M_2-RANKS OF OVERPARTITIONS MODULO 4 AND 8

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Abstract. An overpartition is a partition in which the first occurrence of a number may be overlined. For an overpartition λ, let $\ell(\lambda)$ denote the largest part of λ, and let $n(\lambda)$ denote its number of parts. Then the M_2-rank of an overpartition is defined as

$$M_2\text{-rank}(\lambda) := \left\lceil \frac{\ell(\lambda)}{2} \right\rceil - n(\lambda) + n(\lambda_0) - \chi(\lambda),$$

where $\chi(\lambda) = 1$ if $\ell(\lambda)$ is odd and non-overlined and $\chi(\lambda) = 0$, otherwise. In this paper, we study the M_2-rank differences of overpartitions modulo 4 and 8. Especially, we obtain some relations between the generating functions of the M_2-rank differences modulo 4 and 8 and the second order mock theta functions. Furthermore, we deduce some inequalities on M_2-ranks of overpartitions.

1. Introduction

Let q denote a complex number with $0 < |q| < 1$. Recall that for positive integers n and m,

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k),$$
$$j(x;q) := (x;q)_\infty (q/x;q)_\infty (q;q)_\infty, \quad J_m := (q^m;q^m)_\infty.$$

A partition of n is a nonincreasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of partitions of n. Dyson [5] defined the rank of a partition as its largest part minus the number of parts. Let $N(s,m,n)$ be the number of partitions of n with rank congruent to s modulo m. Atkin and Swinnerton-Dyer [2] deduced the generating functions of the rank differences $N(s,m,mn+d) - N(t,m,mn+d)$ for $m = 5$ or 7 and $0 \le d, s, t < m$. Subsequently, Atkin and Hussain [1] provided the generating functions of the rank differences for $m = 11$. Since then, rank differences of partitions modulo other numbers were widely concerned. For instance, see [7, 8, 12, 14].

Corteel and Lovejoy [4] defined the overpartition of n as a partition of n in which the first occurrence of a number may be overlined. Then the D-rank of an overpartition [9] is introduced as its largest part $\ell(\lambda)$ minus its number of parts $n(\lambda)$. Let λ_0 be the partition with non-overlined odd parts of an overpartition λ. The M_2-rank of an overpartition given by Lovejoy [10] is defined as

$$M_2\text{-rank}(\lambda) := \left\lceil \frac{\ell(\lambda)}{2} \right\rceil - n(\lambda) + n(\lambda_0) - \chi(\lambda),$$

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where $\chi(\lambda)=1$ if $\ell(\lambda)$ is odd and non-overlined and $\chi(\lambda)=0$, otherwise. Let $\overline{N}_2(m,n)$ be the number of overpartitions of n with M_2-rank m, and let $\overline{N}_2(s,m,n)$ be the number of overpartitions of n with M_2-rank congruent to s modulo m. In [10], Lovejoy introduced the generating function of $\overline{N}_2(m,n)$ given by

$$\sum_{n=0}^{\infty}\overline{N}_2(m,n)q^n=\frac{2J_2}{J_1^2}\sum_{n=1}^{\infty}(-1)^{n+1}q^{n^2+2|m|n}\frac{1-q^{2n}}{1+q^{2n}}.$$

Subsequently, Lovejoy and Osburn [11] provided that

$$\sum_{n=0}^{\infty}\overline{N}_2(s,m,n)q^n=\frac{2J_2}{J_1^2}\sum_{n=-\infty}^{\infty}{}'\frac{(-1)^nq^{n^2+2n}(q^{2sn}+q^{2(m-s)n})}{(1+q^{2n})(1-q^{2mn})},\tag{1.1}$$

where the prime means that we omit the term of $n=0$. Notice that

$$\overline{N}_2(m,n)=\overline{N}_2(-m,n),$$
$$\overline{N}_2(s,m,n)=\overline{N}_2(m-s,m,n).$$

Lovejoy and Osburn [11] considered the rank differences $\overline{N}_2(s,m,mn+d)-\overline{N}_2(t,m,mn+d)$ for $m=3$ or 5. Recently, Zhang [15] considered the rank differences of overpartitions modulo 6 and 10.

The aim of this paper is to study the M_2-rank differences of overpartitions modulo 4 and 8. We establish some relations between the generating functions of the M_2-rank differences modulo 4 and 8 and the second order mock theta functions. Furthermore, we obtain some inequalities on the M_2-ranks of overpartitions modulo 4 and 8.

Ramanujan's general theta function $f(a,b)$ is given by

$$f(a,b):=\sum_{n=-\infty}^{\infty}a^{n(n+1)/2}b^{n(n-1)/2}.$$

Then, we have the following two special cases:

$$\varphi(q):=f(q,q)=\sum_{n=-\infty}^{\infty}q^{n^2},$$
$$\psi(q):=f(q,q^3)=\sum_{n=0}^{\infty}q^{\binom{n+1}{2}}.$$

Notice that

$$\varphi(q)=j(-q;q^2)=\frac{J_2^5}{J_1^2J_4^2},\tag{1.2}$$
$$\psi(q)=j(-q;q^4)=\frac{1}{2}j(-1;q)=\frac{J_2^2}{J_1},\tag{1.3}$$
$$\varphi(-q)=j(q;q^2)=\frac{J_1^2}{J_2},$$
$$\psi(-q)=j(q;q^4)=\frac{J_1J_4}{J_2}.$$

The second order mock theta functions $A(q)$ and $B(q)$ are defined as [13]

$$A(q)=\sum_{n=0}^{\infty}\frac{q^{n+1}(-q^2;q^2)_n}{(q;q^2)_{n+1}},\tag{1.4}$$

$$B(q)=\sum_{n=0}^{\infty}\frac{q^n(-q;q^2)_n}{(q;q^2)_{n+1}}. \tag{1.5}$$

Then the main results of this paper are stated as follows.

Theorem 1.1. *We have*

$$\sum_{n=0}^{\infty}\left(\overline{N}_2(0,4,n)-\overline{N}_2(2,4,n)\right)q^n=2qB(q^2)+\frac{J_4^8}{J_2^4J_8^3}-1. \tag{1.6}$$

Theorem 1.2. *For $n\geq 1$,*

$$\overline{N}_2(0,4,n)>\overline{N}_2(2,4,n).$$

Theorem 1.3. *We have*

$$\sum_{n=0}^{\infty}\left(\overline{N}_2(1,8,n)-\overline{N}_2(3,8,n)\right)q^n=-2q^{-1}A(q^8)-2q^4B(q^8)+M_1, \tag{1.7}$$

$$\sum_{n=0}^{\infty}\left(\overline{N}_2(0,8,n)-\overline{N}_2(4,8,n)\right)q^n=4q^{-1}A(q^8)+2q^4B(q^8)+M_2, \tag{1.8}$$

$$\sum_{n=0}^{\infty}\left(\overline{N}_2(2,8,n)-\overline{N}_2(4,8,n)\right)q^n=2q^{-1}A(q^8)-qB(q^2)+q^4B(q^8)+M_3, \tag{1.9}$$

$$\sum_{n=0}^{\infty}\left(\overline{N}_2(0,8,n)-\overline{N}_2(2,8,n)\right)q^n=2q^{-1}A(q^8)+qB(q^2)+q^4B(q^8)+M_4, \tag{1.10}$$

where

$$\begin{aligned}
M_1&:=\frac{q^{-1}J_{16}^{11}}{2J_8^5J_{32}^5}-\frac{q^{-1}J_2J_4^2J_8^2}{2J_1^2J_{16}J_{32}}+\frac{J_2J_8J_{16}^6}{J_1^2J_4^2J_{32}^3}-\frac{8q^5J_2J_{32}^5}{J_1^2J_8J_{16}^2},\\
M_2&:=-\frac{q^{-1}J_{16}^{11}}{J_8^5J_{32}^5}+\frac{q^{-1}J_2J_4^2J_8^2}{J_1^2J_{16}J_{32}}-\frac{J_2J_8J_{16}^6}{J_1^2J_4^2J_{32}^3}+\frac{8q^5J_2J_{32}^5}{J_1^2J_8J_{16}^2}-1,\\
M_3&:=-\frac{q^{-1}J_{16}^{11}}{2J_8^5J_{32}^5}+\frac{q^{-1}J_2J_4^2J_8^2}{2J_1^2J_{16}J_{32}}-\frac{J_2J_8J_{16}^6}{2J_1^2J_4^2J_{32}^3}+\frac{4q^5J_2J_{32}^5}{J_1^2J_8J_{16}^2}-\frac{J_4^8}{2J_2^4J_8^3},\\
M_4&:=-\frac{q^{-1}J_{16}^{11}}{2J_8^5J_{32}^5}+\frac{q^{-1}J_2J_4^2J_8^2}{2J_1^2J_{16}J_{32}}-\frac{J_2J_8J_{16}^6}{2J_1^2J_4^2J_{32}^3}+\frac{4q^5J_2J_{32}^5}{J_1^2J_8J_{16}^2}+\frac{J_4^8}{2J_2^4J_8^3}-1.
\end{aligned}$$

Theorem 1.4. *For $n\geq 1$,*

$$\overline{N}_2(0,8,n)+\overline{N}_2(1,8,n)>\overline{N}_2(3,8,n)+\overline{N}_2(4,8,n).$$

This paper is organized as follows. In Section 2, we state some lemmas which are used to prove the main theorems. In Section 3, we prove Theorems 1.1–1.4.

2. Preliminaries

The following identities are frequently used in this paper.

$$j(x;q)=j(q/x;q),$$
$$j(x;q)=-xj(qx;q).$$

In order to prove the main results, the following lemmas are needed.

Lemma 2.1. *[3, Entry 25] We have*

$$\varphi(q)+\varphi(-q)=2\varphi(q^4), \tag{2.1}$$
$$\varphi(q)-\varphi(-q)=4q\psi(q^8), \tag{2.2}$$
$$\varphi(q)\varphi(-q)=\varphi^2(-q^2), \tag{2.3}$$
$$\varphi(q)\psi(q^2)=\psi^2(q), \tag{2.4}$$
$$\varphi^2(q)-\varphi^2(-q)=8q\psi^2(q^4), \tag{2.5}$$
$$\varphi^2(q)+\varphi^2(-q)=2\varphi^2(q^2), \tag{2.6}$$
$$\varphi^4(q)-\varphi^4(-q)=16q\psi^4(q^2). \tag{2.7}$$

Notice that combining (2.1) and (2.2) yields

$$\varphi(q)=\varphi(q^4)+2q\psi(q^8), \tag{2.8}$$
$$\varphi(-q)=\varphi(q^4)-2q\psi(q^8). \tag{2.9}$$

Meanwhile, from (2.5) and (2.6), it can be shown that

$$\varphi^2(q)=\varphi^2(q^2)+4q\psi^2(q^4), \tag{2.10}$$
$$\varphi^2(-q)=\varphi^2(q^2)-4q\psi^2(q^4). \tag{2.11}$$

Furthermore, we need the following identity [3, Equation (31.10)]

$$\varphi(-q^2)\psi(-q)=\varphi(-q)\psi(q). \tag{2.12}$$

Hickerson and Mortenson [6] defined Appell–Lerch sums as follows.

Definition 2.2. *Let $x,z\in\mathbb{C}^*:=\mathbb{C}/\{0\}$ with neither z nor xz an integral power of q. Then*

$$m(x,q,z)=\frac{-z}{j(z;q)}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{\binom{n+1}{2}}z^n}{1-q^n xz}. \tag{2.13}$$

Proposition 2.3. *[6] For generic $x,z\in\mathbb{C}^*$,*

$$m(x,q,z)=m(x,q,qz), \tag{2.14}$$
$$m(x,q,z)=x^{-1}m(x^{-1},q,z^{-1}). \tag{2.15}$$

Following [6], the term "generic" means that the parameters do not cause poles in the Appell–Lerch sums or in the quotients of theta functions.

Lemma 2.4. *[6, Theorem 3.3] For generic $x,z_0,z_1\in\mathbb{C}^*$,*

$$m(x,q,z_1)-m(x,q,z_0)=\frac{z_0J_1^3 j(z_1/z_0;q)j(xz_0z_1;q)}{j(z_0;q)j(z_1;q)j(xz_0;q)j(xz_1;q)}.$$

Hickerson and Mortenson [6] deduced that

$$A(q)=-m(q,q^4,q^2), \tag{2.16}$$
$$B(q)=-q^{-1}m(1,q^4,q^3). \tag{2.17}$$

Lemma 2.5. *We have*

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+2n}}{1+q^{4n}}=\frac{qJ_1^2}{J_2}B(q^2)+\frac{J_1^2J_4^8}{2J_2^5J_8^3}.$$

Proof. First, by splitting the sum into two sums according to the summation index n modulo 2, we have

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+2n}}{1+q^{4n}}=\sum_{n=-\infty}^{\infty}\frac{q^{4n^2+4n}}{1+q^{8n}}-\sum_{n=-\infty}^{\infty}\frac{q^{4n^2+8n+3}}{1+q^{8n+4}}.$$

Then by (1.2), (1.3), and (2.13), we obtain

$$\sum_{n=-\infty}^{\infty}\frac{q^{4n^2+4n}}{1+q^{8n}}=j(-1;q^8)m(1,q^8,-1)=2\psi(q^8)m(1,q^8,-1),$$
$$\sum_{n=-\infty}^{\infty}\frac{q^{4n^2+8n+3}}{1+q^{8n+4}}=q^{-1}j(-q^4;q^8)m(1,q^8,-q^4)=q^{-1}\varphi(q^4)m(1,q^8,-q^4).$$

Hence,

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+2n}}{1+q^{4n}}=2\psi(q^8)m(1,q^8,-1)-q^{-1}\varphi(q^4)m(1,q^8,-q^4).$$

Furthermore, utilizing (2.1) and (2.2) yields

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+2n}}{1+q^{4n}}\\
&=q^{-1}m(1,q^8,-1)\frac{\varphi(q)-\varphi(-q)}{2}-q^{-1}m(1,q^8,-q^4)\frac{\varphi(q)+\varphi(-q)}{2}\\
&=\frac{q^{-1}}{2}\varphi(q)\left(m(1,q^8,-1)-m(1,q^8,-q^4)\right)-\frac{q^{-1}}{2}\varphi(-q)\left(m(1,q^8,-1)+m(1,q^8,-q^4)\right)\\
&=\frac{q^{-1}}{2}\varphi(q)\left(\frac{J_8^3j(q^4;q^8)^2}{j(-1;q^8)^2j(-q^4;q^8)^2}\right)-\frac{q^{-1}}{2}\varphi(-q)\left(m(1,q^8,-1)+m(1,q^8,-q^4)\right)\\
&=\frac{q^{-1}J_8^3\varphi(q)\varphi^2(-q^4)}{8\varphi^2(q^4)\psi^2(q^8)}-\frac{q^{-1}}{2}\varphi(-q)\left(m(1,q^8,-1)+m(1,q^8,-q^4)+2q^2B(q^2)\right)\\
&\quad+q\varphi(-q)B(q^2),
\end{aligned}\tag{2.18}$$

where the penultimate step follows from Lemma 2.4. Moreover, in view of (2.17), we have

$$\begin{aligned}
&\frac{q^{-1}}{2}\varphi(-q)\left(m(1,q^8,-1)+q^2B(q^2)+m(1,q^8,-q^4)+q^2B(q^2)\right)\\
&=\frac{q^{-1}}{2}\varphi(-q)\left(m(1,q^8,-1)-m(1,q^8,q^6)+m(1,q^8,-q^4)-m(1,q^8,q^6)\right)\\
&=\frac{q^{-1}}{2}\varphi(-q)\left(\frac{q^6J_8^3j(-q^{-6};q^8)j(-q^6;q^8)}{j(q^6;q^8)^2j(-1;q^8)^2}+\frac{q^6J_8^3j(-q^{-2};q^8)j(-q^{10};q^8)}{j(q^6;q^8)^2j(-q^4;q^8)^2}\right)\\
&=\frac{q^{-1}}{2}\varphi(-q)\cdot\frac{J_8^3\psi^2(q^2)}{\psi^2(-q^2)}\cdot\frac{\varphi^2(q^4)+4q^2\psi^2(q^8)}{4\varphi^2(q^4)\psi^2(q^8)}\\
&=\frac{q^{-1}}{2}\varphi(-q)\cdot\frac{J_8^3\psi^2(q^2)}{\psi^2(-q^2)}\cdot\frac{\varphi^2(q^2)}{4\psi^4(q^4)}\\
&=\frac{q^{-1}J_8^3\varphi(-q)\varphi^2(q^2)\psi^2(q^2)}{8\psi^2(-q^2)\psi^4(q^4)},
\end{aligned}$$

where we obtain the second equality by using Lemma 2.4 and the penultimate step follows from (2.4) and (2.10). Substituting the above identity into (2.18), we derive

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+2n}}{1+q^{4n}}\\
&=\frac{q^{-1}J_8^3\varphi(q)\varphi^2(-q^4)}{8\varphi^2(q^4)\psi^2(q^8)}-\frac{q^{-1}J_8^3\varphi(-q)\varphi^2(q^2)\psi^2(q^2)}{8\psi^2(-q^2)\psi^4(q^4)}+q\varphi(-q)B(q^2)\\
&=\frac{q^{-1}J_8^3\varphi(q)\varphi^2(-q^4)}{8\psi^4(q^4)}-\frac{q^{-1}J_8^3\varphi(-q)\varphi^2(q^2)\psi^2(q^2)}{8\psi^2(-q^2)\psi^4(q^4)}+q\varphi(-q)B(q^2)\qquad\text{(by (2.4))}\\
&=\frac{q^{-1}J_8^3\left(\varphi(q)\varphi^2(-q^4)\psi^2(-q^2)-\varphi(-q)\varphi^2(q^2)\psi^2(q^2)\right)}{8\psi^2(-q^2)\psi^4(q^4)}+q\varphi(-q)B(q^2)\\
&=\frac{q^{-1}J_8^3\left(\varphi(q)\varphi^2(-q^2)\psi^2(q^2)-\varphi(-q)\varphi^2(q^2)\psi^2(q^2)\right)}{8\psi^2(-q^2)\psi^4(q^4)}+q\varphi(-q)B(q^2)\qquad\text{(by (2.12))}\\
&=\frac{q^{-1}J_8^3\psi^2(q^2)\left(\varphi(-q)\varphi^2(q)-\varphi(-q)\varphi^2(q^2)\right)}{8\psi^2(-q^2)\psi^4(q^4)}+q\varphi(-q)B(q^2)\qquad\text{(by (2.3))}\\
&=\frac{q^{-1}J_8^3\varphi(-q)\psi^2(q^2)\left(\varphi^2(q)-\varphi^2(q^2)\right)}{8\psi^2(-q^2)\psi^4(q^4)}+q\varphi(-q)B(q^2)\\
&=\frac{J_8^3\varphi(-q)\psi^2(q^2)}{2\psi^2(-q^2)\psi^2(q^4)}+q\varphi(-q)B(q^2)\qquad\text{(by (2.10))}\\
&=\frac{J_1^2J_4^8}{2J_2^5J_8^3}+\frac{qJ_1^2}{J_2}B(q^2).
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.6. *We have*

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+2n}}{1+q^{8n}}=\frac{q^{-1}J_1^2}{J_2}A(q^8)-\frac{q^{-1}J_1^2J_{16}^{11}}{4J_2J_8^5J_{32}^5}+\frac{q^{-1}J_4^2J_8^2}{4J_{16}J_{32}}.$$

Proof. First, we split the sum into four sums in light of the summation index n modulo 4.

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+2n}}{1+q^{8n}}\\
&=\sum_{n=-\infty}^{\infty}\frac{q^{16n^2+8n}}{1+q^{32n}}-\sum_{n=-\infty}^{\infty}\frac{q^{16n^2+16n+3}}{1+q^{32n+8}}+\sum_{n=-\infty}^{\infty}\frac{q^{16n^2+24n+8}}{1+q^{32n+16}}-\sum_{n=-\infty}^{\infty}\frac{q^{16n^2+32n+15}}{1+q^{32n+24}}\\
&=q^8j(-q^{-8};q^{32})m(q^8,q^{32},-q^{-8})-q^3j(-1;q^{32})m(q^8,q^{32},-1)\\
&\quad+j(-q^8;q^{32})m(q^8,q^{32},-q^8)-q^{-1}j(-q^{16};q^{32})m(q^8,q^{32},-q^{16})\\
&=\psi(q^8)m(q^8,q^{32},-q^{-8})-2q^3\psi(q^{32})m(q^8,q^{32},-1)\\
&\quad+\psi(q^8)m(q^8,q^{32},-q^8)-q^{-1}\varphi(q^{16})m(q^8,q^{32},-q^{16})\\
&=\psi(q^8)\left(m(q^8,q^{32},-q^{-8})+m(q^8,q^{32},-q^8)\right)\\
&\quad-q^{-1}\left(2q^4\psi(q^{32})m(q^8,q^{32},-1)+\varphi(q^{16})m(q^8,q^{32},-q^{16})\right),
\end{aligned}\tag{2.19}$$

where the second equality follows from (2.13).

Let

$$X=\psi(q^8)\left(m(q^8,q^{32},-q^{-8})+m(q^8,q^{32},-q^8)\right).$$

Then from (2.16), it can be seen that

$$\begin{aligned}
X &= \psi(q^8)\left(m(q^8,q^{32},-q^{-8})+m(q^8,q^{32},-q^8)+2A(q^8)\right)-2\psi(q^8)A(q^8)\\
&= \psi(q^8)\left(m(q^8,q^{32},-q^{-8})-m(q^8,q^{32},q^{16})+m(q^8,q^{32},-q^8)-m(q^8,q^{32},q^{16})\right)-2\psi(q^8)A(q^8)\\
&= \psi(q^8)\left(\frac{q^{16}J_{32}^3 j(-q^{-24};q^{32})j(-q^{16};q^{32})}{j(q^{16};q^{32})j(-q^{-8};q^{32})j(q^8;q^{32})j(-1;q^{32})}\right.\\
&\quad\left.+\frac{q^{16}J_{32}^3 j(-q^{-8};q^{32})j(-1;q^{32})}{j(q^{16};q^{32})j(-q^8;q^{32})j(q^8;q^{32})j(-q^{16};q^{32})}\right)-2\psi(q^8)A(q^8)\\
&= \psi(q^8)\left(\frac{J_{32}^3 j(-q^{16};q^{32})}{j(-1;q^{32})j(q^8;q^{32})j(q^{16};q^{32})}+\frac{q^8J_{32}^3 j(-1;q^{32})}{j(q^8;q^{32})j(q^{16};q^{32})j(-q^{16};q^{32})}\right)-2\psi(q^8)A(q^8)\\
&= \frac{J_{32}^3\psi(q^8)}{\varphi(-q^{16})\psi(-q^8)}\cdot\frac{\varphi^2(q^{16})+4q^8\psi^2(q^{32})}{2\varphi(q^{16})\psi(q^{32})}-2\psi(q^8)A(q^8)\\
&= \frac{J_{32}^3\psi(q^8)}{\varphi(-q^{16})\psi(-q^8)}\cdot\frac{\varphi^2(q^8)}{2\varphi(q^{16})\psi(q^{32})}-2\psi(q^8)A(q^8)\\
&= \frac{J_{32}^3\varphi^2(q^8)\psi(q^8)}{2\varphi(q^{16})\varphi(-q^{16})\psi(-q^8)\psi(q^{32})}-2\psi(q^8)A(q^8), \qquad (2.20)
\end{aligned}$$

where we derive the third step by utilizing Lemma 2.4 and the penultimate step follows from (2.10).

Set

$$Y = q^{-1}\left(2q^4\psi(q^{32})m(q^8,q^{32},-1)+\varphi(q^{16})m(q^8,q^{32},-q^{16})\right).$$

Then in view of (2.1) and (2.2), we obtain

$$\begin{aligned}
Y &= \frac{q^{-1}}{2}\left((\varphi(q^4)-\varphi(-q^4))m(q^8,q^{32},-1)+(\varphi(q^4)+\varphi(-q^4))m(q^8,q^{32},-q^{16})\right)\\
&= \frac{q^{-1}\varphi(q^4)}{2}\left(m(q^8,q^{32},-1)+m(q^8,q^{32},-q^{16})\right)+\frac{q^{-1}\varphi(-q^4)}{2}\left(m(q^8,q^{32},-q^{16})-m(q^8,q^{32},-1)\right)\\
&= \frac{q^{-1}\varphi(q^4)}{2}\left(m(q^8,q^{32},-1)+m(q^8,q^{32},-q^{16})+2A(q^8)\right)-q^{-1}\varphi(q^4)A(q^8)\\
&\quad-\frac{q^{-1}\varphi(-q^4)}{2}\cdot\frac{J_{32}^3 j(q^8;q^{32})j(q^{16};q^{32})}{j(-1;q^{32})j(-q^8;q^{32})^2j(-q^{16};q^{32})}\\
&= \frac{q^{-1}\varphi(q^4)}{2}\left(m(q^8,q^{32},-1)+m(q^8,q^{32},-q^{16})+2A(q^8)\right)\\
&\quad-q^{-1}\varphi(q^4)A(q^8)-\frac{q^{-1}J_{32}^3\varphi(-q^4)\varphi(-q^{16})\psi(-q^8)}{4\varphi(q^{16})\psi^2(q^8)\psi(q^{32})}, \qquad (2.21)
\end{aligned}$$

where the penultimate step follows from Lemma 2.4. According to (2.16), the first term on the right-hand side of (2.21) becomes

$$\begin{aligned}
&\frac{q^{-1}\varphi(q^4)}{2}\left(m(q^8,q^{32},-1)+m(q^8,q^{32},-q^{16})+2A(q^8)\right)\\
&= \frac{q^{-1}\varphi(q^4)}{2}\left(m(q^8,q^{32},-1)-m(q^8,q^{32},q^{16})+m(q^8,q^{32},-q^{16})-m(q^8,q^{32},q^{16})\right)\\
&= \frac{q^{-1}\varphi(q^4)}{2}\left(\frac{J_{32}^3 j(-q^{16};q^{32})}{j(-1;q^{32})j(q^8;q^{32})j(q^{16};q^{32})}+\frac{q^8J_{32}^3 j(-1;q^{32})}{j(q^8;q^{32})j(q^{16};q^{32})j(-q^{16};q^{32})}\right)\\
&= \frac{q^{-1}J_{32}^3\varphi(q^4)}{2}\cdot\frac{j(-q^{16};q^{32})^2+q^8j(-1;q^{32})^2}{j(-1;q^{32})j(q^8;q^{32})j(q^{16};q^{32})j(-q^{16};q^{32})}
\end{aligned}$$

$$= \frac{q^{-1}J_{32}^3\varphi(q^4)}{2}\cdot\frac{\varphi^2(q^{16})+4q^8\psi^2(q^{32})}{2\varphi(q^{16})\varphi(-q^{16})\psi(-q^8)\psi(q^{32})}$$
$$= \frac{q^{-1}J_{32}^3\varphi(q^4)}{2}\cdot\frac{\varphi^2(q^{8})}{2\varphi(q^{16})\varphi(-q^{16})\psi(-q^8)\psi(q^{32})}$$
$$= \frac{q^{-1}J_{32}^3\varphi(q^4)\varphi^2(q^8)}{4\varphi(q^{16})\varphi(-q^{16})\psi(-q^8)\psi(q^{32})},$$

where the second step follows from Lemma 2.4 and we derive the penultimate equality by using (2.10). Substituting the above identity into (2.21) yields

$$Y = \frac{q^{-1}J_{32}^3\varphi(q^4)\varphi^2(q^8)}{4\varphi(q^{16})\varphi(-q^{16})\psi(-q^8)\psi(q^{32})} - q^{-1}\varphi(q^4)A(q^8) - \frac{q^{-1}J_{32}^3\varphi(-q^4)\varphi(-q^{16})\psi(-q^8)}{4\varphi(q^{16})\psi^2(q^8)\psi(q^{32})}. \tag{2.22}$$

Combining (2.19), (2.20) and (2.22), we arrive at

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+2n}}{1+q^{8n}}$$
$$= \frac{J_{32}^3\varphi^2(q^8)\psi(q^8)}{2\varphi(q^{16})\varphi(-q^{16})\psi(-q^8)\psi(q^{32})} - \frac{q^{-1}J_{32}^3\varphi(q^4)\varphi^2(q^8)}{4\varphi(q^{16})\varphi(-q^{16})\psi(-q^8)\psi(q^{32})} + q^{-1}\varphi(q^4)A(q^8)$$
$$- 2\psi(q^8)A(q^8) + \frac{q^{-1}J_{32}^3\varphi(-q^4)\varphi(-q^{16})\psi(-q^8)}{4\varphi(q^{16})\psi^2(q^8)\psi(q^{32})}$$
$$= \frac{q^{-1}J_{32}^3\varphi^2(q^8)\left(2q\psi(q^8)-\varphi(q^4)\right)}{4\varphi(q^{16})\varphi(-q^{16})\psi(-q^8)\psi(q^{32})} + q^{-1}\left(\varphi(q^4)-2q\psi(q^8)\right)A(q^8)$$
$$+ \frac{q^{-1}J_{32}^3\varphi(-q^4)\varphi(-q^{16})\psi(-q^8)}{4\varphi(q^{16})\psi^2(q^8)\psi(q^{32})}$$
$$= q^{-1}\varphi(-q)A(q^8) - \frac{q^{-1}J_{32}^3\varphi(-q)\varphi^2(q^8)}{4\varphi(q^{16})\varphi(-q^{16})\psi(-q^8)\psi(q^{32})} + \frac{q^{-1}J_{32}^3\varphi(-q^4)\varphi(-q^{16})\psi(-q^8)}{4\varphi(q^{16})\psi^2(q^8)\psi(q^{32})}$$
$$= q^{-1}\varphi(-q)A(q^8) - \frac{q^{-1}J_{32}^3\varphi(-q)\varphi^2(q^8)}{4\varphi(-q^{16})\psi(-q^8)\psi^2(q^{16})} + \frac{q^{-1}J_{32}^3\varphi(-q^4)\varphi(-q^8)}{4\psi(q^8)\psi^2(q^{16})}$$
$$= q^{-1}\varphi(-q)A(q^8) - \frac{q^{-1}J_1^2J_{16}^{11}}{4J_2J_8^5J_{32}^5} + \frac{q^{-1}J_4^2J_8^2}{4J_{16}J_{32}},$$

where the third equality follows from (2.9) and we derive the penultimate equality by using (2.4) and (2.12). Thus, we complete the proof. □

Lemma 2.7. *We have*

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+4n}}{1+q^{8n}} = -\frac{q^4J_1^2}{J_2}B(q^8) + \frac{J_8J_{16}^6}{2J_4^2J_{32}^3} - \frac{4q^5J_{32}^5}{J_8J_{16}^2}.$$

Proof. First, we split the sum into four sums in light of the summation index n modulo 4.

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+4n}}{1+q^{8n}}$$
$$= \sum_{n=-\infty}^{\infty}\frac{q^{16n^2+16n}}{1+q^{32n}} - \sum_{n=-\infty}^{\infty}\frac{q^{16n^2+24n+5}}{1+q^{32n+8}} + \sum_{n=-\infty}^{\infty}\frac{q^{16n^2+32n+12}}{1+q^{32n+16}} - \sum_{n=-\infty}^{\infty}\frac{q^{16n^2+40n+21}}{1+q^{32n+24}}$$
$$= j(-1;q^{32})m(1,q^{32},-1) - q^{-3}j(-q^8;q^{32})m(1,q^{32},-q^8)$$

$$
\begin{aligned}
&+ q^{-4}j(-q^{16};q^{32})m(1,q^{32},-q^{16}) - q^{-3}j(-q^{8};q^{32})m(1,q^{32},-q^{24})\\
&= q^{-4}\left(2q^4\psi(q^{32})m(1,q^{32},-1) + \varphi(q^{16})m(1,q^{32},-q^{16})\right)\\
&\quad - q^{-3}\psi(q^8)\left(m(1,q^{32},-q^8) + m(1,q^{32},-q^{24})\right), \qquad (2.23)
\end{aligned}
$$

where we obtain the second equality by utilizing (2.13).

Let

$$
W = q^{-3}\psi(q^8)\left(m(1,q^{32},-q^8) + m(1,q^{32},-q^{24})\right).
$$

In view of (2.14) and (2.15), we derive

$$
m(1,q^{32},-q^{24}) = m(1,q^{32},-q^8).
$$

Then combining the above identity and (2.17) yields

$$
\begin{aligned}
W &= 2q^{-3}\psi(q^8)m(1,q^{32},-q^8)\\
&= 2q^{-3}\psi(q^8)\left(m(1,q^{32},-q^8) + q^8B(q^8)\right) - 2q^5\psi(q^8)B(q^8)\\
&= 2q^{-3}\psi(q^8)\left(m(1,q^{32},-q^8) - m(1,q^{32},q^{24})\right) - 2q^5\psi(q^8)B(q^8)\\
&= 2q^{-3}\psi(q^8)\frac{q^8J_{32}^3j(-1;q^{32})j(-q^{16};q^{32})}{j(q^8;q^{32})^2j(-q^8;q^{32})^2} - 2q^5\psi(q^8)B(q^8)\\
&= \frac{4q^5J_{32}^3\varphi(q^{16})\psi(q^{32})}{\psi(q^8)\psi^2(-q^8)} - 2q^5\psi(q^8)B(q^8)\\
&= \frac{4q^5J_{32}^3\psi^2(q^{16})}{\psi(q^8)\psi^2(-q^8)} - 2q^5\psi(q^8)B(q^8), \qquad (2.24)
\end{aligned}
$$

where the fourth equality follows from Lemma 2.4 and we obtain the last equality by utilizing (2.4).

Set

$$
Z = q^{-4}\left(2q^4\psi(q^{32})m(1,q^{32},-1) + \varphi(q^{16})m(1,q^{32},-q^{16})\right).
$$

Then according to (2.1) and (2.2), we have

$$
\begin{aligned}
Z &= \frac{q^{-4}}{2}\left((\varphi(q^4)-\varphi(-q^4))m(1,q^{32},-1) + (\varphi(q^4)+\varphi(-q^4))m(1,q^{32},-q^{16})\right)\\
&= \frac{q^{-4}\varphi(q^4)}{2}\left(m(1,q^{32},-1) + m(1,q^{32},-q^{16})\right) + \frac{q^{-4}\varphi(-q^4)}{2}\left(m(1,q^{32},-q^{16}) - m(1,q^{32},-1)\right)\\
&= \frac{q^{-4}\varphi(q^4)}{2}\left(m(1,q^{32},-1) + m(1,q^{32},-q^{16}) + 2q^8B(q^8)\right)\\
&\quad - q^4\varphi(q^4)B(q^8) - \frac{q^{-4}J_{32}^3\varphi(-q^4)\varphi^2(-q^{16})}{8\varphi^2(q^{16})\psi^2(q^{32})}, \qquad (2.25)
\end{aligned}
$$

where the last step follows from Lemma 2.4. From (2.17), the first term on the right-hand side of (2.25) becomes

$$
\begin{aligned}
&\frac{q^{-4}\varphi(q^4)}{2}\left(m(1,q^{32},-1) + m(1,q^{32},-q^{16}) + 2q^8B(q^8)\right)\\
&= \frac{q^{-4}\varphi(q^4)}{2}\left(m(1,q^{32},-1) - m(1,q^{32},q^{24}) + m(1,q^{32},-q^{16}) - m(1,q^{32},q^{24})\right)\\
&= \frac{q^{-4}\varphi(q^4)}{2}\left(\frac{J_{32}^3j(-q^8;q^{32})^2}{j(q^8;q^{32})^2j(-1;q^{32})^2} + \frac{q^8J_{32}^3j(-q^8;q^{32})^2}{j(q^8;q^{32})^2j(-q^{16};q^{32})^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{q^{-4}J_{32}^3\varphi(q^4)\psi^2(q^8)}{2\psi^2(-q^8)}\cdot\frac{\varphi^2(q^{16})+4q^8\psi^2(q^{32})}{4\varphi^2(q^{16})\psi^2(q^{32})}\\
&= \frac{q^{-4}J_{32}^3\varphi(q^4)\psi^2(q^8)}{8\psi^2(-q^8)}\cdot\frac{\varphi^2(q^8)}{\varphi^2(q^{16})\psi^2(q^{32})},
\end{aligned}
$$

where we obtain the second equality by using Lemma 2.4 and the last equality follows from (2.10).

Substituting the above identity into (2.25), we obtain

$$
\begin{aligned}
Z &= \frac{q^{-4}J_{32}^3\varphi(q^4)\varphi^2(q^8)\psi^2(q^8)}{8\psi^2(-q^8)\psi^4(q^{16})} - q^4\varphi(q^4)B(q^8) - \frac{q^{-4}J_{32}^3\varphi(-q^4)\varphi^2(-q^{16})}{8\psi^4(q^{16})} \qquad \text{(by (2.4))}\\
&= \frac{q^{-4}J_{32}^3\varphi(q^4)\varphi^2(q^8)\psi^2(q^8)}{8\psi^2(-q^8)\psi^4(q^{16})} - q^4\varphi(q^4)B(q^8) - \frac{q^{-4}J_{32}^3\varphi(-q^4)\varphi^2(-q^{16})\psi^2(-q^8)}{8\psi^2(-q^8)\psi^4(q^{16})}\\
&= \frac{q^{-4}J_{32}^3\varphi(q^4)\varphi^2(q^8)\psi^2(q^8)}{8\psi^2(-q^8)\psi^4(q^{16})} - q^4\varphi(q^4)B(q^8) - \frac{q^{-4}J_{32}^3\varphi(-q^4)\varphi^2(-q^8)\psi^2(q^8)}{8\psi^2(-q^8)\psi^4(q^{16})} \qquad \text{(by (2.12))}\\
&= \frac{q^{-4}J_{32}^3\psi^2(q^8)\left(\varphi(q^4)\varphi^2(q^8)-\varphi(q^4)\varphi^2(-q^4)\right)}{8\psi^2(-q^8)\psi^4(q^{16})} - q^4\varphi(q^4)B(q^8) \qquad \text{(by (2.3))}\\
&= \frac{q^{-4}J_{32}^3\varphi(q^4)\psi^2(q^8)\left(\varphi^2(q^8)-\varphi^2(-q^4)\right)}{8\psi^2(-q^8)\psi^4(q^{16})} - q^4\varphi(q^4)B(q^8)\\
&= \frac{J_{32}^3\varphi(q^4)\psi^2(q^8)}{2\psi^2(-q^8)\psi^2(q^{16})} - q^4\varphi(q^4)B(q^8), \qquad (2.26)
\end{aligned}
$$

where the last equality follows from (2.11).

Finally, combining (2.23), (2.24), and (2.26) yields

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+4n}}{1+q^{8n}} &= \frac{J_{32}^3\varphi(q^4)\psi^2(q^8)}{2\psi^2(-q^8)\psi^2(q^{16})} - q^4\varphi(q^4)B(q^8) - \frac{4q^5J_{32}^3\psi^2(q^{16})}{\psi(q^8)\psi^2(-q^8)} + 2q^5\psi(q^8)B(q^8)\\
&= -q^4\varphi(-q)B(q^8) + \frac{J_{32}^3\varphi(q^4)\psi^2(q^8)}{2\psi^2(-q^8)\psi^2(q^{16})} - \frac{4q^5J_{32}^3\psi^2(q^{16})}{\psi(q^8)\psi^2(-q^8)}\\
&= -q^4\varphi(-q)B(q^8) + \frac{J_8J_{16}^6}{2J_4^2J_{32}^3} - \frac{4q^5J_{32}^5}{J_8J_{16}^2},
\end{aligned}
$$

where we deduce the penultimate equality by utilizing (2.9). Therefore, we complete the proof. □

3. Main Results

In this section, we prove Theorems 1.1–1.4.

Proof of Theorem 1.1. First, based on (1.1), we deduce

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left(\overline{N}_2(0,4,n)-\overline{N}_2(2,4,n)\right)q^n\\
&= \frac{2J_2}{J_1^2}\left(\sum_{n=-\infty}^{\infty}{}' \frac{(-1)^nq^{n^2+2n}(1+q^{8n})}{(1+q^{2n})(1-q^{8n})} - 2\sum_{n=-\infty}^{\infty}{}' \frac{(-1)^nq^{n^2+6n}}{(1+q^{2n})(1-q^{8n})}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2J_2}{J_1^2} \sideset{}{'}\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}(1-q^{4n})^2}{(1+q^{2n})(1-q^{8n})} \\
&= \frac{2J_2}{J_1^2} \sideset{}{'}\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}(1-q^{2n})}{1+q^{4n}} \\
&= \frac{2J_2}{J_1^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}}{1+q^{4n}} - \frac{J_2}{J_1^2}\left(\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+4n}}{1+q^{4n}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+4n}}{1+q^{4n}} \right) \\
&= \frac{2J_2}{J_1^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}}{1+q^{4n}} - \frac{J_2}{J_1^2}\left(\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+4n}}{1+q^{4n}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2}}{1+q^{4n}} \right) \\
&= \frac{2J_2}{J_1^2}\left(\frac{qJ_1^2 B(q^2)}{J_2} + \frac{J_1^2 J_4^8}{2J_2^5 J_8^3} \right) - \frac{J_2}{J_1^2}\left(\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \right) \\
&= 2qB(q^2) + \frac{J_4^8}{J_2^4 J_8^3} - 1,
\end{aligned}
$$

where we derive the fifth equality by replacing n by $-n$ in the second sum inside the parentheses, and we deduce the penultimate equality by using Lemma 2.5. Thus, we complete the proof. □

For two power series $A_1(q) := \sum_{n=-\infty}^{\infty} a_1(n)q^n$ and $A_2(q) := \sum_{n=-\infty}^{\infty} a_2(n)q^n$, we define that if $a_1(n) \geq a_2(n)$ holds for any integer n, then $A_1(q) \succeq A_2(q)$.

Proof of Theorem 1.2. In view of (1.6), we have

$$\sum_{n=0}^{\infty} \left(\overline{N}_2(0,4,n) - \overline{N}_2(2,4,n)\right) q^n = 2qB(q^2) + \frac{\varphi(q^2)\psi(q^2)J_4}{J_2J_8} - 1.$$

From (1.5), it can be shown that

$$2qB(q^2) \succeq \frac{2q}{1-q^2} = 2\sum_{k=0}^{\infty} q^{2k+1}.$$

Furthermore, observe that

$$\frac{\varphi(q^2)\psi(q^2)J_4}{J_2J_8} - 1 = \frac{\varphi(q^2)\psi(q^2)}{J_8(q^2;q^4)_\infty} - 1 \succeq \frac{1}{1-q^2} - 1 = \sum_{k=1}^{\infty} q^{2k}.$$

Therefore, we complete the proof. □

Proof of Theorem 1.3. Relying on (1.1), we deduce

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \left(\overline{N}_2(1,8,n) - \overline{N}_2(3,8,n)\right) q^n \\
&= \frac{2J_2}{J_1^2} \sideset{}{'}\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}\left(q^{2n} - q^{6n} - q^{10n} + q^{14n}\right)}{(1+q^{2n})(1-q^{16n})} \\
&= \frac{2J_2}{J_1^2} \sideset{}{'}\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+4n}(1-q^{4n})(1-q^{8n})}{(1+q^{2n})(1-q^{16n})} \\
&= \frac{2J_2}{J_1^2} \sideset{}{'}\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+4n}(1-q^{2n})}{1+q^{8n}}
\end{aligned}
$$

$$= \frac{2J_2}{J_1^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+4n}}{1+q^{8n}} - \frac{2J_2}{J_1^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+2n}}{1+q^{8n}},$$

where we obtain the last equality by replacing n by $-n$ in the latter sum. Then in view of Lemmas 2.6 and 2.7, we derive

$$\sum_{n=0}^{\infty}\left(\overline{N}_2(1,8,n)-\overline{N}_2(3,8,n)\right)q^n$$
$$= -2q^{-1}A(q^8) - 2q^4B(q^8) + \frac{J_2J_8J_{16}^6}{J_1^2J_4^2J_{32}^3} - \frac{8q^5J_2J_{32}^5}{J_1^2J_8J_{16}^2} + \frac{q^{-1}J_{16}^{11}}{2J_8^5J_{32}^5} - \frac{q^{-1}J_2J_4^2J_8^2}{2J_1^2J_{16}J_{32}}.$$

Hence, we finish the proof of (1.7).

According to (1.1), we have

$$\sum_{n=0}^{\infty}\left(\overline{N}_2(0,8,n)-\overline{N}_2(4,8,n)\right)q^n$$
$$= \frac{2J_2}{J_1^2}{\sum_{n=-\infty}^{\infty}}'\frac{(-1)^n q^{n^2+2n}(1-q^{8n})^2}{(1+q^{2n})(1-q^{16n})}$$
$$= \frac{2J_2}{J_1^2}{\sum_{n=-\infty}^{\infty}}'\frac{(-1)^n q^{n^2+2n}(1-q^{2n})(1+q^{4n})}{1+q^{8n}}$$
$$= \frac{2J_2}{J_1^2}\left(\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+2n}}{1+q^{8n}} - \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+4n}}{1+q^{8n}}\right.$$
$$\left.+\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+6n}}{1+q^{8n}} - \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+8n}}{1+q^{8n}}\right)$$
$$= \frac{4J_2}{J_1^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+2n}}{1+q^{8n}} - \frac{2J_2}{J_1^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+4n}}{1+q^{8n}}$$
$$-\frac{J_2}{J_1^2}\left(\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+8n}}{1+q^{8n}} + \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+8n}}{1+q^{8n}}\right)$$
$$= \frac{4J_2}{J_1^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+2n}}{1+q^{8n}} - \frac{2J_2}{J_1^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+4n}}{1+q^{8n}}$$
$$-\frac{J_2}{J_1^2}\left(\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+8n}}{1+q^{8n}} + \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2}}{1+q^{8n}}\right)$$
$$= \frac{4J_2}{J_1^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+2n}}{1+q^{8n}} - \frac{2J_2}{J_1^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+4n}}{1+q^{8n}} - \frac{J_2}{J_1^2}\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}$$
$$= \frac{4J_2}{J_1^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+2n}}{1+q^{8n}} - \frac{2J_2}{J_1^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+4n}}{1+q^{8n}} - 1.$$

Then by means of Lemmas 2.6 and 2.7, we arrive at

$$\sum_{n=0}^{\infty}\left(\overline{N}_2(0,8,n)-\overline{N}_2(4,8,n)\right)q^n$$

$$= 4q^{-1}A(q^8) + 2q^4B(q^8) - \frac{q^{-1}J_{16}^{11}}{J_8^5J_{32}^5} + \frac{q^{-1}J_2J_4^2J_8^2}{J_1^2J_{16}J_{32}} - \frac{J_2J_8J_{16}^6}{J_1^2J_4^2J_{32}^3} + \frac{8q^5J_2J_{32}^5}{J_1^2J_8J_{16}^2} - 1,$$

which implies (1.8).

From (1.1), it can be shown that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left(\overline{N}_2(2,8,n) - \overline{N}_2(4,8,n)\right) q^n \\
&= \frac{2J_2}{J_1^2} \sum_{n=-\infty}^{\infty}{}' \frac{(-1)^n q^{n^2+6n}\left(1-q^{4n}\right)^2}{(1+q^{2n})(1-q^{16n})} \\
&= \frac{2J_2}{J_1^2} \sum_{n=-\infty}^{\infty}{}' \frac{(-1)^n q^{n^2+6n}(1-q^{2n})}{(1+q^{4n})(1+q^{8n})} \\
&= \frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+6n}(1-q^{2n})}{1+q^{4n}} + \frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+6n}(1-q^{2n})(1-q^{4n})}{1+q^{8n}} \\
&= \frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+6n}}{1+q^{4n}} - \frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+8n}}{1+q^{4n}} + \frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+6n}}{1+q^{8n}} \\
&\quad - \frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+8n}}{1+q^{8n}} - \frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+10n}}{1+q^{8n}} + \frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+12n}}{1+q^{8n}} \\
&= \frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+6n}}{1+q^{4n}} - \frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+8n}}{1+q^{4n}} + \frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}}{1+q^{8n}} \\
&\quad - \frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+8n}}{1+q^{8n}} - \frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+10n}}{1+q^{8n}} + \frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+12n}}{1+q^{8n}}. \qquad (3.1)
\end{aligned}$$

Notice that

$$\begin{aligned}
\frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+6n}}{1+q^{4n}} &= \frac{J_2}{J_1^2} \left(\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}(1+q^{4n})}{1+q^{4n}} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}}{1+q^{4n}} \right) \\
&= \frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2+2n} - \frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}}{1+q^{4n}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+8n}}{1+q^{4n}} &= \frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2+4n} - \frac{1}{2}, \\
\frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+8n}}{1+q^{8n}} &= \frac{1}{2}, \\
\frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+10n}}{1+q^{8n}} &= \frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2+2n} - \frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}}{1+q^{8n}}, \\
\frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+12n}}{1+q^{8n}} &= \frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2+4n} - \frac{J_2}{J_1^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+4n}}{1+q^{8n}}.
\end{aligned}$$

Hence, substituting the above five identities into (3.1) yields

$$\sum_{n=0}^{\infty}\left(\overline{N}_2(2,8,n)-\overline{N}_2(4,8,n)\right)q^n$$
$$=-\frac{J_2}{J_1^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+2n}}{1+q^{4n}}+\frac{2J_2}{J_1^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+2n}}{1+q^{8n}}-\frac{J_2}{J_1^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+4n}}{1+q^{8n}}$$
$$=-qB(q^2)-\frac{J_4^8}{2J_2^4J_8^3}+2q^{-1}A(q^8)-\frac{q^{-1}J_{16}^{11}}{2J_8^5J_{32}^5}+\frac{q^{-1}J_2J_4^2J_8^2}{2J_1^2J_{16}J_{32}}+q^4B(q^8)-\frac{J_2J_8J_{16}^6}{2J_1^2J_4^2J_{32}^3}+\frac{4q^5J_2J_{32}^5}{J_1^2J_8J_{16}^2},$$

where the last equality follows from Lemmas 2.5, 2.6 and 2.7.

Finally, we utilize (1.8) and (1.9) to obtain (1.10). □

Proof of Theorem 1.4. First, by means of (1.7) and (1.8), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(\overline{N}_2(0,8,n)+\overline{N}_2(1,8,n)-\overline{N}_2(3,8,n)-\overline{N}_2(4,8,n)\right)q^n\\&=2q^{-1}A(q^8)-\frac{q^{-1}J_{32}^3\varphi^2(q^8)}{2\varphi(-q^{16})\psi(-q^8)\psi^2(q^{16})}+\frac{q^{-1}J_{32}^3\varphi(-q^4)\varphi(-q^8)}{2\varphi(-q)\psi(q^8)\psi^2(q^{16})}-1\\&=2q^{-1}A(q^8)-\frac{q^{-1}J_{32}^3\varphi^2(q^8)}{2\varphi(-q^{16})\psi(-q^8)\psi^2(q^{16})}+\frac{q^{-1}J_{32}^3\varphi(q)\varphi(-q^4)\varphi(-q^8)}{2\varphi^2(-q^2)\psi(q^8)\psi^2(q^{16})}-1\\&=2q^{-1}A(q^8)-\frac{q^{-1}J_{32}^3\varphi^2(q^8)}{2\varphi(-q^{16})\psi(-q^8)\psi^2(q^{16})}+\frac{q^{-1}J_{32}^3\varphi(-q^4)\varphi(-q^8)(\varphi(q^4)+2q\psi(q^8))}{2\varphi^2(-q^2)\psi(q^8)\psi^2(q^{16})}-1,\end{aligned}\tag{3.2}$$

where we deduce the second equality by using (2.3) and the last equality follows from (2.8).

Thus,

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(\overline{N}_2(0,8,2n)+\overline{N}_2(1,8,2n)-\overline{N}_2(3,8,2n)-\overline{N}_2(4,8,2n)\right)q^n\\&=\frac{J_{16}^3\varphi(-q^2)\varphi(-q^4)}{\varphi^2(-q)\psi^2(q^8)}-1\\&=\frac{J_{16}^3\varphi^2(q)\varphi(-q^4)}{\varphi^3(-q^2)\psi^2(q^8)}-1\\&=\frac{J_{16}^3\varphi(-q^4)(\varphi^2(q^2)+4q\psi^2(q^4))}{\varphi^3(-q^2)\psi^2(q^8)}-1,\end{aligned}\tag{3.3}$$

where we use (2.3) to obtain the second equality and the last equality follows from (2.10). So,

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(\overline{N}_2(0,8,4n)+\overline{N}_2(1,8,4n)-\overline{N}_2(3,8,4n)-\overline{N}_2(4,8,4n)\right)q^n\\&=\frac{J_8^3\varphi(-q^2)\varphi^2(q)}{\varphi^3(-q)\psi^2(q^4)}-1\\&=\frac{J_2^{15}}{J_1^{10}J_4^3J_8}-1\\&=\frac{\psi^8(q)}{J_1^2J_2J_4^3J_8}-1.\end{aligned}$$

Notice that

$$\frac{\psi^8(q)}{J_1^2J_2J_4^3J_8}-1\succeq\frac{1}{1-q}-1=\sum_{k=1}^{\infty}q^k.$$

Therefore, for $n\geq 1$,

$$\overline{N}_2(0,8,4n)+\overline{N}_2(1,8,4n)>\overline{N}_2(3,8,4n)+\overline{N}_2(4,8,4n). \tag{3.4}$$

Furthermore, according to (3.3), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(\overline{N}_2(0,8,4n+2)+\overline{N}_2(1,8,4n+2)-\overline{N}_2(3,8,4n+2)-\overline{N}_2(4,8,4n+2)\right)q^n\\&=\frac{4J_8^3\varphi(-q^2)\psi^2(q^2)}{\varphi^3(-q)\psi^2(q^4)}\\&=\frac{4J_2^3J_4^5}{J_1^6J_8}\\&=\frac{4\psi^3(q)\psi^3(q^2)}{J_1^3J_4J_8}.\end{aligned}$$

Observe that

$$\frac{4\psi^3(q)\psi^3(q^2)}{J_1^3J_4J_8}\succeq\frac{4}{1-q}=4\sum_{k=0}^{\infty}q^k.$$

Thus, for $n\geq 0$,

$$\overline{N}_2(0,8,4n+2)+\overline{N}_2(1,8,4n+2)>\overline{N}_2(3,8,4n+2)+\overline{N}_2(4,8,4n+2). \tag{3.5}$$

Next, in view of (2.3) and (3.2), we derive

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(\overline{N}_2(0,8,2n+1)+\overline{N}_2(1,8,2n+1)-\overline{N}_2(3,8,2n+1)-\overline{N}_2(4,8,2n+1)\right)q^n\\&=2q^{-1}A(q^4)+\frac{q^{-1}J_{16}^3\varphi^3(-q^4)}{2\varphi^2(-q)\psi(q^4)\psi^2(q^8)}-\frac{q^{-1}J_{16}^3\varphi^2(q^4)}{2\varphi(-q^8)\psi(-q^4)\psi^2(q^8)}\\&=2q^{-1}A(q^4)+\frac{q^{-1}J_{16}^3\varphi^2(q)\varphi^3(-q^4)}{2\varphi^4(-q^2)\psi(q^4)\psi^2(q^8)}-\frac{q^{-1}J_{16}^3\varphi^2(q^4)}{2\varphi(-q^8)\psi(-q^4)\psi^2(q^8)}\\&=2q^{-1}A(q^4)+\frac{q^{-1}J_{16}^3\varphi^3(-q^4)(\varphi^2(q^2)+4q\psi^2(q^4))}{2\varphi^4(-q^2)\psi(q^4)\psi^2(q^8)}-\frac{q^{-1}J_{16}^3\varphi^2(q^4)}{2\varphi(-q^8)\psi(-q^4)\psi^2(q^8)},\end{aligned} \tag{3.6}$$

where the last equality follows from (2.10).

Therefore,

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(\overline{N}_2(0,8,4n+1)+\overline{N}_2(1,8,4n+1)-\overline{N}_2(3,8,4n+1)-\overline{N}_2(4,8,4n+1)\right)q^n\\&=\frac{2J_8^3\varphi^3(-q^2)\psi(q^2)}{\varphi^4(-q)\psi^2(q^4)}\\&=\frac{2J_2^9J_4}{J_1^8J_8}\\&=\frac{2\psi^5(q)\psi(q^2)}{J_1^3J_4J_8}.\end{aligned}$$

Moreover, since

$$\frac{2\psi^5(q)\psi(q^2)}{J_1^3J_4J_8} \succeq \frac{2}{1-q} = 2\sum_{k=0}^{\infty} q^k,$$

we obtain that for $n \geq 0$,

$$\overline{N}_2(0,8,4n+1) + \overline{N}_2(1,8,4n+1) > \overline{N}_2(3,8,4n+1) + \overline{N}_2(4,8,4n+1). \tag{3.7}$$

From (3.6), it can be seen that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left(\overline{N}_2(0,8,4n+3) + \overline{N}_2(1,8,4n+3) - \overline{N}_2(3,8,4n+3) - \overline{N}_2(4,8,4n+3)\right) q^n \\
&= 2q^{-1}A(q^2) + \frac{q^{-1}J_8^3\varphi^2(q)\varphi^3(-q^2)}{2\varphi^4(-q)\psi(q^2)\psi^2(q^4)} - \frac{q^{-1}J_8^3\varphi^2(q^2)}{2\varphi(-q^4)\psi(-q^2)\psi^2(q^4)} \\
&= 2q^{-1}A(q^2) + \frac{q^{-1}J_8^3}{2\psi^2(q^4)}\left(\frac{\varphi^3(q)\varphi(-q^2)}{\varphi^3(-q)\psi(q^2)} - \frac{\varphi^2(q^2)}{\varphi(-q^4)\psi(-q^2)}\right) \\
&= 2q^{-1}A(q^2) + \frac{q^{-1}J_8^3}{2\psi^2(q^4)}\left(\frac{\varphi^3(q)\varphi(-q^2)}{\varphi^3(-q)\psi(q^2)} - \frac{\varphi^2(q^2)}{\varphi(-q^2)\psi(q^2)}\right) \\
&= 2q^{-1}A(q^2) + \frac{q^{-1}J_8^3}{2\psi(q^2)\psi^2(q^4)} \cdot \frac{\varphi^3(q)\varphi^2(-q^2) - \varphi^3(-q)\varphi^2(q^2)}{\varphi^3(-q)\varphi(-q^2)} \\
&= 2q^{-1}A(q^2) + \frac{q^{-1}J_8^3}{2\psi(q^2)\psi^2(q^4)} \cdot \frac{\varphi^4(q)\varphi(-q) - \varphi^3(-q)\varphi^2(q^2)}{\varphi^3(-q)\varphi(-q^2)} \\
&= 2q^{-1}A(q^2) + \frac{q^{-1}J_8^3}{2\psi(q^2)\psi^2(q^4)} \cdot \frac{\varphi^4(q) - \varphi^2(-q)\varphi^2(q^2)}{\varphi^2(-q)\varphi(-q^2)},
\end{aligned}$$

where the second and the penultimate steps follow from (2.3), respectively, and we obtain the third equality by using (2.12). Moreover, in light of (2.6), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left(\overline{N}_2(0,8,4n+3) + \overline{N}_2(1,8,4n+3) - \overline{N}_2(3,8,4n+3) - \overline{N}_2(4,8,4n+3)\right) q^n \\
&= 2q^{-1}A(q^2) + \frac{q^{-1}J_8^3}{4\psi(q^2)\psi^2(q^4)} \cdot \frac{2\varphi^4(q) - \varphi^2(-q)(\varphi^2(q) + \varphi^2(-q))}{\varphi^2(-q)\varphi(-q^2)} \\
&= 2q^{-1}A(q^2) + \frac{q^{-1}J_8^3}{4\psi(q^2)\psi^2(q^4)} \cdot \frac{\varphi^2(q)(\varphi^2(q) - \varphi^2(-q)) + \varphi^4(q) - \varphi^4(-q)}{\varphi^2(-q)\varphi(-q^2)} \\
&= 2q^{-1}A(q^2) + \frac{q^{-1}J_8^3}{4\psi(q^2)\psi^2(q^4)} \cdot \frac{8q\varphi^2(q)\psi^2(q^4) + 16q\psi^4(q^2)}{\varphi^2(-q)\varphi(-q^2)} \\
&= 2q^{-1}A(q^2) + \frac{2J_8^3\varphi^2(q)}{\varphi^2(-q)\varphi(-q^2)\psi(q^2)} + \frac{4J_8^3\psi^3(q^2)}{\varphi^2(-q)\varphi(-q^2)\psi^2(q^4)} \\
&= 2q^{-1}A(q^2) + \frac{2J_2^{11}J_8^3}{J_1^8J_4^5} + \frac{4J_4^9}{J_1^4J_2^3J_8} \\
&= 2q^{-1}A(q^2) + \frac{2\psi^6(q)\psi^2(q^4)}{J_1^2J_2J_4^3J_8} + \frac{4\psi(q)\psi^5(q^2)}{J_1^3J_4J_8},
\end{aligned}$$

where the third equality follows from (2.5) and (2.7). Based on (1.4), we obtain

$$2q^{-1}A(q^2) \succeq \frac{2q}{1-q^2} = 2\sum_{k=0}^{\infty} q^{2k+1}.$$

Meanwhile, observe that

$$\frac{2\psi^6(q)\psi^2(q^4)}{J_1^2J_2J_4^3J_8} \succeq \frac{2}{1-q} = 2\sum_{k=0}^{\infty} q^k,$$
$$\frac{4\psi(q)\psi^5(q^2)}{J_1^3J_4J_8} \succeq \frac{4}{1-q} = 4\sum_{k=0}^{\infty} q^k.$$

Hence, for $n \geq 0$,

$$\overline{N}_2(0,8,4n+3) + \overline{N}_2(1,8,4n+3) > \overline{N}_2(3,8,4n+3) + \overline{N}_2(4,8,4n+3). \tag{3.8}$$

Therefore, combining (3.4), (3.5) (3.7), and (3.8), we prove the theorem. □

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