TWO-ARC-TRANSITIVE GRAPHS OF ODD ORDER – II

CAI HENG LI, JING JIAN LI, AND ZAI PING LU

Abstract. It is shown that each subgroup of odd index in an alternating group of degree at least 10 has all insoluble composition factors to be alternating. A classification is then given of 2-arc-transitive graphs of odd order admitting an alternating group or a symmetric group. This is the second of a series of papers aiming towards a classification of 2-arc-transitive graphs of odd order.

1. Introduction

Let $\Gamma = (V, E)$ be a graph with vertex set V and edge set E, which is finite, simple and undirected. The number of vertices $|V|$ is called the *order* of the graph. A 2-*arc* in Γ is a triple of distinct vertices (α, β, γ) such that β is adjacent to both α and γ. In general, for an integer $s \geqslant 1$, an *s-arc* is a sequence of $s+1$ vertices with any two consecutive vertices adjacent and any three consecutive vertices distinct. A graph Γ is said to be (G, s)-*arc-transitive* if $G \leqslant \mathsf{Aut}\Gamma$ is transitive on both the vertex set and the set s-arcs of Γ, or simply called *s-arc-transitive*. By the definition, an s-arc-transitive graph is also t-arc-transitive for $1 \leqslant t < s$.

The class of s-arc-transitive graphs has been one of the central topics in algebraic graph theory since Tutte's seminal result [18]: there is no 6-arc-transitive cubic graph, refer to [17, 19] and [1, 4, 5, 7, 8, 10, 12, 13, 15], and references therein. A great achievement in the area was due to Weiss [19] who proved that there is no 8-arc-transitive graph of valency at least 3. Later in [9], the first named author proved that there is no 4-arc-transitive graph of odd order. Moreover, it was shown in [9] that an s-arc-transitive graph of odd order with $s = 2$ or 3 is a normal cover of some $(G, 2)$-arc-transitive graph where G is an almost simple group, led to the problem:

Classify $(G, 2)$-arc-transitive graphs of odd order with G almost simple.

This is one of a series of papers aiming to solve this problem, and does this work for alternating groups and symmetric groups. The first one [11] of the series of papers solves the problem for the exceptional groups of Lie type, and the sequel will solve the problem for other families of almost simple groups.

Let $\Gamma = (V, E)$ be a connected $(G, 2)$-arc-transitive graph of odd order, where G is an almost simple group with socle being an alternating group. For the case where G is primitive on V, it is easily deduced from [16] that Γ is one of the complete graphs and the odd graphs. The main result of this paper shows that these are all the graphs we expected.

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Theorem 1.1. *Let G be an almost simple group with socle being an alternating group A_n, and let Γ be a connected $(G,2)$-arc-transitive graph of odd order. Then either*

(i) *Γ is the complete graph $\mathbf{K}_n$, and n is odd; or*
(ii) *Γ is the odd graph $\mathbf{O}_{2^e-1}$, and $n=\binom{2^{e+1}-1}{2^e-1}$ for some integer $e\geqslant 2$.*

Remark. It would be infeasible to extend the classification in Theorem 1.1 to those graphs of even order. This is demonstrated by the work of Praeger-Wang in [16] which presents a description of $(G,2)$-arc-transitive and G-vertex-primitive graphs with socle of G being an alternating group.

As a byproduct, the following result shows that subgroups of alternating groups and symmetric groups of odd index are very restricted: each insoluble composition factor is alternating except for three small exceptions.

Theorem 1.2. *Let G be an almost simple group with socle A_n, and let H be an insoluble proper subgroup of G of odd index. Then $G\in\{\mathrm{A}_n,\mathrm{S}_n\}$ and either*

(i) *every insoluble composition factor of H is an alternating group; or*
(ii) $(G,H)=(\mathrm{A}_7,\mathrm{GL}(3,2))$, $(\mathrm{A}_8,\mathrm{AGL}(3,2))$ *or* $(\mathrm{A}_9,\mathrm{AGL}(3,2))$.

The notation used in the paper is standard, see for example the Atlas [3]. In particular, a positive integer n sometimes denotes a cyclic group of order n, and for a prime p, the symbol p^n sometimes denotes an elementary abelian p-group. For groups A and B, an upward extension of A by B is denoted by $A.B$, and a semi-direct product of A by B is denoted by $A{:}B$.

For a positive integer n and a prime p, let n_p denote the p-part of n, that is, $n=n_pn'$ such that n_p is a power of p and $\gcd(n_p,n')=1$. For a subgroup H of a group G, let $|G:H|=|G|/|H|$, the index of H in G, and denote by $\mathbf{N}_G(H)$ and $\mathbf{C}_G(H)$ the normalizer and the centralizer of H in G, respectively.

2. Examples

We study the graphs which appear in our classification.

It is easily shown that, for an integer $n\geqslant 3$, the complete graph $\mathbf{K}_n$ is $(G,2)$-arc-transitive if and only if G is a 3-transitive permutation group of degree n. Thus, if $n\geqslant 5$ is odd then $\mathbf{K}_n$ is one of the desired graphs.

The second type of example is the odd graph, defined below.

Example 2.1. Let $\Omega=\{1,2,\dots,2m+1\}$, and let $\Omega^{\{m\}}$ consist of m-subsets of Ω. Define a graph (V,E) with vertex set and edge set

$$V=\Omega^{\{m\}},\ E=\{(\alpha,\beta)\mid\alpha\cap\beta=\emptyset\},$$

respectively, which is called an *odd graph* and denoted by $\mathbf{O}_m$.

The graph $\mathbf{O}_m$ has valency $m+1$, and has $\mathrm{Sym}(\Omega)=\mathrm{S}_{2m+1}$ to be the automorphism group, see [6, pp. 147, Corollary 7.8.2]. The order of $\mathbf{O}_m$ is given by

$$|V|=|\Omega^{\{m\}}|=\binom{2m+1}{m}=\frac{(2m+1)!}{m!(m+1)!}.$$

For example, the Petersen graph is $\mathbf{O}_2$, which has order $\binom{5}{2}=10$ and valency 3; $\mathbf{O}_3$ has order $\binom{7}{3}=35$ and valency 4. The former has even order, and the latter has odd order. We next give a necessary and sufficient condition for $\binom{2m+1}{m}$ to be odd.

For a positive integer n, letting $2^{t+1}>n\geqslant 2^t$ for some integer $t\geqslant 0$, set

$$s(n)=\left[\frac{n}{2}\right]+\left[\frac{n}{2^2}\right]+\cdots+\left[\frac{n}{2^i}\right]+\cdots+\left[\frac{n}{2^t}\right],$$

where $[x]$ is the largest integer which is not larger than x. Then $[\frac{n}{2^i}]$ is the number of integers in $\{1,2,\ldots,n\}$ which are divisible by 2^i, and it follows that the 2-part of $n!$ is equal to $2^{s(n)}$. Clearly, $2^{s(n)}=2^{s(n-1)}n_2$ if $n\geqslant 2$, where n_2 is the 2-part of n. We observe that $[\frac{m}{2^i}]+[\frac{n}{2^i}]\leqslant[\frac{m+n}{2^i}]$ for all positive integers i. It follows that

$$s(m)+s(n)\leqslant s(m+n), \tag{2.1}$$

and

$$s(m)+s(n)=s(m+n)\Longleftrightarrow\left[\frac{m}{2^i}\right]+\left[\frac{n}{2^i}\right]=\left[\frac{m+n}{2^i}\right]\ \text{for all } i\geqslant 1. \tag{2.2}$$

Further, if $s(m)+s(n)=s(m+n)$ then at least one of n and m is even.

Let $1\leqslant m\leqslant n$ and $\left[\frac{m}{2^i}\right]+\left[\frac{n}{2^i}\right]=\left[\frac{m+n}{2^i}\right]$ for some $i\geq 1$. Suppose that $a:=\left[\frac{m}{2^i}\right]\neq 0$. Then $b:=\left[\frac{n}{2^i}\right]\geqslant a$. Write $m=a2^i+c$ and $n=b2^i+d$ for $c,d<2^i$. We have

$$\left[\frac{m+n}{2^{i+1}}\right]=\left[\frac{a+b}{2}+\frac{c+d}{2^{i+1}}\right]\geqslant\left[\frac{a+b}{2}\right]\geqslant\left[\frac{a}{2}\right]+\left[\frac{b}{2}\right]=\left[\frac{m}{2^{i+1}}\right]+\left[\frac{n}{2^{i+1}}\right].$$

Noting that $\left[\frac{a+b}{2}\right]\geqslant 1$, if $\left[\frac{m+n}{2^{i+1}}\right]=\left[\frac{m}{2^{i+1}}\right]+\left[\frac{n}{2^{i+1}}\right]$ then $b\geqslant 2$, and so $\left[\frac{n}{2^{i+1}}\right]\neq 0$. Then, using (2.1) and (2.2), we have the following lemma.

Lemma 2.2. *Assume that $s(m+n)=s(m)+s(n)$. If $m\leqslant n$ and $\left[\frac{m}{2^i}\right]\neq 0$ then $\left[\frac{n}{2^{i+1}}\right]\neq 0$; in particular, $m<n$, and $n\geqslant 2^t$ if $\left[\frac{m+n}{2^t}\right]\neq 0$.*

The following is a criterion for $\binom{2m+1}{m}$ to be odd.

Lemma 2.3. *The number $\binom{2m+1}{m}=\frac{(2m+1)!}{m!(m+1)!}$ is odd if and only if $m+1$ is a 2-power.*

Proof. Suppose that $\binom{2m+1}{m}$ is odd. Then $s(2m+1)=s(m)+s(m+1)$. Write $2^k\leqslant m<2^{k+1}$. By Lemma 2.2, $\left[\frac{m+1}{2^{k+1}}\right]\neq 0$, yielding $m+1\geqslant 2^{k+1}$, and so $m+1=2^{k+1}$.

Conversely, we assume $m+1=2^\ell$ for some positive integer ℓ. Since $m=2^\ell-1$ and $2m+1=2^{\ell+1}-1$, we obtain

$$\left[\frac{m}{2^i}\right]=\left[\frac{2^\ell-1}{2^i}\right]=\begin{cases}2^{\ell-i}-1, & \text{for } 1\leqslant i\leqslant \ell-1,\\ 0, & \text{for } i\geqslant \ell.\end{cases}$$

$$\left[\frac{2m+1}{2^i}\right]=\left[\frac{2^{\ell+1}-1}{2^i}\right]=\begin{cases}2^{\ell+1-i}-1, & \text{for } 1\leqslant i\leqslant \ell,\\ 0, & \text{for } i\geqslant \ell+1.\end{cases}$$

Therefore, we have

$$\begin{aligned}s(m)&=(2^{\ell-1}-1)+(2^{\ell-2}-1)+\cdots+(2-1),\\ s(m+1)&=2^{\ell-1}+2^{\ell-2}+\cdots+2+1,\\ s(2m+1)&=(2^{\ell+1-1}-1)+(2^{\ell+1-2}-1)+\cdots+(2-1).\end{aligned}$$

Then $s(m) + s(m+1) = s(2m+1)$, and $\binom{2m+1}{m}$ is odd. □

By the above lemma, we get the following consequence.

Corollary 2.4. *The odd graph $\mathbf{O}_m$ is of odd order if and only if $m+1$ is a 2-power.*

3. Subgroups with odd index in A_n or S_n

Let G be an almost simple group with socle A_n. Then either $G \in \{\mathrm{A}_n, \mathrm{S}_n\}$ or $n = 6$ and $G \in \{\mathrm{PGL}(2,9), \mathrm{M}_{10}, \mathrm{P\Gamma L}(2,9)\}$. In this section, we shall determine the insoluble composition factors of subgroups of G of odd index.

For the natural action of S_n on $\Omega = \{1, 2, \ldots, n\}$ and a subset $\Delta \subseteq \Omega$, the symmetric group $\mathrm{Sym}(\Delta)$ is sometimes identified with a subgroup of S_n. Thus we write the set-stabilizer G_Δ as $(\mathrm{Sym}(\Delta) \times \mathrm{Sym}(\Omega \setminus \Delta)) \cap G$ or simply, $G_\Delta = (\mathrm{S}_m \times \mathrm{S}_{n-m}) \cap G$ if $|\Delta| = m$. Also, $(\mathrm{S}_m \wr \mathrm{S}_k) \cap G$ stands for the stabilizer in G of some partition of Ω into k parts with equal size m.

Based on O'Nan-Scott theorem, the following lemma was first obtained by Liebeck and Saxl [14].

Lemma 3.1 ([14])**.** *Let G have socle $T = \mathrm{A}_n$ with $n \geqslant 5$ and have a maximal subgroup M of odd index. Then one of the following holds:*

1. *$M = (\mathrm{S}_m \times \mathrm{S}_{n-m}) \cap G$ with $1 \leqslant m < \frac{n}{2}$; or*
2. *$M = (\mathrm{S}_m \wr \mathrm{S}_k) \cap G$, where $n = mk$ and $m, k > 1$; or*
3. *$G = \mathrm{A}_7$ and $M \cong \mathrm{SL}(3,2)$, or $G = \mathrm{A}_8$ and $M \cong \mathrm{AGL}(3,2)$; or*
4. *$G = \mathrm{PGL}(2,9)$, M_{10} or $\mathrm{P\Gamma L}(2,9)$, and M is a Sylow 2-subgroup of G.*

In particular, if $G \neq \mathrm{A}_7$ or A_8, then each insoluble composition factor of M is an alternating group.

For a subgroup $X \leqslant \mathrm{S}_n$ fixing a subset $\Delta \subseteq \Omega$, denote by X^Δ the permutation group induced by X on Δ.

Lemma 3.2. *Let $G = \mathrm{S}_n$ or A_n with $n \geqslant 5$, and let H be a subgroup of G with odd index $|G : H| > 1$. Suppose that H normalizes a subgroup $L = \mathrm{Sym}(\Delta_1) \times \cdots \times \mathrm{Sym}(\Delta_t)$ of S_n, where $t \geqslant 2$ and $\Omega = \cup_{i=1}^t \Delta_i$. Then*

1. *$|(L \cap G) : (L \cap H)|$ and $|(L \cap G)^{\Delta_i} : (L \cap H)^{\Delta_i}|$ are odd, where $1 \leqslant i \leqslant t$;*
2. *each composition factor of $L \cap H$ is a composition factor of some $(L \cap H)^{\Delta_i}$.*

Proof. By the assumption LH is a subgroup of S_n, and so $H \leqslant LH \cap G = (L \cap G)H \leqslant G$. Thus $|(L \cap G)H : H|$ is odd. Then $|(L \cap G) : (L \cap H)|$ is odd as $|(L \cap G)H : H| = \frac{|L \cap G|}{|L \cap H|}$.

Let L_i be the kernel of $L \cap G$ acting on Δ_i, where $1 \leqslant i \leqslant t$. Then $L^{\Delta_i} \cong L/L_i$, $(L \cap G)^{\Delta_i} \cong (L \cap G)/(L_i \cap G)$ and $(L \cap H)^{\Delta_i} \cong (L \cap H)(L_i \cap G)/(L_i \cap G)$. Since $|(L \cap G) : (L \cap H)|$ is odd, $|(L \cap G) : (L \cap H)(L_i \cap G)|$ is odd, and so is $|(L \cap G)^{\Delta_i} : (L \cap H)^{\Delta_i}|$, as in part (1).

Let S be a composition factor of $L\cap H$. Since $(L\cap H)^{\Delta_t}\cong(L\cap H)(L_t\cap G)/(L_t\cap G)\cong(L\cap H)/(L_t\cap H)$, it follows that S is a composition factor of one of $(L\cap H)^{\Delta_t}$ and $L_t\cap H$. If S is a composition factor of $(L\cap H)^{\Delta_t}$, then part (2) holds by taking $i=t$. Now let S be a composition factor of $L_t\cap H$, and consider the triple $(L_t, L_t\cap G, L_t\cap H)$. By induction, we may assume that S is a composition factor of $(L_t\cap H)^{\Delta_i}$ for some $i\leqslant t-1$. Since $L_t\cap H\trianglelefteq L\cap H$, we have $(L_t\cap H)^{\Delta_i}\trianglelefteq(L\cap H)^{\Delta_i}$, and thus S is a composition factor of $(L\cap H)^{\Delta_i}$. Then part (2) follows. □

Now we prove Theorem 1.2 for $G=\mathrm{S}_n$.

Lemma 3.3. *Let $G=\mathrm{S}_n$ with $n\geqslant 5$, and let H be an insoluble subgroup of G with odd index $|G:H|>1$. Then each insoluble composition factor of H is an alternating group.*

Proof. We prove this lemma by induction on n. Let S be an insoluble composition factor of H. Take a maximal subgroup M of G with $H\leqslant M$. By Lemma 3.1, either $M=\mathrm{S}_m\times\mathrm{S}_{n-m}$ with $1\leqslant m<n/2$, or $M=\mathrm{S}_m\wr\mathrm{S}_k$ with $mk=n$ and $m,k>1$.

For $M=\mathrm{S}_m\times\mathrm{S}_{n-m}$, Lemma 3.2 works for H and M, which yields that S is a composition factor of a subgroup with odd index in S_k for some $k<n$, and the lemma holds by induction. Thus, let $M=\mathrm{S}_m\wr\mathrm{S}_k$ with $mk=n$ and $m,k>1$ in the following.

Let L be the base subgroup of the wreath product $\mathrm{S}_m\wr\mathrm{S}_k$. Then Lemma 3.2 works for the triple $(L,H,L\cap H)$, and hence the lemma holds by induction if S is a composition factor of $L\cap H$.

Assume that S is not a composition factor of $L\cap H$. Then S is a composition factor of $H/(L\cap H)$. Noting that $HL/L\cong H/(L\cap H)$, it implies that S is a composition factor of HL/L. Consider that pair M/L and HL/L. Since $|G:H|$ is odd, $|M:(HL)|$ and hence $|(M/L):(HL/L)|$ is also odd. Further, $M/L\cong\mathrm{S}_k$. Then, since $k<n$, the lemma holds by induction. □

Now we handle the case $G=\mathrm{A}_n$.

Lemma 3.4. *Let $G=\mathrm{A}_n$ with $n\geqslant 5$. Let H be an insoluble subgroup of G with odd index $|G:H|>1$. Then either*

(i) *(G,H) is one of $(\mathrm{A}_7,\mathrm{GL}(3,2))$, $(\mathrm{A}_8,\mathrm{AGL}(3,2))$ and $(\mathrm{A}_9,\mathrm{AGL}(3,2))$; or*
(ii) *every insoluble composition factor of H is an alternating group.*

Proof. If $n\leqslant 9$ then the lemma is easily shown by checking the subgroups of A_n. In the following, by induction on n, we show (ii) of this lemma always holds for $n\geqslant 10$.

Let $n\geqslant 10$, and let S be an insoluble composition factor of H. Take a maximal subgroup M of A_n with $H\leqslant M$. By Lemma 3.1, $M=(\mathrm{S}_m\times\mathrm{S}_{n-m})\cap\mathrm{A}_n$ with $1\leqslant m<n/2$, or $M=(\mathrm{S}_m\wr\mathrm{S}_k)\cap\mathrm{A}_n$ with $mk=n$ and $m,k>1$.

Suppose that $n=10$. Then $M\cong\mathrm{S}_8$ or $2^4{:}\mathrm{S}_5$. By the Atlas [3], S_8 has no insoluble subgroup of odd index. Then $M\cong 2^4{:}\mathrm{S}_5$, and we have $S=\mathrm{A}_5$. Thus, in the following, we let $n\geqslant 11$, and process in two cases.

Case 1. Let $M=(\mathrm{S}_m\times\mathrm{S}_{n-m})\cap\mathrm{A}_n$. If $m=1$ then $M=\mathrm{A}_{n-1}$ and, since $10\leqslant n-1<n$, S is alternating by induction. Now let $m\geqslant 2$. Writing $M=$

$(\mathrm{Sym}(\Delta)\times\mathrm{Sym}(\Omega\setminus\Delta))\cap\mathrm{A}_n$ with $|\Delta|=m$, we have $M=(\mathrm{Alt}(\Delta)\times\mathrm{Alt}(\Omega\setminus\Delta))\langle\sigma_1\sigma_2\rangle$, where $\sigma_1\in\mathrm{Sym}(\Delta)$ and $\sigma_2\in\mathrm{Sym}(\Omega\setminus\Delta)$ are transpositions. Then $M^{\Delta}\cong\mathrm{S}_m$ and $M^{\Omega\setminus\Delta}\cong\mathrm{S}_{n-m}$. By Lemma 3.2, S is a composition factor of a subgroup with odd index in either S_m or S_{n-m}. Then S is alternating by Lemma 3.3.

Case 2. Let $M=(\mathrm{S}_m\wr\mathrm{S}_k)\cap\mathrm{A}_n$. Let $L=\mathrm{S}_m^k$ be the base group of the wreath product $\mathrm{S}_m\wr\mathrm{S}_k$. Note that S is a composition factor of one of $H/(L\cap H)$ and $L\cap H$.

Assume that S is a composition factor of $H/(L\cap H)$. Then S is a composition factor of HL/L as $HL/L\cong H/(L\cap H)$. It is easily shown that $|(M/L):(HL/L)|$ is odd. Further, since $M/L\cong\mathrm{S}_k$, we know that S is alternating by Lemma 3.3.

Now let S be a composition factor of $L\cap H$. Write $L=\mathrm{Sym}(\Delta_1)\times\cdots\times\mathrm{Sym}(\Delta_k)$, where $|\Delta_i|=m$. Then $L\cap\mathrm{A}_n=(\mathrm{Alt}(\Delta_1)\times\cdots\times\mathrm{Alt}(\Delta_k))\langle\sigma_1\sigma_t,\sigma_2\sigma_t,\ldots,\sigma_{t-1}\sigma_t\rangle$, where $\sigma_i\in\mathrm{Alt}(\Delta_i)$ are transpositions. It follows that $(L\cap\mathrm{A}_n)^{\Delta_i}\cong\mathrm{S}_m$ for $1\leqslant i\leqslant t$. Thus, using Lemmas 3.2 and 3.3, S is an alternating group. □

Finally, if $n=6$ and $G=\mathrm{PGL}(2,9)$, M_{10} or $\mathrm{P\Gamma L}(2,9)$ then, by Lemma 3.1, G has no insoluble proper subgroup of odd index. The proof of Theorem 1.2 now follows from Lemmas 3.3 and 3.4.

4. 2-Arc-transitive graphs

In this section, we assume that $\Gamma=(V,E)$ is a connected $(G,2)$-arc-transitive graph of odd order and valency at least 3, where $G\leqslant\mathsf{Aut}\Gamma$.

4.1. **Stabilizers.** Fix a 2-arc (α,β,γ) of Γ. Let G_α be the stabilizer of α in G. Then G_α acts 2-transitively on the neighborhood $\Gamma(\alpha)$ of α in Γ. Let $G_\alpha^{[1]}$ be the kernel of G_α on $\Gamma(\alpha)$, and let $G_\alpha^{\Gamma(\alpha)}$ be the 2-transitive permutation group induced by G_α on $\Gamma(\alpha)$. Then $G_\alpha^{\Gamma(\alpha)}\cong G_\alpha/G_\alpha^{[1]}$. Clearly, $G_\alpha^{[1]}\trianglelefteq G_{\alpha\beta}$, and

$$(G_\alpha^{[1]})^{\Gamma(\beta)}\trianglelefteq G_{\alpha\beta}^{\Gamma(\beta)}\cong G_{\alpha\beta}^{\Gamma(\alpha)}. \tag{4.3}$$

Let $G_{\alpha\beta}^{[1]}=G_\alpha^{[1]}\cap G_\beta^{[1]}$, the point-wise stabilizer of the 'double star' $\Gamma(\alpha)\cup\Gamma(\beta)$. A fundamental result about 2-arc-transitive graphs characterizes $G_{\alpha\beta}^{[1]}$.

Theorem 4.1. (Thompson-Wielandt Theorem) *$G_{\alpha\beta}^{[1]}$ is a p-group with p prime.*

By definition, we have $G_{\alpha\beta}^{[1]}\trianglelefteq G_\beta^{[1]}\trianglelefteq G_{\beta\gamma}$, and so

$$(G_{\alpha\beta}^{[1]})^{\Gamma(\gamma)}\trianglelefteq(G_\beta^{[1]})^{\Gamma(\gamma)}\trianglelefteq G_{\beta\gamma}^{\Gamma(\gamma)}.$$

Let $O_p((G_\beta^{[1]})^{\Gamma(\gamma)})$ and $O_p(G_{\beta\gamma}^{\Gamma(\gamma)})$ be the maximal normal p-subgroups of $(G_\beta^{[1]})^{\Gamma(\gamma)}$ and $G_{\beta\gamma}^{\Gamma(\gamma)}$, respectively. Then

$$(G_{\alpha\beta}^{[1]})^{\Gamma(\gamma)}\trianglelefteq O_p((G_\beta^{[1]})^{\Gamma(\gamma)})\trianglelefteq O_p(G_{\beta\gamma}^{\Gamma(\gamma)}).$$

Suppose that $(G_{\alpha\beta}^{[1]})^{\Gamma(\gamma)}=1$. Then $G_{\alpha\beta}^{[1]}\leqslant G_\gamma^{[1]}$, and so $G_{\alpha\beta}^{[1]}\leqslant G_{\beta\gamma}^{[1]}$. Noting that $G_{\alpha\beta}^{[1]}\cong G_{\beta\gamma}^{[1]}$, we have $G_{\alpha\beta}^{[1]}=G_{\beta\gamma}^{[1]}$. Then the connectedness of Γ yields that $G_{\alpha\beta}^{[1]}=G_{\alpha'\beta'}^{[1]}$ for each arc (α',β') of Γ, and hence $G_{\alpha\beta}^{[1]}=1$. Thus, if $G_{\alpha\beta}^{[1]}$ is a non-trivial

p-group, then so is $(G_{\alpha\beta}^{[1]})^{\Gamma(\gamma)}$, and then $O_p(G_{\beta\gamma}^{\Gamma(\gamma)}) \neq 1$. Noting that $G_{\alpha\beta}^{\Gamma(\alpha)} \cong G_{\beta\gamma}^{\Gamma(\gamma)}$, we have a useful conclusion.

Lemma 4.1. *Let $\{\alpha,\beta\} \in E$. If $G_{\alpha\beta}^{[1]}$ is a nontrivial p-subgroup, then $G_{\alpha\beta}^{\Gamma(\alpha)}$ has a nontrivial normal p-subgroup, where p is a prime.*

Recall that $G_\alpha^{\Gamma(\alpha)}$ is 2-transitive on $\Gamma(\alpha)$. Inspecting 2-transitive permutation groups (refer to [2, page 194-197, Tables 7.3 and 7.4]), we have the following result.

Lemma 4.2. *Let G be an almost simple group with socle A_n, and $\{\alpha,\beta\} \in E$. Then either G_α is soluble, or $G \in \{\mathrm{A}_n, \mathrm{S}_n\}$ and one of the following holds.*

(1) $\mathsf{soc}(G_\alpha^{\Gamma(\alpha)}) \cong \mathrm{A}_m$ *for some $m \geqslant 5$, and one of the following holds:*
   (i) $G_\alpha^{\Gamma(\alpha)} \cong \mathrm{A}_m$ *or* S_m *for even $m \geqslant 6$, and* $G_{\alpha\beta}^{\Gamma(\alpha)} \cong \mathrm{A}_{m-1}$ *or* S_{m-1}, *respectively;*
   (ii) $G_\alpha^{\Gamma(\alpha)} \cong \mathrm{PSL}(2,5)$ *or* $\mathrm{PGL}(2,5)$, *and* $G_{\alpha\beta}^{\Gamma(\alpha)} \cong \mathrm{D}_{10}$ *or* 5:4, *respectively;*
   (iii) $G_\alpha^{\Gamma(\alpha)} \cong \mathrm{PSL}(2,9).\mathcal{O}$, *and* $G_{\alpha\beta}^{\Gamma(\alpha)} \cong 3^2{:}(4.\mathcal{O})$, *where* $\mathcal{O} \leqslant 2^2$.

(2) $G_\alpha^{\Gamma(\alpha)} \cong 2^4{:}H$, *where* $H = G_{\alpha\beta}^{\Gamma(\alpha)} \cong \mathrm{A}_5$, S_5, $3\times\mathrm{A}_5$, $(3\times\mathrm{A}_5).2$, A_6, S_6, A_7 *or* A_8*; in particular,* $G_{\alpha\beta}^{[1]} = 1$.

Proof. Note that

$$G_\alpha = G_\alpha^{[1]}.G_\alpha^{\Gamma(\alpha)} = (G_{\alpha\beta}^{[1]}.(G_\alpha^{[1]})^{\Gamma(\beta)}).G_\alpha^{\Gamma(\alpha)}. \tag{4.4}$$

Clearly, if $G_\alpha^{\Gamma(\alpha)}$ is insoluble then G_α is insoluble. If $G_\alpha^{\Gamma(\alpha)}$ is soluble then, by (4.3), $(G_\alpha^{[1]})^{\Gamma(\beta)}$ is soluble, and so G_α is soluble by (4.4). Thus G_α is soluble if and only if $G_\alpha^{\Gamma(\alpha)}$ is soluble. To finish the proof of this lemma, we assume that G_α is insoluble in the following; in particular, $G \in \{\mathrm{A}_n, \mathrm{S}_n\}$ by Theorem 1.2. Since Γ is $(G,2)$-arc-transitive, $G_\alpha^{\Gamma(\alpha)}$ is an insoluble 2-transitive permutation group. As $|V|$ is odd, the valency $|\Gamma(\alpha)|$ is even, and so $G_\alpha^{\Gamma(\alpha)}$ is of even degree.

Case 1. First assume that $G_\alpha^{\Gamma(\alpha)}$ is an almost simple 2-transitive permutation group with socle S say. By Theorem 1.2, either $S \cong \mathrm{A}_m$ for some $m \geqslant 5$, or one of the following cases occurs:

(a) $G = \mathrm{A}_7$, $G_\alpha = \mathrm{SL}(3,2)$;
(b) $G = \mathrm{A}_8$, $G_\alpha = \mathrm{AGL}(3,2)$;
(c) $G = \mathrm{A}_9$, $G_\alpha = \mathrm{AGL}(3,2)$.

For (a) and (b), we have that $|V| = 15$, and G is 2-transitive on V, yielding $\Gamma \cong \mathbf{K}_{15}$. Noting that Γ is $(G,2)$-arc-transitive, it follows that $G = \mathrm{A}_7$ or A_8 is 3-transitive on the 15 vertices of Γ, which is impossible.

Suppose that (c) occurs. Let $G_\alpha^{\Gamma(\alpha)}$ be of affine type. Then $G_{\alpha\beta} = \mathrm{SL}(3,2)$; in this case, as a subgroup, $\mathrm{SL}(3,2)$ is self-normalized in A_9. Thus there is no element in G interchanging α and β, which contradicts the arc-transitivity of G on Γ. Thus $G_\alpha^{\Gamma(\alpha)}$ is almost simple. Then $G_\alpha^{[1]} = \mathbb{Z}_2^3$ and $G_\alpha^{\Gamma(\alpha)} \cong \mathrm{SL}(3,2) \cong \mathrm{PSL}(2,7)$. Since Γ has even valency, considering the 2-transitive permutation representations of $\mathrm{SL}(3,2)$, we have $|\varGamma(\alpha)|=8$. Then $G_\alpha^{[1]}$ is not faithful on $\varGamma(\beta)\setminus\{\alpha\}$, and so $G_{\alpha\beta}^{[1]}$ is a non-trivial normal 2-group. By Lemma 4.1, $G_{\alpha\beta}^{\varGamma(\alpha)}$ has a non-trivial 2-subgroup; however, $G_{\alpha\beta}^{\varGamma(\alpha)}\cong\mathbb{Z}_7{:}\mathbb{Z}_3$, a contradiction.

Let $S\cong\mathrm{A}_m$. Note that $\mathrm{A}_5\cong\mathrm{PSL}(2,5)$ and $\mathrm{A}_6\cong\mathrm{PSL}(2,9)$. By the classification of 2-transitive permutation groups (refer to [2, page 197, Table 7.4]), since $|\varGamma(\alpha)|$ is even, either $|\varGamma(\alpha)|=m$ with m even, or $(S,|\varGamma(\alpha)|)$ is one of $(\mathrm{PSL}(2,5),6)$ and $(\mathrm{PSL}(2,9),10)$. Then part (1) follows.

Case 2. Now suppose that $G_\alpha^{\varGamma(\alpha)}$ is an insoluble affine group. Then $|\varGamma(\alpha)|=2^d$ for some positive integer $d\geqslant 3$, and $G_{\alpha\beta}^{\varGamma(\alpha)}\leqslant\mathrm{GL}(d,2)$. In particular, by [19], we have $G_{\alpha\beta}^{[1]}=1$. Since each insoluble composition factor of $G_\alpha^{\varGamma(\alpha)}$ is alternating, by the classification of affine 2-transitive permutation groups (see [2, page 195, Table 7.3]), we conclude that $d=4$ and $G_{\alpha\beta}^{\varGamma(\alpha)}$ is isomorphic to one of A_5 (isomorphic to $\mathrm{SL}(2,4)$), S_5 (isomorphic to $\Sigma\mathrm{L}(2,4)$), $\mathbb{Z}_3\times\mathrm{A}_5$ (isomorphic to $\mathrm{GL}(2,4)$), $(\mathbb{Z}_3\times\mathrm{A}_5).2$ (isomorphic to $\Gamma\mathrm{L}(2,4)$), A_6 (isomorphic to $\mathrm{Sp}(4,2)'$), S_6 (isomorphic to $\mathrm{Sp}(4,2)$), A_7 and A_8 (isomorphic to $\mathrm{GL}(4,2)$). This gives rise to the candidates in part (2). □

Let G be an almost simple group with socle A_n. We next organize our analysis of the candidates for G_α according to the description in Lemma 4.2. Note that $G\in\{\mathrm{A}_n,\mathrm{S}_n\}$ if G_α is insoluble.

4.2. **Almost simple stabilizers.** Assume that $G_\alpha^{\varGamma(\alpha)}$ is almost simple, where $\alpha\in V$. First we consider the candidates in Lemma 4.2 (1)(i).

Lemma 4.3. *Let $\{\alpha,\beta\}\in E$. Assume $G_\alpha^{\varGamma(\alpha)}\cong\mathrm{A}_m$ or S_m, and $G_{\alpha\beta}^{\varGamma(\alpha)}\cong\mathrm{A}_{m-1}$ or S_{m-1}, respectively, where $|\varGamma(\alpha)|=m\geqslant 6$ is even. Then one of the following holds:*

- (i) *$(G_\alpha,G)=(\mathrm{A}_m,\mathrm{A}_{m+1})$ or $(\mathrm{S}_m,\mathrm{S}_{m+1})$, and $\varGamma=\mathbf{K}_{m+1}$, where m is even;*
- (ii) *$G_\alpha=(\mathrm{S}_m\times\mathrm{S}_{m-1})\cap G$, $G=\mathrm{A}_{2m-1}$ or S_{2m-1}, respectively, and $\varGamma=\mathbf{O}_{m-1}$, where m is a power of 2.*

Proof. Since $G_{\alpha\beta}^{\varGamma(\alpha)}$ is almost simple, $G_{\alpha\beta}^{[1]}=1$ by Lemma 4.1, and so

$$G_\alpha=G_\alpha^{[1]}.G_\alpha^{\varGamma(\alpha)}=(G_{\alpha\beta}^{[1]}.(G_\alpha^{[1]})^{\varGamma(\beta)}).G_\alpha^{\varGamma(\alpha)}=(G_\alpha^{[1]})^{\varGamma(\beta)}.G_\alpha^{\varGamma(\alpha)}. \tag{4.5}$$

Since $(G_\alpha^{[1]})^{\varGamma(\beta)}$ is isomorphic to a normal subgroup of $G_{\alpha\beta}^{\varGamma(\alpha)}$, we have $(G_\alpha^{[1]})^{\varGamma(\beta)}=1$, or $(G_\alpha^{[1]})^{\varGamma(\beta)}\cong\mathrm{A}_{m-1}$ or S_{m-1}. It follows that $G_\alpha\cong\mathrm{A}_m$, S_m, $\mathrm{A}_{m-1}\times\mathrm{A}_m$, $(\mathrm{A}_{m-1}\times\mathrm{A}_m).2$ or $\mathrm{S}_{m-1}\times\mathrm{S}_m$.

Case 1. Assume first that $G_\alpha\cong\mathrm{A}_m$ or S_m, where m is even. Since $G=\mathrm{A}_n$ or S_n and $|G:G_\alpha|$ is odd, it follows that either $n=m+1$ and $G_\alpha=\mathrm{S}_m\cap G$, or $n=m+k$, $G=\mathrm{A}_{m+k}$ and $G_\alpha\cong\mathrm{S}_m$ for $k\in\{2,3\}$.

Suppose that $n=m+k$, $G=\mathrm{A}_{m+k}$ and $G_\alpha\cong\mathrm{S}_m$, where $k=2$ or 3. Then $G_{\alpha\beta}\cong\mathrm{S}_{m-1}$ since $\varGamma$ is of valency m. Consider the maximal subgroups of $G=\mathrm{A}_{m+k}$ which contains G_α. By Lemma 3.1, we conclude that G_α is contained in the stabilizer of an m-subset of $\varOmega=\{1,2,\ldots,m+k\}$, say $\varDelta=\{1,2,\ldots,m\}$. Thus we may let $G_\alpha=\mathrm{Alt}(\varDelta).\langle\sigma\rangle$, where $\sigma=(1\ 2)(m+1\ m+k)$. Without loss of generality, we may assume that $G_{\alpha\beta} = \mathrm{Alt}(\Delta \setminus \{m\}).\langle\sigma\rangle$. Let $g \in G$ interchange α and β. Then g normalizes $G_{\alpha\beta}$, and hence g fixes $\Delta \setminus \{m\}$ setwise, and $\sigma^g = (i\ j)(m+1\ m+k)$. It follows that Δ and $\{m+1, m+k\}$ are two orbits of $\langle G_\alpha, g\rangle$, which is a contradiction since $\langle G_\alpha, g\rangle$ should be equal to G. Thus $(G_\alpha, G) = (\mathrm{A}_m, \mathrm{A}_{m+1})$ or $(\mathrm{S}_m, \mathrm{S}_{m+1})$. It then follows that $\Gamma = \mathbf{K}_{m+1}$, as in part (i).

Case 2. Now assume that G_α has a subgroup isomorphic to $\mathrm{A}_m \times \mathrm{A}_{m-1}$. Clearly, $n \geqslant 2m-1$. Recall that $2^{s(l)}$ is the 2-part of $l!$, see Section 2. Then $|G|_2 \geqslant 2^{s(n)-1}$ and $|G_\alpha|_2 \leqslant 2^{s(m)+s(m-1)}$. Since $|G : G_\alpha|$ is odd, $s(m) + s(m-1) \geqslant s(n) - 1 \geqslant s(2m-1) - 1$. By (2.1) given in Section 2, $s(2m-1) \geqslant s(m) + s(m-1)$, and so

$$s(2m-1) \geqslant s(m) + s(m-1) \geqslant s(n) - 1 \geqslant s(2m-1) - 1.$$

Since m is even, $2m$ is divisible by 2^2, and hence $s(2m) \geqslant s(2m-1) + 2$. It follows that $n < 2m$. Therefore, we have

$$n = 2m - 1$$

and $s(2m-1) = s(m) + s(m-1)$. Then m is a power of 2 by Lemma 2.3. Since $|G : G_\alpha|$ is odd, either $G = \mathrm{A}_{2m-1}$ and $G_\alpha = (\mathrm{A}_m \times \mathrm{A}_{m-1}).2$, or $G = \mathrm{S}_{2m-1}$ and $G_\alpha = \mathrm{S}_m \times \mathrm{S}_{m-1}$. That is to say, G_α is the stabilizer of G acting on the set of $(m-1)$-subsets of $\{1, 2, \ldots, 2m-1\}$. It follows since Γ is $(G, 2)$-arc-transitive that $\Gamma = \mathbf{O}_{m-1}$ is an odd graph, as in part (ii). □

Next, we handle the candidates in part (1)(ii-iii) of Lemma 4.2.

Lemma 4.4. *There is no 2-arc-transitive graph corresponding to part* (1)(ii) *of Lemma* 4.2.

Proof. Suppose that $G_\alpha^{\Gamma(\alpha)} \cong \mathrm{PSL}(2,5)$ or $\mathrm{PGL}(2,5)$, and $G_{\alpha\beta}^{\Gamma(\alpha)} \cong \mathrm{D}_{10}$ or 5:4. By Lemma 4.1, $G_{\alpha\beta}^{[1]}$ is a 5-group, and so $|G_\alpha^{[1]}|_2 = |(G_\alpha^{[1]})^{\Gamma(\beta)}|_2$ divides $|G_{\alpha\beta}^{\Gamma(\beta)}|_2$. Thus

$$|G_\alpha|_2 = |G_\alpha^{[1]}|_2 |G_\alpha^{\Gamma(\alpha)}|_2 \leqslant 2^5,$$

that is, a Sylow 2-subgroup of G_α has order a divisor of 2^5. It follows that $G \leqslant \mathrm{S}_7$. Since $G_\alpha^{\Gamma(\alpha)} \cong \mathrm{PSL}(2,5)$ or $\mathrm{PGL}(2,5)$, we conclude that either $G = \mathrm{A}_7$ and $G_\alpha \cong \mathrm{S}_5$, or $G = \mathrm{S}_7$ and $G_\alpha = \mathrm{S}_2 \times \mathrm{S}_5$. Then Γ is an orbital graph of $G = \mathrm{S}_7$ acting on 2-subsets of $\{1, 2, \ldots, 7\}$, which is not 2-arc-transitive. □

Lemma 4.5. *There is no 2-arc-transitive graph corresponding to to part* (1)(iii) *of Lemma* 4.2.

Proof. Suppose that $G_\alpha^{\Gamma(\alpha)} \cong \mathrm{PSL}(2,9).\mathcal{O}$, and $G_{\alpha\beta}^{\Gamma(\alpha)} \cong 3^2{:}(4.\mathcal{O})$, where $\mathcal{O} \leqslant 2^2$. By Lemma 4.1, $G_{\alpha\beta}^{[1]}$ is a 3-group, and so $|G_\alpha^{[1]}|_2 = |(G_\alpha^{[1]})^{\Gamma(\beta)}|_2$ divides $|G_{\alpha\beta}^{\Gamma(\beta)}|_2$. We have

$$|G_\alpha|_2 = |G_\alpha^{[1]}|_2 |G_\alpha^{\Gamma(\alpha)}|_2 \leqslant 2^9,$$

that is, a Sylow 2-subgroup of G_α is of order dividing 2^9. It follows that $G \leqslant \mathrm{A}_{13}$, and further, either $G \leqslant \mathrm{S}_{11}$, or G is one of A_{12} and A_{13}.

Suppose $|G|_2 = 2^9$. Then $G = \mathrm{S}_{11}$, A_{12} or A_{13}, and moreover, $G_\alpha^{\Gamma(\alpha)} \cong \mathrm{PSL}(2,9).2^2$ and $G_\alpha^{[1]} \cong 3^2{:}[2^4]$, and hence

$$G_\alpha = (\mathrm{PSL}(2,9) \times (3^2{:}4)).[2^4].$$

By the Atlas [3], G does not have a subgroup of odd index which contains a normal subgroup $\mathrm{PSL}(2,9)\times(3^2{:}4)$, which is a contradiction. Thus $|G|_2\leqslant 2^8$, and then $G\leqslant \mathrm{A}_{11}$ or S_{10}. Checking the subgroups of G with odd index, we conclude that $\mathrm{A}_7\leqslant G\leqslant \mathrm{S}_7$ and $\mathrm{A}_6\leqslant G_\alpha\leqslant \mathrm{S}_6$. It follows that $\Gamma=\mathbf{K}_7$, which is not possible since Γ should have valency 10. □

4.3. **The affine stabilizers.** Let $\{\alpha,\beta\}\in E$. Assume that $G_\alpha^{\Gamma(\alpha)}$ is an affine 2-transitive permutation group.

Now consider the case where G_α is soluble. By [11], Theorem 1.1 holds for the case where G_α is soluble.

Lemma 4.6. *If G_α is soluble, then Γ has valency 4, and either*

(i) *$n=5$ and Γ is the complete graph $\mathbf{K}_5$, or*
(ii) *$n=7$ and Γ is the odd graph $\mathbf{O}_3$ of order 35.*

We now consider the candidates for $G_\alpha^{\Gamma(\alpha)}$ in part (2) of Lemma 4.2.

Lemma 4.7. *There is no 2-arc-transitive graph corresponding to part (2) of Lemma 4.2.*

Proof. Suppose that $G_\alpha^{\Gamma(\alpha)}\cong 2^4{:}H$ is affine and described as in part (2) of Lemma 4.2. Let $\{\alpha,\beta\}\in E$. Since $G_{\alpha\beta}^{[1]}=1$, (4.3) yields that $G_\alpha^{[1]}$ is isomorphic to a normal subgroup of $H=G_{\alpha\beta}^{\Gamma(\alpha)}$. Then the outer automorphism group of $G_\alpha^{[1]}$ has order at most 4. It follows that G_α has a (minimal) normal subgroup N which is regular on $\Gamma(\alpha)$, and thus

$$G_\alpha=N{:}G_{\alpha\beta},\ \mathbf{C}_{G_\alpha}(N)=N\times G_\alpha^{[1]}.$$

Moreover, $|G_\alpha^{[1]}|_2$ is a divisor of $|G_{\alpha\beta}^{\Gamma(\beta)}|_2=|H|_2$, and then $|G|_2=|G_\alpha|_2$ is a divisor of $2^4|H|_2^2$. In particular, $2^6\leqslant|G|_2\leqslant 2^{16}$, and then $8\leqslant n\leqslant 19$.

Consider the natural action of G_α on $\Omega=\{1,2,\dots,n\}$, and choose a G_α-orbit Δ such that N is nontrivial on Δ. Let $|\Delta|=m$. Then m is even, and $|G_\alpha^\Delta|_2=|\mathrm{S}_m|_2$ or $|\mathrm{A}_m|_2$ by Lemma 3.2.

Let K be the kernel of G_α acting on Δ. Then $K\cap N=1$ as N is a minimal normal subgroup of G_α, and so $K\leq \mathbf{C}_{G_\alpha}(N)=N\times G_\alpha^{[1]}$. It follows that $K\leq G_\alpha^{[1]}$, and hence G_α^Δ is insoluble. In particular, $m\geqslant 6$.

Case 1. Suppose that K is soluble. Then $|K|_2=1$, and $2^4|H|_2||G_\alpha^{[1]}|_2=|G_\alpha|_2=|G_\alpha^\Delta|_2=|\mathrm{S}_m|_2$ or $|\mathrm{A}_m|_2$. Recalling that $|G_\alpha|_2=|G|_2=|\mathrm{S}_n|_2$ or $|\mathrm{A}_n|_2$, we have $n\leqslant m+3$. If N is transitive on Δ, then $m=|N|=16$, yielding $|G_\alpha|_2=2^{15}$ or 2^{14}, which is impossible. Thus N is intransitive on Δ, and then $G_\alpha^\Delta\lesssim \mathrm{S}_\ell\wr\mathrm{S}_k$, where $\ell,k>1$, $m=\ell k$ and ℓ is the size of each N-orbit. In particular, $\ell=2$, 4 or 8.

For $\ell=4$ or 8, since $m=\ell k\leqslant n\leqslant 19$, we have $m=16$, which yields a contradiction as above. Therefore, $\ell=2$ and, since G_α^Δ is insoluble, $5\leqslant k\leqslant 9$. Then G_α has exactly one insoluble composition factor, and thus $|G_\alpha|_2=|G_\alpha^\Delta|_2=2^4|H|_2$. This implies that $k=5$, $m=10$, and $|G_\alpha|_2=2^7$ or 2^8. Then $G=\mathrm{A}_{11}$ or A_{10}, and

$G_\alpha = 2^4{:}\mathrm{S}_5$ which is faithful on Δ. Thus $G_{\alpha\beta} \cong \mathrm{S}_5$, which has two orbits on Δ of equal size 5.

Let $g \in G$ with $(\alpha, \beta)^g = (\beta, \alpha)$. Then g normalizes $G_{\alpha\beta}$, fixes $\Omega \setminus \Delta$ and either interchanges or fixes those two $G_{\alpha\beta}$-orbits on Δ. It follows that $g \in G_\alpha$, a contradiction.

Case 2. Suppose that K is insoluble. In this case, G_α is intransitive on Ω, and K has a normal subgroup L isomorphic to A_r, where $r \in \{5, 6, 7, 8\}$. Choose a G_α-orbit Δ' such that L is faithful on Δ'. Then $m' := |\Delta'| \geqslant r$, and $19 \geqslant n \geqslant m+m' \geqslant m+r$.

Note that $2^4|H|_2 \leqslant |G_\alpha^\Delta|_2 \leqslant 2^5|H|_2$, and $|G_\alpha^\Delta|_2 = |\mathrm{S}_m|_2$ or $|\mathrm{A}_m|_2$. If $r = 8$ then $m \geqslant 12$, and so $n \geq m + r \geqslant 20$, a contradiction. Suppose $r = 7$. Then $m \geqslant 8$ and $n \geqslant 15$, and so $|G|_2 \geqslant 2^{10}$. It follows that $|G|_2 = 2^{10}$ and $m = 8$; however, in this case, $G_\alpha^\Delta \cong 2^4{:}\mathrm{A}_7$, which can not be contained in a group isomorphic to S_8. For $r = 6$ and $H \cong \mathrm{A}_6$, we get a similar contradiction as above. Suppose that $r = 6$ and $H \cong \mathrm{S}_6$. Then $2^8 \leqslant |G_\alpha^\Delta|_2 \leqslant 2^9$, and thus $10 \leqslant m \leqslant 13$, yielding $n \geqslant 16$. This leads to $|G_\alpha|_2 \geqslant 2^{14}$, which is impossible.

By the above argument, we have $r = 5$ and $|G_\alpha|_2 = 2^8$, 2^9 or 2^{10}, and then $n \leqslant 15$. On the other hand, $2^6 \leqslant |G_\alpha^\Delta|_2 \leqslant 2^8$, we have $m \leqslant 11$, yielding $m = 10$ and $n = 15$. It follows that $G = \mathrm{A}_{15}$ and $G_\alpha = (\mathrm{Alt}(\Delta') \times 2^4{:}\mathrm{S}_5)\langle\sigma\tau\rangle$, where σ is a transposition in $\mathrm{Sym}(\Delta')$ and τ is a product of five disjoint transpositions in $\mathrm{Sym}(\Delta')$. Then both G_α and $G_{\alpha\beta}$ have two orbits Δ' and Δ on Ω. Thus there is no element $g \in \mathbf{N}_G(G_{\alpha\beta})$ such that $\langle G_\alpha, g\rangle$ is transitive on Ω, a contradiction. □

4.4. **Proof of Theorem 1.1.** Let G be an almost simple group with socle A_n, and let Γ be $(G, 2)$-arc-transitive.

The sufficiency is obvious since the complete graphs $\mathbf{K}_n$ and the odd graphs are clearly 2-arc-transitive under the action of A_n.

The necessity has been established in several lemmas, explained below. By Lemma 4.2, the vertex stabilizer G_α is either soluble or divided into two parts (1)-(2), according to $G_\alpha^{\Gamma(\alpha)}$ being almost simple or affine. For the case where $G_\alpha^{\Gamma(\alpha)}$ is almost simple, Lemmas 4.3-4.5 show that Γ is a complete graph or an odd graph. For the affine case, Lemmas 4.6-4.7 verify the theorem. □

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Cai Heng Li, Department of Mathematics, Southern University of Science and Technoogy, Shenzhen 518055, P. R. China

E-mail address: `lich@sustech.edu.cn`

Jing Jian Li, School of Mathematics and Information Sciences, Guangxi University, Nanning 530004, P. R. China.

E-mail address: `lijjhx@gxu.edu.cn`

Zai Ping Lu, Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, P. R. China

E-mail address: `lu@nankai.edu.cn`