Upper bounds for the MD-numbers and characterization of extremal graphs[1]

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Abstract

For an edge-colored graph G, we call an edge-cut M of G monochromatic if the edges of M are colored with the same color. The graph G is called monochromatic disconnected if any two distinct vertices of G are separated by a monochromatic edge-cut. For a connected graph G, the monochromatic disconnection number (or MD-number for short) of G, denoted by $md(G)$, is the maximum number of colors that are allowed in order to make G monochromatic disconnected. For graphs with diameter one, they are complete graphs and so their MD-numbers are 1. For graphs with diameter at least 3, we can construct 2-connected graphs such that their MD-numbers can be arbitrarily large; whereas for graphs G with diameter two, we show that if G is a 2-connected graph then $md(G) \leq 2$, and if G has a cut-vertex then $md(G)$ is equal to the number of blocks of G. So, we will focus on studying 2-connected graphs with diameter two, and give two upper bounds of their MD-numbers depending on their connectivity and independent numbers, respectively. We also characterize the $\lfloor \frac{n}{2} \rfloor$-connected graphs (with large connectivity) whose MD-numbers are 2 and the 2-connected graphs (with small connectivity) whose MD-numbers achieve the upper bound $\lfloor \frac{n}{2} \rfloor$ (these graphs are called extremal graphs). For graphs with connectivity less than $\frac{n}{2}$, we show that if the connectivity of a graph is linear in its order n, then its MD-number is upper bounded by a constant, and this suggests us to leave a conjecture that for a k-connected graph G, $md(G) \leq \lfloor \frac{n}{k} \rfloor$.

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1 Introduction

Let G be a graph and let $V(G)$, $E(G)$ denote the vertex-set and the edge-set of G, respectively. We use $|G|$ and $||G||$ to denote the number of vertices and the number of edges of G, respectively, and call them the order and the size of G. If there is no confusion, we also use n and m to denote $|G|$ and $||G||$, respectively, throughout this paper. Let S and F be a vertex subset and an edge subset of G, respectively. Then $G - S$ is the graph obtained from G by deleting the vertices of S together with the edges incident with vertices of S, and $G - F$ is the graph whose vertex-set is $V(G)$ and edge-set is $E(G) - F$. Let $G[S]$ and $G[F]$ be the subgraphs of G induced, respectively, by S and F. We use $[r]$ to denote the set $\{1, 2, \cdots, r\}$ of positive integers. If $r = 0$, then set $[r] = \emptyset$. For all other terminology and notation not defined here we follow Bondy and Murty [4].

For a graph G, let $\Gamma : E(G) \to [r]$ be an *edge-coloring* of G that allows a same color to be assigned to adjacent edges. For an edge e of G, we use $\Gamma(e)$ to denote the color of e. If H is a subgraph of G, we also use $\Gamma(H)$ to denote the set of colors on the edges of H and use $|\Gamma(H)|$ to denote the number of colors in $\Gamma(H)$. For an edge-colored graph G and a vertex v of G, the *color-degree* of v, denoted by $d^c(v)$, is the number of colors appearing on the edges incident with v.

The three main colored connection colorings: rainbow connection coloring [8], proper connection coloring [5] and proper-walk connection coloring [3], monochromatic connection coloring [6], have been well-studied in recent years. As a counterpart concept of the rainbow connection coloring, rainbow disconnection coloring was introduced in [7] by Chartrand et al. in 2018. Subsequently, the concepts of monochromatic disconnection coloring and proper disconnection coloring were also introduced in [12] and [1, 9]. We refer to [2] for the philosophy of studying these so-called global graph colorings. More details on the monochromatic disconnection coloring can be found in [13]. We will further study this coloring in this paper and get some deeper and stronger results.

For an edge-colored graph G, we call an edge-cut M a *monochromatic edge-cut* if the edges of M are colored with the same color. If there is a monochromatic uv-cut with color i, then we say that color i *separates* u and v. We use $C_\Gamma(u, v)$ to denote the set of colors in $\Gamma(G)$ that separate u and v, and let $c_\Gamma(u, v) = |C_\Gamma(u, v)|$.

An edge-coloring of a graph is called a *monochromatic disconnection coloring* (or *MD-coloring for short*) if each pair of distinct vertices of the graph has a monochromatic edge-cut separating them, and the graph is called *monochromatic disconnected.* For a connected graph G, the *monochromatic disconnection number (or MD-number for short)* of G, denoted by $md(G)$, is defined as the *maximum* number of colors that are allowed in order to make G monochromatic disconnected. An *extremal MD-coloring* of G is an MD-coloring that uses $md(G)$ colors. If H is a subgraph of G and Γ is an edge-coloring of G, we call Γ an edge-coloring *restricted* on H.

The following terminology and notation are needed in the sequel. Let G and H be two graphs. The *union* of G and H is the graph $G \cup H$ with vertex-set $V(G) \cup V(H)$ and edge-set $E(G) \cup E(H)$. The *intersect* of G and H is the graph $G \cap H$ with vertex-set $V(G) \cap V(H)$ and edge-set $E(G) \cap E(H)$. The *Cartesian product* of G and H is the graph $G \square H$ with $V(G \square H) = \{(u, v) : u \in V(G), v \in V(H)\}$, (u, v) and (x, y) are adjacent in $G \square H$ if either ux is an edge of G and $v = y$, or vy is an edge of H and $u = x$. If G and H are vertex-disjoint, then let $G \vee H$ denote the *join* of G and H which is obtained from G and H by adding an edge between every vertex of G and every vertex of H.

For a graph G, a *pendent vertex* of G is a vertex with degree one. The *ends* of G is the set of pendent vertices, and the *internal vertex set* of G is the set of vertices with degree at least two. We use $end(G)$ and $I(G)$ to denote the ends of G and the internal vertex set of G, respectively. The *independent number* of G, denoted by $\alpha(G)$, is the order of a maximum independent set of G. For two vertices u, v of G, we use $N(u)$ to denote the *neighborhood* of u in G, and $N(u, v)$ to denote the set of common neighbors of u and v in G. The distance between u and v in G is denoted by $d(u, v)$, and the diameter of G is denoted by $diam(G)$. We call a cycle C (path P) a *t-cycle* (*t-path*) if $|C| = t$ ($||P|| = t$). If t is even (odd), then we call the path an *even (odd) path* and the cycle an *even (odd) cycle*. A 3-cycle is also called a *triangle*. A *matching-cut* of G is an edge-cut of G, which also forms a matching in G.

In [12, 13] we got the following results, which are restated for our later use.

Lemma 1.1. *[12]*

1. *If a connected graph G has r blocks $B_1, \cdots, B_r$, then $md(G) = \sum_{i \in [r]} md(B_i)$ and $md(G) = n - 1$ if and only if G is a tree.*

2. *$md(G) = \lfloor \frac{|G|}{2} \rfloor$ if G is a cycle, and $md(G) = 1$ if G is a complete multipartite graph and G is not a star.*

3. *If H is a connected spanning subgraph of G, then $md(H) \geq md(G)$. Thus, $md(G) \leq n - 1$.*

4. *If G is connected, then $md(v \vee G) = 1$.*

5. *If v is neither a cut-vertex nor a pendent vertex of G and Γ is an extremal MD-coloring of G, then $\Gamma(G) \subseteq \Gamma(G - v)$, and thus, $md(G) \leq md(G - v)$.*

Theorem 1.2. *[12] If G is a 2-connected graph, then $md(G) \leq \lfloor \frac{n}{2} \rfloor$.*

Theorem 1.3. *[13] If G_1 and G_2 are connected graphs, then $md(G_1 \square G_2) = md(G_1) + md(G_2)$.*

Lemma 1.4. *[13] If G has a matching-cut, then $md(G) \geq 2$.*

We will list some easy observations in the following, which will be used many times throughout this paper. Suppose Γ is an MD-coloring of G. If H is a subgraph of G, then Γ is an MD-coloring restricted on H. Every triangle of G is monochromatic. If G is a 4-cycle, then its opposite edges have the same color. If G is a 5-cycle, then there are two adjacent edges having the same color.

Let V be a set of vertices and let $\mathcal{E} \subseteq 2^V$. Then a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is a *linear hypergraph* if $|E_i| \geq 2$ and $|E_i \cap E_j| \leq 1$ for any $E_i, E_j \in \mathcal{E}$. The *size* of $\mathcal{H}$ is the number of hyperedges in $\mathcal{H}$. A *hyperedge-coloring* of $\mathcal{H}$ assigns each hyperedge a positive integer. A linear hypergraph $\mathcal{H}$ (say the size of $\mathcal{H}$ is k) is a *linear hypercycle* if there is a sequence of hyperedges of $\mathcal{H}$, say $E_1, \cdots, E_k$, and there exist k distinct vertices $v_1, \cdots, v_k$ of $\mathcal{H}$, such that $E_1 \cap E_k = \{v_k\}$ and $E_i \cap E_{i+1} = \{v_i\}$ for $i \in [k-1]$. If we delete a hyperedge from a linear hypercycle and then delete the vertices only in this hyperedge, then we call the resulting hypergraph a *linear hyperpath*. A linear hypercycle (linear hyperpath) is called a *linear hyper k-cycle (linear hyper k-path)* if the size of this linear hypercycle (linear hyperpath) is k.

2 Preliminaries

We need some more preparations before proceeding to our main results.

Lemma 2.1. *For two connected graphs G_1 and G_2, if $md(G_1 \cap G_2) = 1$ then $md(G_1 \cup G_2) = md(G_1) + md(G_2) - 1$.*

Proof. Let $G = G_1 \cup G_2$ and Γ be an extremal MD-coloring of G. Then $|\Gamma(G_1 \cap G_2)| = 1$ and Γ is an MD-coloring restricted on G_1 (and also G_2). So, $md(G_1 \cup G_2) = |\Gamma(G_1)| + |\Gamma(G_2)| - |\Gamma(G_1 \cap G_2)| \leq md(G_1) + md(G_2) - 1$. On the other hand, since $E(G_1 \cap G_2)$ is monochromatic under any MD-coloring of $G_1 \cup G_2$, let Γ_i be an MD-coloring of G_i for $i \in [2]$ such that $\Gamma_1(G_1 \cap G_2) = \Gamma_2(G_1 \cap G_2) = \Gamma(G_1) \cap \Gamma(G_2)$. Let Γ' be an edge-coloring of $G_1 \cup G_2$ such that $\Gamma'(e) = \Gamma_i(e)$ if $e \in E(G_i)$, and let w be a vertex of $G_1 \cap G_2$. Then for any two vertices u, v of $G_1 \cup G_2$, if $u, v \in V(G_i)$, then $C_{\Gamma_i}(u, v) \subseteq C_{\Gamma'}(u, v)$; if $u \in V(G_1) - V(G_2)$ and $v \in V(G_2) - V(G_1)$, then $(C_{\Gamma_1}(u, w) \cup C_{\Gamma_2}(v, w)) \subseteq C_{\Gamma'}(u, v)$. So, Γ' is an MD-coloring of G, i.e., $md(G_1 \cup G_2) \geq |\Gamma(G_1 \cup G_2)| = md(G_1) + md(G_2) - 1$. Therefore, $md(G_1 \cup G_2) = md(G_1) + md(G_2) - 1$. ∎

Lemma 2.2. *Let G be a connected graph and let G' be a graph obtained from G by replacing an edge $e = ab$ with a path P. Then $md(G') \geq md(G) + \left\lfloor \frac{||P||-1}{2} \right\rfloor$.*

Proof. Let Γ be an extremal MD-coloring of G. Let $||P|| = t$ and let $P = ae_1c_1 \cdots e_tb$. Let Γ' be an edge-coloring of G' such that $\Gamma(f) = \Gamma'(f)$ when $f \in E(G) - e$, $\Gamma'(e_i) = \Gamma'(e_{t+1-i}) = |\Gamma(G)| + i$ for $i \in [\lfloor \frac{t-1}{2} \rfloor]$, $\Gamma(e) = \Gamma'(e_{\frac{t+1}{2}})$ when t is odd, and $\Gamma(e) = \Gamma'(e_{\frac{t}{2}}) = \Gamma'(e_{\frac{t}{2}+1})$ when t is even. It is easy to verify that Γ' is an MD-coloring of G'. Thus, $md(G') \geq md(G) + \left\lfloor \frac{||P||-1}{2} \right\rfloor$. ∎

Lemma 2.3. *Suppose u, v are nonadjacent vertices of G and Γ is an extremal MD-coloring of G. Let $C_\Gamma(u, v) = \{t\}$ and e an extra edge, and let Γ' be an edge-coloring of $G \cup e$ that is obtained from Γ by coloring the added edge e with color t. Then Γ' is an MD-coloring of $G \cup e$ and $md(G) = md(G \cup e)$.*

Proof. Let H_i be the graph obtained from G by deleting all the edges with color i. Let $G' = G \cup e$. If Γ' is not an MD-coloring of G', then there are two vertices x, y of G' such that $C_{\Gamma'}(x, y) = \emptyset$. If $t \in C_\Gamma(x, y)$, since x, y are in different components of H_t, we have $t \in C_{\Gamma'}(x, y)$, a contradiction. If $t \notin C_\Gamma(x, y)$, then let $j \in C_\Gamma(x, y)$. Then there are two components D_1, D_2 of H_j such that $x \in V(D_1)$ and $y \in V(D_2)$. Since j does not separate x, y in G', the edge e connects D_1 and D_2, say $u \in V(D_1)$ and $v \in V(D_2)$. Thus, the color j separates u, v in G, which contradicts that $C_\Gamma(u, v) = \{t\}$. Therefore, Γ' is an MD-coloring of G'. Since $|\Gamma'(G')| = |\Gamma(G)|$ and Γ is an extremal MD-coloring of G, we have $md(G') \geq md(G)$. Since G is a connected spanning subgraph of G', by Lemma 1.1 (3) we have $md(G) \geq md(G')$. So, $md(G) = md(G')$. ∎

Suppose Γ is an MD-coloring of G and G_i is the subgraph of G induced by the set of edges with color i, which, in what follows, is called the *color i induced subgraph* of G. Then for any component D_1 of G_i and any component D_2 of G_j, we have $|V(D_1) \cap V(D_2)| \leq 1$; otherwise, suppose $u, v \in V(D_1) \cap V(D_2)$. Then $C_\Gamma(u, v) = \emptyset$, a contradiction. We use $\mathcal{H}_\Gamma$ to denote a hyperedge-colored hypergraph with vertex-set $V(G)$ and hyperedge-set $\{V(D) \mid D$ is a component of some $G_i\}$, and the hyperedge F has color i if F corresponds to a component of G_i. Let H_Γ be a graph with $V(H_\Gamma) = V(G)$ and

$$E(H_\Gamma) = \{uv \mid u, v \text{ are in the same component of some } G_i\}.$$

Then each hyperedge of $\mathcal{H}_\Gamma$ corresponds to a clique of H_Γ, and any two hyperedges of $\mathcal{H}_\Gamma$ (any two cliques of H_Γ) share at most one vertex. Thus, $\mathcal{H}_\Gamma$ is a linear hypergraph. If F is a hyperedge of $\mathcal{H}_\Gamma$ and $u, v \in F$, then $c_\Gamma(u, v) = 1$. According to Lemma 2.3, we have the following result.

Lemma 2.4. *If Γ is an extremal MD-coloring of G, then $md(G) = md(H_\Gamma)$.*

Suppose Γ is an MD-coloring of G and $\mathcal{C}$ is a hyper k-cycle of $\mathcal{H}_\Gamma$. Then there is a k-cycle C of H_Γ such that any adjacent edges of C have different colors. Thus, $t \neq 3, 5$. Moreover, if $k = 4$, then the opposite hyperedges of $\mathcal{C}$ have the same color.

3 Graphs with diameter two

In this section, we show that $md(G) \leq 2$ for a 2-connected graph G if $diam(G) \leq 2$. However, for any integer $d \geq 3$, we can construct a 2-connected graph G such that

$diam(G) = d$ and $md(G)$ can be arbitrarily large. Thus, it makes sense to focus on studying the graphs with diameter two, since graphs with diameter 1 are complete graphs and their MD-numbers are 1.

Theorem 3.1. *Suppose G is a graph with $diam(G) = 2$. Then*

1. *if G has a cut-vertex, then $md(G)$ is equal to the number of blocks of G;*
2. *if G is a 2-connected graph, then $md(G) \leq 2$;*
3. *if any two nonadjacent vertices of G has at least two common neighbors, then $md(G) \leq 2$, and the equality holds if and only if $G = K_s \square K_t$, where $s, t \geq 2$.*

Proof. The proof of statement (1) goes as follows. If v is a cut-vertex of G and $diam(G) = 2$, then v connects every vertex of $V(G - v)$. Thus, for each block D of G, $D - v$ is connected and $D = (D - v) \vee v$, i.e., $md(D) = 1$. Therefore, $md(G)$ is equal to the number of blocks of G.

Next, for the proof of statement (2) suppose Γ is an MD-coloring of G with $|\Gamma(G)| \geq 3$. Then each hypercycle (hyperpath) of the above mentioned hypergraph $\mathcal{H}_\Gamma$ is a linear hypercycle (linear hyperedge). We now prove that there is a rainbow hyper 3-path (the colors of the three hyperedges are pairwise differently) in $\mathcal{H}_\Gamma$. Since $\mathcal{H}_\Gamma$ does not have hyper 3-cycle, the union of three consecutive hyperedges forms a hyper 3-path. If every vertex z of G has $d^c(z) \leq 2$, then there is a rainbow hyper 3-path in $\mathcal{H}_\Gamma$. If there is a vertex x of G with $d^c(x) \geq 3$, then there are three hyperedges, say D_1, D_2 and D_3, such that x is the common vertex of them. Then the colors of D_1, D_2 and D_3 are pairwise differently. Since G is a 2-connected graph, there is a vertex w of $V(D_1) - \{x\}$ with $d^c(w) \geq 2$ (otherwise, x is a cut-vertex of G, a contradiction). Then there is a hyperedge F of $\mathcal{H}_\Gamma$, such that w is a common vertex of F and D_1. Thus, either $F \cup D_1 \cup D_2$ or $F \cup D_1 \cup D_3$ is a rainbow hyper 3-path.

Let $\mathcal{P}$ be a rainbow hyper 3-path of $\mathcal{H}_\Gamma$ and let $V(D_i) \cap V(D_{i+1}) = \{u_i\}$ for $i \in [2]$. Let $u \in V(D_1) - \{u_1\}$ and $v \in V(D_3) - \{u_2\}$. We use $\mathcal{P}_{u,v}$ to denote a minimum hyperpath connecting u and v. Since $diam(G) = 2$, the size of $\mathcal{P}_{u,v}$ is either one or two. Let $\mathcal{C} = \mathcal{P}_{u,v} \cup \mathcal{P}$. If $\mathcal{P}_{u,v}$ is a hyperedge, then $\mathcal{C}$ is a hyper 4-cycle. Since D_1 and D_3 are opposite hyperedges of $\mathcal{C}$ and they have different colors, a contradiction. If $\mathcal{P}_{u,v}$ is a hyper 2-path, then let F_1, F_2 be hyperedges of $\mathcal{P}_{u,v}$, and let $V(F_1) \cap V(F_2) = \{u_3\}$. If $u_3 \notin \{u_1, u_2\}$, then $\mathcal{C}$ is a hyper 5-cycle, a contradiction. If $u_3 \in \{u_1, u_2\}$, then $\mathcal{C}$ contains a hyper 3-cycle, a contradiction.

Finally, we show statement (3). It is obvious that $diam(G) \leq 2$, and G is a 2-connected graph when $n \geq 3$. So, $md(G) \leq 2$. Suppose $G = K_s \square K_t$ and $s, t \geq 2$. Then $|N(u, v)| = 2$ for any nonadjacent vertices u and v of G. By Lemma 1.1 (2) and Theorem 1.3, we have $md(G) = md(K_s) + md(K_t) = 2$.

Suppose $md(G) = 2$. Then $n \geq 3$ and G is a 2-connected graph. Let Γ be an extremal MD-coloring of G and let G_1, G_2 be the induced subgraphs of G colored by the colors 1 and 2, respectively. Since $md(G) = 2$, we have $d^c(v) \leq 2$ for each $v \in V(G)$. If $d^c(v) = 1$, by symmetry, suppose v is in a component D of G_1. Since $md(G) = 2$, we have $D \neq G$, i.e., there exists a vertex u in $V(G) - V(D)$. Then u, v are nonadjacent and $N(u, v) \subseteq D$. Let $\{a, b\} \subseteq N(u, v)$. Since $\Gamma(va) = \Gamma(vb) = 1$, we have $va \cup vb \cup ua \cup ub$ is a monochromatic 4-cycle, i.e., $u \in V(D)$, a contradiction. Thus, $d^c(v) = 2$ for each $v \in V(G)$. We use D_u^1 and D_u^2 to denote the components of G_1 and G_2, respectively, such that $V(D_u^1) \cap V(D_u^2) = u$.

Suppose there are t components of G_1 and s components of G_2. Since G is a 2-connected graph, we have $s, t \geq 2$. Otherwise, if $s = 1$, then for each vertex v of G_1, v is a cut-vertex, a contradiction. We label the t components of G_1 by the numbers in $[t]$ and label the s components of G_2 by the numbers in $[s]$, respectively. We use $l_1(D)$ to denote the label of a component D of G_1, and use $l_2(F)$ to denote the label of a component F of G_2. For a vertex u of G, since $d^c(u) = 2$, we use $(l_1(D_u^1), l_2(D_u^2))$ to denote u. For two vertices u, v of G, let $u = (i, j)$ and let $v = (s, t)$. In order to show $G = K_s \square K_t$, we need to show that uv is an edge of G when $i = s$ and $j \neq t$, or $i \neq s$ and $j = t$, and u, v are nonadjacent vertices when $i \neq s$ and $j \neq t$. If $i \neq s$ and $j \neq t$, then $v \notin V(D_u^1 \cup D_u^2)$. Since $N(u) \subseteq V(D_u^1 \cup D_u^2)$, u, v are nonadjacent vertices of G. If, by symmetry, $i = s$ and $j \neq t$, then $D_u^1 = D_v^1$. Let $u' \in V(D_u^2) - \{u\}$. Then u', v are nonadjacent. Since $N(v) \subseteq V(D_v^1 \cup D_v^2)$ and $N(u') \subseteq V(D_{u'}^1 \cup D_{u'}^2)$, we have

$$2 \leq |N(v, u')| \leq |V(D_v^1 \cup D_v^2) \cap V(D_{u'}^1 \cup D_{u'}^2)| = |D_v^1 \cap D_{u'}^2| + |D_{u'}^1 \cap D_v^2| \leq 2.$$

Thus, $D_v^1 \cap D_{u'}^2 \subseteq N(v, u')$. Since $D_v^1 \cap D_{u'}^2 = \{u\}$, we have uv is an edge of G. ∎

Remark 1. *Suppose $G = \bigcup_{i \in [r]} L_i$, where $L_1, \cdots, L_r$ are r (≥ 2) internal disjoint odd paths with an order $2k_i + 2$ for each $i \in [r]$, and they have the same ends $\{u, v\}$. Let $L_i = ue_1^i x_1^i e_2^i x_2^i \cdots x_{2k_i}^i e_{2k_i+1}^i v$. Let $c_0 = 1$ and $c_i = \Sigma_{j=0}^i k_j$. If $k_i \geq 1$ for $i \in [r]$, then let Γ be an edge-coloring of G such that $\Gamma(e_j^i) = \Gamma(e_{2k_i+2-j}^i) = c_{i-1} + j$ and $\Gamma(e_{k_i+1}^i) = 1$ for each $i \in [r]$ and $j \in [k_i]$. Then Γ is an MD-coloring of G with $|\Gamma(G)| = \frac{|G|}{2}$. Since G is a 2-connected graph, we have $md(G) = \frac{|G|}{2}$. If $k_i = 1$ for each $i \in [r]$, then G is a 2-connected graph with $diam(G) = 3$ and $md(G) = r + 1$. Therefore, there exist 2-connected graphs with diameter three, but their MD-numbers can be arbitrarily large.*

Let A_n be a graph with $V(A_n) = \{v_1, \cdots, v_{\lceil \frac{n}{2} \rceil}\} \cup \{u_1, \cdots, u_{\lfloor \frac{n}{2} \rfloor}\}$ and $E(A_n) = \{v_i v_j : i, j \in [\lceil \frac{n}{2} \rceil]\} \cup \{u_i u_j : i, j \in [\lfloor \frac{n}{2} \rfloor]\} \cup \{v_i u_i : i \in [\lfloor \frac{n}{2} \rfloor]\}$. Then $\{v_i u_i : i \in [\lfloor \frac{n}{2} \rfloor]\}$ is a matching-cut of G. If n is an odd integer, then let

$$\mathcal{A}_n = \{A_n - E \mid E \text{ is either an emptyset or a matching of } A_n[\{v_1, \cdots, v_{\frac{n-1}{2}}\}]\}.$$

In the following theorem, we characterize extremal $\lfloor \frac{n}{2} \rfloor$-connected graphs, i.e., the $\lfloor \frac{n}{2} \rfloor$-connected graphs with MD-number two.

Theorem 3.2. *Suppose G is a $\lfloor \frac{n}{2} \rfloor$-connected graph and $n \geq 4$. Then $md(G) \leq 2$ and*

1. *if n is even, then $md(G) = 2$ if and only if $G = A_n$;*

2. *if n is odd, then $md(G) = 2$ if and only if $G \in \mathcal{A}_n$.*

Proof. Since $|N(x)| + |N(y)| \geq n - 1$ for any two nonadjacent vertices x and y, we have $diam(G) \leq 2$. So, $md(G) \leq 2$.

It is obvious that G is a $\lfloor \frac{n}{2} \rfloor$-connected graph if $G = A_n$ or $G \in \mathcal{A}_n$. Moreover, by Lemma 1.4 and Theorem 3.1, we have $md(G) = 2$.

Now suppose G is a $\lfloor \frac{n}{2} \rfloor$-connected graph and $md(G) = 2$. Since $n \geq 4$, G is a 2-connected graph. We distinguish the following cases for our proof.

Case 1. n is even.

For any two nonadjacent vertices u, v of G, $|N(u) \cap N(v)| \geq 2$. By Theorem 3.1 (3), $G = K_s \square K_t$, where $s, t \geq 2$. We need to prove that at least one of s, t equals two. Suppose H_1, H_2 are two cliques of order s, t, respectively, and $V(H_1) \cap V(H_2) = \{u\}$. Then $N(u) \subseteq V(H_1 \cup H_2)$, i.e., $s + t - 2 \geq \frac{n}{2}$. Since $n = st$, we have $t(s-2) \leq 2(s-2)$. Thus, either $s = 2$ or $t = 2$.

Case 2. n is odd.

Say $n = 2k + 1$ for some integer k. Suppose Γ is an extremal MD-coloring of G and G_1, G_2 are the colors $1, 2$ induced subgraphs, respectively.

Subcase 2.1 Every vertex v of G has $d^c(v) = 2$.

Suppose there are components D, F of G_1, G_2, respectively, such that $V(D) \cap V(F) = \emptyset$. Then let $u \in V(D)$ and $v \in V(F)$. Since $d^c(u) = d^c(v) = 2$, there are components D' of G_1 and F' of G_2, such that $V(D) \cap V(F') = \{u\}$ and $V(F) \cap V(D') = \{v\}$. Since $V(D) \cup V(F') - \{u\}$ and $V(D') \cup V(F) - \{v\}$ are vertex-cuts of G, we have $|V(D) \cup V(F')| \geq k + 1$ and $|V(D') \cup V(F)| \geq k + 1$. Since $|V(D') \cap V(F')| \leq 1$, we have $n \geq |V(D) \cup V(F')| + |V(D') \cup V(F)| - |V(D') \cap V(F')| \geq 2k + 1 = n$, i.e., $D \cup D' \cup F \cup F' = G$. Then u is a cut-vertex of G, a contradiction. Therefore, for each component D of G_1 and each component F of G_2, we have $|V(G) \cap V(F)| = 1$. Then since $d^c(v) = 2$ for each $v \in V(G)$, any two components of G_1 (and also G_2) have the same order, say s (the order is t). Then $s, t > 2$; otherwise, suppose $s = 2$, i.e., G_1 is a matching. Since n is odd, we have $V(G) - V(G_1) \neq \emptyset$. Thus, each vertex v of $V(G) - V(G_1)$ has $d^c(v) = 1$, a contradiction. For a vertex x of G, let D_1, D_2 be the components of G_1, G_2, respectively, containing x. Then $D_1 \cup D_2 - \{x\}$ is a vertex-cut of G, i.e., $s + t - 2 \geq k$. However, $2k + 1 = n = st$ and $s, t > 3$, a contradiction.

Subcase 2.2 There is a vertex v of G with $d^c(v) = 1$.

Suppose D is the component of G_1 containing v. Then since $D - \{v\}$ is a vertex cut of G, we have $|D| \geq k + 1$. Since the set of vertices of D with color-degree two is a vertex-cut of G, there are at least k vertices of D, say $v_1, \cdots, v_k$, such that $d^c(v_i) = 2$ for

$i \in [k]$. Let F_i be the component of G_2 containing v_i and let $U = \bigcup_{i\in[k]}(V(F_i) - \{v_i\})$. Then $|U| \geq k$. Since $n \geq |D| + |U| \geq 2k + 1 = n$, we have $|D| = k + 1$, $|U| = k$, and $|F_i| = 2$ for $i \in [k]$. Moreover, $N(v) = \{v_1, \cdots, v_k\}$. Let $V(F_i) - \{v_i\} = \{u_i\}$. For $i, j \in [k]$, if u_iu_j is not an edge of G, then $U - \{u_i, u_j\} + v_j$ is a vertex-cut of G with order $k-1$, which contradicts that G is k-connected. For each v_i, if there are two vertices v_j, v_l such that v_iv_j and v_iv_l are not edges of G, then $V(D) - \{v_i, v_j, v_l\} + u_i$ is a vertex-cut of G with order $k-1$, which contradicts that G is k-connected. Therefore, v_i connects all but at most one vertex of $D - v$. So, $G \in \mathcal{A}_n$. ∎

4 Upper bounds

In this section, we give two upper bounds of the monochromatic disconnection number of a graph G, one of which depends on the connectivity of G, and the other depends on the independent number of G. Note that for a k-connected graph G, when $k = 2$ (small) and $k \geq \lfloor \frac{n}{2} \rfloor$ (large), from Theorems 1.2 and 3.2 we know that $md(G) \leq \lfloor \frac{n}{k} \rfloor$. This suggests us to make the following conjecture.

Conjecture 4.1. *Suppose G is a k-connected graph. Then $md(G) \leq \lfloor \frac{n}{k} \rfloor$.*

Suppose P is a k-path. Then $md(K_r \square P) = md(K_r) + md(P) = k + 1$. Since $n = |K_r \square P| = r(k + 1)$ and $K_r \square P$ is an r-connected graph, the bound is sharp for $k \geq 2$ if the conjecture is true.

The *mean distance* of a connected graph G is defined as $\mu(G) = \binom{n}{2}^{-1}\Sigma_{u,v\in V(G)}d(u, v)$. Plesník in [14] posed the problem of finding sharp upper bounds on $\mu(G)$ for k-connected graphs. Favaron et al. in [11] proved that if G is a k-connected graph of order n, then

$$\mu(G) \leq \left\lfloor \frac{n + k - 1}{k} \right\rfloor \cdot \frac{n - 1 - \frac{k}{2} \left\lfloor \frac{n-1}{k} \right\rfloor}{n - 1}, \tag{1}$$

and the bound is sharp when n is even. If n is odd and $k \geq 3$, then Dankelmann et al. in [10] proved that $\mu(G) \leq \frac{n}{2k+1} + 30$ and this bound is, apart from an additive constant, best possible.

The following result gives a relationship between the monochromatic disconnection number and the connectivity of a graph, which means that if the connectivity of a graph is linear in the order of the graph, then the monochromatic disconnection number of the graph is upper bounded by a constant.

Theorem 4.2. *For any $0 < \varepsilon < \frac{1}{2}$, there is a constant $C = C(\varepsilon) < \frac{(1+\varepsilon)^2}{4\varepsilon^2(1-\varepsilon)}$, such that for any εn-connected graph G, $md(G) \leq C$.*

Proof. Suppose Γ is an extremal MD-coloring of G and $V(G) = \{v_1, \cdots, v_n\}$. We use (i, j) to denote an unordered integer pair in this proof. For each color i of $\Gamma(G)$, let

$$S_i = \{(j, l) : \text{ the color } i \text{ separates } v_j \text{ and } v_l\}.$$

Then $\Sigma_{i\in\Gamma}|S_i| = \Sigma_{j\neq l} c_\Gamma(v_j, v_l)$.

Claim 4.3. *$|S_i| \geq k(n-k)$ for each $i \in \Gamma(G)$.*

Proof. Let $\varepsilon n = k$. The result holds obviously for $k = 1$. Thus, let $k \geq 2$. For each $i \in \Gamma(G)$, let G_i be the color i induced subgraph of G, and let H_i be the graph obtained from G by deleting all the edges with color i. Then H_i is a disconnected graph. Suppose there is a component D of H_i with $|D| > n - k$. Let $U = \{v_j \mid v_j \in V(D) \cap V(G_i)\}$. For a component B of G_i, if $V(B) \cap V(D) \neq \emptyset$, then $|V(B) \cap V(D)| = 1$. Since B contains at least one vertex of $V(G - D)$, we have $|U| \leq |V(G - D)| < k$. Since $|D| > n - k = n(1 - \varepsilon) > \varepsilon n = k$, U is a proper subset of $V(D)$. So, U is a vertex-cut of G. Since $|U| < k$ and G is k-connected, this yields a contradiction. Thus, for each $i \in \Gamma(G)$, there is no component of H_i with order greater than $n - k$.

We partition the components of H_i into r parts such that r is minimum and the number of vertices in each part is at most $n - k$. Suppose the r parts have $n_1, \cdots, n_r$ vertices, respectively. Then $\sum_{j\in[r]} n_j = n$. If $r \geq 4$, then since r is minimum, $n_l + n_j > n - k$ for each $l, j \in [r]$. Thus,

$$n(r-1) = (r-1)\sum_{t\in[r]} n_t = \sum_{l,j\in[r]} (n_l + n_j) > \binom{r}{2}(n-k),$$

and then $r(n-k) < 2n$. Since $k < \frac{n}{2}$, this yields a contradiction. Therefore, r is equal to 2 or 3. If $r = 2$, then $|S_i| \geq n_1 \cdot n_2 \geq k(n-k)$. If $r = 3$, then there is an n_l such that $k \leq n_l \leq n - k$, say $l = 1$. Otherwise, $n_j < k$ for each $j \in [3]$, then $n = \sum_{j\in[3]} n_j < n$, a contradiction. Thus, $|S_i| > n_1 \cdot (n_2 + n_3) \geq k(n-k)$. ∎

By the inequality (1) above, we have

$$\begin{aligned}
\mu(G) &\leq \left\lfloor \frac{n+k-1}{k} \right\rfloor \cdot \frac{n-1-\frac{k}{2}\left\lfloor\frac{n-1}{k}\right\rfloor}{n-1} = \left\lfloor \frac{n+k-1}{k} \right\rfloor \cdot \left(1 - \frac{k}{2(n-1)}\left\lfloor \frac{n-1}{k} \right\rfloor\right) \\
&\leq \left(\frac{n+k-1}{k}\right) \cdot \left[1 - \frac{k}{2(n-1)}\left(\frac{n-1}{k} - 1\right)\right] \\
&= \frac{n+k-1}{k} \cdot \frac{n+k-1}{2(n-1)} < \frac{(n+k)^2}{2k(n-1)}.
\end{aligned}$$

Since $\sum_{i,j} d(v_i, v_j) = \mu(G) \cdot \binom{n}{2}$, we have $\sum_{i,j} d(v_i, v_j) < \frac{(n+k)^2 n}{4k}$. It is obvious that $d(v_i, v_j) \geq c_\Gamma(v_i, v_j)$ for any two vertices v_i, v_j of G. Thus,

$$md(G) \leq \frac{\Sigma_{i\in\Gamma}|S_i|}{k(n-k)} = \frac{\sum_{i,j} c_\Gamma(v_i, v_j)}{k(n-k)} \leq \frac{\sum_{i,j} d(u, v)}{k(n-k)} < \frac{(n+k)^2 n}{4k^2(n-k)} = \frac{(1+\varepsilon)^2}{4\varepsilon^2(1-\varepsilon)}.$$

The proof is thus complete. ∎

Remark 2. *Since* $\varepsilon < \frac{1}{2}$, *we have* $\frac{(1+\varepsilon)^2}{4\varepsilon^2(1-\varepsilon)} < (\frac{3}{2})^2/2\varepsilon^2 = \frac{9}{8\varepsilon^2}$. *This means that when the connectivity of a graph increases, its* MD*-number could decrease, and the upper bound is* 4 *when* ε *is getting to* $\frac{1}{2}$.

The following result gives a relationship between the monochromatic disconnection number and the independent number of a graph.

Theorem 4.4. *If* G *is a* 2*-connected graph, then* $md(G) \leq \alpha(G)$. *The bound is sharp.*

Proof. Let P be a path and let $t \geq 2$ be an integer. Since $\alpha(K_t \Box P) = |P| = md(K_t \Box P)$, the bound is sharp if the result holds.

The proof proceeds by induction on the order n of a graph G. If $n \leq 2\alpha(G)$, then since G is a 2-connected graph, $md(G) \leq \alpha(G)$. If G has a vertex v such that $G - v$ is still 2-connected, then by Lemma 1.1 (5), we know $md(G - v) \geq md(G)$. Since $\alpha(G - v) \leq \alpha(G)$, by induction, we have $md(G) \leq md(G - v) \leq \alpha(G - v) \leq \alpha(G)$. Thus, we only need to consider the graph G with the property that $G - v$ is not a 2-connected graph for any vertex v of G.

Let u be a vertex of G such that $G - u$ has a maximum component. Let $\mathcal{B} = \{D_1, \cdots, D_s\}$ be the set of components of $G-u$ and let D_r be a maximum component. Let S be the set of cut-vertices of $G - u$. The *block-tree* of $G - u$, denoted by T, is a bipartite graph with bipartition $\mathcal{B}$ and S, and a block D_i has an edge with a cut-vertex v in T if and only if D_i contains v. Then the leaves of T are blocks, say $D_{k_1}, \cdots, D_{k_l}$. Since G is 2-connected, there is a vertex v_i of $D_{k_i} - S$ such that u connects v_i in G for $i \in [l]$. We use $P_{i,j}$ to denote the subpath of T from D_{k_i} to D_{k_j}. We now prove that T is a path and D_i is an edge for $i \neq r$. If T is not a path, then $l \geq 3$. There are two leaves of T, say D_{k_1} and D_{k_2}, such that $D_r \in V(P_{1,2})$. Then $G - v_3$ has a component containing $V(D_r) \cup \{u\}$, which contradicts that D_r is maximum. Thus, T is a path. Suppose $r \neq j$ and D_j is not an edge, i.e., D_j is a 2-connected graph. Since T is a path, we have $W = V(D_j) - S - \{v_1, \cdots, v_l\} \neq \emptyset$. Let $u' \in W$. Then $G - u'$ has a component containing $V(D_r) \cup \{u\}$, which contradicts that D_r is maximum. Thus, D_i is an edge for $i \neq r$.

Without loss of generality, suppose $V(D_i) \cap V(D_{i+1}) = \{u_i\}$ for $i \in [s-1]$. Then, D_1, D_s are leaves of T, D_i is an edge for $i \neq r$ and $S = \{u_1, \cdots, u_{s-1}\}$. Let $u_0 \in V(D_1 - S)$ and $u_s \in V(D_s - S)$ be two vertices adjacent to u.

Let $P_1 = \bigcup_{i<r} D_i$ and let $P_2 = \bigcup_{i=r+1}^{s} D_i$. Then P_1 and P_2 are paths. There is an independent set U_i of P_i such that $U_i \cap V(D_r) = \emptyset$ and $|U_i| = \left\lceil \frac{|P_i|-1}{2} \right\rceil$ for $i \in [2]$. Let U be a maximum independent set of D_r. Then $U \cup U_1 \cup U_2$ is an independent set of

$G-u$, i.e.,

$$\alpha(G) \geq \alpha(G-v) \geq |U \cup U_1 \cup U_2| = \alpha(D_r) + \left\lceil \frac{|P_1|-1}{2} \right\rceil + \left\lceil \frac{|P_2|-1}{2} \right\rceil$$
$$\geq \alpha(D_r) + \left\lceil \frac{|P_1|+|P_2|-2}{2} \right\rceil = \alpha(D_r) + \left\lceil \frac{s-1}{2} \right\rceil.$$

Let $P = \{uu_0, uu_s\} \cup (\bigcup_{i\neq r} D_i)$ and let $G' = D_r \cup P$. Then P is an $(s+1)$-path and G' is a 2-connected spanning subgraph of G. By Lemma 1.1 (3), we have $md(G) \leq md(G')$. Let Γ be an extremal MD-coloring of G'. Then Γ is an MD-coloring restricted on D_r and P. We call D_r and each edge of P the *joints* of G'. Let C be the set of colors $c \in \Gamma(G')$ such that c is in at least two joints of G'. For $c \in C$, we use n_c to denote the number of joints of G having edges colored with c. Then $md(G') = |\Gamma(G')| = |\Gamma(D_r)| + ||P|| - \Sigma_{c\in C}(n_c - 1)$. Since there is a color c of $C_\Gamma(u_{r-1}, u_r)$ that separates u_{r-1} and u_r, we have $c \in \Gamma(D_r) \cap \Gamma(P)$. By the same reason, for each $e \in E(P)$, either $\Gamma(e) = \Gamma(f)$ for an edge f of $P - e$, or $\Gamma(e) \subsetneq \Gamma(D_r)$. Thus, $\Sigma_{c\in C}(n_c - 1) \geq \lceil \frac{s+2}{2} \rceil$. Therefore,

$$md(G) \leq md(G') = |\Gamma(D_r)| + ||P|| - \Sigma_{c\in C}(n_c - 1)$$
$$\leq \alpha(D_r) + s + 1 - \left\lceil \frac{s+2}{2} \right\rceil = \alpha(D_r) + \left\lfloor \frac{s}{2} \right\rfloor$$
$$= \alpha(D_r) + \left\lceil \frac{s-1}{2} \right\rceil \leq \alpha(G).$$

The proof is thus complete. ∎

5 Characterization of extremal 2-connected graphs

We knew that $md(G) \leq 2$ if G is a $\lfloor \frac{n}{2} \rfloor$-connected graph and $md(G) \leq \lfloor \frac{n}{2} \rfloor$ if G is a 2-connected graph. We have characterized extremal $\lfloor \frac{n}{2} \rfloor$-connected graphs in Theorem 3.2. In this section, we characterize extremal 2-connected graphs, i.e., the 2-connected graphs with MD-number $\lfloor \frac{n}{2} \rfloor$.

For a 2-connected graph G, we use $\mathcal{E} = (L_0; L_1, \cdots, L_t)$ to denote an ear-decomposition of G, where L_0 is a 2-connected subgraph of G and L_i is a path for $i \in [t]$. Let $Z_{\mathcal{E}} = \{L_i \mid i > 0 \text{ and } end(L_i) \subseteq V(L_0)\}$.

If C is a cycle of G and $v \in V(G) - V(C)$, then we use $\kappa(v, C)$ to denote the maximum number of vv_i-path P_i of G, such that $V(P_i) \cap V(P_j) = \{v\}$ and $V(P_i) \cap V(C) = \{v_i\}$. We call $H = C \cup (\bigcup_{i=1}^{\kappa(v,C)} P_i)$ a (v, C)-*umbrella* of G (or an *umbrella* for short) if $\kappa(v, C) \geq 3$. The vertices $v_1, \cdots, v_{\kappa(v,C)}$ divide C into $\kappa(v, C)$ paths, say $P'_1, \cdots, P'_{\kappa(v,C)}$. We call P_i a *spoke* of H and call P'_i a *rim* of H. If the size of each spoke is odd and the size of each rim is even, then we call the (v, C)-umbrella a *uniform* (v, C)-*umbrella* (or *uniform umbrella* for short).

A graph G is called a *θ-graph* if G is the union of three internal disjoint paths T_1, T_2 and T_3 with $end(T_1) = end(T_2) = end(T_3)$. If each T_i is an even path, then we call G an *even θ-graph* and call each T_i a *route*.

Suppose $\mathcal{E} = (L_0; L_1, \cdots L_t)$ is an ear-decomposition of G. Then the concept *normal ear-decomposition* of G is defined as follows.

• If $|G|$ is even, then $\mathcal{E}$ is a normal ear-decomposition of G if L_0 is a cycle.

• If $|G|$ is odd and G is not a bipartite graph, then $\mathcal{E}$ is a normal ear-decomposition of G if L_0 is an odd cycle.

• If $|G|$ is odd and G is a bipartite graph, then $\mathcal{E}$ is a normal ear-decomposition of G if L_0 is either an umbrella or an even θ-graph. Moreover, if L_0 is an even θ-graph, then for each $L_i \in Z_{\mathcal{E}}$, $end(L_i)$ is contained in one route.

Lemma 5.1. *If G is a 2-connected graph, then G has a normal ear-decomposition.*

Proof. If n is even or G is a nonbipartite graph with n odd, then G has a normal ear-decomposition. If G is a bipartite graph and n is odd, then let $\mathcal{E} = \{L_0; L_1, \cdots, L_t\}$ be an ear-decomposition of G with L_0 an even cycle. Since $n = |L_0| + \Sigma_{i\in[t]}(|L_i| - 2)$ and n is odd, there is an even path among the ears, say L_i. Since $H = \bigcup_{l=0}^{i-1} L_i$ is a 2-connected bipartite graph, there is an even cycle C of H containing $end(L_i)$. Moreover, $end(L_i)$ divides C into two even paths. So, $L'_0 = C \cup L_i$ is an even θ-graph, say the three routes are T_1, T_2 and T_3. Let $\mathcal{E}' = \{L'_0; L'_1, \cdots, L'_s\}$ be an ear-decomposition of G and let $end(L'_j) = \{u_j, v_j\}$ for $j \in [s]$. If the ends of each L'_j in $Z_{\mathcal{E}'}$ are contained in one route, then $\mathcal{E}'$ is a normal ear-decomposition of G. Otherwise, suppose $L'_j \in Z_{\mathcal{E}'}$, $u_j \in I(T_1)$ and $v_j \in I(T_2)$. Then $\kappa(u_j, T_2 \cup T_3) \geq 3$, i.e., there is a $(u_j, T_2 \cup T_3)$-umbrella, say M. Then there is a normal ear-decomposition of G containing M. ∎

Lemma 5.2. *Suppose G is a 2-connected graph with $md(G) = \left\lfloor \frac{n}{2} \right\rfloor$. Let $\mathcal{E} = (L_0; L_1, \cdots, L_t)$ be an ear-decomposition of G with L_0 a 2-connected subgraph of G and $end(L_i) = \{a_i, b_i\}$ for $i \in [t]$. Then we have the following results.*

1. *If H is a 2-connected subgraph of G, then each extremal MD-coloring of G is an extremal MD-coloring restricted on H, and $md(H) = \left\lfloor \frac{|H|}{2} \right\rfloor$.*

2. *If n is even, then G is a bipartite graph and L_i is an odd path for $i \in [t]$.*

3. *If n is odd, then when $|L_0|$ is even, exact one of $\{||L_1||, \cdots, ||L_t||\}$ is even; when $|L_0|$ is odd, L_i is an odd path for $i \in [t]$.*

Proof. Let Γ be an extremal MD-coloring of G. Then for each $i \in [t]$, $\Gamma(L_i) \cap \Gamma(\bigcup_{l=0}^{i-1} L_l) \neq \emptyset$; otherwise, $C_\Gamma(a_i, b_i) = \emptyset$, a contradiction. Moreover, each color of $\Gamma(L_i) - \Gamma(\bigcup_{l=0}^{i-1} L_l)$ is used on at least two edges of L_i. Otherwise, suppose

$p \in \Gamma(L_i) - \Gamma(\bigcup_{l=0}^{i-1} L_l)$ and color p is only used on one edge $e = xy$ of L_i. Then since $\Gamma(\bigcup_{l=0}^{i} L_l) - e$ is connected, $C_\Gamma(x, y) = \emptyset$, a contradiction. Therefore,

$$\begin{aligned}\left\lfloor \frac{n}{2} \right\rfloor &= md(G) = |\Gamma(L_0)| + \Sigma_{i=1}^{t}|\Gamma(L_i) - \Gamma(\bigcup_{l=0}^{i-1} L_l)| \\ &\le md(L_0) + \Sigma_{i=1}^{t} \left\lfloor \frac{||L_i|| - 1}{2} \right\rfloor \\ &\le \left\lfloor \frac{|L_0|}{2} \right\rfloor + \Sigma_{i=1}^{t} \left\lfloor \frac{||L_i|| - 1}{2} \right\rfloor \\ &\le \left\lfloor \frac{|L_0|}{2} + \Sigma_{i\in[t]} \frac{||L_i|| - 1}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor.\end{aligned}$$

Then $|\Gamma(L_0)| = md(L_0) = \left\lfloor \frac{|L_0|}{2} \right\rfloor$ and $|\Gamma(L_i)| = \left\lfloor \frac{||L_i||-1}{2} \right\rfloor$ for each $i \in [t]$. So, Γ is an extremal MD-coloring restricted on L_0, and $md(L_0) = \left\lfloor \frac{|L_0|}{2} \right\rfloor$. Moreover, $|\Gamma(L_i) \cap \Gamma(\bigcup_{l=0}^{i-1} L_l)| = 1$ when L_i is an odd path.

If G is not a bipartite graph, n is even and L_0 an odd cycle, then the above inequality does not hold. Thus, G is a bipartite graph when n is even. Moreover, L_i is an odd path for each $i \in [t]$. If n and $|L_0|$ are odd, then L_i is an odd path for $i \in [t]$. If n is odd and $|L_0|$ is even, then exact one of $\{||L_1||, \cdots, ||L_t||\}$ is even. ∎

For a normal ear-decomposition $\mathcal{E} = \{L_0; L_1, \cdots, L_t\}$ of a 2-connected graph G, if L_0 is an odd cycle and $L_i \in Z_\mathcal{E}$, then $end(L_i)$ divides L_0 into an odd path and an even path, which are denoted by $f_o(\mathcal{E}, i)$ and $f_e(\mathcal{E}, i)$, respectively. If L_0 is an even cycle, $L_i \in Z_\mathcal{E}$ and $e \in E(L_0)$, then we use $g(\mathcal{E}, i, e)$ to denote the subpath of L_0 with ends $end(L_i)$ and $g(\mathcal{E}, i, e)$ contains e. We define a function $f(\mathcal{E}, i, j)$ for $0 \le i < j \le t$ as follows.

$$f(\mathcal{E}, i, j) = \begin{cases} f_o(\mathcal{E}, j) & i = 0, L_j \in Z_\mathcal{E} \text{ and } L_0 \text{ is an odd cycle;} \\ g(\mathcal{E}, i, e) & i = 0, L_j \in Z_\mathcal{E} \text{ and } L_0 \text{ is an even cycle with } e \in E(L_0); \\ a_jPb_j & i = 0, L_j \in Z_\mathcal{E}, L_0 \text{ is an umbrella}, P \text{ is either a spoke or a rim of} \\ & L_0 \text{ such that } end(L_j) \subseteq V(P); \\ a_jTb_j & i = 0, L_j \in Z_\mathcal{E}, L_0 \text{ is an even } \theta\text{-graph}, \ T \text{ is one of the three} \\ & \text{routes such that } end(L_i) \subseteq V(T); \\ a_jL_ib_j & i > 0 \text{ and } end(L_j) \subseteq V(L_i); \\ K_4 & \textit{otherwise.} \end{cases}$$

If L_0 is not an even cycle, then the function depends only on $\mathcal{E}, i$ and j. If L_0 is an even cycle and $i = 0$, then the function also depends on e. Thus, we need to fix an edge e of L_0 in advance if L_0 is an even cycle.

Lemma 5.3. *If G is a uniform umbrella or an even θ-graph other than $K_{2,3}$, then $|G|$ is odd and $md(G) = \left\lfloor \frac{|G|}{2} \right\rfloor$.*

Proof. It is obvious that $|G|$ is odd. Fix an integer $k \geq 3$. Suppose G' is either a minimum even θ-graph other than $K_{2,3}$, or a minimum uniform umbrella with k spokes.

If G' is a minimum even θ-graph other than $K_{2,3}$, then G' and one of its extremal MD-colorings are depicted in Figure 1 (1), which implies $md(G') = 3 = \left\lfloor \frac{|G'|}{2} \right\rfloor$.

If G' is a minimum uniform umbrella with k spokes, then each spoke is an edge and each rim is a 2-path. Suppose the k spokes are $e_1 = vv_1, \cdots, e_k = vv_k$, and the k rims are $P_1 = v_1f_1u_1f_2v_2, \cdots, P_k = v_kf_{2k-1}u_kf_{2k}v_1$. We color each e_i with i. The colors of the edges of P_i obey the rule that opposite edges of any 4-cycle have the same color (see Figure 1). Since $k \geq 3$, we know that for v_1, $\{e_1, f_2, f_{2k-1}\}$ is a monochromatic

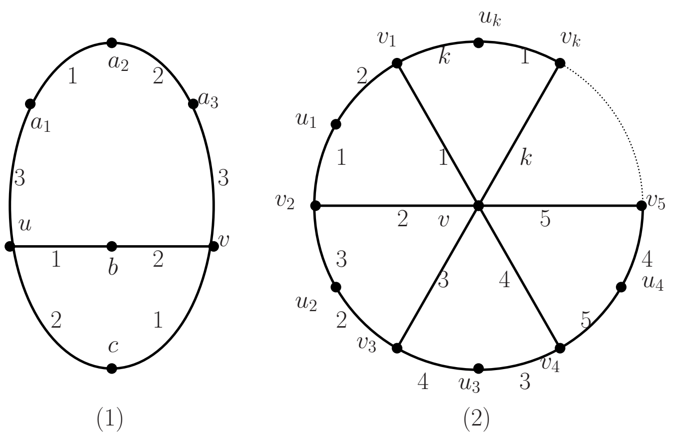

Figure 1: Extremal MD-colorings of the minimum even θ-graph and the minimum uniform umbrella.

v_1v-cut (it is also a monochromatic v_1v_i-cut for $i \neq 1$, and a monochromatic v_1u_i-cut for $i \neq \{1, 2, k\}$), $\{e_2, f_1, f_4\}$ is a monochromatic v_1u_1-cut and $\{e_k, f_{2k}, f_{2k-3}\}$ is a monochromatic v_1u_k-cut. By symmetry, the edge-coloring is an MD-coloring of G' with k colors. Since G' is 2-connected and $|G'| = 2k+1$, we have $md(G') = k = \left\lfloor \frac{|G'|}{2} \right\rfloor$.

Suppose G is a uniform umbrella with k spokes (an even θ-graph other than $K_{2,3}$). Then G is obtained from G' by replacing some edges with odd paths, respectively. W.l.o.g., suppose G is obtained from G' by replacing one edge with an odd path P. Then by Lemma 2.2, we have $md(G) \geq md(G') + \left\lfloor \frac{||P||-1}{2} \right\rfloor = \left\lfloor \frac{|G|}{2} \right\rfloor$, i.e., $md(G) = \left\lfloor \frac{|G|}{2} \right\rfloor$. The proof is thus complete. ∎

Lemma 5.4. *If G is a bipartite graph of odd order and $md(G) = \lfloor \frac{n}{2} \rfloor$, then each umbrella of G is a uniform umbrella.*

Proof. Suppose G is a bipartite graph of odd order and $md(G) = \lfloor \frac{n}{2} \rfloor$. Let H be a (v, C)-umbrella of G. We show that H is a uniform umbrella.

If $\kappa(v, C) = 3$, then let R_1, R_2 and R_3 be spokes of H and R_i be a vv_i-path. Then C is divided into three paths by vertices v_1, v_2 and v_3 (say, the three paths are W_1, W_2

and W_3, such that $end(W_1) = \{v_1, v_2\}$, $end(W_2) = \{v_2, v_3\}$ and $end(W_3) = \{v_1, v_3\}$). If each R_i is an odd path, then since G is a bipartite graph, each W_i is an even path, H be a uniform (v, C)-umbrella of G. If, by symmetry, R_1 is an even path and R_2, R_3 are odd paths, then W_1, W_3 are odd paths and W_2 is an even path. Then since $(W_1 \cup W_3 \cup R_2 \cup R_3; R_1, W_2)$ is an ear-decomposition of H containing even paths R_1 and W_2, by Lemma 5.2 (1) and (3) this yields a contradiction. If, by symmetry, R_1 is an odd path and R_2, R_3 are even paths, then H is a uniform $(v_1, R_2 \cup R_3 \cup W_2)$-umbrella. If each R_i is an even path, then $(C; R_1 \cup R_2, R_3)$ is an ear-decomposition of H containing two even paths, a contradiction.

If $\kappa(v, C) \geq 4$, then let R_1, R_2, R_3, R_4 be four spokes of H (let R_i be a vv_i path for $i \in [4]$). Then C is divided into two paths by v_2 and v_3 (say, the two paths are Y_1 and Y_2). W.l.o.g., suppose R_1 is an even path. Then $(Y_1 \cup R_2 \cup R_3; Y_2, R_4, R_1)$ is an ear-decomposition of H. Since $md(H) = \left\lfloor \frac{|H|}{2} \right\rfloor$ and R_1 is an even path, by Lemma 5.2 (3), Y_2 is an odd path. Since H is a bipartite graph, either R_2 or R_3 is an even path (say R_2). Then $(C \cup R_3 \cup R_4; R_1, R_2)$ is an ear-decomposition of H containing two even paths, a contradiction. So, each spoke of H is an odd path. Since H is a bipartite graph, each rim of H is an even path. ∎

Suppose $\mathcal{E} = (L_0; L_1, \cdots L_t)$ is an ear-decomposition of G. Then $\mathcal{E}$ can have the following possible properties.

Q: If $end(L_j) \cap I(L_i) \neq \emptyset$, then $end(L_j) \subseteq V(L_i)$.

R: If $end(L_j) \cap I(f(\mathcal{E}, k, i)) \neq \emptyset$, then $f(\mathcal{E}, k, j)$ is a proper subpath of $f(\mathcal{E}, k, i)$.

The concept *standard ear-decomposition* of G is defined as follows.

• If $|G|$ is even, then $\mathcal{E}$ is a standard ear-decomposition of G if L_0 is an even cycle.

• If $|G|$ is odd and G is not a bipartite graph, then $\mathcal{E}$ is a standard ear-decomposition of G if L_0 is an odd cycle and $f_e(\mathcal{E}, i) \cap f_e(\mathcal{E}, j) \neq \emptyset$ for $L_i, L_j \in Z_{\mathcal{E}}$.

• If $|G|$ is odd and G is a bipartite graph, then $\mathcal{E}$ is a standard ear-decomposition of G if L_0 is either a uniform umbrella or a even θ-graph other than $K_{2,3}$. Moreover, for each $L_i \in Z_{\mathcal{E}}$, if L_0 is a uniform umbrella, then $end(L_i)$ is contained in either a rim or a spoke; if L_0 is an even θ-graph other than $K_{2,3}$, then $end(L_i)$ is contained in one route.

Therefore, a standard ear-decomposition of G is also a normal ear-decomposition of G.

Lemma 5.5. *If $\mathcal{E} = (L_0; L_1, \cdots, L_t)$ is a standard ear-decomposition of G and $\mathcal{E}$ has properties **Q** and **R**, then there exist integers $0 \leq k < r \leq t$ such that $end(L_r) \subseteq V(L_k)$, and $d(u) = 2$ for each $u \in I(f(\mathcal{E}, k, r)) \cup I(L_r)$.*

Proof. For $i \in [t]$, let $end(L_i) = \{a_i, b_i\}$. We use m_r (n_r) to demote the minimum integer such that $a_r \in V(L_{m_r})$ ($b_r \in V(L_{n_r})$). Since $I(L_0) = V(L_0)$, we have $a_i \in$

$I(L_{m_r})$ and $b_r \in I(L_{n_r})$. Since $\mathcal{E}$ has property **Q**, we know for each $i \in [t]$, either $end(L_i) \subseteq V(L_{m_i})$, or $end(L_i) \subseteq V(L_{n_i})$. Let l_i be the minimum integer such that $end(L_i) \subseteq V(L_{l_i})$.

Let D be a digraph with vertex-set $V(D) = \{s_0, s_1, \cdots, s_t\}$ and arc-set $A(D) = \{(s_i, s_j) \mid f(\mathcal{E}, i, j) \neq K_4\}$. We use d_j to denote the length of a minimum directed path from s_0 to s_j. If $end(L_j) \cap I(L_i) \neq \emptyset$, then $d_j = d_i + 1$. Let $U = \{j \mid d_j \text{ is maximum}\}$. If $j \in U$, then $d_G(u) = 2$ for each $u \in I(L_j)$.

Let i be an integer in U such that $|f(\mathcal{E}, l_i, i)|$ is minimum. If there is a vertex v of $I(f(\mathcal{E}, l_i, i))$ such that $d_G(v) \geq 3$, then there is a path L_k such that $v \in end(L_k) \cap I(f(\mathcal{E}, l_i, i))$. Since $\mathcal{E}$ has property **R**, $f(\mathcal{E}, l_i, k)$ is a proper subpath of $f(\mathcal{E}, l_i, i)$, i.e., $|f(\mathcal{E}, l_i, k)| < |f(\mathcal{E}, l_i, i)|$. Since $|f(\mathcal{E}, l_i, i)|$ is minimum, we have $k \notin U$. Then there is a path, say L_p, such that $end(L_p) \cap I(L_k) \neq \emptyset$. Thus, $d_p > d_k = d_i$, a contradiction. Hence, $d_G(u) = 2$ for each $u \in I(f(\mathcal{E}, l_i, i))$. ∎

Theorem 5.6. *Suppose G is a 2-connected graph and $\mathcal{E} = (L_0; L_1, \cdots L_t)$ is a normal ear-decomposition of G. Then $md(G) = \lfloor \frac{n}{2} \rfloor$ if and only if $\mathcal{E}$ is a standard ear-decomposition of G that has properties* **Q** *and* **R**, *L_i is an odd path for each $i \in [t]$, and $f(\mathcal{E}, i, j)$ is an odd path if $f(\mathcal{E}, i, j) \neq K_4$.*

Proof. For $i \in [t]$, let $end(L_i) = \{a_i, b_i\}$.

For the necessity, suppose $md(G) = \lfloor \frac{n}{2} \rfloor$. If n is even, then L_0 is an even cycle. By Lemma 5.2 (2), G is a bipartite graph and L_i is an odd path for $i \in [t]$. Since $f(\mathcal{E}, i, j) \cup L_j$ is an even cycle, $f(\mathcal{E}, i, j)$ is an odd path. If n is odd, then since $\mathcal{E}$ is normal, $|L_0|$ is odd. By Lemma 5.2 (3), L_i is an odd path for $i \in [t]$. Suppose there are integers i, j such that $f(\mathcal{E}, i, j)$ is an even path. If $i = 0$ and L_0 is an odd cycle, then $f(\mathcal{E}, i, j) = f_o(i, j)$ is an odd path, a contradiction. If $i > 0$ and L_0 is an odd cycle, then $H = L_j \cup (\bigcup_{c=0}^{i} L_c)$ is a 2-connected subgraph of G and $(L_0; L_1 \cdots, L_{i-1}, L_i \cup L_j - I(f(\mathcal{E}, i, j)), f(\mathcal{E}, i, j))$ is an ear-decomposition of H with L_0 an odd cycle and $f(\mathcal{E}, i, j)$ an even path, and by Lemma 5.2 (1) and (3) this yields a contradiction. If L_0 is an umbrella or an even θ-graph other than $K_{2,3}$, then G is a bipartite graph. Since $f(\mathcal{E}, i, j) \cup L_j$ is an even cycle and L_j is an odd path, $f(\mathcal{E}, i, j)$ is an odd path. Thus, $f(\mathcal{E}, i, j)$ is an odd path if n is odd.

We need to prove that $\mathcal{E}$ is standard and $\mathcal{E}$ has properties **Q** and **R** below.

Claim 5.7. *$\mathcal{E}$ is standard.*

Proof. If n is even, then since G is a bipartite graph, L_0 is an even cycle. Thus, $\mathcal{E}$ is standard.

If G is not a bipartite graph and n is odd, then L_0 is an odd cycle. Suppose $\mathcal{E}$ is not a standard ear-decomposition of G. Then there are paths L_i and L_j of $Z_{\mathcal{E}}$ such that $E(f_e(\mathcal{E}, i)) \cap E(f_e(\mathcal{E}, j)) = \emptyset$. Let $D = L_i \cup L_j \cup [L_0 - I(f_e(\mathcal{E}, i) \cup f_e(\mathcal{E}, j))]$.

Then D is 2-connected subgraph of $L_0 \cup L_j \cup L_i$. Since $(D; f_e(\mathcal{E}, i), f_e(\mathcal{E}, j))$ is an ear-decomposition of $L_0 \cup L_i \cup L_j$ and $f_e(\mathcal{E}, i), f_e(\mathcal{E}, j)$ are even paths, by Lemma 5.2 (1) and (3) this yields a contradiction. Thus, $\mathcal{E}$ is standard.

If G is a bipartite graph, n is odd and L_0 is an even θ-graph, then $L_0 \neq K_{2,3}$. Otherwise L_0 is a 2-connected subgraph of G with $md(L_0) = 1 < \left\lfloor \frac{|L_0|}{2} \right\rfloor$ (by Lemma 1.1 (2)), and by Lemma 5.2 (1) this yields a contradiction. Thus, $\mathcal{E}$ is standard.

If G is a bipartite graph, n is odd and L_0 is an umbrella, then suppose the rims of L_0 are $W_1, \cdots, W_k$, where $k \geq 3$ and W_i is a $v_i v_{i+1}$-path for $i \in [k-1]$. Suppose the spokes are $R_1, \cdots, R_k$, where R_i is a vv_i-path. Let $C = \bigcup_{i \in [k]} W_i$. Since $md(G) = \left\lfloor \frac{n}{2} \right\rfloor$, by Lemma 5.4, L_0 is a uniform umbrella, i.e., each W_i is an even path and each R_i is an odd path. Suppose there is a path L_i of $Z_{\mathcal{E}}$ such that $end(L_i)$ is neither contained in any spoke nor contained in any rim. If $a_i \in I(R_j)$ and $b_i \in V(L_0) - V(R_j)$, then a_i divides R_j into two subpaths $R_j^1 = vL_j a_i$ and $R_j^2 = a_i L_j v_j$. Since $k \geq 3$, w.l.o.g., let $b_i \notin I(W_k)$. Then $H_s = R_j^s \cup L_i \cup (\bigcup_{l \neq k} W_l) \cup (\bigcup_{l \neq j} R_l)$ is a 2-connected graph for $s \in [2]$. Since L_j is an odd path, one of R_j^1 and R_j^2 is an even path, say R_j^1. Since $(H_2; W_k, R_j^1)$ is an ear-decomposition of $L_0 \cup L_i$ and W_k, R_j^1 are even paths, by Lemma 5.2 (1) and (3) this yields a contradiction. If $end(L_i) \subseteq V(C)$, then since G is a bipartite graph, L_i is an odd path and each W_j is an even path, we have $|end(L_i) \cap \{v_1, \cdots, v_k\}| \leq 1$. Therefore, there is a rim W_j such that a_i divides W_j into two odd paths $W_j^1 = v_j W_j a_i$ and $W_j^2 = a_i W_j v_{j+1}$. (w.l.o.g., suppose $1 \leq j < k$). Since there is no rim containing $end(L_i)$, we have $b_i \notin V(W_j)$. Note that $end(L_i)$ divides C into two subpaths C^1 and C^2 such that $v_j \in V(C^1)$ and $v_{j+1} \in V(C^2)$. Since $k \geq 3$, by symmetry, suppose $|C^1 \cap \{v_1, \cdots, v_k\}| \geq 2$. Then there is an integer $l \in [k] - \{j+1\}$ such that C^1 contains v_j and v_l. Then there is an ear-decomposition $(C'; P_1', P_2', \cdots)$ of $L_0 \cup L_i$ such that $C' = C^1 \cup L_i, P_1' = R_j \cup R_l$ and $P_2' = W_j^2 \cup R_{j+1}$. Since P_1' and P_2' are even paths, by Lemma 5.2 (3) this yields a contradiction. Thus $\mathcal{E}$ is standard. ∎

Claim 5.8. *$\mathcal{E}$ has property* **Q**.

Proof. Let m_i (n_i) be the minimum integer such that $a_i \in V(L_{m_i})$ ($b_i \in V(L_{n_i})$). Since $I(L_0) = V(L_0)$, we have $a_i \in I(L_{m_i})$ and $b_i \in I(L_{n_i})$.

Suppose $\mathcal{E}$ does not have property **Q**. Then there are integers $0 \leq j < r \leq t$ such that $a_r \in I(L_j)$ and $b_r \notin V(L_j)$. Since $b_r \in I(L_{n_r})$, by symmetry, suppose $j > n_r$. For convenience, let $n_r = i$. Since L_j is an odd path, let $a_j L_j a_r$ be an even path. Let $l = \max\{m_j, n_j, n_r\}$ and $H = L_j \cup L_r \cup (\bigcup_{h=0}^{l} L_h)$. Then H is a 2-connected graph with an ear-decomposition $(L_0; L_1, \cdots, L_l, a_r L_j b_j \cup L_r, a_j L_j a_r)$. If L_0 is an odd cycle, or a uniform umbrella, or an even θ-graph other than $K_{2,3}$, then since $|L_0|$ is odd and $a_j L_j a_r$ is an even path, by Lemma 5.2 (1) and (3) this yields a contradiction. If L_0 is an even cycle, then by Lemma 5.2 (1) and (2) this yields a contradiction. ∎

Claim 5.9. *$\mathcal{E}$ has property* **R**.

Proof. If $\mathcal{E}$ does not have property $\mathbf{R}$, then there are integers r, i, j such that $end(L_j) \cap I(f(\mathcal{E}, r, i)) \neq \emptyset$ and $f(\mathcal{E}, r, j)$ is not a subpath of $f(\mathcal{E}, r, i)$. Since $\mathcal{E}$ has property $\mathbf{Q}$, $f(\mathcal{E}, r, j)$ is a subpath of L_r. Then $end(L_i)$ and $end(L_j)$ appear alternately on $L = f(\mathcal{E}, r, i) \cup f(\mathcal{E}, r, j)$, say a_i, a_j, b_i, b_j are consecutively on L. Here, L is a subpath of the path L_r if $r > 0$; L is a subpath of either a rim or a spoke of L_r if $r = 0$ and L_0 is a uniform umbrella; L is a subpath of a route if $r = 0$ and L_0 is an even θ-graph other than $K_{2,3}$; L is a subpath of a cycle L_r if $r = 0$ and L_0 is a cycle. Let $W^1 = a_i L a_j, W^2 = a_j L b_i$ and $W^3 = b_i L b_j$. Since $f(\mathcal{E}, r, i)$ and $f(\mathcal{E}, r, j)$ are odd paths, either W^1, W^3 are even paths and W^2 is an odd path, or W^2 is an even path and W^1, W^3 are odd paths. Let $H = (\bigcup_{l=0}^{r} L_l) \cup L_i \cup L_j$.

Suppose W^1, W^3 are even paths and W^2 is an odd path. Let H' be a graph obtained from H by removing W^1 and W^3. Then H' is a 2-connected graph. Since $(H'; W^1, W^3)$ is an ear-decomposition of H and W^1, W^3 are even paths, by Lemma 5.2 this yields a contradiction.

Suppose W^2 is an even path and W^1, W^3 are odd paths. Let H_i be a graph obtain from H by removing W^i for $i \in [3]$. It is obvious that each H_i is a 2-connected graph. If L_0 is an even cycle, then $(H_2; W^2)$ is an ear-decomposition of G, and by Lemma 5.2 (1) and (2) this yields a contradiction. If $r = 0$ and L_0 is an odd cycle, then $P = L_0 - I(L)$ is an even path and $C = H_2 - I(P)$ is an even cycle. Since $(C; P, W^2)$ is an ear-decomposition of H and P, W^2 are even paths, by Lemma 5.2 (1) and (3) this yields a contradiction. If $r = 0$ and L_0 is an even θ-graph, then suppose T_1, T_2 and T_3 are routes of L_0, and suppose L is a subpath of T_1. Then $(H_2 - I(T_2); T_2, W^2)$ is an ear-decomposition of H and T_2, W^2 are even paths, a contradiction. If $r = 0$ and L_0 is a uniform umbrella, then there is a rim W of L_0 such that L is not a subpath of W. Then $(H_2 - I(W); W, W^2)$ is an ear-decomposition of H and W, W^2 are even paths, a contradiction. If $r > 0$ and n is odd, then $(L_0; \cdots, W^2)$ is an ear-decomposition of H. Since $|L_0|$ is odd and W^2 is an even path, by Lemma 5.2 (1) and (3) this yields a contradiction. ∎

Now for the sufficiency, suppose $\mathcal{E} = (L_0; L_1, \cdots, L_t)$ satisfies all conditions of the theorem, i.e., $\mathcal{E}$ is a standard ear-decomposition of G that has properties $\mathbf{Q}$ and $\mathbf{R}$, L_i is an odd path for $i \in [t]$, and $f(\mathcal{E}, j, i)$ is an odd path when $f(\mathcal{E}, j, i) \neq K_4$. Recall the definitions of digraph D, set U and integer l_i in Lemma 5.5. We choose an integer r from U such that $|f(\mathcal{E}, l_r, r)|$ is minimum. For convenience, let $l = l_r$. Then for each vertex u of $I(f(\mathcal{E}, l, r)) \cup I(L_r)$, we have $d_G(u) = 2$. The proof proceeds by induction on t. By Lemmas 1.1 (2) and 5.3, the result holds for $t = 0$.

If L_r is not an edge, then let G' be a graph obtained from G by replacing $f(\mathcal{E}, l, r)$ with an edge $f = a_r b_r$, let $G'_1 = G' - I(L_r)$ and $G'_2 = L_r \cup f$. Let $L = [L_l - I(f(\mathcal{E}, l, r)) - E(f(\mathcal{E}, l, r))] \cup f$. Let $\mathcal{E}'$ be an ear-decomposition of G'_1 obtained from $\mathcal{E}$ by removing L_r, and then replacing L_l with L. If $l > 0$, then since $f(\mathcal{E}, l, r)$ is an odd path, L is an odd path and $\mathcal{E}'$ satisfies all the conditions. If $l = 0$ and L_l is a uniform umbrella (an odd cycle or an even cycle), then L is also a uniform umbrella (an odd cycle, an even cycle), i.e., $\mathcal{E}'$ satisfies all the conditions in this case. If $l = 0$ and L_l is an even θ-graph, then $\mathcal{E}'$ satisfies all the conditions except for $L = K_{2,3}$. Thus, $\mathcal{E}'$ satisfies all the conditions unless $L = K_{2,3}$.

If $L \neq K_{2,3}$, then $\mathcal{E}'$ satisfies all the conditions. Since the number of paths in $\mathcal{E}'$ is $t-1$, by the induction hypothesis we have $md(G'_1) = \left\lfloor \frac{|G'_1|}{2} \right\rfloor$. Since G'_2 is an even cycle, we have $md(G'_2) = \frac{|G'_2|}{2}$. Thus, by Lemma 2.1, $md(G') = md(G'_1) + md(G'_2) - 1 = \left\lfloor \frac{|G'|}{2} \right\rfloor$. Since G is a graph obtained from G' by replacing f with the odd path $f(\mathcal{E}, l, r)$, by Lemma 2.2 we have $md(G) \geq md(G') + \left\lfloor \frac{||f(\mathcal{E},l,r)||-1}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor$. Therefore, $md(G) = \left\lfloor \frac{n}{2} \right\rfloor$.

If $L = K_{2,3}$, then $l = 0$ and $r = 1$. Since $r \in U$, d_r is maximum and $d_r = 1$ (the definition d_r is in the proof of Lemma 5.5). Thus, $L_i \in Z_{\mathcal{E}}$ for each $i \in [t]$. Let T_1, T_2 and T_3 be routes of L_0 with $|T_1| \leq |T_2| \leq |T_3|$. Then T_1 and T_2 are 2-paths and $f(\mathcal{E}, 0, r)$ is a subpath of T_3 with $|f(\mathcal{E}, 0, r)| = |T_3| - 1$. Since $L_0 \neq K_{2,3}$, we have $|f(\mathcal{E}, 0, r)| = |T_3| - 1 \geq 4$. For each L_i, if $end(L_i) \cap I(T_j) \neq \emptyset$ for $j \in [2]$, then $|f(\mathcal{E}, 0, i)| = 2 < |f(\mathcal{E}, l, r)|$, a contradiction; if $end(L_i) = end(T_3)$, then $f(\mathcal{E}, 0, i)$ is an even path, a contradiction. Thus, $f(\mathcal{E}, 0, i)$ is a proper subpath of T_3 and $|f(\mathcal{E}, 0, i)| = |f(\mathcal{E}, 0, r)|$ for each $i \in [t]$. If $end(L_i) \neq end(L_r)$ for $i, j \in [t]$, then $end(L_i) \cap I(f(\mathcal{E}, 0, r)) \neq \emptyset$ and $f(\mathcal{E}, 0, i)$ is not a proper subpath of $f(\mathcal{E}, 0, r)$, i.e., $\mathcal{E}$ does not have property **R**, a contradiction. Therefore, $end(L_i) = end(L_j)$ for each $i, j \in [t]$. Let $H = T_2 \cup T_3 \cup (\bigcup_{i \in [t]} L_i)$. Then H is a graph constructed in Remark 1. Thus, $md(H) = \frac{|H|}{2}$. Suppose Γ is an extremal MD-coloring of H (see Remark 1). Let $T_1 = ue_1ae_2v$ and $T_2 = uf_1bf_2v$. Since $G = H \cup T_1$, let Γ' be an edge-coloring of G such that $\Gamma(e) = \Gamma'(e)$ for each $e \in E(H)$, and $\Gamma(e_1) = \Gamma'(f_2)$ and $\Gamma(e_2) = \Gamma'(f_1)$. Then Γ' is an MD-coloring of G with $\left\lfloor \frac{n}{2} \right\rfloor$ colors, i.e., $md(G) = \left\lfloor \frac{n}{2} \right\rfloor$.

If L_r is an edge, then replace L_l by $L_l \cup L_r - I(f(\mathcal{E}, l, r))$ and replace L_r by $f(\mathcal{E}, l, r)$. Then the new ear-decomposition also satisfies all the conditions. Moreover, d_r is maximum and $|f(\mathcal{E}, l_r, r)| = 2$ is minimum in the new ear-decomposition. Since L_r is not an edge in the new ear-decomposition, this case has been discussed above. ∎

Remark 3. *Recalling the proof of Lemma 5.1, we can find a normal ear-decomposition for a given 2-connected graph in polynomial time. For a normal ear-decomposition $\mathcal{E}$ of G, deciding whether $\mathcal{E}$ satisfies all the conditions of Theorem 5.6 can be done in polynomial time. Thus, given a 2-connected graph G, deciding whether $md(G) = \left\lfloor \frac{n}{2} \right\rfloor$ is polynomially solvable.*

Corollary 5.10. *If G is a 2-connected graph with $md(G) = \left\lfloor \frac{|G|}{2} \right\rfloor$, then G is a planar graph.*

Proof. By Theorem 5.6, there is a standard ear-decomposition $\mathcal{E} = \{L_0; L_1, \cdots, L_t\}$ of G that has properties **Q** and **R**. Since G is a planar graph if G is a cycle, an umbrella or a θ-graph, the result holds for $t = 0$. Our proof proceeds by induction on t. Suppose $t > 0$. By Lemma 5.5, there are integers k, i such that $f(\mathcal{E}, k, i)$ is a path of order at least two, and $d_G(u) = 2$ for each $u \in I(f(\mathcal{E}, k, i)) \cup I(L_i)$. Let G' be a graph obtained from G by removing L_i. By Lemma 5.2 (1), $md(G') = \left\lfloor \frac{|G'|}{2} \right\rfloor$. By the induction hypothesis, G' is a planar graph. Since $d_G(u) = 2$ for each $u \in I(f(\mathcal{E}, k, i))$, there is a face F of G' such that $f(\mathcal{E}, k, i)$ is a subpath of F. Therefore, L_i can be embedded in F and G is a planar graph. ∎

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