(C)| = 4$. From Corollary 3.1 and Proposition 3.1 we know $|W_2(C)| = 1$ and $|W_3(C)| = 3$. Then, $c(wu_3) = c(wu_5)$ while $c(wu_4) \neq c(wu_2)$ or $c(wu_2) = c(wu_4)$ while $c(wu_3) \neq c(wu_1)$, a contradiction. If $c(wu_{a-3}) \neq c(wu_\ell)$, then $(u_1, u_{a-2}) \in DP_w$. Furthermore, if $u_p \neq u_1$, then $c(u_1u_{a-2}) \in \{c(u_1u_\ell), c(u_{a-2}u_{a-3})\}$; otherwise, $wu_2\overrightarrow{C}u_{a-2}u_1\overleftarrow{C}u_pu_{a-1}\overrightarrow{C}u_{p-1}w$ is a requested cycle. Since $c(wu_{a-2}) \neq c(wu_\ell)$, we have $c(u_1u_{a-2}) = c(u_{a-2}u_{a-3}) = c_w$. Thus, $|R_C(w)| = 2$, a contradiction. If $u_p = u_1$, then since $|R_C(w)| \geq 3$, we have $(u_1, u_{a-2}) \in DP_w$. Since $wu_2\overrightarrow{C}u_{a-2}u_1u_{a-1}\overrightarrow{C}u_\ell w$ is of length $\ell + 1$ and contains v_1, we have $c(u_1u_{a-2}) \in \{c(u_{a-1}u_1), c(u_{a-2}u_{a-3})\}$. Then, $c(u_1u_{a-2}) = c(u_{a-2}u_{a-1}) = c(u_{a-1}u_\ell)$. Thus, there exist joint monochromatic triangles in (G, c), a contradiction. Hence, the assertion follows.

Since C_{u_1} is proper, according to Lemma 3.12, if $|P^3_{C_{u_1}}(u_1)| = 2$, the result follows. In the following suppose $|P^3_{C_{u_1}}(u_1)| \neq 2$.

In the first case suppose $|P^3_{C_{u_1}}(u_1)| \geq 3$. Then, $|P^2_{C_{u_1}}(u_1)| = 1$; otherwise, $c(wu_3) = c(u_2u_3)$, a contradiction. If $c(u_1, P^1_{C_{u_1}}(u_1)) = c_w$, then according to Lemma 3.11, the result follows. If $c(u_1, P^1_{C_{u_1}}(u_1)) \neq c_w$, then $c(u_1u_i) \notin \{c_w, c(u_iu_{i+1})\}$ for $i = 4, 5$. Hence, $c(wu_4) = c(wu_3) = c(wu_\ell)$, which means $|P_C^3(w)| = 2$, a contradiction.

In the second case suppose $|P_C^3(w)| = 1$ and $|P^3_{C_{u_1}}(u_1)| \leq 1$. When $c(wu_3) \neq c_w$, C_{u_2} is proper. Since $wu_3u_2u_4\overrightarrow{C}u_1w$ is a cycle of length $\ell + 1$ and contains v_1, we have $c(u_2u_4) \in \{c(u_2u_3), c(u_4u_5)\}$. If $c(u_2u_4) = c(u_4u_5)$, then the result holds. If $c(u_2u_4) = c(u_2u_3) \neq c(u_4u_5)$, then $u_2 \in W_3(C_{u_2})$ with $|P^3_{C_{u_2}}(u_2)| \geq 2$. Thus, according to Lemmas 3.11 and 3.12, the result follows. When $c(wu_3) = c_w$, from our assertion we know $R_C(w) = P_C^1(w) = V(u_4\overrightarrow{C}u_1)$. Since for $u_k \in V(u_5\overrightarrow{C}u_\ell)$, both $wu_3\overrightarrow{C}u_{k-1}u_1u_2u_k\overrightarrow{C}u_\ell w$ and $wu_{k-1}\overleftarrow{C}u_2u_k\overrightarrow{C}u_1$ are of length $\ell + 1$ and contain v_1, we have $c(u_2u_k) = c(u_ku_{k+1})$. The result thus follows.

In the third case suppose $P_C^3(w) = \emptyset$ and $|P^3_{C_{u_1}}(u_1)| \leq 1$. Apparently, if $|P^3_{C_{u_1}}(u_1)| = 1$, the result holds. In the following suppose $P^3_{C_{u_1}}(u_1) = \emptyset$. If $u_3\overrightarrow{C}u_2$ is rainbow, from Proposition 3.4, there is a requested cycle in (G, c). If $u_4 \notin R_C(w)$, then assuming $c(wu_4) = c(uu_p)$, we have $u_p \in R_C(w)$. Since $c(u_1u_k) = c(wu_k)$, we have $c(wu_k) \neq c_w$ for $u_k \in P_C^1(w)$. Since $wu_2u_3u_1\overleftarrow{C}u_pu_{a-1}\overrightarrow{C}u_{p-1}w$ is of length $\ell + 1$ and contains v_1, we have $c(u_1u_3) = c(u_1u_\ell)$, that is, $c(wu_3) = c(wu_\ell)$. Then, $u_4\overrightarrow{C}u_1$ is rainbow. According to Proposition 3.4, there is a requested cycle in (G, c). ∎

Proposition 3.5. *Let (G, c) be an edge-colored complete graph on $n \geq 3$ vertices such that $\delta^c(G) \geq \frac{n+1}{2}$, and not contain joint monochromatic triangles. For any $w \in W_3(C)$,* *if $Q_C(w) = P_C^2(w)$, then (G, c) is properly vertex-pancyclic.*

Proof. Suppose, to the contrary, that there exists a vertex v which is contained in a proper ℓ-cycle C in (G, c) for some ℓ with $4 \leq \ell \leq n-1$, but no proper cycle of length $\ell+1$ in (G, c) contains vertex v. According to Lemma 3.9, suppose that there is a vertex $w \in W_3(C)$ such that $|P_C^3(w)| \leq 2$.

According to Lemmas 3.10, 3.11, 3.12 and 3.13, there is a vertex set V_1 of size $\ell - 4$ such that $|\mathcal{C}(u, V(C) \setminus V_1) \cap \mathcal{C}(u, V_1)| \geq 2$ for $u \in V_1$, and V_1 has the DP_w. Then, there is a vertex $u_p \in V_1$ with $d^c_{V(C)}(u_p) \leq \frac{\ell+1}{2}$. If $W_3(C) = \{w\}$, $d^c_{V(C)}(u_p) \leq \frac{\ell+1}{2} < \frac{n+1}{2}$, a contradiction. Thus, $|W_3(C)| \geq 2$. Then, we give $(G[V_1], c)$ a coloring orientation. Assume u_p is the maximum out-degree in $D(G[V_1])$.

If $W_3(C) = \{w, w'\}$, then $d^+_D(u_p) = \frac{\ell-5}{2}$; otherwise, $d^c(u_p) < \ell - 5 - \frac{\ell-5}{2} + 4 = \frac{\ell+3}{2} \leq \frac{n+1}{2}$, a contradiction. Since the average out-degree of D is $\frac{\ell-5}{2}$, we have $d^+_D(u) = \frac{\ell-5}{2}$ for $u \in V_1$. Therefore, $V_1 \subseteq P_C^3(w')$; otherwise, the color of $w'u$ is a used color in $\mathcal{C}(u, V(C))$. Then, there is a vertex whose color degree less that $\frac{n+1}{2}$. Thus, $P_C^3(w') = u_5 \overrightarrow{C} u_\ell$ or $P_C^3(w') = u_3 \overrightarrow{C} u_{\ell-2}$; otherwise, according to Lemma 3.1, (G, c) has joint monochromatic triangles. Then, we might as well suppose $P_C^3(w') = u_5 \overrightarrow{C} u_\ell$. Then, $w'u_3u_4$ is monochromatic. According to Lemma 3.9, we have that $u_3 \in W_3(C_{u_3})$ with $|P^3_{C_{u_3}}(u_3)| \leq 1$ or $u_4 \in W_3(C_{u_4})$ with $|P^3_{C_{u_4}}(u_4)| \leq 1$. (Note that $C_{u_i} = u_{i-1}w'u_{i+1}\overrightarrow{C}u_{i-1}$, $i = 3, 4$). Thus, there is a vertex $u_i \in V_1$ such that $\{c(u_3u_i), c(u_4u_i)\} \cap \{c(u_iu_{i+1}), c(u_iu_{i-1})\} \neq \emptyset$. Then, $d^c(u_i) < \ell - 5 - \frac{\ell-5}{2} + 3 = \frac{\ell+1}{2} \leq \frac{n+1}{2}$, a contradiction.

If $W_3(C) = \{w, w', w''\}$, then we can suppose that wv_1v_i is not monochromatic, $i = 2, \ell$. Then $u_p \in P_C^3(w') \cap P_C^3(w'')$; otherwise, $d^c(u_p) \leq \ell - 5 - \frac{\ell-5}{2} + 1 + 3 = \frac{\ell+3}{2} < \frac{n+1}{2}$, a contradiction. Thus, $\frac{\ell-5}{2} \leq d^+_D(u_p) \leq \frac{\ell-4}{2}$; otherwise, $d^c(u_p) \leq \ell - 5 - \frac{\ell-3}{2} + 5 = \frac{\ell+3}{2} < \frac{n+1}{2}$, a contradiction. When $d^c(u_p) = \frac{\ell-5}{2}$, since the average out-degree of D is $\frac{\ell-5}{2}$, we have $d^+_D(u) = \frac{\ell-5}{2}$ for $u \in V_1$. Therefore, $V_1 \subseteq P_C^3(w') \cap P_C^3(w'')$. Thus, $P_C^3(w') \cap P_C^3(w'') = u_5 \overrightarrow{C} u_\ell$ or $P_C^3(w') \cap P_C^3(w'') = u_3 \overrightarrow{C} u_{\ell-2}$. Then, there is a vertex $u \in V_1$ such that $d^c(u) < \frac{n+1}{2}$, a contradiction. When $d^c(u_p) = \frac{\ell-4}{2}$, there is a distinct vertex u' in V_1 such that $d^+(u') \geq \frac{\ell-5}{2}$. Thus, $\{u_p, u'\} \subseteq P_C^3(w') \cap P_C^3(w'')$. According to the claim of Corollary 3.1, we have $|P_C^3(w') \cap P_C^3(w'')| \geq 3$. According to Lemma 3.9, we might as well have $|P_C^3(w')| \leq 1$ and $w''v_1v_i$ is monochromatic, $i = 2, \ell$. Then, $c(w'u_p) \subseteq \{c(u_pu_{p+1}), c(u_pu_{p-1})\}$. Thus, $d^c(u_p) \leq \ell - 5 - \frac{\ell-4}{2} + 1 + 3 = \frac{\ell+2}{2} < \frac{n+1}{2}$, a contradiction. ∎

Eventually, combining Propositions 3.1 through 3.5, we get the proof of our main result Theorem 2.2.

4 Concluding remarks

Many years have passed since Conjecture 2 was proposed by Fujita and Magnant. However Conjecture 1 has not been solved, yet. Fujita and Magnant proved the existence of properly colored cycles of lengths 3,4, and at least 5 passing though a given vertex. Inspired by this idea, Chen et al. in [8] proved that Conjecture 2 is true by adding an extra condition "without monochromatic triangles" and they used induction and contradiction technique to prove it. They analysed the relationship between a properly colored cycle of a fixed length and the vertices that are not on the cycle, and divided this vertices into two categories. Recently, Li in [19] investigated the existence of properly colored cycles in edge-colored complete graphs when monochromatic triangles are forbidden, to obtain a vertex-pancyclic analogous result combined with a characterization of all the exceptions.

Based on Theorem 2.1, we add a looser condition that the edge-colored complete graphs can have monochromatic triangles but do not have any two joint monochromatic triangles and show that Conjecture 2 is true under this condition. We show this also using induction and contradiction technique and some claims proved in [8]. Our results, although improving the known result Theorem 2.1 and partially solving Conjecture 2, are still far from a solution for Conjecture 2. Up to now, we have not found any better method to solve it. Further study is needed.

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