# SYMMETRIC GRAPHS OF PRIME VALENCY ASSOCIATED WITH SOME ALMOST SIMPLE GROUPS

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ABSTRACT. A graph is symmetric if its automorphism group acts transitively on both the vertex and the arc sets. Stimulated by Lorimer's work on symmetric graphs of prime valency, we make a further investigation on the structural properties of the automorphism groups of connected symmetric graphs with prime valency. Also, as an application, we give a classification for symmetric graphs of prime valency arising from a class of almost simple groups.

KEYWORDS. Symmetric graph, quasiprimitive group, almost simple group.

## 1. INTRODUCTION

All graphs considered in this paper are assumed to be finite, simple and undirected.

Let $\Gamma = (V, E)$ be a connected graph, and denote by $\mathrm{Aut}(\Gamma)$ the automorphism group of $\Gamma$. An arc in $\Gamma$ is an ordered pair of adjacent vertices. For a subgroup $G \leqslant \mathrm{Aut}(\Gamma)$, the graph $\Gamma$ is called $G$-vertex-transitive, $G$-edge-transitive or $G$-symmetric if $G$ acts transitively on the vertex set, the edge set or the arc set of $\Gamma$, respectively.

Interest in symmetric graphs stems from Tutte's work on cubic graphs [20]. Since then, symmetric graphs have received considerable attention in the literature. There are too many published results in this field to be adequately summarised here. What interests us is Lorimer's works [15, 16] on symmetric graphs of prime valency.

Let $\Gamma = (V, E)$ be a connected $G$-symmetric graph of prime valency $r$, where $G \leqslant \mathrm{Aut}(\Gamma)$. Assume that $G$ contains a normal subgroup $N$ which is intransitive on $V$. Then each $N$-orbit on $V$ is an independent set of $\Gamma$. The normal quotient of $\Gamma$ with respect to $(G, N)$, denoted by $\Gamma_{(G,N)}$, is the graph with vertex set $V_N := \{\alpha^N \mid \alpha \in V\}$ and edge set $E_N := \{\{\alpha^N, \beta^N\} \mid \{\alpha, \beta\} \in E\}$. Lorimer [15] proved that either $\Gamma$ is bipartite, or $G$ has subgroups $G_1$ and $G_2$ such that $G_2$ is normal in $G_1$, $G_1/G_2$ is simple, and either $G_2$ is regular on $V$ or $\Gamma_{(G_1,G_2)}$ is $G_1/G_2$-symmetric and of valency $r$. Thus those graphs arising from simple groups play an important role in the study of symmetric graphs of prime valency. However, we are more interested in the gap between $G$ and $G_1$, or how to choose $G_1$ in $G$. For simplicity, we assume further that each minimal normal subgroup of $G$ has at most two orbits on $V$. In Section 4 of this paper, for non-bipartite $\Gamma$, we prove $G_1$ may be chosen as the socle $\mathrm{soc}(G)$ of $G$, see Lemma 4.1. We also give a similar version for bipartite $\Gamma$ in Lemma 4.2. The first result of this paper is summarized as follows.

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**Theorem 1.1.** *Let $\Gamma = (V, E)$ be a connected $G$-symmetric graph of prime valency $r \geqslant 3$. Assume that each minimal normal subgroup of $G$ has at most two orbits on $V$ and $\Gamma$ is not isomorphic to the complete bipartite graph $\mathsf{K}_{r,r}$. Then $\mathrm{soc}(G)$ is the unique minimal normal subgroup of $G$, and one of the following holds.*

(1) $\mathrm{soc}(G)$ *is simple;*
(2) $\mathrm{soc}(G) \cong T^k$ *for a simple group $T$ and integer $k > 1$, and either*
  (i) $\mathrm{soc}(G)$ *has a normal subgroup $T^{k-1}$ which is semiregular but not transitive on each $\mathrm{soc}(G)$-orbit on $V$; or*
  (ii) $(T, r)$ *is one of* $(\mathrm{A}_r, r)$, $(\mathrm{PSL}(n, q), \frac{q^n-1}{q-1})$, $(\mathrm{PSL}(2, 11), 11)$ *or* $(\mathrm{M}_{23}, 23)$, $k \geqslant 4$, *$\Gamma$ is bipartite, $\mathrm{soc}(G) = L_1 \times L_2$, where $L_1$ and $L_2$ are isomorphic and semiregular but intransitive on each $\mathrm{soc}(G)$-orbit on $V$.*

Suppose that $\Gamma$ and $G$ are described as in Theorem 1.1. If $\Gamma$ has order a power of 2, then one can read off the graph $\Gamma$ from [14, Theorem 1.1]. This stimulates us to classify $\Gamma$ under the assumption that $|V|$ is a product of two prime powers.

**Theorem 1.2.** *Let $\Gamma = (V, E)$ be a connected $G$-symmetric graph of prime valency $r \geqslant 3$. Assume that each minimal normal subgroup of $G$ has at most two orbits on $V$, and that one minimal normal subgroup of $G$ has an orbit of length $q^a p^b$, where $a$, $b$ are positive integers, and $p$, $q$ are distinct primes. Then $G$ is almost simple, and $\Gamma$ is isomorphic to one of the following graphs:*

(1) *ten of cubic graphs defined in* [3]*:* F010 (*Petersen graph*), F020$A$, F030 (*Tutte's* 8*-cage*), F028, F040, F056$B$, F056$C$, F110, F112$A$ *and* F182$D$*;*
(2) *Hoffman-Singleton graph, the point-hypeplane incidence graph of the projective geometry* $\mathrm{PG}(n-1, t^e)$, *the incidence graph of the generalized quadrangle* $\mathrm{GQ}(4, 2^{2^i})$, *where $t$ is a prime and* $(n, t^e) = (3, 2)$, $(3, 4)$, $(4, t)$ *or* $(6, 2)$, $i \geqslant 1$*;*
(3) *the complete graph* $\mathsf{K}_{q^a p^b}$, *and the graphs in Examples* 3.1-3.5*;*
(4) *the standard double covers of Petersen graph, Hoffman-Singleton graph and* $\mathsf{K}_{q^a p^b}$.

## 2. Preliminaries

For a finite group $G$ and $H, K < G$ with $|K : (H \cap K)| = 2$ and $\cap_{g\in G} H^g = 1$, define a graph $\mathrm{Cos}(G, H, K)$ on $[G : H] := \{Hx \mid x \in G\}$ such that $\{Hx, Hy\}$ is an edge if and only if $yx^{-1} \in HKH \setminus H$. Then the group $G$ can be viewed as a subgroup of $\mathrm{Aut}(\mathrm{Cos}(G, H, K))$, where $G$ acts on $[G : H]$ by right multiplication. It is easily shown that $\mathrm{Cos}(G, H, K)$ is $G$-symmetric and, for $x \in K \setminus H$, the edge $\{H, Hx\}$ and the arc $(H, Hx)$ have stabilizers $K$ and $H \cap K$ in $G$, respectively. Moreover, $\mathrm{Cos}(G, H, K)$ is connected if and only if $\langle H, K\rangle = G$.

Now let $\Gamma = (V, E)$ be a connected $G$-symmetric graph, and $\{\alpha, \beta\} \in E$. Then there is some $g \in G$ such that $(\alpha, \beta)^g = (\beta, \alpha)$. Since $\Gamma$ is connected, we have $G = \langle g, G_\alpha\rangle$. Replacing with a power of $g$, the element $g$ may be chosen as a 2-element. Note further that such $g$ is contained in the edge-stabilizer $G_{\{\alpha,\beta\}}$ but not in the arc-stabilizer $G_{\alpha\beta}$. Then we have a simple fact as follows.

**Lemma 2.1.** *Let $\Gamma = (V, E)$ be a connected $G$-symmetric graph, and $\{\alpha, \beta\} \in E$. Then $|G_{\{\alpha,\beta\}} : G_{\alpha\beta}| = 2$ and $\langle G_\alpha, G_{\{\alpha,\beta\}}\rangle = G$; in particular, $G_{\alpha\beta}$ has even index in its normalizer $\mathbf{N}_G(G_{\alpha\beta})$.*

Considering the bijection $\alpha^g \mapsto G_\alpha g$, $\forall g \in G$, we have an isomorphism from $\Gamma$ to $\mathrm{Cos}(G, G_\alpha, G_{\{\alpha,\beta\}})$. Then the following lemma holds.

**Lemma 2.2.** *Let $\Gamma = (V, E)$ be a regular graph of valency $d$, and $G \leqslant \mathrm{Aut}(\Gamma)$. Then $\Gamma$ is a connected $G$-symmetric graph if and only if $\Gamma \cong \mathrm{Cos}(G, H, K)$ for some $H, K < G$ with $\cap_{g\in G} H^g = 1$, $|H : (H \cap K)| = d$, $|K : (H \cap K)| = 2$ and $\langle H, K\rangle = G$.*

In the following, we assume that $\Gamma = (V, E)$ is a connected $G$-vertex-transitive graph. For a vertex $\alpha \in V$, denote by $G_\alpha^{\Gamma(\alpha)}$ the permutation group induced by $G_\alpha$ on $\Gamma(\alpha)$, the neighborhood of $\alpha$ in $\Gamma$. Let $G_\alpha^{[1]}$ be the kernel of this action. Then $G_\alpha^{\Gamma(\alpha)} \cong G_\alpha/G_\alpha^{[1]}$. Take an edge $\{\alpha, \beta\}$ of $\Gamma$, and set $G_{\alpha\beta}^{[1]} = G_\alpha^{[1]} \cap G_\beta^{[1]}$. Then

$$G_\alpha^{[1]}/G_{\alpha\beta}^{[1]} \cong (G_\alpha^{[1]})^{\Gamma(\beta)} \trianglelefteq (G_\beta^{\Gamma(\beta)})_\alpha,\ G_{\alpha\beta}/G_{\alpha\beta}^{[1]} \lesssim (G_\alpha^{\Gamma(\alpha)})_\beta \times (G_\beta^{\Gamma(\beta)})_\alpha,$$
$$G_\alpha^{[1]} = G_{\alpha\beta}^{[1]}.(G_\alpha^{[1]})^{\Gamma(\beta)},\ G_\alpha = (G_{\alpha\beta}^{[1]}.(G_\alpha^{[1]})^{\Gamma(\beta)}).G_\alpha^{\Gamma(\alpha)}. \tag{2.1}$$

Note, $(G_\beta^{\Gamma(\beta)})_\alpha \cong (G_\alpha^{\Gamma(\alpha)})_\beta$ if $\Gamma$ is $G$-symmetric.

**Lemma 2.3.** *Let $\Gamma = (V, E)$ be a connected $G$-vertex-transitive graph, and $\alpha \in V$. Assume that $G_\alpha^{\Gamma(\alpha)}$ is soluble. Then $G_\alpha$ is soluble.*

*Proof.* It suffices to show that $G_\alpha^{[1]}$ is soluble. Assume that $\Gamma$ has $n$ vertices $\alpha_1 = \alpha$, $\alpha_2$, $\ldots$, $\alpha_n$. Since $\Gamma$ is $G$-vertex-transitive, it is easily shown that every $G_{\alpha_i}^{\Gamma(\alpha_i)}$ is soluble. For $1 \leqslant i \leqslant n$, set $G_{\alpha_1\cdots\alpha_i} = \cap_{1\leqslant j\leqslant i} G_{\alpha_j}$ and $G_{\alpha_1\cdots\alpha_i}^{[1]} = \cap_{1\leqslant j\leqslant i} G_{\alpha_j}^{[1]}$. Since $\Gamma$ is connected, relabelling if necessary, we can assume that each vertex $\alpha_{i+1}$ is adjacent to some $\alpha_j \in \{\alpha_1, \ldots, \alpha_i\}$, where $1 \leqslant i < n$. Then $G_{\alpha_1\cdots\alpha_i}^{[1]} \leqslant G_{\alpha_{i+1}}$, and so $G_{\alpha_1\cdots\alpha_i}^{[1]}/G_{\alpha_1\cdots\alpha_i\alpha_{i+1}}^{[1]} \cong G_{\alpha_1\cdots\alpha_i}^{[1]}\mathrm{G}_{\alpha_{i+1}}^{[1]}/\mathrm{G}_{\alpha_{i+1}}^{[1]} \cong (G_{\alpha_1\cdots\alpha_i}^{[1]})^{\Gamma(\alpha_{i+1})} \leqslant G_{\alpha_{i+1}}^{\Gamma(\alpha_{i+1})}$. This implies that $G_{\alpha_1\cdots\alpha_i}^{[1]}/G_{\alpha_1\cdots\alpha_i\alpha_{i+1}}^{[1]}$ is soluble. Thus we have a normal series $1 = G_{\alpha_1\cdots\alpha_n}^{[1]} \trianglelefteq \cdots \trianglelefteq G_{\alpha_1\cdots\alpha_i\alpha_{i+1}}^{[1]} \trianglelefteq G_{\alpha_1\cdots\alpha_i}^{[1]} \trianglelefteq \cdots \trianglelefteq G_{\alpha_1\alpha_2}^{[1]} \trianglelefteq G_{\alpha_1}^{[1]} = G_\alpha^{[1]}$ whose factors are soluble. Then $G_\alpha^{[1]}$ is soluble, and the Lemma follows. □

Assume that $\Gamma$ is $G$-symmetric and of prime valency $r \geqslant 3$. Then $G_\alpha^{\Gamma(\alpha)}$ is a transitive permutation group of prime degree $r$. Thus either $G_\alpha^{\Gamma(\alpha)} \leqslant \mathrm{AGL}(1, r)$, or $G_\alpha^{\Gamma(\alpha)}$ is 2-transitive on $\Gamma(\alpha)$, refer to [6, p. 99, Corollary 3.5B]. For a prime $s$, let $\mathrm{O}_s(G_\alpha)$ be the largest normal $s$-subgroup of $G_\alpha$. Then we have the following fact.

**Lemma 2.4.** *Let $\Gamma = (V, E)$ be a connected $G$-symmetric graph of prime valency $r \geqslant 3$, and $\{\alpha, \beta\} \in E$. Assume that $\mathrm{O}_s(G_\alpha) \not\leqslant G_{\alpha\beta}^{[1]}$ for some prime $s$ not equal to $r$. Then $s$ is a divisor of $r - 1$.*

*Proof.* Since $\mathrm{O}_s(G_\alpha)$ is normal in $G_\alpha$, all $\mathrm{O}_s(G_\alpha)$-orbits on $\Gamma(\alpha)$ have equal length, which is a power of $s$. It follows that $\mathrm{O}_s(G_\alpha) \leqslant G_\alpha^{[1]}$, and so $\mathrm{O}_s(G_\alpha)$ is a normal subgroup of $G_{\alpha\beta}$. Since $\mathrm{O}_s(G_\alpha) \not\leqslant G_{\alpha\beta}^{[1]}$, we have $1 \neq (\mathrm{O}_s(G_\alpha))^{\Gamma(\beta)} \trianglelefteq (G_{\alpha\beta})^{\Gamma(\beta)} = (G_\beta^{\Gamma(\beta)})_\alpha$. Recall that $G_\beta^{\Gamma(\beta)} \leqslant \mathrm{AGL}(1, r)$ or $G_\beta^{\Gamma(\beta)}$ is 2-transitive on $\Gamma(\beta)$. It follows that each $(G_\beta^{\Gamma(\beta)})_\alpha$-orbit on $\Gamma(\beta) \setminus \{\alpha\}$ has length a divisor of $r-1$. Then, considering the action of $(\mathrm{O}_s(G_\alpha))^{\Gamma(\beta)}$ on a given $(G_\beta^{\Gamma(\beta)})_\alpha$-orbit, the lemma follows. □

For normal subgroups of $G$, by [15, Theorem 9] and [16, Theorem 1], we have the following fact.

**Lemma 2.5.** *Assume that $\Gamma = (V, E)$ is a $G$-symmetric graph of prime valency $r \geqslant 3$, and $N$ is a normal subgroup of $G$. Then either $N$ is semiregular on $V$, or $N_\alpha$ is transitive on $\Gamma(\alpha)$ and $N$ has at most two orbits on $V$, where $\alpha \in V$.*

At the end of this section, we list a result on permutation groups. Recall that a permutation group on a nonempty set $\Omega$ is quasiprimitive if each of its minimal normal subgroup is transitive on $\Omega$. Let $\mathrm{soc}(G)$ be the subgroup of a finite group $G$ generated by its minimal normal subgroups, called the socle of $G$. Then the following lemma is easily shown, see also the second paragraph of Section 3 in [18].

**Lemma 2.6.** *Let $G$ be a quasiprimitive group on a finite set $\Omega$. Then either $\mathrm{soc}(G)$ is the minimal normal subgroup of $G$, or $G$ has exactly two minimal normal subgroups. Moreover, for the latter case, if $N$ is a minimal normal subgroup of $G$ then $\mathrm{soc}(G) = N \times \mathbf{C}_G(N)$, $N \cong \mathbf{C}_G(N)$, and both $N$ and $\mathbf{C}_G(N)$ are insoluble and regular on $\Omega$.*

## 3. Examples

In this section, we present some graphs involved in Theorem 1.2.

Recall the standard double cover of a graph $\Sigma = (U, E')$, denoted by $\Sigma^{(2)}$, which is a bipartite graph defined on $U \times \{1, 2\}$ such that $\{(u_1, 1), (u_2, 2)\}$ is an edge of $\Sigma^{(2)}$ if and only if $\{u_1, u_2\} \in E'$. It is well known that $\Sigma^{(2)}$ is connected if and only if $\Sigma$ is a connected non-bipartite graph. Using standard double covers, we may construct some desired graphs.

Assume that $\Sigma = (U, E')$ is a connected $G$-symmetric non-bipartite graph (of prime valency $r$). Then $\Sigma^{(2)}$ is a connected $(G \times \langle\tau\rangle)$-symmetric graph (of valency $r$), where $\tau$ is defined as $(u, 1) \leftrightarrow (u, 2)$, and $G$ acts on $U \times \{1, 2\}$ by $(u, i)^g = (u^g, i)$ for $u \in U$, $i \in \{1, 2\}$ and $g \in G$. If $G = G_0{:}\langle\sigma\rangle$ for an almost simple subgroup $G_0$ and some involution $\sigma \in G$, then $G_0{:}\langle\sigma\tau\rangle$ is almost simple and acts transitively on the arc set of $\Sigma^{(2)}$. For example, the complete graph $\mathsf{K}_n$ has automorphism group $\mathrm{S}_n$, and its standard double cover $\mathsf{K}_n^{(2)}$ admits an $\mathrm{S}_n$ acting transitively on the arcs.

We next construct several examples using the coset graphs. Let $G$ be an almost simple group with socle $T$, and $L < H < G$ such that $T \not\leqslant H$, $|H : L|$ is a prime $r$ and $|\mathbf{N}_G(L) : L|$ is even. Take $L < K \leqslant \mathbf{N}_G(L)$ with $|K : L| = 2$. (Note that such $K$ always exists.) If $G = \langle H, K\rangle$ then, by Lemma 2.2, we have a connected $G$-symmetric graph $\mathrm{Cos}(G, H, K)$ of valency $r$ and order $|G : H|$. Further, such a graph $\mathrm{Cos}(G, H, K)$ is bipartite if and only if $|T : (T \cap H)| \neq |G : H|$. The following examples only involve those almost simple groups which have socle $\mathrm{PSL}(2, t)$ (with $t$ a prime) or some simple groups included in the Atlas [5]. Thus we can get the subgroup structures of them from [2, 5] and, sometimes, computation using GAP [19].

**Example 3.1.** Let $G$ be one of the almost simple groups listed in the first column of Table 3.1. Checking the subgroups of $G$, we conclude that there are $H, K < G$ which satisfy the conditions in Lemma 2.2. Thus, for each triple $(G, H, K)$ listed in Table 3.1, the coset graph $\mathrm{Cos}(G, H, K)$ is connected, $G$-symmetric and of valency $r$, where $r$ is the prime listed in the fifth column of Table 3.1. □

| $G$ | $H$ | $K$ | $\|G:H\|$ | $r$ | Remark |
|---|---|---|---|---|---|
| $\mathrm{A}_5, \mathrm{S}_5$ | $\mathbb{Z}_5, \mathrm{D}_{10}$ | $\mathbb{Z}_2, \mathbb{Z}_4$ | $2^2 \cdot 3$ | 5 | |
| $\mathrm{S}_5$ | $\mathbb{Z}_5$ | $\mathbb{Z}_2$ | $2^3 \cdot 3$ | 5 | bipartite |
| $\mathrm{A}_6, \mathrm{A}_6.\mathbb{Z}_2^i$ | $\mathrm{D}_{10}, \mathbb{Z}_5{:}[2^{i+1}]$ | $\mathbb{Z}_4, [2^{i+2}]$ | $2^2 \cdot 3^2$ | 5 | $i \in \{1, 2\}$ |
| $\mathrm{A}_6.\mathbb{Z}_2^i$ | $\mathbb{Z}_5{:}[2^i]$ | $[2^{i+1}]$ | $2^3 \cdot 3^2$ | 5 | $i \in \{1, 2\}$, bipartite |
| $\mathrm{A}_6.\mathbb{Z}_2^i$ | $\mathbb{Z}_5{:}[2^i]$ | $[2^{i+1}]$ | $2^3 \cdot 3^2$ | 5 | $i \in \{0, 1, 2\}$ |
| $\mathrm{A}_6.\mathbb{Z}_2$ | $\mathbb{Z}_5$ | $\mathbb{Z}_2$ | $2^4 \cdot 3^2$ | 5 | bipartite |
| $\mathrm{M}_{12}, \mathrm{M}_{12}.2$ | $\mathbb{Z}_{11}{:}\mathbb{Z}_5, \mathbb{Z}_{11}{:}\mathbb{Z}_{10}$ | $\mathbb{Z}_{10}, \mathrm{D}_{20}$ | $2^6 \cdot 3^3$ | 11 | |
| $\mathrm{M}_{12}.2$ | $\mathbb{Z}_{11}{:}\mathbb{Z}_5$ | $\mathrm{D}_{10}$ | $2^7 \cdot 3^3$ | 11 | bipartite |
| $\mathrm{PSL}(2, 8)$ | $\mathrm{D}_{14}$ | $\mathbb{Z}_2^2$ | $2^2 \cdot 3^2$ | 7 | |
| $\mathrm{PSL}(3, 3).o$ | $\mathbb{Z}_{13}{:}\mathbb{Z}_{3o}$ | $\mathrm{D}_{6o}$ | $2^4 \cdot 3^2$ | 13 | $o \in \{1, 2\}$ |
| $\mathrm{PSL}(3, 3).2$ | $\mathbb{Z}_{13}{:}\mathbb{Z}_3$ | $\mathrm{D}_6$ | $2^5 \cdot 3^2$ | 13 | bipartite |
| $\mathrm{PSL}(3, 3).o$ | $\mathbb{Z}_{13}{:}\mathbb{Z}_o$ | $\mathbb{Z}_2 \times \mathbb{Z}_o$ | $2^4 \cdot 3^3$ | 13 | $o \in \{1, 2\}$ |
| $\mathrm{PSL}(3, 3).2$ | $\mathbb{Z}_{13}$ | $\mathbb{Z}_2$ | $2^5 \cdot 3^3$ | 13 | bipartite |
| $\mathrm{PSL}(3, 5)$ | $\mathbb{Z}_{31}{:}\mathbb{Z}_3$ | $\mathbb{Z}_6, \mathrm{D}_6$ | $2^5 \cdot 5^3$ | 31 | |
| $\mathrm{PSL}(3, 5).2$ | $\mathbb{Z}_{31}{:}\mathbb{Z}_6$ | $\mathbb{Z}_{12}, \mathrm{D}_{12}$ | $2^5 \cdot 5^3$ | 31 | |
| $\mathrm{PSL}(3, 5).2$ | $\mathbb{Z}_{31}{:}\mathbb{Z}_3$ | $\mathrm{D}_{12}$ | $2^5 \cdot 5^3$ | 31 | bipartite |
| $\mathrm{PSU}(3, 3)$ | $\mathbb{Z}_7$ | | $2^5 \cdot 3^3$ | 7 | |
| $\mathrm{PSU}(3, 4)$ | $\mathbb{Z}_{13}{:}\mathbb{Z}_3$ | | $2^6 \cdot 5^2$ | 13 | |
| $\mathrm{PSU}(3, 7)$ | $\mathbb{Z}_{43}{:}\mathbb{Z}_3$ | | $2^7 \cdot 7^3$ | 43 | |
| $\mathrm{PGL}(2, t)$ | $\mathrm{D}_{t-1} < \mathrm{PSL}(2, t)$ | $\mathrm{D}_8$ | $t(t + 1)$ | $\frac{t-1}{4}$ | prime $t = 2 \cdot 3^a - 1, a$ odd prime |
| $\mathrm{PGL}(2, t)$ | $\mathrm{D}_{t+1} < \mathrm{PSL}(2, t)$ | $\mathrm{D}_8$ | $t(t - 1)$ | $\frac{t+1}{4}$ | prime $t = 2 \cdot p^{2^i} + 1, p$ odd prime |
| $\mathrm{PSL}(2, t)$ | $\mathbb{Z}_t{:}\mathbb{Z}_l$ | $\mathrm{D}_{2l}$ | $\frac{(t+1)(t-1)}{2l}$ | $t$ | $l \mid \frac{t-1}{2}, t = 2^a - 1, a$ odd prime |
| $\mathrm{PGL}(2, t)$ | $\mathbb{Z}_t{:}\mathbb{Z}_{2l} \nleq \mathrm{PSL}(2, t)$ | $\mathrm{D}_{4l}$ | $\frac{(t+1)(t-1)}{2l}$ | $t$ | $l \mid \frac{t-1}{2}, t = 2^a - 1, a$ odd prime |
| $\mathrm{PGL}(2, t)$ | $\mathbb{Z}_t{:}\mathbb{Z}_l < \mathrm{PSL}(2, t)$ | $\mathrm{D}_{2l}$ | $\frac{(t+1)(t-1)}{l}$ | $t$ | $l \mid \frac{t-1}{2}, t = 2^a - 1, a$ odd prime |

TABLE 3.1.

**Example 3.2.** (1) Let $G = \mathrm{PGL}(2, t)$, where $t \in \{19, 29, 59, 61\}$. Then $G$ has a subgroup $H \cong \mathrm{A}_5$ contained in $\mathrm{PSL}(2, t)$. Let $\mathrm{A}_4 \cong L < H$ and $K = \mathbf{N}_G(L)$. Checking the subgroups of $G$ (refer to [2, Theorem 2]), we conclude that $\mathbf{N}_G(L) \cong \mathrm{S}_4$ and $G = \langle H, K\rangle$ (also confirmed by GAP). Then $\mathrm{Cos}(G, H, K)$ is a connected $G$-symmetric bipartite graph of valency 5 and order $\frac{t(t^2-1)}{60}$.

(2) Let $G = \mathrm{PSL}(2, 31)$. Then $G$ has a maximal subgroup $H$ and a subgroup $K$ with $H \cong \mathrm{A}_5$, $K \cong \mathrm{S}_4$, $H \cap K \cong \mathrm{A}_4$ and $G = \langle H, K\rangle$, confirmed by GAP. Thus $\mathrm{Cos}(G, H, K)$ is a connected $G$-symmetric graph of valency 5 and order $31 \cdot 8$.

**Example 3.3.** Let $G$ be an almost simple group with socle $T = \mathrm{PSU}(3, 3)$. Take $\mathrm{PSL}(2, 7) \cong H < T$, $\mathrm{S}_4 \cong L < H$, and $K = \mathbf{N}_G(L)$. Then $K = L \times \mathbb{Z}_2$ and $G = \langle H, K\rangle$, confirmed by GAP. Thus $\mathrm{Cos}(G, H, K)$ is a connected $G$-symmetric graph of valency 7 and order $2^3 \cdot 3^2$.

**Example 3.4.** Let $G$ be an almost simple group with socle $T = \mathrm{PSU}(5, 2)$. Take $\mathrm{PSL}(2, 11) \cong H < T$, $\mathrm{A}_5 \cong L < H$ and $K = \mathbf{N}_G(L)$. Then $K \cong \mathrm{S}_5$ and $G = \langle H, K\rangle$, confirmed by GAP. This yields that $\mathrm{Cos}(G, H, K)$ is a connected $G$-symmetric bipartite graph of valency 11 and order $2^9 \cdot 3^4$.

**Example 3.5.** Let $X$ be an almost simple group with socle $T = \mathrm{M}_{12}$. Then $T$ has two conjugacy classes of subgroups $\mathrm{PSL}(2, 11)$: one of them say $\mathcal{C}_1$ consists of maximal subgroups, and the other one say $\mathcal{C}_2$ comes from the subgroups of maximal subgroups of $T$ isomorphic to $M_{11}$. (Confirmed by GAP.) Note that $\mathrm{PSL}(2, 11)$ has two conjugacy classes of maximal subgroups isomorphic to $\mathrm{A}_5$. Take $H_i \in \mathcal{C}_i$, $i = 1, 2$. For each $i$, let $L_{i1}, L_{i2} < H_i$ such that $L_{i1} \cong L_{i2} \cong \mathrm{A}_5$ but $L_{i1}$ and $L_{i2}$ are not conjugate in $H_i$. Computation shows that

(i) $L_{11}$ and $L_{12}$ are conjugate in $T$, $L_{11}$ is self-normalizing in $T$ and $\mathbf{N}_X(L_{11}) \cong \mathrm{S}_5$;

(ii) $L_{21}$ and $L_{22}$ are not conjugate in $T$, $\mathbf{N}_X(L_{2j}) = \mathbf{N}_T(L_{2j}) \cong \mathrm{S}_5$ for $j = 1, 2$, and there is $\sigma \in \mathbf{N}_X(H_2)$ such that $L_{21}^\sigma = L_{22}$.

(1) Let $G = X$ and $K_1 = \mathbf{N}_X(L_{11})$. Then $G = \langle H_1, K_1\rangle$, confirmed by GAP, and so $\mathrm{Cos}(G, H_1, K_1)$ is a connected $G$-symmetric bipartite graph of valency 11 and order 288.

(2) Let $G = T$ and $K_{2j} = \mathbf{N}_T(L_{2j})$ for $j = 1, 2$. Confirmed by GAP, we have that $G = \langle H_2, K_{21}\rangle = \langle H_2, K_{22}\rangle$. Then we get two connected $G$-symmetric graphs $\mathrm{Cos}(G, H_2, K_{21})$ and $\mathrm{Cos}(G, H_2, K_{22})$, which have valency 11 and order 144. It is easily shown that the $\sigma$ in (ii) induces an isomorphism from $\mathrm{Cos}(G, H_2, K_{21})$ to $\mathrm{Cos}(G, H_2, K_{22})$ by $H_2 g \mapsto H_2 g^\sigma$, $\forall g \in G$. □

## 4. Proof of Theorem 1.1

Assume that $\Gamma = (V, E)$ is a connected graph of prime valency $r \geqslant 3$, and let $G \leqslant \mathrm{Aut}(\Gamma)$. (Note that $\Gamma$ has even order.) If $\Gamma$ is $G$-symmetric and $\alpha \in V$ then, by [4, Lemma 1.1], $r$ is the largest prime divisor of $|G_\alpha|$ and $|G_\alpha|$ is indivisible by $r^2$, see also [15, Theorem 8].

**Lemma 4.1.** *Let $\Gamma = (V, E)$ be a connected $G$-symmetric graph of prime valency $r \geqslant 3$. Assume that $G$ is quasiprimitive on $V$. Then $\mathrm{soc}(G)$ is a minimal normal subgroup of $G$, and one of the following holds:*

1. *$\mathrm{soc}(G)$ is either simple or regular on $V$;*
2. *$\mathrm{soc}(G) \cong T^k$ for a nonabelian simple group $T$ and integer $k \geqslant 2$, $\Gamma$ is $\mathrm{soc}(G)$-arc-transitive, and a normal subgroup $T^{k-1}$ of $\mathrm{soc}(G)$ is semiregular but not transitive on $V$.*

*Proof.* By Lemma 2.6, $G$ has at most two minimal normal subgroups. Suppose $G$ has two minimal normal subgroups say $N$ and $M$. Then, by [18, Theorem 2], $G_\alpha$ does not act 2-transitively on $\Gamma(\alpha)$, where $\alpha \in V$. Thus $G_\alpha^{\Gamma(\alpha)}$ is soluble, and so $G_\alpha$ is soluble by Lemma 2.3. Note that $N$ and $M$ are insoluble and regular on $V$. Thus $N \times M \leqslant G = NG_\alpha$, yielding $M \lesssim NG_\alpha/N \cong G_\alpha$, a contradiction. Therefore, $G$ has a unique minimal normal subgroup say $N$.

If $N$ is regular on $V$ or simple then part (1) of this lemma follows. Thus we assume next that $N$ is irregular on $V$, and write $N = T_1 \times \cdots T_k$ for some integer $k \geqslant 2$ and isomorphic nonabelian simple groups $T_i$. In particular, $N$ is insoluble.

By Lemma 2.5, $\Gamma$ is $N$-symmetric. Then $r$ is a divisor of $|N|$, and so each $|T_i|$ is divisible by $r$. Let $L = T_2 \times \cdots T_k$. Suppose that $L$ is transitive on $V$. Then $N = LN_\alpha$, and thus $T_1 \cong N/L \cong N_\alpha/(L \cap N_\alpha)$. This implies that $r$ is not a divisor of $|L \cap N_\alpha|$ as $|N_\alpha|$ is indivisible by $r^2$. Then, by Lemma 2.5, $L$ is regular on $V$, and thus $L \cap N_\alpha = 1$ and $T_1 \cong N_\alpha$. In particular, since $T_1$ is insoluble, $N_\alpha^{\Gamma(\alpha)}$ is a 2-transitive group of degree $r$. Then, by [18, Theorem 2], $G$ is of type III(b)(i) described as in [18, Section 2]. This implies that $N_\alpha \leqslant R_1 \times \cdots \times R_k$, where each $R_i$ is properly contained in $T_i$. Then we get a contradiction by noting that $N_\alpha \cong T_1$. Therefore, $L$ is intransitive on $V$. Since $\Gamma$ is not bipartite, by Lemma 2.5, we have part (2) of this lemma. □

**Lemma 4.2.** *Let $\Gamma = (V, E)$ be a connected $G$-symmetric bipartite graph of prime valency $r \geqslant 3$. Let $G^+$ be the subgroup of $G$ which preserves the bipartition of $\Gamma$. Suppose that every minimal normal subgroup of $G$ contained in $G^+$ is transitive on both parts of $\Gamma$. If $\Gamma \not\cong \mathsf{K}_{r,r}$, then $\mathrm{soc}(G^+)$ is a minimal normal subgroup of $G$, $\mathrm{soc}(G) = \mathrm{soc}(G^+)$ or $\mathrm{soc}(G^+) \times \mathbb{Z}_2$, either $\mathrm{soc}(G^+)$ is abelian or $G^+$ has at most two minimal normal subgroups, and one of the following holds.*

1. *$\mathrm{soc}(G^+)$ is either simple or semiregular on $V$.*
2. *$\mathrm{soc}(G^+) \cong T^k$ for some nonabelian simple group $T$ and integer $k \geqslant 2$, $\Gamma$ is $\mathrm{soc}(G^+)$-edge-transitive, and either*
   - (i) *a normal subgroup $T^{k-1}$ of $\mathrm{soc}(G^+)$ is semiregular but not transitive on each part of $\Gamma$; or*
   - (ii) *$k = 2l$ for some $l > 1$, $\mathrm{soc}(G_\alpha^{\Gamma(\alpha)}) \cong T$ for $\alpha \in V$, $(T, r)$ is one of $(\mathrm{A}_r, r)$, $(\mathrm{PSL}(n, q), \frac{q^n-1}{q-1})$, $(\mathrm{PSL}(2, 11), 11)$ or $(\mathrm{M}_{23}, 23)$, $\mathrm{soc}(G^+) = L_1 \times L_2$, where $L_1$ and $L_2$ are minimal normal subgroups of $G^+$, which are isomorphic and semiregular but intransitive on each part of $\Gamma$.*

*Proof.* We assume that $\Gamma$ is not a complete bipartite graph. Then $G^+$ is faithful on both parts $U$ and $W$ of $\Gamma$, see [7, Lemma 5.2].

Suppose that $G^+$ contains two minimal normal subgroups of $G$, say $N$ and $M$. Then $N$ and $M$ centralize each other, and thus they are regular on both parts $U$ and $W$ of $\Gamma$. This implies that $N$ and $M$ are not abelian; otherwise, we have $N = M$ by [6, Theorem4.2A], a contradiction. Since $N$ is transitive on $U$ and $W$, we have $G^+ = NG_\alpha$ for $\alpha \in V$. Then $M \cong (MN)N \leqslant G^+/N \cong G_\alpha$, and in particular, $G_\alpha$ is insoluble. Then $G_\alpha^{\Gamma(\alpha)}$ is 2-transitive on $\Gamma(\alpha)$, and thus [17, Theorem 2.1] works here. Combining with [17, Theorem 2.3], we conclude that $G^+$ contains a minimal normal subgroup say $L$ other than $M$ and $N$. Then $G$ has three minimal normal subgroups which centralize every other and are regular on both $U$ and $W$, which is impossible. Therefore, $G^+$ contains a unique minimal normal subgroups of $G$. Note that $\mathbf{C}_{G^+}(N)$ is normal in $G$. It follows that $\mathbf{C}_{G^+}(N) = 1$ or $N$.

Let $N$ be the minimal normal subgroups of $G$ contained in $G^+$. Write $N = T_1 \times \cdots \times T_k$ for isomorphic simple groups $T_i$ and integer $k \geqslant 1$. Take an arbitrary minimal normal subgroup $L$ of $G^+$. Let $g \in G \backslash G^+$. Then $L^g$ is also a minimal normal subgroup of $G^+$, and $LL^g$ is normal in $G$. By the choice of $N$, we have $N \leqslant LL^g$. If $L$ is abelian then $LL^g \leqslant \mathbf{C}_{G^+}(N) = N$, and so $N = LL^g$. If $L^g = L$ then $N \leqslant LL^g = L$, and so $N = LL^g$. Assume that $L$ is nonabelian and $L^g \neq L$. Write $L = S_1 \times \cdots \times S_l$ for isomorphic nonabelian simple groups $S_i$. Then $LL^g = S_1 \times \cdots \times S_l \times S_1^g \times \cdots \times S_l^g$. Since $N$ is normal in $LL^g$, each direct factor of $N$ is contained in $\{S_1, \ldots, S_l, S_1^g, \ldots, S_l^g\}$. Then $1 \neq N = (L \cap N) \times (L^g \cap N)$. It follows from the minimality of $L$ and $L^g$ that $L$ or $L^g$ is contained in $N$, and then $N = LL^g$. The above argument yields $N = \mathrm{soc}(G^+)$. Similarly, if $N$ is nonabelian then it is easily shown that a minimal normal subgroup of $G^+$ contained in $N$ must be $L$ or $L^g$.

Suppose that $N \neq \mathrm{soc}(G^+)$. Let $M$ be a minimal normal subgroup of $G$ with $N \neq M$. Then $M \not\leqslant G^+$, and $M$ interchanges $U$ and $W$. Thus $M \cap G^+$ has index 2 in $M$. Noting that $M$ is a product of isomorphic simple groups, it implies that $M \cong \mathbb{Z}_2^m$ for some integer $m \geqslant 1$. On the other hand, $M \cap G^+$ is normal in $G$, yielding $M \cap G^+ = 1$. Then $M \cong \mathbb{Z}_2$. Noting that $G = G^+ \times M$, we have $\mathrm{soc}(G) \leqslant N\mathbf{C}_G(N) = N(\mathbf{C}_{G^+}(N) \times M) = N \times M$. Thus $\mathrm{soc}(G) = \mathrm{soc}(G^+) \times \mathbb{Z}_2$.

To finish the proof, we assume further that $N = T_1 \times \cdots \times T_k$ is not semiregular on $V$, where $k \geqslant 2$ and $T_i$ are isomorphic nonabelian simple groups. Then $\Gamma$ is $N$-edge-transitive, and so $r$ is a divisor of each $|T_i|$. Let $K = T_2 \times \cdots \times T_k$. If $K$ is intransitive on each of $U$ and $W$ then, by [7, Lemma 5.1], $K$ is semiregular on $V$. Now suppose that $K$ is transitive on $U$. Then $N = KN_\alpha$ for $\alpha \in U$, and hence $T_1 \cong N_\alpha/(K \cap N_\alpha)$. Recalling that $|N_\alpha|$ is not divisible by $r^2$, it follows that $|K \cap N_\alpha|$ is indivisible by $r$. Then $K$ is intransitive on $W$; otherwise, $\Gamma$ should be $K$-edge-transitive, and hence $|K \cap N_\alpha|$ has a divisor $r$, a contradiction. By [7, Lemma 5.5], $K$ has $r$-orbits on $W$. Clearly, $T_1$ acts transitively on these $r$-orbits, and hence $T_1$ has a subgroup of index $r$. By [8], $(T_1, r)$ is one of $(\mathrm{A}_r, r)$, $(\mathrm{PSL}(n,q), \frac{q^n-1}{q-1})$, $(\mathrm{PSL}(2,11), 11)$ and $(\mathrm{M}_{23}, 23)$.

By [7, Lemma 5.5], $K \cap N_\alpha$ fixes $\Gamma(\alpha)$ point-wise. Thus, since $T_1 \cong N_\alpha/(K \cap N_\alpha)$, we conclude that $N_\alpha^{[1]} = K \cap N_\alpha$ and $N_\alpha^{\Gamma(\alpha)} \cong T_1$. Since $N_\alpha$ is normal in $G_\alpha$, we know that $N_\alpha^{\Gamma(\alpha)}$ is normal in $G_\alpha^{\Gamma(\alpha)}$, and hence $\mathrm{soc}(G_\alpha^{\Gamma(\alpha)}) \cong T_1$. Suppose that $\mathrm{soc}(G^+)$ is a minimal normal subgroup of $G^+$. Then $G^+$ is quasiprimitive on both $U$ and $W$. By [17, Theorem 2.3], $G^+$ is of type III(b)(i) (given as in [18, Section 2]) on each of $U$ and $W$. In particular, $N_\alpha \leqslant R_1 \times \cdots \times R_k$, where each $R_i$ is properly contained in $T_i$, which is impossible. Thus $\mathrm{soc}(G^+)$ is not a minimal normal subgroup of $G^+$, and then $\mathrm{soc}(G^+) = L \times L^g$, where $L$ is a minimal normal subgroup of $G^+$ and $g \in G \setminus G^+$. By [17, Theorem 2.1], we conclude that neither $L$ nor $L^g$ is transitive on $U$ or $W$. Then by [7, Theorem 1.1], both $L$ and $L^g$ are semiregular on $V$. Without loss of generality, we may choose $L$ as a subgroup of the above $K$. Since $K$ is transitive on $U$, we get $k - 1 > 1$; otherwise $L = K$ is transitive on $U$, a contradiction. Thus $k = 2l$ for some $l > 1$, and the lemma follows. $\square$

**Remark 4.3.** Let $G$ and $\Gamma = (V, E)$ be as in Lemma 4.2 with $G = G^+ \times \mathbb{Z}_2$. Set $G = G^+ \times \langle\sigma\rangle$, and take a $G^+$-orbit $U$ on $V$. Then $V = U \cup U^\sigma$ and $U \cap U^\sigma = \emptyset$. Define a graph $\Sigma$ on $U$ with edge set $\{\{u_1, u_2\} \mid \{u_1, u_2^\sigma\} \in E\}$. Then $\Sigma$ is $G^+$-symmetric and of valency $r$, and $\Gamma \cong \Sigma^{(2)}$.

**Proof of Theorem 1.1**. Let $\Gamma = (V, E)$ be a connected $G$-symmetric graph of prime valency $r \geqslant 3$. Assume that $\Gamma \ncong \mathsf{K}_{r,r}$, and that each minimal normal subgroup of $G$ has at most two orbits on $V$. If $\Gamma$ is not a bipartite graph then our result is true by Lemma 4.1. Suppose that $\Gamma$ is bipartite. Then $|V| > 4$ as $\Gamma$ has valency $r \geqslant 3$. By Lemma 4.2, $\mathrm{soc}(G^+)$ is a minimal normal subgroup of $G$, and $\mathrm{soc}(G) = \mathrm{soc}(G^+)$ or $\mathrm{soc}(G^+) \times \mathbb{Z}_2$. If $\mathrm{soc}(G) = \mathrm{soc}(G^+) \times \mathbb{Z}_2$ then $G$ has a normal subgroup of order 2, which forces that $|V| = 4$, a contradiction. Thus our result follows. $\square$.

Let $(\Gamma, G)$ be a pair described as in Theorem 1.2. Clearly, $\Gamma$ is not a complete bipartite graph, and $\mathrm{soc}(G)$ is not a group of order a prime power. Note that a nonabelian simple group has order divisible by three distinct primes. Then, by Lemmas 4.1 and 4.2, the following result holds.

**Corollary 4.4.** *Assume that* $\Gamma = (V, E)$ *and* $G$ *are described as in* Theorem 1.2. *Then* $G$ *is almost simple.*

## 5. The graphs

In this section, we determine the graphs in Theorem 1.2. In view of Corollary 4.4, we shall work in this section with the following assumptions.

**Hypothesis 5.1.** Let $\Gamma=(V,E)$ be a connected $G$-symmetric graph of prime valency $r\geqslant 3$, where $G$ is an almost simple group with socle $T$. Let $a,b$ be positive integers, and let $p,q$ be distinct primes. Assume that $T$ has at most two orbits on $V$, $|T:T_\alpha|=q^ap^b$ and $T_\alpha$ is transitive on $\Gamma(\alpha)$, where $\alpha\in V$. If $\Gamma$ is bipartite then let $G^+$ be the subgroup of $G$ preserving the bipartition.

Let $(T,\Gamma)$ be as in Hypothesis 5.1. Then $T$ has a subgroup $T_\alpha$ of index a product of two prime powers. Combining with the following remark, we can read off all possible candidates for $(T,T_\alpha)$ from [11, Theorem 1.1].

**Remark 5.2.** Let $T$ be a finite nonabelian simple group, and $X<T$ such that $|T:X|$ is a product of two prime powers. In [11], the authors gave a classification of such pairs $(T,X)$. But their classification is not complete. We complete their classification by listing the omissions in the following.

(1) When they deal with $\mathrm{PSL}(m,p^f)$ in [11, Lemma 4.1], the authors use [11, Lemma 2.2] to estimate the number of prime divisors of $\frac{p^{mf}-1}{p^f-1}$. Checking their proof, we find the case, where $n$ is a square of some prime $t$ and $f$ is a power of $t$, is ignored in [11, Lemma 2.2]. Thus the groups $\mathrm{PSL}(t^2,p^{t^i})$ and their parabolic subgroups should be added into [11, Table 4.1]. For example, $\mathrm{PSL}(3^2,2^3)$ has a subgroup of index $\frac{8^{27}-1}{8-1}=73\cdot 262657$. Note that the last line of [11, Table 4.1] is just a special case for $\mathrm{PSL}(t^2,p^{t^i})$.

(2) By [11, Lemma 2.2], the pair $(\mathrm{PSL}(6,2),\mathbb{Z}_2^5{:}\mathrm{GL}(5,2))$ should be added into [11, Table 4.1]. Checking the subgroups of $H$ or $K$ given in [11, Theorem 1.1], we find that the following pairs are missing from the tables of [11]:

| | |
|---|---|
| [11, Table 3.1]: | $(\mathrm{A}_9,\mathrm{A}_7),(\mathrm{A}_9,\mathbb{Z}_2^3{:}\mathrm{PSL}(3,2))$ |
| [11, Table 4.1]: | $(\mathrm{PSL}(4,3),\mathbb{Z}_2^4{:}\mathrm{S}_5),(\mathrm{PSL}(3,3),\mathbb{Z}_{13}),(\mathrm{PSL}(2,8),\mathrm{D}_{14}),(\mathrm{PSL}(2,8),\mathbb{Z}_2^3)$, $(\mathrm{PSL}(2,7),\mathrm{A}_4)$ |
| [11, Table 4.2]: | $(\mathrm{PSp}(6,2),\mathrm{A}_8),(\mathrm{PSp}(6,2),\mathrm{A}_7),(\mathrm{PSp}(4,3),\mathrm{A}_6),(\mathrm{PSp}(4,3),\mathrm{A}_5)$, $(\mathrm{PSp}(4,3),\mathrm{D}_{10}),(\mathrm{PSp}(4,3),\mathbb{Z}_5),(\mathrm{PSp}(4,3),[3^4]),(\mathrm{PSp}(4,3),\mathbb{Z}_3^3{:}\mathrm{A}_4)$, $(\mathrm{PSp}(4,3),\mathbb{Z}_3^3{:}\mathrm{S}_3),(\mathrm{PSp}(4,3),[3^3]{:}\mathbb{Z}_6)$, |
| [11, Table 4.3]: | $(\mathrm{PSU}(3,3),[3^3]),(\mathrm{PSU}(3,3),\mathbb{Z}_7),(\mathrm{PSU}(4,3),\mathbb{Z}_3^4{:}\mathrm{A}_6)$, $(\mathrm{PSU}(3,7),[7^3]{:}\mathbb{Z}_3),(\mathrm{PSU}(3,8),[2^9]{:}\mathbb{Z}_7),(\mathrm{PSU}(5,2),\mathbb{Z}_3^4{:}\mathrm{A}_5)$ |
| [11, Table 4.5]: | $(\Omega^+(8,2),[2^9]{:}\mathrm{PSL}(3,2))$ |

□

Note that $G_\alpha$ induces a transitive permutation group $G_\alpha^{\Gamma(\alpha)}$ of prime degree $r$. Then, by [6, p. 99], $(\mathrm{soc}(G_\alpha^{\Gamma(\alpha)}),r)$ is one of $(\mathbb{Z}_r,r)$, $(\mathrm{A}_r,r)$, $(\mathrm{PSL}(2,11),11)$, $(\mathrm{M}_{11},11)$, $(\mathrm{M}_{23},23)$ and $(\mathrm{PSL}(d,t^f),\frac{t^{fd}-1}{t^f-1})$, where $t$ and $d$ are primes.

5.1. **Graphs with insoluble vertex-stabilizers.** In this subsection, we assume that Hypothesis 5.1 holds, and that $G_\alpha$ is insoluble for some $\alpha\in V$. Let $\beta\in\Gamma(\alpha)$.

Since $G_\alpha$ is insoluble, $G_\alpha^{\Gamma(\alpha)}$ is 2-transitive. By [21], either $G_{\alpha\beta}^{[1]}=1$, or $G_{\alpha\beta}^{[1]}$ is a non-trivial $t$-group and $\mathrm{soc}(G_\alpha^{\Gamma(\alpha)})=\mathrm{PSL}(d,t^f)$. (Note that we deal with the group $\mathrm{A}_5$ as the projective special linear group $\mathrm{PSL}(2,4)$.) It follows from (2.1) that $G_\alpha$ has one or two insoluble composition factors, which are given as follows:

| $\mathrm{soc}(G_\alpha^{\Gamma(\alpha)})$ | $\mathrm{A}_r$ | $\mathrm{PSL}(2,11)$ | $\mathrm{M}_{11}$ | $\mathrm{M}_{23}$ | $\mathrm{PSL}(d,t^f)$ |
|---|---|---|---|---|---|
| Others if exists | $\mathrm{A}_{r-1}$ | $\mathrm{A}_5$ | $\mathrm{A}_6$ | $\mathrm{M}_{22}$ | $\mathrm{PSL}(d-1,t^f)$ |

In particular, if $G_\alpha$ has two insoluble composition factors then they are not isomorphic. Since $G$ is almost simple, $G/T$ is soluble. Note that $G/T \geqslant TG_\alpha/T \cong G_\alpha/(T \cap G_\alpha) = G_\alpha/T_\alpha$. It follows that $G_\alpha/T_\alpha$ is soluble, and then $T_\alpha$ inherits all insoluble composition factors of $G_\alpha$. Combining with Remark 5.2, we next work out those groups $H$ and $K$ listed in [11, Tables 3.1, 3.2, 4.1-4.5 and 5.1] which meet the conditions that $T_\alpha$ satisfies.

**Lemma 5.3.** *Assume that Hypothesis* 5.1 *holds and $G_\alpha$ is insoluble. If $T$ is* (*isomorphic to*) *an alternating group* $\mathrm{A}_n$ *for some $n \geqslant 5$, then one of the following holds.*

(1) $T = \mathrm{A}_n$ *and* $\Gamma = \mathsf{K}_n$, *or* $G \cong \mathrm{S}_n$ *and* $\Gamma = \mathsf{K}_n^{(2)}$.
(2) $G \cong \mathrm{S}_8$ *and* $\Gamma$ *is the point-plane incidence graph of the projective geometry* $\mathrm{PG}(3,2)$.

*Proof.* Assume that $T = \mathrm{A}_n$ for some $n \geqslant 5$. Appealing to [11, Table 3.1] and the Remark 5.2, we conclude that one of the following holds:

(i) $T = \mathrm{A}_n$ for $n \geqslant 6$, $T_\alpha = \mathrm{A}_{n-1}$ and $r = n-1$;
(ii) $T = \mathrm{A}_7$, $T_\alpha = \mathrm{PSL}(2,7)$, $|T : T_\alpha| = 3 \cdot 5$ and $r = 7$;
(iii) $T = \mathrm{A}_8 \cong \mathrm{PSL}(4,2)$, $T_\alpha = \mathbb{Z}_2^3{:}\mathrm{PSL}(3,2)$, $|T : T_\alpha| = 3 \cdot 5$ and $r = 7$;
(iv) $T = \mathrm{A}_8$, $\mathbb{Z}_3 \times \mathrm{A}_5 \leqslant T_\alpha \leqslant (\mathbb{Z}_3 \times \mathrm{A}_5){:}\mathbb{Z}_2$, and $r = 5$;
(v) $T = \mathrm{A}_n$ for $n \geqslant 7$, $T_\alpha = \mathrm{S}_{n-2}$, $|T : T_\alpha| = \frac{n(n-1)}{2}$ and $r = n-2$;
(vi) $T = \mathrm{A}_9$, $T_\alpha = \mathrm{A}_7$, $|T : T_\alpha| = 2^3 \cdot 3^2$ and $r = 7$;
(vii) $T = \mathrm{A}_9$, $T_\alpha = \mathbb{Z}_2^3.\mathrm{PSL}(3,2)$, $|T : T_\alpha| = 3^3 \cdot 5$ and $r = 7$.

Suppose that case (iv) holds. Then $T_\alpha$ has a characteristic subgroup $N = \mathbb{Z}_3$. Since $T_\alpha \trianglelefteq G_\alpha$, we know that $N \trianglelefteq G_\alpha$, and so $\mathrm{O}_3(G_\alpha) \neq 1$. By Lemma 2.4, $\mathrm{O}_3(G_\alpha) \leqslant G_{\alpha\beta}^{[1]}$; however $G_{\alpha\beta}^{[1]}$ is a 2-group by [21], a contradiction.

Suppose that case (v) holds. Then the action of $T$ on each $T$-orbit (on $V$) is equivalent to that on $\Omega = \{\{i,j\} \mid 1 \leqslant i < j \leqslant n\}$ in the natural action of $\mathrm{A}_n$. Thus $|\Gamma(\alpha)|$ should be the length of some $\mathrm{S}_{n-2}$-orbits on $\Omega$, which is equal to 1, $2(n-2)$ or $\frac{(n-2)(n-3)}{2}$. Noting that $|\Gamma(\alpha)|$ is a prime, we get a contradiction.

Suppose that case (vi) or (vii) holds. Then $G = \mathrm{A}_9$ or $\mathrm{S}_9$. Let $\{\alpha,\beta\} \in E$. Considering the natural actions of $G$, $G_\alpha$ and $G_{\alpha\beta}$ on $\Omega = \{1,2,\ldots,9\}$, we conclude that $G_\alpha$ has an orbit say $\Delta$ of length 7 or 8, and $G_{\alpha\beta}$ has an orbit $\Delta'$ on $\Delta$ of length 6 or 8 on $\Omega$, respectively. Then each $g \in \mathbf{N}_G(G_{\alpha\beta})$ fixes both $\Delta'$ and $\Omega \setminus \Delta'$ set-wise. In particular, a 2-element in $\mathbf{N}_G(G_{\alpha\beta})$ fixes a point in $\Omega \setminus \Delta$. Thus $G$ cannot be generated by $G_\alpha$ and any 2-element in $\mathbf{N}_G(G_{\alpha\beta})$, a contradiction.

For cases (i)-(iii), $T$ is 2-transitive on every $T$-orbit. If either $\Gamma$ is not bipartite or the actions of $T$ on its orbits are equivalent, then we have $T = \mathrm{A}_n$ and $\Gamma \cong \mathsf{K}_n$ or $\mathsf{K}_n^{(2)}$. Assume that $\Gamma$ is bipartite and the actions of $T$ on its orbits are not equivalent. For (iii), $G = \mathrm{S}_8 \cong \mathrm{PSL}(4,2).\mathbb{Z}_2$, and the resulting graph $\Gamma$ is the point-plane incidence graph of the projective geometry $\mathrm{PG}(3,2)$. Note that, for $\beta \in \Gamma(\alpha)$, the stabilizers $T_\alpha$ and $T_\beta$ are isomorphic but not conjugate in $T$. For (ii), $G = \mathrm{A}_7$ and $T_\alpha \cong \mathrm{PSL}(3,2)$, again the resulting graph $\Gamma$ is the point-plane incidence graph of $\mathrm{PG}(3,2)$. If (i) occurs then $n = 6$ and, similarly, $T_\alpha$ and $T_\beta$ do not intersect at a subgroup of index 5 in $T_\alpha$, again a contradiction. $\square$

**Lemma 5.4.** *Assume that Hypothesis* 5.1 *holds and $G_\alpha$ is insoluble. If $T$ is a sporadic simple group, then one of the following holds.*

1. $G = \mathrm{M}_{12}$ *and* $\Gamma \cong \mathsf{K}_{12}$, *or* $G = \mathrm{M}_{12}.2$ *and* $\Gamma \cong \mathsf{K}_{12}^{(2)}$, *or* $G = \mathrm{M}_{24}$ *and* $\Gamma \cong \mathsf{K}_{24}$;
2. $\Gamma$ *is isomorphic to one of the graphs given in Example* 3.5.

*Proof.* Assume that $T$ is a sporadic simple group. Then [11, Table 3.2] works here. Combining with the restrictions on $T_\alpha$, we conclude that $(T, T_\alpha)$ is one of $(\mathrm{M}_{11}, \mathrm{PSL}(2, 11))$, $(\mathrm{M}_{12}, \mathrm{M}_{11})$, $(\mathrm{M}_{12}, \mathrm{PSL}(2, 11))$, $(\mathrm{M}_{22}, \mathrm{A}_7)$, $(\mathrm{M}_{24}, \mathrm{M}_{23})$ and $(\mathrm{M}_{23}, \mathbb{Z}_2^4{:}\mathrm{A}_7)$. First, for $(\mathrm{M}_{11}, \mathrm{PSL}(2, 11))$, the resulting graph is $\mathsf{K}_{12}$; for $(\mathrm{M}_{12}, \mathrm{M}_{11})$, the resulting graph is $\mathsf{K}_{12}$ or $\mathsf{K}_{12}^{(2)}$; for $(\mathrm{M}_{24}, \mathrm{M}_{23})$, the resulting graph is $\mathsf{K}_{24}$.

Suppose that $(T, T_\alpha) = (\mathrm{M}_{12}, \mathrm{PSL}(2, 11))$. Then $r = 11$, and $T_{\alpha\beta} \cong \mathrm{A}_5$ for $\beta \in \Gamma(\alpha)$. Assume that $\Gamma$ is bipartite. Then $G = T.2$ and $G_\alpha = T_\alpha$. If $T_\alpha$ is not maximal in $T$ then by Example 3.5 (ii), $\mathbf{N}_G(T_{\alpha\beta}) < T$, and thus $\langle G_\alpha, G_{\{\alpha,\beta\}}\rangle \leqslant \langle G_\alpha, \mathbf{N}_G(T_{\alpha\beta})\rangle \leqslant T \neq G$, which contradicts the connectedness of $\Gamma$. Thus $T_\alpha$ is maximal in $T$, and $\Gamma$ is isomorphic to the graph given in Example 3.5 (1). Now assume that $\Gamma$ is not bipartite. Then $|G : G_\alpha| = |T : T_\alpha| = 144$. If $G \neq T$ then $G_\alpha = T_\alpha.2 \cong \mathrm{PGL}(2, 11)$, which has no transitive permutation representation of degree 11, a contradiction. Therefore, $G = T$, and then $\Gamma$ is isomorphic to the graph given in Example 3.5 (2).

Suppose that $(T, T_\alpha) = (\mathrm{M}_{22}, \mathrm{A}_7)$. Then $r = 7$, $G_\alpha = T_\alpha$ and $G_{\alpha\beta} \cong \mathrm{A}_6$ for $\beta \in \Gamma(\alpha)$. From computation by GAP, we get $\mathbf{N}_G(G_{\alpha\beta}) = G_{\alpha\beta}$, which contradicts Lemma 2.1. Suppose that $(T, T_\alpha) = (\mathrm{M}_{23}, \mathbb{Z}_2^4{:}\mathrm{A}_7)$. Then we have $r = 7$, and $G_\alpha^{\Gamma(\alpha)} = \mathrm{A}_7$ or $\mathrm{S}_7$. By [21], $G_{\alpha\beta}^{[1]} = 1$ for $\beta \in \Gamma(\alpha)$. Then $G_\alpha^{[1]} \cong (G_\alpha^{[1]})^{\Gamma(\beta)} \trianglelefteq (G_\beta^{\Gamma(\beta)})_\alpha \cong (G_\alpha^{\Gamma(\alpha)})_\beta$ and $G_\alpha = (G_\alpha^{[1]})^{\Gamma(\beta)}.G_\alpha^{\Gamma(\alpha)}$, see (2.1). It follows that $G_\alpha = \mathrm{A}_7$, $\mathrm{A}_7 \times \mathrm{A}_6$, $\mathrm{S}_7$, $(\mathrm{A}_7 \times \mathrm{A}_6){:}\mathbb{Z}_2$ or $\mathrm{S}_6 \times \mathrm{S}_7$. Thus $G_\alpha$ has no normal subgroup of the form $\mathbb{Z}_2^4{:}\mathrm{A}_7$, a contradiction. This completes the proof. □

**Lemma 5.5.** *Assume that Hypothesis* 5.1 *holds and $G_\alpha$ is insoluble. If $T$ is a projective special linear group, then one of the following holds.*

1. $\Gamma$ *is isomorphic to one of the graphs in Example* 3.2;
2. $\Gamma$ *is the point-hyperplane incidence graph of* $\mathrm{PG}(n-1, t^e)$, *where $t$ is a prime and* $(n, t^e) = (3, 4)$, $(4, t)$ *or* $(6, 2)$.

*Proof.* Assume that $T$ is a projective special linear group $\mathrm{PSL}(n, s^e)$, where $s$ is a prime. Then [11, Table 4.1] is applicable. If $n = 2$ then, combining with [9, Theorem 1.1], we get part (1) of this lemma.

Assume that $n > 2$ in the following. Then either $(T, T_\alpha)$ is one of $(\mathrm{PSL}(5, 2), \mathbb{Z}_2^6{:}(\mathrm{S}_3 \times \mathrm{PSL}(3, 2)))$ and $(\mathrm{PSL}(4, 3), \mathbb{Z}_2^4{:}\mathrm{S}_5)$, or $\mathrm{soc}(T_\alpha^{\Gamma(\alpha)}) = \mathrm{PSL}(n-1, s^e)$. For $(T, T_\alpha) = (\mathrm{PSL}(5, 2), \mathbb{Z}_2^6{:}(\mathrm{S}_3 \times \mathrm{PSL}(3, 2)))$, we have $r = 7$ and $|V| = 5 \cdot 31$ or $2 \cdot 5 \cdot 31$; however, by [13, Theorem 1.1], no desired graph exists in this case.

Suppose that $(T, T_\alpha) = (\mathrm{PSL}(4, 3), \mathbb{Z}_2^4{:}\mathrm{S}_5)$. Then $|T : T_\alpha| = 3^5 \cdot 13$ is odd, and so $\Gamma$ is bipartite and $G \neq T$. Note that $T_\alpha$ is contained in a maximal subgroup of $G^+$ of odd index. By the Atlas [5], we conclude that $G = \mathrm{PGO}_6^+(3)$, $G^+ = T$ and $G_\alpha = T_\alpha \leqslant M \cong \mathrm{PSU}_4(2){:}\mathbb{Z}_2 \times \mathbb{Z}_2$. Since $|\mathbf{N}_G(T_{\alpha\beta}) : T_{\alpha\beta}|$ is even and $|T : T_{\alpha\beta}| = 3^5 \cdot 5 \cdot 13$, we know that $|G : \mathbf{N}_G(T_{\alpha\beta})|$ is odd. Then $\mathbf{N}_G(T_{\alpha\beta}) \leqslant M \cong \mathrm{PSU}_4(2){:}\mathbb{Z}_2 \times \mathbb{Z}_2$. Thus $\mathbf{N}_G(T_{\alpha\beta}) = \mathbf{N}_M(T_{\alpha\beta}) \cong \mathbf{N}_L(T_{\alpha\beta}) \times \mathbb{Z}_2 \cong T_{\alpha\beta} \times \mathbb{Z}_2$, where $L \cong \mathrm{PSU}_4(2){:}\mathbb{Z}_2$. On the other hand, $\mathbf{N}_G(T_{\alpha\beta}) \geqslant \mathbf{N}_M(T_{\alpha\beta}) \gtrsim T_{\alpha\beta} \times \mathbb{Z}_2$. It follows that $\mathbf{N}_G(T_{\alpha\beta}) = \mathbf{N}_M(T_{\alpha\beta})$, and then $\langle T_\alpha, \mathbf{N}_G(T_{\alpha\beta})\rangle \leqslant M \neq G$, a contradiction.

Suppose next that $n \geqslant 3$ and $\mathrm{soc}(T_\alpha^{\Gamma(a)}) = \mathrm{PSL}(n-1, s^e)$. Then either $n = 3$, $s^e \in \{5, 7, 11\}$ and $r = s$, or $s = t$ and $r = \frac{t^{e(n-1)}-1}{t^e-1}$. Suppose that the former case holds. Then, by [11, Table 4.1] and Remark 5.2, we conclude that $\mathrm{O}_s(T_\alpha) \neq 1$. Noting that $T_\alpha^{\Gamma(\alpha)}$ is almost simple, it follows that $\mathrm{O}_s(T_\alpha) \leqslant T_\alpha^{[1]}$. Since $T_\alpha^{\Gamma(\alpha)} \cong T_\alpha/T_\alpha^{[1]}$, we know that $|T_\alpha|$ has a divisor $r^2$, which contradicts [4, Lemma 1.1]. Therefore, $s = t$ and $r = \frac{t^{e(n-1)}-1}{t^e-1}$. In particular, $n-1$ is a prime as $r$ is a prime. Then either $n = 3$ or $n$ is even.

By [11, Table 4.1] and Remark 5.2, without loss of generality, we may assume that $T_\alpha$ is contained in the stabilizer $M$ in $T$ of some projective point. Suppose that $T_\alpha \neq M$. Then $\frac{t^{en}-1}{t^e-1}$ is a power of some prime by [11, Table 4.1]. It follows that $n$ is a prime, and hence $n = 3$. Then $r = \frac{t^{e(n-1)}-1}{t^e-1} = t^e + 1$. Since $r$ is a prime, we get $t = 2$ and $e = 2^i$ for some positive integer $i$. Thus $\frac{t^{en}-1}{t^e-1} = 2^{2e} + 2^e + 1 = (2^e + \sqrt{2^e} + 1)(2^e - \sqrt{2^e} + 1)$, which is not a prime power, a contradiction. Therefore, $T_\alpha = M$, and $|T : T_\alpha| = \frac{t^{en}-1}{t^e-1}$. In particular, $T$ is 2-transitive on each orbit of $T$. Suppose that $T$ is transitive on $V$. Then $T$ is 2-transitive on $V$, and so $\Gamma$ is a complete graph. Thus $\Gamma$ has valency $\frac{t^{en}-1}{t^e-1} - 1$, and then $\frac{t^{en}-1}{t^e-1} - 1$ is a prime. This yields that $e = 1$ and $n = 2$, a contradiction as $n \geqslant 3$. Therefore, $T$ has two orbits on $V$, and $\Gamma$ is bipartite. If the actions of $T$ on its orbits are equivalent, then $\Gamma$ is the standard double cover of the complete graph of valency $\frac{t^{en}-1}{t^e-1} - 1$, which leads to a similar contradiction as above. Thus $\Gamma$ the actions of $T$ on its orbits are not equivalent. It follows that $\Gamma$ is the point-hyperplane incidence graph of $\mathrm{PG}(n-1, t^e)$.

Recalling that either $n = 3$ or $n$ is even, by [11, Lemme 2.2] and Remark 5.2, either $n \leqslant 4$ or $(n, t^e) = (6, 2)$. Assume that $n = 3$. Then $r = t^e + 1$ and, since $r$ is a prime, $t = 2$ and $e = 2^i$ for some positive integer $i$. In this case, $|T : T_\alpha| = (2^{2^i} + 2^{2^{i-1}} + 1)(2^{2^i} - 2^{2^{i-1}} + 1)$. If $i > 1$ then $2^{2^i} + 2^{2^{i-1}} + 1 = (2^{2^{i-1}} + 2^{2^{i-2}} + 1)(2^{2^{i-1}} - 2^{2^{i-2}} + 1)$, yielding that $|T : T_\alpha|$ has at least three distinct prime divisors, a contradiction. Thus $i = 1$, and $\Gamma$ is the point-line incidence graph of $\mathrm{PG}(2, 4)$. Assume that $n = 4$. Then $r = t^{2e} + t^e + 1$. If $e$ is even then $r = t^{2e} + t^e + 1 = (t^e + \sqrt{t^e} + 1)(t^e - \sqrt{t^e} + 1)$, yielding a contradiction. By [11, Lemme 2.2], we have $e = 1$. It follows that $\Gamma$ is the point-plane incidence graph of $\mathrm{PG}(3, t)$. $\square$

**Lemma 5.6.** *Assume that Hypothesis* 5.1 *holds and $G_\alpha$ is insoluble. If $T$ is a projective symplectic group, then $\Gamma$ is the point-line incidence graph of the generalized quadrangle* $\mathrm{GQ}(4, 2^{2^i})$ *associated with* $\mathrm{PSp}(4, 2^{2^i})$*, where $i \geqslant 1$.*

*Proof.* Assume that $T$ is a projective symplectic group. Check the subgroups $H$ and $K$ in [11, Table 4.2] which have an almost quotient with socle $\mathrm{A}_r$, $\mathrm{PSL}(2, 11)$, $\mathrm{M}_{11}$, $\mathrm{M}_{23}$ or $\mathrm{PSL}(d, t^f)$. Then, recalling that $T_\alpha$ has no two isomorphic insoluble composition factors, we conclude that one of the following holds:

(i) $(T, T_\alpha)$ is one of $(\mathrm{PSp}(n, t), \mathrm{S}_{n+1})$, $(\mathrm{PSp}(n, t), \mathrm{A}_{n+1})$ and $(\mathrm{PSp}(6, 2), \mathbb{Z}_2^6{:}\mathrm{PSL}(3, 2))$, where $(n, t) \in \{(4, 3), (6, 2)\}$;
(ii) $T = \mathrm{PSp}(4, t^f)$, $|T : T_\alpha| = (t^f + 1)(t^{2f} + 1)$, $T_\alpha$ is a point or (isotropic) line stabilizer, $\mathrm{soc}(T_\alpha^{\Gamma(\alpha)}) = \mathrm{PSL}(2, t^f)$ and $r = t^f + 1$.

For the last pair in (i), we have $G = T$, and then $|V| = |T : T_\alpha| = 135$, which is impossible as $\Gamma$ has odd valency. For the other pairs in (i), we have $G = \mathrm{PSp}(n,t).o$ and $G_\alpha = T_\alpha$ or $T_\alpha.o$, where $o \in \{1, 2\}$ and $o = 1$ if $n = 6$. Confirmed by GAP, $\langle G_\alpha, \mathbf{N}_G(G_{\alpha\beta})\rangle < G$, a contradiction.

Assume that (ii) holds. Since $|T : T_\alpha|$ is a product of two prime powers, $f$ is a power of 2. Further, either $f = 1$ and $r = t \in \{5, 7, 11\}$, or $r = t^f + 1$. The former yields that $|T_\alpha|$ is divisible by $r^2$, a contradiction. Thus $r = t^f + 1$, yielding $t = 2$ and $f = 2^i$ for some $i \geqslant 1$ as $r$ is a prime. In particular, $|T : T_\alpha|$ is odd, and $\Gamma$ is bipartite. Suppose the actions of $T$ on its orbits are equivalent. Then $|\Gamma(\alpha)|$ should be one of the subdegrees of $T$ as a primitive rank three group of degree $(t^f + 1)(t^{2f} + 1)$. Thus $|\Gamma(\alpha)| = t^{3f}$ or $t^f(t^f + 1)$, which is not a prime, a contradiction. Then those two actions of $T$ are not equivalent. It follows that $\Gamma$ is the point-line incidence graph of the generalized quadrangle associated with $\mathrm{PSp}(4, t^f)$. □

**Lemma 5.7.** *Assume that Hypothesis* 5.1 *holds and $G_\alpha$ is insoluble. If $T$ is a projective special unitary group, then one of the following holds.*

(1) $\Gamma$ *is isomorphic to the graphs in Examples* 3.3 *and* 3.4*.*
(2) $\Gamma$ *is isomorphic to the Hoffman-Singleton graph or its standard double cover, and* $G = \mathrm{PSU}(3, 5).2$ *for the latter.*

*Proof.* Assume that $T$ is a projective special unitary group. Inspecting [11, Table 4.3] and Remark 5.2, one of the following holds:

(i) $(T, T_\alpha)$ is one of $(\mathrm{PSU}(3, 3), \mathrm{PSL}(2, 7))$, $(\mathrm{PSU}(3, 5), \mathrm{A}_7)$, $(\mathrm{PSU}(5, 2), \mathrm{PSL}(2, 11))$ and $(\mathrm{PSU}(4, 3), \mathrm{A}_7)$;
(ii) $(T, T_\alpha)$ is one of $(\mathrm{PSU}(5, 2), [2^8]{:}(3\times\mathrm{A}_5))$, $(\mathrm{PSU}(5, 2), [2^8]{:}\mathrm{A}_5)$, $(\mathrm{PSU}(5, 2), \mathbb{Z}_3^4{:}\mathrm{S}_5)$ and $(\mathrm{PSU}(5, 2), \mathbb{Z}_3^4{:}\mathrm{A}_5)$;
(iii) $(T, T_\alpha) = (\mathrm{PSU}(3, t^f), \frac{t^f+1}{(t^f+1,3)}.\mathrm{PSL}(2, t^f).(2, t^f + 1))$, $t^f \neq 2$ and $t^{2f} - t^f + 1$ is a power of some prime;
(iv) $T = \mathrm{PSU}(4, t^f)$, $T_\alpha$ is contained in the stabilizer of a totally singular 2-space, $|\mathrm{O}_t(T_\alpha)| = t^{4f}$ and $\mathrm{soc}(T_\alpha^{\Gamma(\alpha)}) = \mathrm{PSL}(2, t^{2f})$;

Assume that $(T, T_\alpha) = (\mathrm{PSU}(3, 3), \mathrm{PSL}(2, 7))$. Then $r = 7$ and $T_{\alpha\beta} \cong \mathrm{S}_4$. Confirmed by GAP, $\mathbf{N}_T(T_{\alpha\beta}) = T_{\alpha\beta}$. By Lemma 2.1, $\Gamma$ is not $T$-symmetric, and then $G = T.2$ and $\Gamma$ is bipartite. Thus $G_\alpha = T_\alpha$ and $G_{\alpha\beta} = T_{\alpha\beta}$. Again by GAP, $\mathbf{N}_G(T_{\alpha\beta}) \cong \mathrm{S}_4 \times \mathbb{Z}_2$. It follows that $\Gamma$ is isomorphic to the graph in Example 3.3.

Assume that $(T, T_\alpha) = (\mathrm{PSU}(3, 5), \mathrm{A}_7)$. If $\Gamma$ is not bipartite then $\Gamma$ is isomorphic to the Hoffman-Singleton graph which has order 50 and valency 7 (see [5, pp. 34]), and either $G = T$ or $(G, G_\alpha) = (\mathrm{PSU}(3, 5).2, \mathrm{S}_7)$. Suppose that $\Gamma$ is bipartite. Then $G = T.2$ and $G_\alpha = T_\alpha \cong \mathrm{A}_7$. By the Atlas [5], $T$ has three conjugacy classes of subgroups $\mathrm{A}_7$, and the subgroups $\mathrm{A}_6$ from these $\mathrm{A}_7$ form three conjugacy classes of subgroups in $T$. Thus two non-conjugate subgroups $\mathrm{A}_7$ do not intersect at a subgroup $\mathrm{A}_6$. It follows that the actions of $T$ on its orbits are equivalent. This implies that $\Gamma$ is in fact the standard double cover of the Hoffman-Singleton graph.

Assume that $(T, T_\alpha) = (\mathrm{PSU}(5, 2), \mathrm{PSL}(2, 11))$. Then $r = 11$ and $T_{\alpha\beta} \cong \mathrm{A}_5$. Computation shows that $\mathbf{N}_T(T_{\alpha\beta}) = T_{\alpha\beta}$. It follows from 2.1 that $G \neq T$. Thus $G = T.2$, and $G_\alpha = T_\alpha$ or $G_\alpha = T_\alpha.2 = \mathrm{PGL}(2, 11)$. Noting that $\mathrm{PGL}(2, 11)$ has no subgroup of index 11, we have $G_\alpha = T_\alpha$. Then $\Gamma$ is isomorphic to the graph in Example 3.4.

We next exclude the remaining cases. First, for the last pair in (i), $T_\alpha$ is maximal in $T$ and $T_{\alpha\beta} \cong \mathrm{A}_6$ for $\beta \in \Gamma(\alpha)$. Let $X = T.\mathrm{D}_8$ with socle $T$. Computation by GAP shows that $\mathbf{N}_X(T_\alpha) \cong \mathrm{S}_7$ and $\mathbf{N}_X(T_{\alpha\beta}) \cong \mathrm{S}_6$, yielding $\mathbf{N}_X(T_{\alpha\beta}) \leqslant \mathbf{N}_X(T_\alpha)$. Noting that $\mathrm{A}_6 \cong T_{\alpha\beta} \trianglelefteq G_{\alpha\beta} \leqslant \mathbf{N}_X(T_{\alpha\beta}) \cong \mathrm{S}_6$, it follows that $T_{\alpha\beta}$ is characteristic in $G_{\alpha\beta}$, and then $T_{\alpha\beta}$ is normal in $\mathbf{N}_G(G_{\alpha\beta})$. Thus $\mathbf{N}_G(G_{\alpha\beta}) \leqslant \mathbf{N}_X(T_{\alpha\beta}) \leqslant \mathbf{N}_X(T_\alpha)$. Noting that $G_\alpha \leqslant \mathbf{N}_X(T_\alpha)$, we have $\langle G_\alpha, \mathbf{N}_G(G_{\alpha\beta})\rangle \leqslant \mathbf{N}_X(T_\alpha) \neq G$, a contradiction.

For the first two pairs in (ii), we have $G = T$ or $T.2$, and thus $|\mathrm{O}_2(G_\alpha)| = 2^8$ or $2^9$. In this case, $r = 5$ and, by [13, Theorem 3.4], either $|\mathrm{O}_2(G_\alpha)| \leqslant 2^6$ or $|\mathrm{O}_2(G_\alpha)| = 2^{20}$, a contradiction. For the second two pairs in (ii), we have $r = 5$ and $\mathrm{O}_3(G_\alpha) \neq 1$. By Lemma 2.4, $\mathrm{O}_3(G_\alpha) \leqslant G_{\alpha\beta}^{[1]}$; however $G_{\alpha\beta}^{[1]}$ is a 2-group by [21], a contradiction.

Suppose that (iii) holds. Then $\mathrm{soc}(T_\alpha^{\Gamma(\alpha)}) = \mathrm{PSL}(2, t^f)$, and either $r = t^f \in \{5, 7, 11\}$ or $r = t^f + 1$. If $r = t^f + 1$ then $|T_\alpha|$ has a divisor $r^2$, a contradiction. If $r = t^f \in \{5, 11\}$ then $t^{2f} - t^f + 1$ is not a prime power. Thus we have $r = t^f = 7$, and then $G_\alpha^{\Gamma(\alpha)} = T_\alpha^{\Gamma(\alpha)} = \mathrm{PSL}(2, 7)$. By [21], $G_{\alpha\beta}^{[1]} = 1$ for $\beta \in \Gamma(\alpha)$. Then $G_\alpha^{[1]} \cong (G_\alpha^{[1]})^{\Gamma(\beta)}$, which is a normal subgroup of $\mathrm{S}_4$. It follows that $\mathrm{O}_2(T_\alpha) \leqslant \mathrm{O}_2(G_\alpha) \leqslant \mathbb{Z}_2^2$. However, case (iii) says that $|\mathrm{O}_2(T_\alpha)| \geqslant 8$, a contradiction.

Suppose that (iv) holds. Then $\mathrm{soc}(T_\alpha^{\Gamma(\alpha)}) = \mathrm{PSL}(2, t^{2f})$, and $r = t^{2f} + 1$. Since $r$ is a prime, $t = 2$ and $f = 2^i$. In this case, by [11, Table 4.3], $T_\alpha$ has index a divisor of $t^f + 1$ in a maximal subgroup $[t^{4f}]{:}\mathrm{PSL}(2, t^{2f}){:}(t^f - 1)$ of $T$. This forces $T_\alpha = [t^{4f}]{:}\mathrm{PSL}(2, t^{2f}){:}(t^f - 1)$, and then $T$ is primitive on each of its orbits. Take $\beta \in \Gamma(\alpha)$. If $G_{\alpha\beta}^{[1]} = 1$ then, since $(G_\alpha^{[1]})^{\Gamma(\beta)} \trianglelefteq (G_\beta^{\Gamma(\beta)})_\alpha \cong (G_\alpha^{\Gamma(\alpha)})_\beta$ and $G_\alpha = (G_\alpha^{[1]})^{\Gamma(\beta)}.G_\alpha^{\Gamma(\alpha)}$ (see (2.1), we have $|\mathrm{O}_t(G_\alpha)| = |\mathrm{O}_t((G_\alpha^{[1]})^{\Gamma(\beta)})| \leqslant t^{2f}$, which is impossible as $[t^{4f}] = \mathrm{O}_t(T_\alpha) \leqslant \mathrm{O}_t(G_\alpha)$. Thus $G_{\alpha\beta}^{[1]} \neq 1$. Then, by [21, Theorem 4.6], $G$ acts transitively on the set of 4-arcs of $\Gamma$. This implies that $\Gamma$ is a graph listed in [10, Tables 1 and 2], which is impossible. $\square$

**Lemma 5.8.** *Assume that Hypothesis* 5.1 *holds and $G_\alpha$ is insoluble. Then $T$ is not a simple orthogonal group of dimension no less than* 7.

*Proof.* Suppose that $T$ is a simple orthogonal group of dimension no less than 7. By [11, Lemma 4.4, 4.5] and Remark 5.2, recalling the limitations on $T_\alpha$, we have

(i) $(T, T_\alpha) = (\Omega(7, 3), \mathbb{Z}_2^6{:}\mathrm{A}_7)$; or
(ii) $(T, T_\alpha) = (\Omega(7, 3), [2^5]{:}\mathrm{S}_5)$; or
(iii) $(T, T_\alpha) = (\Omega^+(8, 2), [2^9]{:}\mathrm{PSL}(3, 2))$; or
(iv) $(T, T_\alpha) = (\Omega(7, t^f), \mathbb{Z}_2 \times \mathrm{PSL}(4, t^f))$.

For (iv), we have $r = \frac{t^{4f}-1}{t^f-1} = (t^f + 1)(t^{2f} + 1)$, which is not a prime. For (i), we have $r = 7$ and $G_{\alpha\beta}^{[1]} = 1$, and so we get $\mathrm{O}_2(G_\alpha) = 1$ by (2.1), a contradiction as $\mathrm{O}_2(G_\alpha) \geqslant \mathrm{O}_2(T_\alpha) \cong \mathbb{Z}_2^6$. Thus case (ii) or (iii) holds. Then $G = T$ or $T.2$. Noting that $|T : T_\alpha|$ is odd and $\Gamma$ has odd valency, we have $G = T.2$ and $G_\alpha = T_\alpha$. In particular, $r = 5$ or 7 and $\mathrm{O}_2(G_\alpha) = [2^5]$ or $[2^9]$ respectively, which is impossible by [13, Theorem 3.4].

$\square$

**Lemma 5.9.** *Assume that Hypothesis* 5.1 *holds and* $G_\alpha$ *is insoluble. Then* $T$ *is not a simple exceptional group of Lie type.*

*Proof.* Suppose that $T$ is a simple exceptional group of Lie type. Then, inspecting [11, Table 5.1], we have $(T, T_\alpha) = (\mathrm{G}_2(3), \mathbb{Z}_2^3{:}\mathrm{PSL}(3,2))$. Thus $r = 7$ and $T_{\alpha\beta} \cong \mathbb{Z}_2^3{:}\mathrm{S}_4$ for some $\beta \in \Gamma(\alpha)$. In this case, $|T : T_\alpha| = 3^5 \cdot 13$, and thus $\Gamma$ is bipartite. It follows that $G = T.2$, $G_\alpha = T_\alpha$ and $G_{\alpha\beta} = T_{\alpha\beta}$. Computation by GAP, we get $\mathbf{N}_G(T_{\alpha\beta}) \leqslant \mathbf{N}_G(T_\alpha)$. This implies that $\langle G_\alpha, \mathbf{N}_G(G_{\alpha\beta})\rangle \leqslant \mathbf{N}_G(T_\alpha) \neq G$, a contradiction. □

5.2. **Graphs with soluble vertex-stabilizers.** Let $\Gamma = (V, E)$ be a connected $G$-symmetric graph of prime valency $r \geqslant 3$. Assume that Hypothesis 5.1 holds, and $G_\alpha$ is soluble. Note that $T_\alpha$ is transitive on $\Gamma(\alpha)$, see Lemma 2.5. By [12, Lemma 7], $T_\alpha^{\Gamma(\alpha)} = \mathbb{Z}_r$ if and only if $T_\alpha = \mathbb{Z}_r$. In particular, if $r = 3$ then $T_\alpha^{\Gamma(\alpha)} \not\cong \mathrm{A}_4$ or $\mathrm{A}_4 \times \mathbb{Z}_2$.

**Lemma 5.10.** *Assume that Hypothesis* 5.1 *holds and* $r = 3$*. Then* $\Gamma$ *is isomorphic to one of* $\mathbf{O}_3$, $\mathbf{O}_3^{(2)}$, *Tutte's* 8*-cage,* F020$A$, F040, F028, F056$B$, F056$C$, F110, F112$A$ *and* F182$D$.

| Row | $T$ | $T_\alpha$ | $\lvert T : T_\alpha\rvert$ |
|---|---|---|---|
| 1 | $\mathrm{A}_5$ | $\mathbb{Z}_3$ | $2^2 \cdot 5$ |
| | | $\mathrm{S}_3$ | $2 \cdot 5$ |
| 2 | $\mathrm{A}_6$ | $\mathrm{S}_4$ | $3 \cdot 5$ |
| 3 | $\mathrm{PSL}(2,25)$ | $\mathrm{S}_4$ | $5^2 \cdot 13$ |
| 4 | $\mathrm{PSL}(2,7)$ | $\mathbb{Z}_3$ | $2^3 \cdot 7$ |
| | | $\mathrm{S}_3$ | $2^2 \cdot 7$ |
| 5 | $\mathrm{PSL}(2,13)$ | $\mathrm{D}_{12}$ | $13 \cdot 7$ |
| 6 | $\mathrm{PSL}(2,11)$ | $\mathrm{D}_{12}$ | $11 \cdot 5$ |

TABLE 5.2.

*Proof.* Note that $G_\alpha \cong \mathbb{Z}_3$, $\mathrm{S}_3$, $\mathrm{D}_{12}$, $\mathrm{S}_4$ or $\mathrm{S}_4{\times}\mathbb{Z}_2$, refer to [1]. Noting $T_\alpha$ is normal in $G_\alpha$, it follows that $T_\alpha \cong \mathbb{Z}_3$, $\mathrm{S}_3$, $\mathrm{D}_{12}$, $\mathrm{S}_4$ or $\mathrm{S}_4{\times}\mathbb{Z}_2$. Checking those soluble subgroups of $T$ given in [11, Tables 3.1, 3.2, 4.1-4.5 and 5.1] and Remark 5.2, $(T, T_\alpha)$ is listed in Table 5.2. Note that $\mathrm{A}_6 \cong \mathrm{PSL}(2,9)$. By [9, Theorem 1.1], the graph $\Gamma$ exists for each case except the case where $(T, T_\alpha) \cong (\mathrm{A}_5, \mathbb{Z}_3)$ or $(\mathrm{PSL}(2,7), \mathbb{Z}_3)$. For these two exceptions, $(G, G_\alpha) \cong (\mathrm{S}_5, \mathbb{Z}_3)$, $(\mathrm{A}_5, \mathbb{Z}_3)$, $(\mathrm{S}_5, \mathrm{S}_3)$, $(\mathrm{PSL}(2,7), \mathbb{Z}_3)$, $(\mathrm{PSL}(2,7).\mathbb{Z}_2, \mathrm{S}_3)$ or $(\mathrm{PSL}(2,7).\mathbb{Z}_2, \mathbb{Z}_3)$. With the help of GAP, every above case gives rise to graphs except the first case. Then the lemma follows from checking those graphs in [3] which has order $|T : T_\alpha|$ or $2|T : T_\alpha|$. Note that the pair $(\mathrm{A}_5, \mathrm{S}_3)$ gives Petersen graph or its standard double cover, and $(\mathrm{A}_6, \mathrm{S}_4)$ gives the Tutte's 8-cage which is isomorphic to the incidence graph of $\mathrm{GQ}(4,2)$. □

**Lemma 5.11.** *Assume that Hypothesis* 5.1 *holds,* $G_\alpha$ *is soluble and* $r \geqslant 5$*. Then* $\Gamma$ *is isomorphic to one of* $\mathsf{K}_n$, $\mathsf{K}_n^{(2)}$ *and the graphs given in Example* 3.1.

*Proof.* In this case, $\mathbb{Z}_r \leqslant G_\alpha^{\Gamma(\alpha)} \leqslant \mathrm{AGL}(1,r)$ and $G_{\alpha\beta}^{[1]} = 1$. By [22, Proposition 2.7], we may write $G_\alpha = \mathbb{Z}_r{:}(\mathbb{Z}_k \times \mathbb{Z}_l)$, where $l$ is a divisor of $r-1$ and $k$ is a divisor of $l$. Checking those soluble subgroups in [11, Tables 3.1, 3.2, 4.1-4.5 and 5.1] and Remark 5.2 which have a normal Sylow $r$-subgroup and abelian Hall $r'$-subgroups,

| Row | $T$ | $T_\alpha$ | $\lvert T:T_\alpha\rvert$ | $r$ |
|---|---|---|---|---|
| 1 | $\mathrm{A}_5$ | $\mathrm{D}_{10}$<br>$\mathbb{Z}_5$ | $2\cdot 3$<br>$2^2\cdot 3$ | 5<br>5 |
| 2 | $\mathrm{A}_6$ | $\mathrm{D}_{10}$<br>$\mathbb{Z}_5$ | $2^2\cdot 3^2$<br>$2^3\cdot 3^2$ | 5<br>5 |
| 3 | $\mathrm{M}_{11}$ | $\mathbb{Z}_{11}{:}\mathbb{Z}_5$ | $2^4\cdot 3^2$ | 11 |
| 4 | $\mathrm{M}_{12}$ | $\mathbb{Z}_{11}{:}\mathbb{Z}_5$ | $2^6\cdot 3^3$ | 11 |
| 5 | $\mathrm{PSL}(2,8)$ | $\mathrm{D}_{14}$ | $2^2\cdot 3^2$ | 7 |
| 6 | $\mathrm{PSL}(3,3)$ | $\mathbb{Z}_{13}{:}\mathbb{Z}_3$<br>$\mathbb{Z}_{13}$ | $2^4\cdot 3^2$<br>$2^4\cdot 3^3$ | 13<br>13 |
| 7 | $\mathrm{PSL}(3,5)$ | $\mathbb{Z}_{31}{:}\mathbb{Z}_3$ | $2^5\cdot 5^3$ | 31 |
| 8 | $\mathrm{PSp}(4,3)$ | $\mathbb{Z}_5{:}\mathbb{Z}_4$<br>$\mathrm{D}_{10}$<br>$\mathbb{Z}_5$ | $2^4\cdot 3^4$<br>$2^5\cdot 3^4$<br>$2^6\cdot 3^4$ | 5<br>5<br>5 |
| 9 | $\mathrm{PSU}(3,3)$ | $\mathbb{Z}_7{:}\mathbb{Z}_3$<br>$\mathbb{Z}_7$ | $2^5\cdot 3^2$<br>$2^5\cdot 3^3$ | 7<br>7 |
| 10 | $\mathrm{PSU}(3,4)$ | $\mathbb{Z}_{13}{:}\mathbb{Z}_3$ | $2^6\cdot 5^2$ | 13 |
| 11 | $\mathrm{PSU}(3,7)$ | $\mathbb{Z}_{43}{:}\mathbb{Z}_3$ | $2^7\cdot 7^3$ | 43 |
| 12 | $\mathrm{PSU}(5,2)$ | $\mathbb{Z}_{11}{:}\mathbb{Z}_5$ | $2^{10}\cdot 3^5$ | 11 |

TABLE 5.3.

either $(T,T_\alpha)$ is given as in Table 5.3, or $T=\mathrm{PSL}(2,t^f)$ and one of the following holds:

(i) $t=2$, $f=2^i>4$, $\mathbb{Z}_{2^f-1}\leqslant T_\alpha\leqslant \mathrm{D}_{2(2^f-1)}$;
(ii) $t=2$, $f$ is an odd prime, $\mathbb{Z}_{2^f+1}\leqslant T_\alpha\leqslant \mathrm{D}_{2(2^f+1)}$;
(iii) $f=1$, $T_\alpha=\mathrm{D}_{t-1}$, $\frac{t+1}{2}$ is a prime power;
(iv) $f=1$, $T_\alpha=\mathrm{D}_{t+1}$, $\frac{t-1}{2}$ is a prime power;
(v) $f=1$, $t=r$, $T_\alpha=\mathbb{Z}_r{:}\mathbb{Z}_l$, where $l$ is a divisor of $\frac{r-1}{2}$.

Assume that $(T,T_\alpha)$ is one of the pairs in Table 5.3. For the pair $(\mathrm{A}_5,\mathrm{D}_{10})$, the graph $\Gamma$ is $\mathsf{K}_6$ or $\mathsf{K}_6^{(2)}$. Let $(T,T_\alpha)\neq(\mathrm{A}_5,\mathrm{D}_{10})$, and let $X$ be an almost simple group with socle $T$. With the help of GAP and the Atlas [5], we first search the subgroups $H$ of $X$ such that $T\not\leqslant H$, $T_\alpha\trianglelefteq H$ and $H$ has a subgroup $L$ of index $r$, and then compute $\mathbf{N}_X(L)$. If $|\mathbf{N}_X(L):L|$ is even and $\mathbf{N}_X(L)\not\leqslant H$ then choose $K\leqslant \mathbf{N}_X(L)$ with $|K:L|=2$ and $T\leqslant\langle H,K\rangle$. By such a process, the pairs in Rows 3, 8 and 12 are excluded, and the remaining pairs produce the scattered graphs in Example 3.1.

Now we deal with the cases (i)-(v). Case (i) yields that $r=2^{2^i}-1$, and thus $r$ is divisible by 3, a contradiction. For case (ii), we have $r=2^f+1$, since $f$ is an odd prime, $r$ is divisible by 3, a contradiction.

Assume that case (iii) occurs. Then $r=\frac{t-1}{2}$ or $\frac{t-1}{4}$. Suppose that $r=\frac{t-1}{2}$. Then $\frac{t+1}{2}=2^e$ for some integer $e\geqslant 1$, and $t=2^{e+1}-1$. Thus $r=\frac{t-1}{2}=2^e-1$, and since $r$ is a prime no less than 5, it follows that $e$ is an odd prime. Then $t=2^{e+1}-1=(2^{\frac{e+1}{2}}-1)(2^{\frac{e+1}{2}}+1)$, which contradicts that $t$ is a prime. Therefore, $r=\frac{t-1}{4}$, and hence $\frac{t+1}{2}=s^a$ for some odd prime $s$ and integer $a\geqslant 1$. Then $t=2s^a-1$, and $r=\frac{s^a-1}{2}$. It follows that $s=3$ and $a$ is an odd prime, in particular, $t\not\equiv\pm1\ (\mathrm{mod}\ 8)$. Noting that $T_{\alpha\beta}\cong\mathbb{Z}_2^2$ for $\beta\in\Gamma(\alpha)$, we have $\mathbf{N}_T(T_{\alpha\beta})\cong \mathrm{A}_4$ by checking the subgroups of $\mathrm{PSL}(2,t)$. It follows from Lemma 2.1, $G\neq T$. Then $G=\mathrm{PGL}(2,t)$ and $\mathbf{N}_G(T_{\alpha\beta})\cong\mathrm{D}_8$. This implies that $\Gamma$ is a bipartite graph and isomorphic one of the graphs given in Example 3.1.

Assume that case (iv) occurs. Then $r = \frac{t+1}{2}$ or $\frac{t+1}{4}$. Suppose that $r = \frac{t+1}{2}$. Then $\frac{t-1}{2} = 2^e$ for some integer $e \geqslant 1$, and $t = 2^{e+1} + 1$. Since $t$ is a prime, $e+1$ is a power of 2. Noting that $r = \frac{t+1}{2} = 2^e + 1$, since $r$ is a prime, $e$ is a power of 2. It follows that $e = 1$, and so $r = 3$, which is not the case. Therefore, $r = \frac{t+1}{4}$, and then $\frac{t-1}{2} = p^e$ for some odd prime $p$ and integer $e \geqslant 1$. Thus $r = \frac{p^e+1}{2}$. Since $r$ is a prime, $e$ is a power of 2. In particular, $t \not\equiv \pm 1 \pmod 8$. Then a similar argument as above yields that $\Gamma$ is constructed as in Example 3.1.

Finally, assume that case (v) holds. If $l = \frac{r-1}{2}$ then $|T : T_\alpha| = t + 1$, yielding $t = 2^a p^b - 1$ for some odd prime $p$, and $\Gamma \cong \mathsf{K}_{t+1}$ or $\mathsf{K}_{t+1}^{(2)}$. Assume $l < \frac{r-1}{2}$. Then $r + 1 = 2^a$ and, since $r$ is a prime, $a$ is an odd prime. Checking the subgroups of $\mathrm{PSL}(2,t)$ and $\mathrm{PGL}(2,t)$, we conclude that $\Gamma \cong (G, H, K)$ with $(G, H, K)$ described as in the last three lines of Table 3.1. $\square$

5.3. **The proof of Theorem 1.2.** Assume that $\Gamma$ and $G$ satisfy the assumptions in Theorem 1.2. Then, by Corollary 4.4, $G$ is almost simple. Thus we may let $\Gamma$, $G$ and $T$ be as in Hypothesis 5.1. Take an edge $\{\alpha, \beta\}$ of $\Gamma$. If $T_\alpha$ is soluble then Theorem 1.2 is true by Lemmas 5.10 and 5.11. Suppose that $T_\alpha$ is insoluble. By Lemmas 5.8 and 5.9, $T$ is neither an orthogonal group of dimension no less than 7 nor an exceptional group of Lie type. Combining with [11, Theorem 1.2], we know that $T$ is one of alternating, sporadic and classical groups (of dimension less than 7). Noting the isomorphisms among finite simple groups, Theorem 1.2 follows from Lemmas 5.3-5.7.

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