Descent Generating Polynomials and the Hermite–Biehler Theorem

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Abstract. Brenti conjectured that, for any finite Coxeter group, the descent generating polynomial has only real zeros, and he left the type D case open. Dilks, Petersen, and Stembridge proposed a companion conjecture, which states that, for any irreducible finite Weyl group, the affine descent generating polynomial has only real zeros, and they left the type B and type D cases open. By developing the theory of $\mathbf{s}$-Eulerian polynomials, Savage and Visontai confirmed the type D case of the former conjecture and the type B case of the latter conjecture. In this paper, we give an analytic approach to these two combinatorial conjectures. In particular, based on the Hermite–Biehler theorem and the theory of linear transformations preserving Hurwitz stability, we obtain the Hurwitz stability of certain polynomials related to the descent generating polynomials of type D, and thus give an alternative proof of Savage and Visontai's results. This new approach also enables us to prove Hyatt's conjectures on the interlacing property of half Eulerian polynomials of type B and type D, and to prove that the h-polynomial of certain subcomplexes of Coxeter complexes of type D has only real zeros. We further study the Hurwitz stability of certain polynomials related to the affine descent generating polynomials of type D, and completely confirm Dilks, Petersen, and Stembridge's conjecture.

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1 Introduction

Let $\mathfrak{S}_n$ denote the set of permutations of $[n] = \{1, 2, \ldots, n\}$. For $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathfrak{S}_n$, let

$$\mathrm{Des}\,(\sigma) = \{i \in [n-1] : \sigma_i > \sigma_{i+1}\}$$

denote the set of descents of σ, and let $\mathrm{des}\,(\sigma) = |\mathrm{Des}(\sigma)|$. The Eulerian polynomials $S_n(z)$ are usually defined as the descent generating function over $\mathfrak{S}_n$, namely,

$$S_n(z) = \sum_{\sigma \in \mathfrak{S}_n} z^{\mathrm{des}\,(\sigma)}.$$

These polynomials are not only of interest in combinatorics, but also of significance in geometry, see [30]. For example, the coefficients of Eulerian polynomials can be interpreted as the h-vectors of the Coxeter complexes of type A, or as the even Betti numbers of certain toric varieties, see [20, 33, 35].

There are many interesting generalizations of Eulerian polynomials, see [14, 19, 26, 32, 38] and references therein. In this paper, we focus on the study of descent generating polynomials for finite Coxeter groups [14] and affine descent generating polynomials for irreducible finite Weyl groups [19]. The combinatorial aspects and geometric aspects of these polynomials have been extensively studied. Here we will explore their analytic aspects. Frobenius [21] first showed that the classical Eulerian polynomials have only real zeros. Brenti [14] conjectured that the descent generating polynomial for every finite Coxeter group has only real zeros. Dilks, Petersen, and Stembridge [19] conjectured that the affine descent generating polynomial for every irreducible finite Weyl group has only real zeros. Various examples, techniques and developments on unimodality, log-concavity, and real-rootedness in combinatorics can be found in [11, 12, 13, 28, 34]. The main objective of this paper is to use the Hermite–Biehler theorem to study Brenti's conjecture as well as Dilks, Petersen, and Stembridge's conjecture.

Let us first give an overview of Brenti's conjecture and Dilks, Petersen, and Stembridge's conjecture. We assume that the reader is familiar with Coxeter groups and root systems, see [5, 24]. Let W be a finite Coxeter group generated by $\{s_1, s_2, \ldots, s_n\}$. The length of each $\sigma \in W$ is defined as the number of generators in one of its reduced expressions, denoted $\ell(\sigma)$. We say that i is a descent of σ if $\ell(\sigma s_i) < \ell(\sigma)$. Let $\mathrm{Des}\,(\sigma)$ denote the descent set of σ, and let $\mathrm{des}(\sigma) = |\mathrm{Des}\,(\sigma)|$ denote the descent number. Let $W(z)$ denote the descent generating polynomial of W, namely,

$$W(z) \;=\; \sum_{\sigma \in W} z^{\mathrm{des}\,(\sigma)}.$$

In a geometric context, this polynomial is also the h-polynomial of the Coxeter complex of W, for more information see [36, 37]. We use $A_n(z)$ (resp. $B_n(z)$, $C_n(z)$ or $D_n(z)$) to represent $W(z)$ when W is of type A_n (resp. B_n, C_n or D_n). Note that the polynomial $A_n(z)$ is just the classical Eulerian polynomial $S_n(z)$. The following result was conjectured by Brenti [14] and then proved by Savage and Visontai [32].

Theorem 1.1 ([32, Theorem 3.15])**.** *For any finite Coxeter group W, the descent generating polynomial $W(z)$ has only real zeros.*

Dilks, Petersen, and Stembridge [19] proposed a companion conjecture to Brenti's conjecture. Suppose that W is an irreducible finite Weyl group generated by $\{s_1, s_2, \ldots, s_n\}$. Let s_0 be the reflection corresponding to the highest root. For each $\sigma \in W$, we say that i is an affine descent of σ if either $i \in \mathrm{Des}\,(\sigma)$ for $1 \le i \le n$, or $i = 0$ and $\ell(\sigma s_0) > \ell(\sigma)$. Let $\widetilde{\mathrm{Des}}\,(\sigma)$ denote the set of affine descents of σ, and let $\widetilde{\mathrm{des}}\,(\sigma) = |\widetilde{\mathrm{Des}}\,(\sigma)|$. It is worth mentioning that the affine descents were first introduced by Cellini [15] for finite Weyl groups, for further developments see [16, 17, 22, 27, 29]. Analogous to the definition of $W(z)$, let $\widetilde{W}(z)$ to be the affine descent generating polynomial of W, namely

$$\widetilde{W}(z) \;=\; \sum_{\sigma \in W} z^{\widetilde{\mathrm{des}}\,(\sigma)}.$$

Similarly, we use $\widetilde{A}_n(z)$ (resp. $\widetilde{B}_n(z)$, $\widetilde{C}_n(z)$ or $\widetilde{D}_n(z)$) to represent $\widetilde{W}(z)$ when W is of type A_n (resp. B_n, C_n or D_n). Dilks, Petersen, and Stembridge obtained many interesting properties of $\widetilde{W}(z)$, such as a connection with the h-polynomial of the reduced Steinberg torus. They also showed that the affine Eulerian polynomials have unimodal coefficients. Furthermore, Dilks, Petersen, and Stembridge proposed the following conjecture.

Conjecture 1.2 ([19, Conjecture 4.1])**.** *For any irreducible finite Weyl group W, the affine Eulerian polynomial $\widetilde{W}(z)$ has only real zeros.*

Dilks, Petersen, and Stembridge [19] remarked that the affine descent generating polynomials $\widetilde{A}_n(z)$ and $\widetilde{C}_n(z)$ are both multiples of the classical Eulerian polynomial $S_n(z)$ and hence the above conjecture is true for the groups of type A and C, see also [22, 29]. They also computed the affine Eulerian polynomials for all the exceptional groups [19, Table 1], and checked that the above conjecture also holds for these groups. Dilks, Petersen, and Stembridge [19] left the type B and type D cases open. Savage and Visontai's novel approach to Brenti's conjecture also enables them to settle the above conjecture for the groups of type B.

In Section 2 we shall give a brief overview of Savage and Visontai's approach to Brenti's conjecture and Dilks, Petersen, and Stembridge's conjecture. One can see that the notion of interlacing polynomials plays an important role in their work. Given two real-rooted polynomials $f(z)$ and $g(z)$ with positive leading coefficients, let $\{r_i\}$ be the set of zeros of $f(z)$ and $\{s_j\}$ the set of zeros of $g(z)$. We say that $g(z)$ interlaces $f(z)$, denoted $g(z) \preceq f(z)$, if $\deg f(z) = \deg g(z)$ or $\deg f(z) = \deg g(z) + 1$, and

$$\cdots \leq s_2 \leq r_2 \leq s_1 \leq r_1.$$

Savage and Visontai [32] obtained the following result.

Theorem 1.3. *Both $D_n(z)$ and $\widetilde{B}_n(z)$ have only real zeros. Moreover, there holds*

$$D_n(z) \preceq \widetilde{B}_n(z).$$

This paper was motivated by understanding Theorem 1.3 from the viewpoint of the Hermite–Biehler theorem, a basic result in the Routh–Hurwitz theory [31]. Before stating the Hermite–Biehler theorem, let us first recall some related definitions and notations. Let $\mathbb{C}$ denote the field of complex numbers, and let $\mathbb{C}[z]$ denote the set of all polynomials in z with complex coefficients. A polynomial $f(z) \in \mathbb{C}[z]$ is said to be Hurwitz stable (respectively, weakly Hurwitz stable) if $p(z) \neq 0$ whenever $\operatorname{Re} z \geq 0$ (respectively, $\operatorname{Re}, z > 0$), where $\operatorname{Re} z$ denotes the real part of z. A useful criterion for determining stability was given by Hurwitz [25], which we shall explain below.

Given a polynomial $p(z) = \sum_{k=0}^{n} a_{n-k} z^k$, for any $1 \le k \le n$ let

$$\Delta_k(p) = \det \begin{pmatrix} a_1 & a_3 & a_5 & \dots & a_{2k-1} \\ a_0 & a_2 & a_4 & \dots & a_{2k-2} \\ 0 & a_1 & a_3 & \dots & a_{2k-3} \\ 0 & a_0 & a_2 & \dots & a_{2k-r} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_k \end{pmatrix}_{k\times k}.$$

These determinants are known as the Hurwitz determinants of $p(z)$. Hurwitz showed that the stability of $p(z)$ is uniquely determined by the signs of $\Delta_k(p)$.

Theorem 1.4 ([25]). *Suppose that $p(z) = \sum_{k=0}^{n} a_{n-k} z^k$ is a real polynomial with $a_0 > 0$. Then $p(z)$ is Hurwitz stable if and only if the corresponding Hurwitz determinants $\Delta_k(p) > 0$ for any $1 \le k \le n$.*

The above result is usually called the Routh-Hurwitz stability criterion since it is equivalent to the Routh test, for more historical background see [31, p. 393].

Suppose that

$$f(z) = \sum_{k=0}^{n} a_k z^k.$$

Let

$$f^E(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} a_{2k} z^k \quad \text{and} \quad f^O(z) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{2k+1} z^k. \tag{1}$$

The Hermite–Biehler theorem establishes a connection between the Hurwitz stability of $f(z)$ and the interlacing property of $f^E(z)$ and $f^O(z)$.

Theorem 1.5 ([10, Theorem 4.1], [31, p. 197]). *Let $f(z)$ be a polynomial with real coefficients, and let $f^E(z)$ and $f^O(z)$ be defined as in (1). Suppose that $f^E(z)f^O(z) \not\equiv 0$. Then $f(z)$ is weakly Hurwitz stable if and only if $f^E(z)$ and $f^O(z)$ have only real and non-positive zeros, and $f^O(z) \preceq f^E(z)$.*

Combining Theorems 1.3 and 1.5, we know that the polynomial

$$P_n(z) = D_n(z^2) + \frac{1}{2z} \widetilde{B}_n(z^2) \tag{2}$$

is weakly Hurwitz stable. The construction of $P_n(z)$ is based on the relations among $A_n(z), B_n(z), C_n(z), D_n(z)$ and $\widetilde{B}_n(z)$. We are thus motivated to prove the Hurwitz stability of $P_n(z)$ without using the interlacing property of $D_n(z)$ and $\widetilde{B}_n(z)$. This eventually leads to a new proof of Theorem 1.3, which will be given in Section 3. We would like to point out that Borcea and Brändén's work [6] on the characterization of linear operators preserving Hurwitz stability is critical to our approach. This new approach also has some advantages. As will be shown in Section 4, we can use the Hurwitz stability of $P_n(z)$ to prove the interlacing property of the half Eulerian polynomials of type B and type D conjectured by Hyatt [26]. Zaslavsky [41] initiated the study of subcomplexes of Coxeter complexes, which were further studied by Stembridge [36]. In Section 5, we shall investigate the Hurwitz stability of some polynomials closely related to $P_n(z)$, and then prove that the h-polynomial of Coxeter subcomplex of type D has only real zeros.

It is natural to consider whether the real-rootedness of $\widetilde{D}_n(z)$ can be proved in the same manner. Precisely, we aim to look for some Hurwitz stable polynomial $f(z)$ such that $\widetilde{D}_n(z)$ appears as $f^E(z)$ or $f^O(z)$. We noticed the following remarkable identity

$$\widetilde{D}_n(z) = \widetilde{B}_n(z) - 2nz\, D_{n-1}(z), \tag{3}$$

due to Dilks, Petersen, and Stembridge [19]. By inspection of (2) and (3), we are led to study the Hurwitz stability of the polynomial

$$Q_n(z) = 2P_n(z) - 2nzP_{n-1}(z). \tag{4}$$

It is easy to verify that

$$Q_n^E(z) = 2D_n(z) - n\widetilde{B}_{n-1}(z) \quad \text{and} \quad Q_n^O(z) = \frac{\widetilde{D}_n(z)}{z}.$$

Computer experiments suggest that $Q_n(z)$ is weakly Hurwitz stable, which will be proved in Section 6. Although we could not give a proof of the Hurwitz stability of $Q_n(z)$ similar to our proof of the Hurwitz stability of $P_n(z)$, we are able to prove both $Q_n^O(z)$ and $Q_n^E(z)$ have only real and non-positive zeros and moreover $Q_n^O(z) \preceq Q_n^E(z)$. Thus we completely confirm Conjecture 1.2. It should be mentioned that introducing $Q_n(z)$ is essential for proving that $\widetilde{D}_n(z)$ has only real zeros, and its benefits will be clear in Section 6.

2 Savage and Visontai's proof of Theorem 1.3

The aim of this section is to give an overview of Savage and Visontai's proof of Theorem 1.3. We will also recall some related results which will be used in subsequent sections.

Let us begin with the definition of $\mathbf{s}$-inversion sequences. Given a sequence $\mathbf{s} = (s_1, s_2, \ldots)$ of positive integers, an n-dimensional $\mathbf{s}$-inversion sequence is a sequence $\mathbf{e} = (e_1, \ldots, e_n) \in \mathbb{N}^n$ such that $e_i < s_i$ for each $1 \le i \le n$. Denote the set of n-dimensional $\mathbf{s}$-inversion sequences by $\mathfrak{I}_n^{(\mathbf{s})}$. To prove Theorem 1.3, Savage and Visontai [32] introduced a statistic asc_D on inversion sequences $\mathbf{e} = (e_1, \ldots, e_n) \in \mathfrak{I}_n^{(\mathbf{s})}$ for $\mathbf{s} = (2, 4, 6, \ldots)$, which is defined as the cardinality of the following set

$$\mathrm{Asc}_D(\mathbf{e}) = \{i \in [n-1] : \frac{e_i}{i} < \frac{e_{i+1}}{i+1}\} \cup \{0 : \text{if } e_1 + e_2/2 \ge 3/2\}.$$

Savage and Visontai [32] further showed that the descent generating polynomial $D_n(z)$ can be interpreted as the generating function of the statistic asc_D over $\mathfrak{I}_n^{(2,4,6,\ldots)}$, precisely,

$$2D_n(z) = \sum_{\mathbf{e} \in \mathfrak{I}_n^{(2,4,6,\ldots)}} z^{\mathrm{asc}_D(\mathbf{e})}.$$

Let $T_n(z) = 2D_n(z)$. Clearly, $T_n(z)$ has only real zeros if and only if $D_n(z)$ has only real zeros. To prove that $T_n(z)$ has only real zeros, Savage and Visontai introduced the following refinement of $T_n(z)$:

$$T_{n,i}(z) \;=\; \sum_{\mathbf{e} \in \mathfrak{I}_n^{(2,4,6,\ldots)}} \chi(e_n = i)\, z^{\mathrm{asc}_D(\mathbf{e})}\,,$$

where $\chi(\varphi)$ is 1 if the statement φ is true and 0 otherwise. Note that

$$2D_n(z) = T_n(z) = \sum_{i=0}^{2n-1} T_{n,i}(z). \tag{5}$$

They showed that, for any $n \ge 3$ and $0 \le i \le 2n-1$, these refined polynomials satisfy the following simple recurrence relation:

$$T_{n,i}(z) = z \sum_{j=0}^{\lceil \frac{n-1}{n} i \rceil - 1} T_{n-1,j}(z) + \sum_{j=\lceil \frac{n-1}{n} i \rceil}^{2n-3} T_{n-1,j}(z), \tag{6}$$

where $\lceil t \rceil$ represents the smallest integer larger than or equal to t.

By using the theory of compatible polynomials developed by Chudnovsky and Seymour [18], Savage and Visontai inductively proved that the polynomials satisfying such recurrence relations are compatible, and hereby proved that $T_n(z)$ has only real zeros. Let us recall some related concepts. Suppose that $f_1(z), \ldots, f_m(z)$ are polynomials with real coefficients. These polynomials are said to be *compatible* if, for any nonnegative numbers $c_1, \ldots, c_m$, the polynomial

$$c_1 f_1(z) + c_2 f_2(z) + \cdots + c_m f_m(z)$$

has only real zeros, and they are said to be *pairwise compatible* if, for all $1 \leq i < j \leq m$, the polynomials $f_i(z)$ and $f_j(z)$ are compatible. The following remarkable lemma shows that how the two concepts are related.

Lemma 2.1 ([18, Lemma 2.2])**.** *The polynomials $f_1(z), \ldots, f_m(z)$ with positive leading coefficients are pairwise compatible if and only if they are compatible.*

Given a polynomial sequence $(f_1(z), \ldots, f_m(z))$ with real coefficients, define another polynomial sequence $(g_1(z), \ldots, g_{m'}(z))$ by the equations

$$g_k(z) = \sum_{\ell=1}^{t_k - 1} z f_\ell(z) + \sum_{\ell=t_k}^{m} f_\ell(z), \quad \text{for } 1 \leq k \leq m', \tag{7}$$

where $1 \leq t_1 \leq \ldots \leq t_{m'} \leq m+1$. Savage and Visontai obtained the following useful result.

Theorem 2.2 ([32, Theorem 2.3])**.** *Given a sequence of real polynomials $f_1(z), \ldots, f_m(z)$ with positive leading coefficients, let $g_1(z), \ldots, g_{m'}(z)$ be defined as in (7). If, for all $1 \leq i < j \leq m$,*

(1) $f_i(z)$ and $f_j(z)$ are compatible, and

(2) $z f_i(z)$ and $f_j(z)$ are compatible,

then, for all $1 \leq i < j \leq m'$,

(1') $g_i(z)$ and $g_j(z)$ are compatible, and

(2') $zg_i(z)$ and $g_j(z)$ are compatible.

As pointed out by Savage and Visontai, the description of the above theorem can be simplified by using the notion of interlacing if the polynomials $f_1(z), \ldots, f_m(z)$ have only nonnegative coefficients. Interlacing of two polynomials is closely related to compatibility in the sense of the following, due to Wagner [40].

Theorem 2.3 ([40, Lemma 3.4])**.** *Suppose that $f(z)$ and $g(z)$ are two polynomials with nonnegative coefficients. Then the following statements are equivalent:*

(1) $f(z)$ interlaces $g(z)$, namely $f(z) \preceq g(z)$;

(2) $f(z)$ and $g(z)$ are compatible, and $zf(z)$ and $g(z)$ are compatible.

Parallel to the concept of pairwise compatibility, we say that a sequence of real polynomials $f_1(z), \ldots, f_m(z)$ with positive leading coefficients is *pairwise interlacing* if $f_i(z) \preceq f_j(z)$ for all $1 \leq i < j \leq m$. The following result provides an alternative description of Theorem 2.2 when all the polynomials involved have only nonnegative coefficients.

Theorem 2.4. *Given a polynomial sequence $(f_1(z), \ldots, f_m(z))$ with nonnegative coefficients, let $g_1(z), \ldots, g_{m'}(z)$ be polynomials defined as in (7). If $(f_1(z), \ldots, f_m(z))$ is pairwise interlacing, then so is $(g_1(z), \ldots, g_{m'}(z))$.*

Based on (6) and Theorem 2.4, Savage and Visontai obtained the following result.

Theorem 2.5. *For $n \geq 4$, the sequence $(T_{n,0}(z), T_{n,0}(z), \ldots, T_{n,2n-1}(z))$ is pairwise interlacing.*

By (5) and (6) it follows that

$$D_n(z) = \frac{1}{2} T_{n+1,0}(z). \tag{8}$$

Savage and Visontai also proved that

$$\widetilde{B}_n(z) = T_{n+1,n+1}(z). \tag{9}$$

Theorem 1.3 immediately follows from (8), (9), and Theorem 2.5.

3 A new proof of Theorem 1.3

The aim of this section is to give an alternative proof of Theorem 1.3 different from the former approach given by Savage and Visontai. To this end, we need to prove the weak Hurwitz stability of $P_n(z)$ defined by (2) without using the interlacing property of $D_n(z)$ and $\widetilde{B}_n(z)$.

Let us first recall some formulas on the Eulerian polynomials. For the Eulerian polynomials of type A and B, it is known that

$$\frac{A_{n-1}(z)}{(1-z)^{n+1}} = \sum_{i\geq 0}(i+1)^n z^i, \tag{10}$$

and

$$\frac{B_n(z)}{(1-z)^{n+1}} = \sum_{i\geq 0}(2i+1)^n z^i, \tag{11}$$

see [14] and references therein.

By (10) and (11), we have

$$\begin{aligned}(z+1)^{n+1}A_{n-1}(z) &= (1-z^2)^{n+1}\sum_{i\geq 0}(i+1)^n z^i \\ &= (1-z^2)^{n+1}\left(2^n z\sum_{i\geq 0}(i+1)^n z^{2i} + \sum_{i\geq 0}(2i+1)^n z^{2i}\right),\end{aligned}$$

leading to the following identity,

$$(z+1)^{n+1}A_{n-1}(z) = 2^n z A_{n-1}(z^2) + B_n(z^2). \tag{12}$$

It is well known that $A_n(z)$ has only negative real zeros, and hence $(z+1)^{n+1}A_{n-1}(z)$ is weakly Hurwitz stable for any $n \geq 1$. By Theorem 1.5, the identity (12) implies that $B_n(z) \preceq A_{n-1}(z)$.

For the Eulerian polynomials of type D, Stembridge [36, Lemma 9.1] discovered that $D_n(z)$ has a close connection with the Eulerian polynomials of type A and type B:

$$D_n(z) = B_n(z) - n2^{n-1}zA_{n-2}(z). \tag{13}$$

For the affine Eulerian polynomials of type B and type C, Dilks, Petersen, and Stembridge established the following identities:

$$\widetilde{C}_n(z) = 2^n z A_{n-1}(z), \quad \text{(by [19, Corollary 5.7])}$$
$$2\widetilde{C}_n(z) = \widetilde{B}_n(z) + 2nzC_{n-1}(z), \quad \text{(by [19, Proposition 6.1])}$$
$$B_n(z) = C_n(z) \quad \text{(by [19, Proposition 6.3])}.$$

It is readily seen that

$$\widetilde{B}_n(z) = 2z\left(2^n A_{n-1}(z) - nB_{n-1}(z)\right). \tag{14}$$

The first main result of this section is as follows.

Theorem 3.1. *Let $P_n(z)$ be defined by (2). Then for any $n \geq 2$,*

$$P_n(z) = (z+1)^{n+1} A_{n-1}(z) - nz(z+1)^n A_{n-2}(z). \tag{15}$$

Proof. We have

$$\begin{aligned}
P_n(z) =& D_n(z^2) + \frac{1}{2z}\widetilde{B}_n(z^2) \quad \text{(by (2))} \\
=& \big(B_n(z^2) - n2^{n-1}z^2 A_{n-2}(z^2)\big) \quad \text{(by (13))} \\
& + z\big(2^n A_{n-1}(z^2) - nB_{n-1}(z^2)\big) \quad \text{(by (14))} \\
=& \left(2^n z A_{n-1}(z^2) + B_n(z^2)\right) - nz\big(2^{n-1} z A_{n-2}(z^2) + B_{n-1}(z^2)\big) \\
=& (z+1)^{n+1} A_{n-1}(z) - nz(z+1)^n A_{n-2}(z) \quad \text{(by (12))}.
\end{aligned}$$

This completes the proof. $\square$

By the above theorem, to prove the weak Hurwitz stability of $P_n(z)$ it is sufficient to prove the weak Hurwitz stability of

$$\hat{P}_n(z) = (z+1)A_{n-1}(z) - nzA_{n-2}(z). \tag{16}$$

To prove that $\hat{P}_n(z)$ is weakly Hurwitz stable, we shall use a deep theory on linear operators preserving weak Hurwitz stability, which was developed by Borcea and Brändén [6]. The notion of Hurwitz stability admits an extension from univariate polynomials to multivariate polynomials. Let $\mathbb{C}[z_1, z_2, \ldots, z_n]$

denote the set of polynomials in $z_1, z_2, \ldots, z_n$. We say that $f(z_1, z_2, \ldots, z_n) \in \mathbb{C}[z_1, z_2, \ldots, z_n]$ is weakly Hurwitz stable if $f(z_1, z_2, \ldots, z_n) \neq 0$ for all tuples $(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ with $\operatorname{Re} z_i > 0$ for $1 \leq i \leq n$. Let $\mathbb{C}_m[z]$ denote the set of polynomials over $\mathbb{C}$ with degree less than or equal to m. Borcea and Brändén obtained the following characterization theorem. For related results, see [6, 7, 8].

Theorem 3.2 ([7, Theorem 3.3])**.** *Let $m \in \mathbb{N}$ and $T : \mathbb{C}_m[z] \to \mathbb{C}[z]$ be a linear operator. Then T preserves weak Hurwitz stability if and only if either*

(a) T has range of dimension at most one and is of the form $T(f) = \alpha(f)P$, where α is a linear functional on $\mathbb{C}_m[z]$ and P is a weakly Hurwitz stable polynomial, or

(b) The polynomial

$$T[(zw+1)^m] = \sum_{k=0}^{m} \binom{m}{k} T(z^k) w^k$$

is weakly Hurwitz stable in two variables z, w.

The polynomial $T[(zw+1)^m]$ is called the algebraic symbol of the linear operator T.

We proceed to prove the weak Hurwitz stability of $\hat{P}_n(z)$ defined in (16).

Theorem 3.3. *For any positive integer $n \geq 2$ the polynomial $\hat{P}_n(z)$ is weakly Hurwitz stable.*

Proof. It is known that the Eulerian polynomials $A_n(z)$ satisfy the following recurrence relation:

$$\begin{aligned} A_n(z) &= (nz+1)\, A_{n-1}(z) - z(z-1)A'_{n-1}(z) \\ &= (n+1)(zA_{n-1}(z)) - (z-1)(zA_{n-1}(z))', \end{aligned}$$

with the initial condition $A_0(z) = 1$. Thus, we find that

$$\hat{P}_n(z) = nz\,(zA_{n-2}(z)) - \left(z^2 - 1\right)(zA_{n-2}(z))'.$$

This formula could be restated as

$$\hat{P}_n(z) = T(zA_{n-2}(z)),$$

where

$$T = nz - (z^2 - 1)\frac{d}{dz}$$

denotes the operator acting on $\mathbb{C}_n[z]$. It is easy to see that T is a linear operator.

The algebraic symbol of T is given by

$$T[(zw+1)^n] = n(z+w)(zw+1)^{n-1}.$$

It is easy to see that both $(z+w)$ and $(zw+1)$ are weakly Hurwitz stable in variables z, w. Thus $T[(zw+1)^n]$ is weakly Hurwitz stable in variables z, w.

By Theorem 3.2, the linear operator T preserves stability. The weak Hurwitz stability of $\hat{P}_n$ immediately follows from that of $zA_{n-2}(z)$. This completes the proof. $\square$

Now we are to prove Theorem 1.3.

Proof of Theorem 1.3. Combining Theorems 3.1 and 3.3, we get that $P_n(z)$ is weakly Hurwitz stable. Now Theorem 1.3 immediately follows from (2) and Theorem 1.5. $\square$

4 Hyatt's conjectures

In this section we aim to use the Hurwitz stability of $\hat{P}_n(z)$ defined in (16) to prove some conjectures proposed by Hyatt [26] during his study of descent generating polynomials of finite Coxeter groups.

Let us first give an overview of Hyatt's conjectures. Recall that the Coxeter group $\mathfrak{B}_n$ of type B of rank n can be regarded as the group of all bijections σ of the set $\pm[n] = \{\pm 1, \pm 2, \ldots, \pm n\}$ such that $\sigma(-i) = -\sigma(i)$ for all $i \in \pm[n]$. We usually write σ in one-line notation $(\sigma_1, \sigma_2, \ldots, \sigma_n)$, where $\sigma_i = \sigma(i)$. The half Eulerian polynomials of type B are given by

$$B_n^+(z) = \sum_{\sigma \in \mathfrak{B}_n : \sigma_n > 0} z^{\mathrm{des}_B(\sigma)} \quad \text{and} \quad B_n^-(z) = \sum_{\sigma \in \mathfrak{B}_n : \sigma_n < 0} z^{\mathrm{des}_B(\sigma)}.$$

The Coxeter group $\mathfrak{D}_n$ of type D of rank n is composed of those even signed permutations of $\mathfrak{B}_n$. In the same manner, the half Eulerian polynomials of type D are defined as

$$D_n^+(z) \;=\; \sum_{\sigma\in\mathfrak{D}_n:\sigma_n>0} z^{\operatorname{des}_D(\sigma)} \quad \text{and} \quad D_n^-(z) \;=\; \sum_{\sigma\in\mathfrak{D}_n:\sigma_n<0} z^{\operatorname{des}_D(\sigma)}.$$

Hyatt proposed the following conjecture, which has been confirmed by himself in [26].

Conjecture 4.1 ([26, Corollaries 4.6 and 4.8])**.** *(i) For $n \geq 1$, $B_n^+(z)$ interlaces $z^nB_n^+(1/z)$ and thus $B_n(z) = B_n^+(z) + z^nB_n^+(1/z)$ has only real zeros.*

(ii) For $n \geq 2$, $D_n^+(z)$ interlaces $z^nD_n^+(1/z)$ and thus $D_n(z) = D_n^+(z) + z^nD_n^+(1/z)$ has only real zeros.

We proceed to prove Hyatt's conjecture on the half Eulerian polynomials. The following result establishes a connection between the classical Eulerian polynomials and the half-Eulerian polynomials of type B.

Theorem 4.2. *For any $n \geq 1$, we have*

$$(z+1)^nA_{n-1}(z) = B_n^+(z^2) + \frac{1}{z}B_n^-(z^2). \tag{17}$$

Proof. From the equality [3, (7.5)]

$$\frac{B_n^+(z)}{(1-z)^n} = \sum_{i\geq 0}\left((2i+1)^n - (2i)^n\right)z^i \tag{18}$$

and (11) as well as the fact $B_n(z) = B_n^+(z) + B_n^-(z)$, we get that

$$\frac{B_n^-(z)}{(1-z)^n} = \sum_{i\geq 1}\left((2i)^n - (2i-1)^n\right)z^i. \tag{19}$$

By (10), we obtain that

$$\begin{aligned}
(z+1)^n A_{n-1}(z) &= (1-z^2)^n(1-z)\sum_{i\geq 0}(i+1)^n z^i \\
&= (1-z^2)^n\left(\sum_{i\geq 0}(i+1)^n z^i - \sum_{i\geq 0}(i+1)^n z^{i+1}\right) \\
&= (1-z^2)^n\left(\sum_{i\geq 0}(2i+1)^n z^{2i} - \sum_{i\geq 1}(2i)^n z^{2i}\right) \\
&\quad + (1-z^2)^n\left(\sum_{i\geq 1}(2i)^n z^{2i-1} - \sum_{i\geq 1}(2i-1)^n z^{2i-1}\right).
\end{aligned}$$

The desired identity then immediately follows from (18) and (19). $\square$

Note that Athanasiadis and Savvidou [3, Proposition 7.2] obtained that $B_n^+(z)$ is the even part of $(z+1)^n A_{n-1}(z)$. As remarked by Athanasiadis and Savvidou [3, Remark 7.3], similar formula can be derived from [1, Theorem 4.4], see also Athanasiadis [2, Proposition 2.2].

For the half-Eulerian polynomials of type D, we have the following result.

Theorem 4.3. *Let $\hat{P}_n(z)$ be defined as in (16). Then, for any $n \geq 2$,*

$$(z+1)^{n-1}\hat{P}_n(z) = D_n^+(z^2) + \frac{1}{z}D_n^-(z^2). \tag{20}$$

Proof. First, we prove the following identity:

$$\widetilde{B}_n(z) = 2(zD_n^+(z) + D_n^-(z)). \tag{21}$$

Recall that, if each σ of $\underset{\sim}{\mathfrak{B}_n}$ is taken as a signed permutation, then the descent statistics $\mathrm{des}_D\sigma$ and $\widetilde{\mathrm{des}}_B$ have the following combinatorial characterization:

$$\mathrm{des}_D(\sigma) = \chi(\sigma_1+\sigma_2<0) + |\{i\in[n-1] : \sigma_i > \sigma_{i+1}\}|,$$

and

$$\widetilde{\mathrm{des}}_B(\sigma) = \chi(\sigma_1<0) + |\{i\in[n-1] : \sigma_i > \sigma_{i+1}\}| + \chi(\sigma_{n-1}+\sigma_n>0),$$

where $\chi(\cdot)$ is 1 if the statement is true and 0 otherwise.

As shown by Savage and Visontai [32], under the involution

$$(\sigma_1, \sigma_2, \ldots, \sigma_n) \mapsto (-\sigma_n, \ldots, -\sigma_2, -\sigma_1),$$

the statistic $\widetilde{\mathrm{des}}_B$ has the same distribution over $\mathfrak{B}_n$ as the statistic

$$\widetilde{stat}_B\,(\sigma) = \chi(\sigma_1 + \sigma_2 < 0) + |\{i \in [n-1] : \sigma_i > \sigma_{i+1}\}| + \chi(\sigma_n > 0).$$

Hence,

$$\begin{aligned}
\widetilde{B}_n(z) &= \sum_{\sigma \in \mathfrak{B}_n^+} z^{\widetilde{stat}_B\,(\sigma)} + \sum_{\sigma \in \mathfrak{B}_n^-} z^{\widetilde{stat}_B\,(\sigma)} \\
&= z \sum_{\sigma \in \mathfrak{B}_n^+} z^{\mathrm{des}_D \sigma} + \sum_{\sigma \in \mathfrak{B}_n^-} z^{\mathrm{des}_D\,(\sigma)}.
\end{aligned}$$

Taking the involution on $\mathfrak{B}_n$:

$$(\sigma_1, \sigma_2, \ldots, \sigma_n) \mapsto (-\sigma_1, \sigma_2, \ldots, \sigma_n),$$

we get

$$\sum_{\sigma \in \mathfrak{B}_n^+} z^{\mathrm{des}_D\,(\sigma)} = 2D_n^+(z) \quad \text{and} \quad \sum_{\sigma \in \mathfrak{B}_n^-} z^{\mathrm{des}_D\,(\sigma)} = 2D_n^-(z),$$

which can be shown by an elementary but tedious analysis of cases. This completes the proof of (21).

It is known that

$$D_n(z) = D_n^+(z) + D_n^-(z). \tag{22}$$

Therefore,

$$\begin{aligned}
(z+1)^{n-1}\hat{P}_n(z) &= \frac{1}{z+1} P_n(z) \quad \text{(by (15) and (16))} \\
&= \frac{1}{z+1}\left(D_n(z^2) + \frac{1}{2z}\widetilde{B}_n(z^2) \right) \quad \text{(by (2))} \\
&= \frac{1}{z+1}\Big(D_n^+(z^2) + D_n^-(z^2) \quad \text{(by (22))} \\
&\quad + \frac{1}{z}\left(z^2 D_n^+(z^2) + D_n^-(z^2) \right) \Big) \quad \text{(by (21))} \\
&= D_n^+(z^2) + \frac{1}{z} D_n^-(z^2),
\end{aligned}$$

as desired. The proof is complete. □

To prove Hyatt's conjecture, we also need the following identities:

$$B_n^-(z) = z^n B_n^+(1/z), \tag{23}$$

$$D_n^-(z) = z^n D_n^+(1/z), \tag{24}$$

which have been proven by Hyatt [26]. See also [3, Lemma 7.1] for the type B case.

Now we can prove Hyatt's conjecture on half Eulerian polynomials.

Theorem 4.4. *(i) For $n \geq 1$, we have $B_n^+(z) \preceq z^n B_n^+(1/z)$.*

(ii) For $n \geq 2$, we have $D_n^+(z) \preceq z^n D_n^+(1/z)$.

Proof. Let us first prove (i). Since $(z+1)^n A_{n-1}(z)$ has only non-positive real zeros, Theorem 1.5 together with (17) implies that $B_n^+(z) \preceq B_n^-(z)$. By (23), this shows that $B_n^+(z) \preceq z^n B_n^+(1/z)$. The proof is complete.

In the same manner, we can prove (ii). Note that, by Theorem 3.3, the polynomial $(z+1)^{n-1}\hat{P}_n(z)$ is weakly Hurwitz stable. Thus $D_n^+(z) \preceq D_n^-(z)$ by (20) and Theorem 1.5. In view of (24), we get $D_n^+(z) \preceq z^n D_n^+(1/z)$. This completes the proof of (ii). □

5 h-Polynomials of subcomplexes of type D

Given a Coxeter group W, it is known that the h-polynomial of the correpsonding Coxeter complex is just the Eulerian polynomial $W(z)$, see Björner [4, Theorem 2.1]. Zaslavsky [41] first considered the subcomplexes of the Coxeter complex, which are composed of faces fixed by a given group element. It is often the case that these subcomplexes are isomorphic to Coxeter complexes of smaller rank. But that is not the case for the Coxeter groups of type D. Stembridge [36, Lemma 9.1] proved that the h-polynomials of such subcomplexes are of the following form:

$$D_n^{\langle \ell \rangle}(z) = B_n(z) - (n-\ell)2^{n-1} z A_{n-2}(z), \tag{25}$$

where ℓ is a nonnegative integer $\ell \leq n$.

Note that $D_n^{\langle 0 \rangle}(z) = D_n(z)$ and $D_n^{\langle n \rangle}(z) = B_n(z)$. Since both $D_n(z)$ and $B_n(z)$ have only real zeros, it is natural to consider whether $D_n^{\langle \ell \rangle}(z)$ has only real zeros for any $0 \leq \ell \leq n$. The main result of this sections is as follows.

Theorem 5.1. *For any $n \geq 2$ and $0 \leq \ell \leq n$, the polynomial $D_n^{\langle \ell \rangle}(z)$ has only real zeros.*

As in Section 3, we are to prove the above theorem by using the Hermite–Biehler theorem. To this end, we hope that $D_n^{\langle \ell \rangle}(z)$ also appears as $f^E(z)$ or $f^O(z)$ for some stable polynomial $f(z)$ as $D_n(z)$ does in (2). By (12), we see that

$$(z+1)^{n+1}A_{n-1}(z) = D_n^{\langle n \rangle}(z^2) + 2^n z A_{n-1}(z^2).$$

By the proof of Theorem 3.1, we have

$$(z+1)^{n+1}A_{n-1}(z) - nz(z+1)^n A_{n-2}(z) = \\ D_n^{\langle 0 \rangle}(z^2) + z\big(2^n A_{n-1}(z^2) - nB_{n-1}(z^2)\big).$$

Observing the above two identities, we are motivated to consider the weak Hurwitz stability of the polynomial

$$P_n^{\langle \ell \rangle}(z) = D_n^{\langle \ell \rangle}(z^2) + z\left(2^n A_{n-1}(z^2) - (n-\ell)B_{n-1}(z^2)\right). \tag{26}$$

We obtain the following result.

Theorem 5.2. *For any $0 \leq \ell \leq n$, let $P_n^{\langle \ell \rangle}(z)$ be defined as in (26). Then, we have*

$$P_n^{\langle \ell \rangle}(z) = (z+1)^{n+1}A_{n-1}(z) - (n-\ell)z(z+1)^n A_{n-2}(z). \tag{27}$$

Proof. By (25), the left hand side of (27) is equal to

$$\begin{aligned}
&\big(B_n(z^2) - (n-\ell)2^{n-1}z^2 A_{n-2}(z^2)\big) + z\left(2^n A_{n-1}(z^2) - (n-\ell)B_{n-1}(z^2)\right) \\
&= \big(2^n z A_{n-1}(z^2) + B_n(z^2)\big) - (n-\ell)z\big(2^{n-1}zA_{n-2}(z^2) + B_{n-1}(z^2)\big) \\
&= (z+1)^{n+1}A_{n-1}(z) - (n-\ell)z(z+1)^n A_{n-2}(z),
\end{aligned}$$

where the last equality follows from (12). This completes the proof. □

Thus, to prove that $P_n^{\langle \ell \rangle}(z)$ is weakly Hurwitz stable, it is sufficient to prove that

$$\hat{P}_n^{\langle k \rangle}(z) = (z+1)A_{n-1}(z) + kzA_{n-2}(z)$$

is weakly Hurwitz stable for any $k \geq -n$.

Theorem 5.3. *For any positive integer $n \geq 2$ and any real number $k \geq -n$, the polynomial $\hat{P}_n^{\langle k \rangle}(z)$ is weakly Hurwitz stable.*

Proof. Following the proof of Theorem 3.3, it is not hard to show that

$$\hat{P}_n^{\langle k \rangle}(z) = T^{\langle k \rangle}(zA_{n-2}(z)),$$

where

$$T^{\langle k \rangle} = (nz + n + k) - (z^2 - 1)\frac{d}{dz}$$

denotes the linear operator acting on $\mathbb{C}_n[z]$. The algebraic symbol of $T^{\langle k \rangle}$ is given by

$$\begin{aligned} T^{\langle k \rangle}[(zw+1)^n] &= (zw+1)^{n-1}\left((k+n)(zw+1) + n(z+w)\right) \\ &= n(zw+1)^n \left(\frac{z+w}{zw+1} + \frac{k+n}{n}\right). \end{aligned}$$

We claim that

$$\frac{z+w}{zw+1} + \frac{k+n}{n}$$

is weakly Hurwitz stable in variables z, w if $k \geq -n$. To prove this, let

$$z = \frac{x-1}{x+1}, \qquad w = \frac{y-1}{y+1}.$$

Note that $\operatorname{Re} z > 0$ if and only if $|x| > 1$. It is obvious that

$$\frac{z+w}{zw+1} = \frac{xy-1}{xy+1}.$$

If $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$, then $|x| > 1$ and $|y| > 1$, and hence $|xy| > 1$. Therefore, we have $\operatorname{Re} \frac{xy-1}{xy+1} > 0$ and thus $\operatorname{Re} \frac{z+w}{zw+1} > 0$. Moreover, it is clear that $zw + 1 \neq 0$ whenever $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$. It follows that $T^{\langle k \rangle}[(zw+1)^n]$ is weakly Hurwitz stable in variables z, w.

By Theorem 3.2, the linear operator $T^{\langle k \rangle}$ preserves stability. The weak Hurwitz stability of $\hat{P}_n^{\langle k \rangle}(z)$ immediately follows from that of $zA_{n-2}(z)$. This completes the proof. □

Now we can prove Theorem 5.1.

Proof of Theorem 5.1. By Theorems 5.2 and 5.3, for any positive integer $n \geq 2$ and any nonnegative integer $\ell \leq n$, the polynomial $P_n^{\langle \ell \rangle}(z)$ is weakly Hurwitz stable. By (26) and Theorem 1.5, we obtain that $D_n^{\ell}(z)$ has only real zeros. This completes the proof of Theorem 5.1. $\square$

6 Affine descent generating polynomials of type D

The aim of this section is to prove that $\widetilde{D}_n(z)$ has only real zeros for any $n \geq 3$. Our main result is as follows.

Theorem 6.1. *Let $Q_n(z)$ be defined as in (4). Then both $Q_n^O(z) = \widetilde{D}_n(z)/z$ and $Q_n^E(z)$ have only real and non-positive zeros and moreover $Q_n^O(z) \preceq Q_n^E(z)$. Consequently, $Q_n(z)$ is weakly Hurwitz stable.*

Before proving Theorem 6.1, let us first note several lemmas which will be used later. The first result is due to Haglund, Ono, and Wagner [23].

Lemma 6.2 ([23, Lemma 8])**.** *Let $f_1(z), \ldots, f_m(z)$ be real-rooted polynomials with nonnegative coefficients, and let $a_1, \ldots, a_m \geq 0$ and $b_1, \ldots, b_m \geq 0$ be such that $a_i b_{i+1} \geq b_i a_{i+1}$ for all $1 \leq i \leq m-1$. If the sequence $(f_1(z), \ldots, f_m(z))$ is pairwise interlacing, then*

$$\sum_{i=1}^{m} a_i f_i(z) \preceq \sum_{i=1}^{m} b_i f_i(z).$$

The second result is easy to prove, which might be considered a well-known result.

Lemma 6.3 ([9, Lemma 2.3], [39, Proposition 3.5])**.** *Let $g(z)$ and $\{f_i(z)\}_{i=1}^n$ be real-rooted polynomials with positive leading coefficients, and let $F(z) = f_1(z) + f_2(z) + \cdots + f_n(z)$. Then*

(1) if $f_i(z) \preceq g(z)$ for each $1 \leq i \leq n$, then $F(z)$ is real-rooted with $F(z) \preceq g(z)$;

(2) if $g(z) \preceq f_i(z)$ for each $1 \leq i \leq n$, then $F(z)$ is real-rooted with $g(z) \preceq F(z)$.

The third result gives an expression of $Q_n^E(z)$ in terms of $T_{n-1,i}(z)$, as well as that of $Q_n^O(z)$.

Lemma 6.4. *We have*

$$Q_n^E(z) = \sum_{j=0}^{n-2} \big((n-j-1)z + j + 1\big)\big(T_{n-1,j}(z) + T_{n-1,n-1+j}(z)\big), \tag{28}$$

$$zQ_n^O(z) = \sum_{j=0}^{n-2} \big((n-j-1)z + j + 1\big)\big(zT_{n-1,j}(z) + T_{n-1,n-1+j}(z)\big). \tag{29}$$

Proof. By (3), we have

$$\begin{aligned} Q_n^E(z) &= 2D_n(z) - n\widetilde{B}_{n-1}(z) \\ &= T_{n+1,0}(z) - nT_{n,n}(z) \quad \text{(by (8) and (9))} \\ &= \sum_{i=0}^{2n-1} T_{n,i} - nT_{n,n}(z). \quad \text{(by (6))} \end{aligned}$$

Note that, by (6), for each $0 \le i \le n-1$ there holds

$$T_{n,i}(z) = z\sum_{j=0}^{i-1} T_{n-1,j}(z) + \sum_{j=i}^{2n-3} T_{n-1,j}(z),$$

and for each $n \le i \le 2n-1$ there holds

$$T_{n,i}(z) = z\sum_{j=0}^{i-2} T_{n-1,j}(z) + \sum_{j=i-1}^{2n-3} T_{n-1,j}(z).$$

Thus

$$\begin{aligned} Q_n^E(z) = &\sum_{i=0}^{n-1} \left(z\sum_{j=0}^{i-1} T_{n-1,j}(z) + \sum_{j=i}^{2n-3} T_{n-1,j}(z) \right) \\ &+ \sum_{i=n}^{2n-1} \left(z\sum_{j=0}^{i-2} T_{n-1,j}(z) + \sum_{j=i-1}^{2n-3} T_{n-1,j}(z) \right) \\ &- n\left(z\sum_{j=0}^{n-2} T_{n-1,j}(z) + \sum_{j=n-1}^{2n-3} T_{n-1,j}(z) \right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{n-1}\sum_{j=0}^{i-1} zT_{n-1,j}(z) + \sum_{i=0}^{n-1}\sum_{j=i}^{2n-3} T_{n-1,j}(z) \\
&\quad + \sum_{i=n}^{2n-1}\sum_{j=0}^{i-2} zT_{n-1,j}(z) + \sum_{i=n}^{2n-1}\sum_{j=i-1}^{2n-3} T_{n-1,j}(z) \\
&\quad - \sum_{j=0}^{n-2} nzT_{n-1,j}(z) - \sum_{j=0}^{n-2} nT_{n-1,n-1+j}(z).
\end{aligned}
$$

For the above four double summations, we have

$$
\sum_{i=0}^{n-1}\sum_{j=0}^{i-1} zT_{n-1,j}(z) = \sum_{j=0}^{n-2}\sum_{i=j+1}^{n-1} zT_{n-1,j}(z) = \sum_{j=0}^{n-2}(n-j-1)zT_{n-1,j}(z),
$$

$$
\begin{aligned}
\sum_{i=0}^{n-1}\sum_{j=i}^{2n-3} T_{n-1,j}(z) &= \sum_{i=0}^{n-1}\sum_{j=i}^{n-2} T_{n-1,j}(z) + \sum_{i=0}^{n-1}\sum_{j=n-1}^{2n-3} T_{n-1,j}(z) \\
&= \sum_{j=0}^{n-2}\sum_{i=0}^{j} T_{n-1,j}(z) + \sum_{i=0}^{n-1}\sum_{j=0}^{n-2} T_{n-1,n-1+j}(z) \\
&= \sum_{j=0}^{n-2}(j+1)T_{n-1,j}(z) + \sum_{j=0}^{n-2} nT_{n-1,n-1+j}(z),
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i=n}^{2n-1}\sum_{j=0}^{i-2} zT_{n-1,j}(z) &= \sum_{i=n}^{2n-1}\sum_{j=0}^{n-2} zT_{n-1,j}(z) + \sum_{i=n}^{2n-1}\sum_{j=n-1}^{i-2} zT_{n-1,j}(z) \\
&= \sum_{j=0}^{n-2} nzT_{n-1,j}(z) + \sum_{j=n-1}^{2n-3}\sum_{i=j+2}^{2n-1} zT_{n-1,j}(z) \\
&= \sum_{j=0}^{n-2} nzT_{n-1,j}(z) + \sum_{j=n-1}^{2n-3} (2n-j-2)zT_{n-1,j}(z) \\
&= \sum_{j=0}^{n-2} nzT_{n-1,j}(z) + \sum_{j=0}^{n-2}(n-1-j)zT_{n-1,n-1+j}(z),
\end{aligned}
$$

$$
\sum_{i=n}^{2n-1}\sum_{j=i-1}^{2n-3} T_{n-1,j}(z) = \sum_{j=n-1}^{2n-3}\sum_{i=n}^{j+1} T_{n-1,j}(z) = \sum_{j=0}^{n-2}(j+1)T_{n-1,n-1+j}(z).
$$

Therefore,

$$Q_n^E(z)=\sum_{j=0}^{n-2}\big((n-j-1)z+j+1\big)\big(T_{n-1,j}(z)+T_{n-1,n-1+j}(z)\big)$$

as desired. The second formula can be proved in the same manner. This completes the proof. $\square$

Lemma 6.4 implies that both $Q_n^E(z)$ and $Q_n^O(z)$ are polynomials in z with nonnegative coefficients. By Theorem 2.3, to show that $Q_n^O(z) \preceq Q_n^E(z)$, or equivalently $Q_n^E(z) \preceq zQ_n^O(z)$, it suffices to prove both $Q_n^E(z)$ and $zQ_n^E(z)$ are compatible with $zQ_n^O(z)$. Equivalently, we only need to show that both

$$Q_n^E(z)+czQ_n^O(z)$$

and

$$czQ_n^E(z)+zQ_n^O(z)$$

have only real zeros for any $c\geq 0$. If for any $c \geq 0$ we let $(K_{n,i}^{\langle c\rangle}(z))_{i=0}^{2n-1}$ be the polynomial sequence given by

$$K_{n,i}^{\langle c\rangle}(z) \;=\begin{cases} T_{n,i}(z)+cT_{n,n+i}(z), & \text{if } 0\leq i\leq n-1\\ czT_{n,i-n}(z)+T_{n,i}(z), & \text{if } n\leq i\leq 2n-1\end{cases} \tag{30}$$

and let $(L_{n,i}^{\langle c\rangle}(z))_{i=0}^{2n-1}$ be the polynomial sequence given by

$$L_{n,i}^{\langle c\rangle}(z) \;=\begin{cases} K_{n,i}^{\langle 1\rangle}(z)+cK_{n,n+i}^{\langle 1\rangle}(z), & \text{if } 0\leq i\leq n-1\\ czK_{n,i-n}^{\langle 1\rangle}(z)+K_{n,i}^{\langle 1\rangle}(z), & \text{if } n\leq i\leq 2n-1\end{cases} \tag{31}$$

then, by Lemma 6.4, we have

$$Q_n^E(z)+czQ_n^O(z)=\sum_{j=0}^{n-2}\big((n-j-1)z+j+1\big)L_{n-1,j}^{\langle c\rangle}(z), \tag{32}$$

$$czQ_n^E(z)+zQ_n^O(z)=\sum_{j=0}^{n-2}\big((n-j-1)z+j+1\big)L_{n-1,n-1+j}^{\langle c\rangle}(z). \tag{33}$$

Before proving that both $Q_n^E(z)+czQ_n^O(z)$ and $czQ_n^E(z)+zQ_n^O(z)$ have only real zeros, let us note the following interesting result.

Theorem 6.5. *For $c \geq 0, n \geq 4$ and $0 \leq i \leq 2n-1$, we have*

$$K^{\langle c\rangle}_{n,i}(z) \;=\; z \sum_{j=0}^{\lceil \frac{n-1}{n} i\rceil -1} K^{\langle c\rangle}_{n-1,j}(z) + \sum_{j=\lceil \frac{n-1}{n} i\rceil}^{2n-3} K^{\langle c\rangle}_{n-1,j}(z) \tag{34}$$

and

$$L^{\langle c\rangle}_{n,i}(z) \;=\; z \sum_{j=0}^{\lceil \frac{n-1}{n} i\rceil -1} L^{\langle c\rangle}_{n-1,j}(z) + \sum_{j=\lceil \frac{n-1}{n} i\rceil}^{2n-3} L^{\langle c\rangle}_{n-1,j}(z), \tag{35}$$

where $K^{\langle c\rangle}_{n,i}(z)$ and $L^{\langle c\rangle}_{n,i}(z)$ are defined by (30) and (31) respectively.

Proof. We first prove (34). We use some matrix techniques to give a proof. Let

$$\mathcal{K}_n = (K^{\langle c\rangle}_{n,0}(z), K^{\langle c\rangle}_{n,1}(z), \ldots, K^{\langle c\rangle}_{n,2n-1}(z))^t,$$
$$\mathcal{T}_n = (T_{n,0}(z), T_{n,1}(z), \ldots, T_{n,2n-1}(z))^t,$$

where the symbol ^{t} denotes the matrix transpose. From (30) it follows that

$$\mathcal{K}_n = \begin{pmatrix} I_n & cI_n \\ zI_n & cI_n \end{pmatrix} \mathcal{T}_n, \tag{36}$$

where I_n is the identity matrix of order n. Note that the recurrence relation (6) can be rewritten as

$$T_{n,i}(z) = z \sum_{j=0}^{i-1} T_{n-1,j}(z) + \sum_{j=i}^{2n-3} T_{n-1,j}(z),$$
$$T_{n,n+i}(z) = z \sum_{j=0}^{n+i-2} T_{n-1,j}(z) + \sum_{j=n+i-1}^{2n-3} T_{n-1,j}(z),$$

where $0 \leq i \leq n-1$. Therefore, we get

$$\mathcal{T}_n = \begin{pmatrix} A & B \\ zB & A \end{pmatrix} \mathcal{T}_{n-1}, \tag{37}$$

where

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z & 1 & \cdots & 1 \\ z & z & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ z & z & \cdots & z \end{pmatrix}_{n\times(n-1)}$$

and B is an $n \times (n-1)$ matrix with all entries equal to 1. One can compute that

$$\begin{aligned}\begin{pmatrix} I_n & cI_n \\ czI_n & I_n \end{pmatrix}\begin{pmatrix} A & B \\ zB & A \end{pmatrix} &= \begin{pmatrix} A+czB & cA+B \\ czA+zB & A+czB \end{pmatrix} \\ &= \begin{pmatrix} A & B \\ zB & A \end{pmatrix}\begin{pmatrix} I_{n-1} & cI_{n-1} \\ czI_{n-1} & I_{n-1} \end{pmatrix}. \end{aligned} \tag{38}$$

Combining (36), (37), and (38), we obtain

$$\mathcal{K}_n = \begin{pmatrix} A & B \\ zB & A \end{pmatrix}\mathcal{K}_{n-1},$$

which is equivalent to (34). The proof of (35) can be done exactly in the same way. This completes the proof. □

With the above recurrence relation, we obtain a result analogous to Theorem 2.5.

Theorem 6.6. *For $n \geq 4$ and $c \geq 0$, both $(K_{n,i}^{\langle c\rangle}(z))_{i=0}^{2n-1}$ and $(L_{n,i}^{\langle c\rangle}(z))_{i=0}^{2n-1}$ are pairwise interlacing, where $K_{n,i}^{\langle c\rangle}(z)$ and $L_{n,i}^{\langle c\rangle}(z)$ are defined by (30) and (31) respectively.*

Proof. We first prove the case of $(K_{n,i}^{\langle c\rangle}(z))_{i=0}^{2n-1}$. We may assume that $c > 0$ since $(K_{n,i}^{\langle 0\rangle}(z))_{i=0}^{2n-1}$ is just $(T_{n,i}(z))_{i=0}^{2n-1}$. We use induction on n. For $n = 4$, by using (30), we can directly compute the polynomials $K_{4,i}^{\langle c\rangle}(z)$ for $0 \leq i \leq 7$. The eight polynomials are listed below:

$$K_{4,0}^{\langle c\rangle}(z) = (10c+2)z^3 + (28c+22)z^2 + (10c+22)z + 2,$$

$$K_{4,1}^{\langle c\rangle}(z) = (14c+4)z^3 + (28c+24)z^2 + (6c+20)z,$$

$$K_{4,2}^{\langle c\rangle}(z) = (20c+6)z^3 + (24c+28)z^2 + (4c+14)z,$$

$$K_{4,3}^{\langle c\rangle}(z) = 2cz^4 + (22c+10)z^3 + (22c+28)z^2 + (2c+10)z,$$

$$K_{4,4}^{\langle c\rangle}(z) = 2cz^4 + (22c+10)z^3 + (22c+28)z^2 + (2c+10)z,$$

$$K_{4,5}^{\langle c\rangle}(z) = 4cz^4 + (24c+14)z^3 + (20c+28)z^2 + 6z,$$

$$K_{4,6}^{\langle c\rangle}(z) = 6cz^4 + (28c+20)z^3 + (14c+24)z^2 + 4z,$$

$$K_{4,7}^{\langle c\rangle}(z) = (10c+2)z^4 + (28c+22)z^3 + (10c+22)z^2 + 2z.$$

To prove that $K_{4,i}^{\langle c\rangle}(z) \preceq K_{4,j}^{\langle c\rangle}(z)$ for any $0 \le i < j \le 7$, by Theorem 1.5 it suffices to show that $zK_{4,i}^{\langle c\rangle}(z^2) + K_{4,j}^{\langle c\rangle}(z^2)$ is weakly Hurwitz stable for any $i < j$ with $i, j \in \{0, 1, 2, 3, 5, 6\}$. Let

$$H_{i,j}(z) = \frac{zK_{4,i}^{\langle c\rangle}(z^2) + K_{4,j}^{\langle c\rangle}(z^2)}{z^{m_{i,j}}},$$

where $m_{i,j}$ is the largest nonnegative integer k such that

$$z^k \mid (zK_{4,i}^{\langle c\rangle}(z^2) + K_{4,j}^{\langle c\rangle}(z^2)).$$

We proceed to show that $H_{i,j}(z)$ is Hurwitz stable for any $i < j$ with $i, j \in \{0, 1, 2, 3, 5, 6\}$. By Theorem 1.4, we only need to show that all the Hurwitz determinants of $H_{i,j}(z)$ are positive for any $c > 0$. It is straightforward to compute these Hurwitz determinants with the aid of a computer. As shown in the appendix, for any $i < j$ with $i, j \in \{0, 1, 2, 3, 5, 6\}$, all the Hurwitz determinants of $H_{i,j}(z)$ are polynomials in c with nonnegative coefficients, and hence are positive for $c > 0$. This establishes the weak Hurwitz stability of $zK_{4,i}^{\langle c\rangle}(z^2) + K_{4,j}^{\langle c\rangle}(z^2)$. Thus we get the desired result for $n = 4$. Then, by Theorems 2.4 and 6.5, we obtain the pairwise interlacing property of $(K_{n,i}^{\langle c\rangle}(z))_{i=0}^{2n-1}$ for any $n \ge 4$.

For the case of $(L_{n,i}^{\langle c\rangle}(z))_{i=0}^{2n-1}$, we can directly compute the polynomials $L_{4,i}^{\langle c\rangle}(z)$ for $0 \le i \le 7$. The eight polynomials are listed below:

$$L_{4,0}^{\langle c\rangle}(z) = 2cz^4 + (32c+12)z^3 + (50c+50)z^2 + (12c+32)z + 2,$$

$$L_{4,1}^{\langle c\rangle}(z) = 4cz^4 + (38c+18)z^3 + (48c+52)z^2 + (6c+26)z,$$
$$L_{4,2}^{\langle c\rangle}(z) = 6cz^4 + (48c+26)z^3 + (38c+52)z^2 + (4c+18)z,$$
$$L_{4,3}^{\langle c\rangle}(z) = (12c+2)z^4 + (50c+32)z^3 + (32c+50)z^2 + (2c+12)z,$$
$$L_{4,4}^{\langle c\rangle}(z) = (12c+2)z^4 + (50c+32)z^3 + (32c+50)z^2 + (2c+12)z,$$
$$L_{4,5}^{\langle c\rangle}(z) = (18c+4)z^4 + (52c+38)z^3 + (26c+48)z^2 + 6z,$$
$$L_{4,6}^{\langle c\rangle}(z) = (26c+6)z^4 + (52c+48)z^3 + (18c+38)z^2 + 4z,$$
$$L_{4,7}^{\langle c\rangle}(z) = 2cz^5 + (32c+12)z^4 + (50c+50)z^3 + (12c+32)z^2 + 2z.$$

To prove that $L_{4,i}^{\langle c\rangle}(z) \preceq L_{4,j}^{\langle c\rangle}(z)$ for any $0 \le i < j \le 7$, by Theorem 1.5 it suffices to show that $zL_{4,i}^{\langle c\rangle}(z^2)+L_{4,j}^{\langle c\rangle}(z^2)$ is weakly Hurwitz stable for any $i<j$ with $i,j \in \{0,1,2,3,5,6\}$. Similarly, for any $i<j$ with $i,j \in \{0,1,2,3,5,6\}$, let $n_{i,j}$ be the largest nonnegative integer k such that

$$z^k \mid (zL_{4,i}^{\langle c\rangle}(z^2) + L_{4,j}^{\langle c\rangle}(z^2)).$$

Let

$$\tilde{H}_{i,j}(z) = \frac{zL_{4,i}^{\langle c\rangle}(z^2) + L_{4,j}^{\langle c\rangle}(z^2)}{z^{n_{i,j}}}.$$

Now it suffices to show that $\tilde{H}_{i,j}(z)$ is Hurwitz stable for any $i<j$ with $i,j \in \{0,1,2,3,5,6\}$. As shown in the appendix, each Hurwitz determinant of $\tilde{H}_{i,j}(z)$ is a product of some polynomial in c with nonnegative coefficients and some even power of $(c-1)$, and hence it is positive for $c \neq 0$ and $c \neq 1$. Therefore, the polynomial $zL_{4,i}^{\langle c\rangle}(z^2) + L_{4,j}^{\langle c\rangle}(z^2)$ is weakly Hurwitz stable for any $0 \le i < j \le 7$ and $c \neq 0, 1$. While for $c = 0$ or $c = 1$ we can directly check the pairwise interlacing property of $(L_{4,i}^{\langle c\rangle}(z))_{i=0}^{7}$. Again by Theorems 2.4 and 6.5, we obtain the pairwise interlacing property of $(L_{n,i}^{\langle c\rangle}(z))_{i=0}^{2n-1}$ for any $n \ge 4$. This completes the proof. □

Now we are in the position to prove Theorem 6.1.

Proof of Theorem 6.1. As discussed before, we only need to prove that both $Q_n^E(z) + czQ_n^O(z)$ and $czQ_n^E(z) + zQ_n^O(z)$ have only real zeros for any $c \ge 0$.

By Theorem 6.6, we know that, for any $n \geq 4$, both $(L^{\langle c \rangle}_{n-1,j}(z))_{j=0}^{n-2}$ and $(L^{\langle c \rangle}_{n-1,j}(z))_{j=n-1}^{2n-3}$ are pairwise interlacing. Let $m = n-1$ and define $a_i = n-i$, $b_i = i$ and $f_i(z) = L^{\langle c \rangle}_{n-1,i-1}(z)$ for $1 \leq i \leq n-1$ in Theorem 6.2. Since $a_i b_{i+1} - b_i a_{i+1} = n > 0$, it is immediate that

$$\sum_{i=1}^{n-1}(n-i)L^{\langle c \rangle}_{n-1,i-1}(z) \preceq \sum_{i=1}^{n-1} i\, L^{\langle c \rangle}_{n-1,i-1}(z).$$

Since all the zeros of these two polynomials are real and nonpositive, we get

$$\sum_{i=1}^{n-1} iL^{\langle c \rangle}_{n-1,i-1}(z) \preceq z\sum_{i=1}^{n-1}(n-i)L^{\langle c \rangle}_{n-1,i-1}(z).$$

Further, by Lemma 6.3, we know that $\sum_{i=1}^{n-1}\big((n-i)z+i\big)L^{\langle c \rangle}_{n-1,i-1}(z)$ has only real zeros. Thus by (32) the polynomial $Q_n^E(z) + czQ_n^O(z)$ has only real zeros for any $c \geq 0$. To prove that $czQ_n^E(z) + zQ_n^O(z)$ has only real zeros for any $c \geq 0$, by virtue of (33), the same arguments as before apply except that taking $f_i(z) = L^{\langle c \rangle}_{n-1,n-2+i}(z)$ for $1 \leq i \leq n-1$ in Lemma 6.2. This completes the proof. $\square$

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Appdendix

In this section we shall list all Hurwitz determinants which are used in the proof of Theorem 6.6.

The following table presents a list of the Hurwitz determinants of $H_{i,j}(z)$ for any $i < j$ with $i, j \in \{0, 1, 2, 3, 5, 6\}$.

(i,j)	$H_{i,j}(z)$	$\Delta_k(H_{i,j}(z))$
$(0,1)$	$(10c+2)z^6+(14c+4)z^5+(28c+22)z^4$ $+(28c+24)z^3+(10c+22)z^2+(6c+20)z+2$	$\Delta_1 = 14c+4$, $\Delta_2 = 4\left(28c^2+31c+10\right)$, $\Delta_3 = 96\left(21c^3+41c^2+30c+8\right)$, $\Delta_4 = 192\left(32c^4+71c^3+67c^2+34c+8\right)$, $\Delta_5 = 36864c^2(c+1)^3$, $\Delta_6 = 73728c^2(c+1)^3$.

(i,j)	$H_{i,j}(z)$	$\Delta_k(H_{i,j}(z))$
$(0,2)$	$(10c+2)z^6+(20c+6)z^6+(28c+22)z^4$ $+(24c+28)z^3+(10c+22)z^2+(4c+14)z+2$	$\Delta_1 = 20c+6,\ \Delta_2 = 4\left(80c^2+70c+19\right),$ $\Delta_3 = 32\left(140c^3+240c^2+171c+47\right),$ $\Delta_4 = 64\left(240c^4+640c^3+906c^2+693c+211\right),$ $\Delta_5 = 6144(c+1)^3\left(10c^2+5c+19\right),$ $\Delta_6 = 12288(c+1)^3\left(10c^2+5c+19\right).$
$(0,3)$	$2cz^7+(10c+2)z^6+(22c+10)z^5+(28c+22)z^4$ $+(22c+28)z^3+(10c+22)z^2+(2c+10)z+2$	$\Delta_1 = 10c+2,\ \Delta_2 = 4\left(41c^2+25c+5\right),$ $\Delta_3 = 8\left(324c^3+401c^2+205c+41\right),$ $\Delta_4 = 576\left(49c^4+101c^3+94c^2+45c+9\right),$ $\Delta_5 = 1152\left(144c^5+445c^4+641c^3\right.$ $\left.+530c^2+245c+49\right),$ $\Delta_6 = 331776(c+1)^4\left(c^2+c+1\right),$ $\Delta_7 = 663552(c+1)^4\left(c^2+c+1\right).$
$(0,5)$	$4cz^7+(10c+2)z^6+(24c+14)z^5+(28c+22)z^4$ $+(20c+28)z^3+(10c+22)z^2+6z+2$	$\Delta_1 = 10c+2,\ \Delta_2 = 4\left(32c^2+25c+7\right),$ $\Delta_3 = 8\left(248c^3+372c^2+245c+63\right),$ $\Delta_4 = 64\left(240c^4+556c^3+609c^2+360c+91\right),$ $\Delta_5 = 128\left(1200c^5+4080c^4\right.$ $\left.+6218c^3+5469c^2+2802c+647\right),$ $\Delta_6 = 12288(c+1)^3\left(25c^2+20c+19\right),$ $\Delta_7 = 24576(c+1)^3\left(25c^2+20c+19\right).$
$(0,6)$	$6cz^7+(10c+2)z^6+(28c+20)z^5+(28c+22)z^4$ $+(14c+24)z^3+(10c+22)z^2+4z+2$	$\Delta_1 = 10c+2,\ \Delta_2 = 4\left(28c^2+31c+10\right),$ $\Delta_3 = 8\left(292c^3+552c^2+387c+98\right),$ $\Delta_4 = 192\left(32c^4+71c^3+67c^2+34c+8\right),$ $\Delta_5 = 384\left(160c^5+448c^4+453c^3+193c^2+38c+8\right),$ $\Delta_6 = 73728c^2(c+1)^3,\ \Delta_7 = 147456c^2(c+1)^3.$
$(1,2)$	$(14c+4)z^5+(20c+6)z^4+(28c+24)z^3$ $+(24c+28)z^2+(6c+20)z+(4c+14)$	$\Delta_1 = 20c+6,\ \Delta_2 = 32\left(7c^2+5c+1\right),$ $\Delta_3 = 64\left(64c^3+82c^2+41c+8\right),\ \Delta_4 = 3072(c+1)^4,$ $\Delta_5 = 6144(c+1)^4(2c+7).$
$(1,3)$	$2cz^6+(14c+4)z^5+(22c+10)z^4+(28c+24)z^3$ $+(22c+28)z^2+(6c+20)z+(2c+10)$	$\Delta_1 = 14c+4,\ \Delta_2 = 4\left(63c^2+45c+10\right),$ $\Delta_3 = 32\left(91c^3+117c^2+66c+16\right),$ $\Delta_4 = 64\left(647c^4+1150c^3+975c^2+500c+124\right),$ $\Delta_5 = 6144(c+1)^3\left(19c^2+5c+10\right),$ $\Delta_6 = 12288(c+1)^3\left(19c^3+100c^2+35c+50\right).$
$(1,5)$	$4cz^6+(14c+4)z^5+(24c+14)z^4+(28c+24)z^3$ $+(20c+28)z^2+(6c+20)z+6$	$\Delta_1 = 14c+4,\ \Delta_2 = 28\left(8c^2+7c+2\right),$ $\Delta_3 = 64\left(42c^3+68c^2+49c+14\right),$ $\Delta_4 = 384\left(72c^4+147c^3+154c^2+98c+28\right),$ $\Delta_5 = 55296(c+1)^3\left(3c^2+c+2\right),$ $\Delta_6 = 331776(c+1)^3\left(3c^2+c+2\right).$
$(1,6)$	$6cz^6+(14c+4)z^5+(28c+20)z^4+(28c+24)z^3$ $+(14c+24)z^2+(6c+20)z+4$	$\Delta_1 = 14c+4,\ \Delta_2 = 8\left(28c^2+31c+10\right),$ $\Delta_3 = 192\left(21c^3+41c^2+30c+8\right),$ $\Delta_4 = 768\left(32c^4+71c^3+67c^2+34c+8\right),$ $\Delta_5 = 147456c^2(c+1)^3,\ \Delta_6 = 589824c^2(c+1)^3.$
$(2,3)$	$2cz^6+(20c+6)z^5+(22c+10)z^4+(24c+28)z^3$ $+(22c+28)z^2+(4c+14)z+(2c+10)$	$\Delta_1 = 20c+6,\ \Delta_2 = 4\left(98c^2+69c+15\right),$ $\Delta_3 = 96\left(8c^3+18c^2+19c+7\right),$ $\Delta_4 = 192\left(8c^4+54c^3+191c^2+210c+73\right),$ $\Delta_5 = 36864(c+1)^3,\ \Delta_6 = 73728(c+1)^3(c+5).$
$(2,5)$	$4cz^6+(20c+6)z^5+(24c+14)z^4+(24c+28)z^3$ $+(20c+28)z^2+(4c+14)z+6$	$\Delta_1 = 20c+6,\ \Delta_2 = 12\left(32c^2+26c+7\right),$ $\Delta_3 = 192\left(8c^3+18c^2+19c+7\right),$ $\Delta_4 = 1152\left(8c^3+46c^2+57c+21\right),$ $\Delta_5 = 147456(c+1)^3,\ \Delta_6 = 884736(c+1)^3.$

(i,j)	$H_{i,j}(z)$	$\Delta_k(H_{i,j}(z))$
$(2,6)$	$6cz^6+(20c+6)z^5+(28c+20)z^4+(24c+28)z^3$ $+(14c+24)z^2+(4c+14)z+4$	$\Delta_1 = 20c+6,\ \Delta_2 = 8\left(52c^2+50c+15\right),$ $\Delta_3 = 64\left(76c^3+158c^2+130c+39\right),$ $\Delta_4 = 256\left(108c^4+326c^3+493c^2+380c+114\right),$ $\Delta_5 = 55296(c+1)^3\left(2c^2+c+3\right),$ $\Delta_6 = 221184(c+1)^3\left(2c^2+c+3\right).$
$(3,5)$	$2cz^7+4cz^6+(22c+10)z^5+(24c+14)z^4$ $+(22c+28)z^3+(20c+28)z^2+(2c+10)z+6$	$\Delta_1 = 4c,\ \Delta_2 = 4c(10c+3),$ $\Delta_3 = 24c\left(32c^2+26c+7\right),$ $\Delta_4 = 192c\left(8c^3+18c^2+19c+7\right),$ $\Delta_5 = 1152c\left(8c^3+46c^2+57c+21\right),$ $\Delta_6 = 73728c(c+1)^3,\ \Delta_7 = 442368c(c+1)^3.$
$(3,6)$	$2cz^7+6cz^6+(22c+10)z^5+(28c+20)z^4$ $+(22c+28)z^3+(14c+24)z^2+(2c+10)z+4$	$\Delta_1 = 6c,\ \Delta_2 = 4c(19c+5),$ $\Delta_3 = 16c\left(94c^2+85c+25\right),$ $\Delta_4 = 64c\left(211c^3+432c^2+345c+100\right),$ $\Delta_5 = 256c\left(456c^4+1403c^3+1926c^2+1305c+350\right),$ $\Delta_6 = 12288c(c+1)^3\left(19c^2+20c+25\right),$ $\Delta_7 = 49152c(c+1)^3\left(19c^2+20c+25\right).$
$(5,6)$	$4cz^7+6cz^6+(24c+14)z^5+(28c+20)z^4$ $+(20c+28)z^3+(14c+24)z^2+6z+4$	$\Delta_1 = 6c,\ \Delta_2 = 4c(8c+1),$ $\Delta_3 = 16c\left(32c^2+20c+5\right),$ $\Delta_4 = 128c\left(24c^3+43c^2+29c+7\right),$ $\Delta_5 = 512c\left(84c^4+240c^3+278c^2+151c+32\right),$ $\Delta_6 = 12288c(c+1)^4,$ $\Delta_7 = 49152c(c+1)^4.$

We proceed to list the Hurwitz determinants of $\tilde{H}_{i,j}(z)$ for any $i<j$ with $i,j \in \{0,1,2,3,5,6\}$.

(i,j)	$\tilde{H}_{i,j}(z)$	$\Delta_k(\tilde{H}_{i,j}(z))$
$(0,1)$	$2cz^8+4cz^7+(32c+12)z^6$ $+(38c+18)z^5+(50c+50)z^4+(48c+52)z^3$ $+(12c+32)z^2+(6c+26)z+2$	$\Delta_1 = 4c, \Delta_2 = 4c(13c+3),$ $\Delta_3 = 24c\left(65c^2+42c+9\right),$ $\Delta_4 = 48c\left(208c^3+373c^2+266c+69\right),$ $\Delta_5 = 4608c(c+1)^2\left(33c^2+56c+25\right),$ $\Delta_6 = 9216c(c+1)^2\left(64c^3+169c^2+160c+53\right),$ $\Delta_7 = 3538944c(c+1)^6,$ $\Delta_8 = 7077888c(c+1)^6.$
$(0,2)$	$2cz^8+6cz^7+(32c+12)z^6$ $+(48c+26)z^5+(50c+50)z^4+(38c+52)z^3$ $+(12c+32)z^2+(4c+18)z+2$	$\Delta_1 = 6c,\ \Delta_2 = 4c(24c+5),$ $\Delta_3 = 8c\left(408c^2+285c+65\right),$ $\Delta_4 = 16c\left(4352c^3+6488c^2+3825c+845\right),$ $\Delta_5 = 256c\left(5440c^4+12997c^3+13593c^2+7075c+1495\right),$ $\Delta_6 = 512c\left(12288c^5+39872c^4+62735c^3\right.$ $\left.+58395c^2+30425c+6725\right),$ $\Delta_7 = 393216c(c+1)^4\left(64c^2+35c+85\right),$ $\Delta_8 = 786432c(c+1)^4\left(64c^2+35c+85\right).$
$(0,3)$	$2cz^8+(12c+2)z^7+(32c+12)z^6$ $+(50c+32)z^5+(50c+50)z^4+(32c+50)z^3$ $+(12c+32)z^2+(2c+12)z+2$	$\Delta_1 = 12c+2,\ \Delta_2 = 4\left(71c^2+36c+6\right),$ $\Delta_3 = 8\left(971c^3+1002c^2+426c+71\right),$ $\Delta_4 = 16\left(10728c^4+18719c^3+14610c^2+5826c+971\right),$ $\Delta_5 = 2304\left(1027c^5+2706c^4+3323c^3\right.$ $\left.+2317c^2+894c+149\right),$ $\Delta_6 = 4608\left(3456c^6+13109c^5+23358c^4\right.$ $\left.+24781c^3+16283c^2+6162c+1027\right),$ $\Delta_7 = 31850496(c+1)^5\left(c^2+c+1\right),$ $\Delta_8 = 63700992(c+1)^5\left(c^2+c+1\right).$

(i,j)	$\tilde{H}_{i,j}(z)$	$\Delta_k(\tilde{H}_{i,j}(z))$
$(0,5)$	$2cz^8+(18c+4)z^7+(32c+12)z^6$ $+(52c+38)z^5+(50c+50)z^4+(26c+48)z^3$ $+(12c+32)z^2+6z+2$	$\Delta_1 = 18c+4,\ \Delta_2 = 472c^2+268c+48,$ $\Delta_3 = 8\left(1160c^3+1301c^2+633c+128\right),$ $\Delta_4 = 16\left(13584c^4+23096c^3+19257c^2\right.$ $\left.+9013c+1856\right),$ $\Delta_5 = 256\left(6528c^5+16733c^4+21137c^3\right.$ $\left.+16995c^2+8375c+1856\right),$ $\Delta_6 = 512\left(39168c^6+153216c^5+272383c^4\right.$ $\left.+295147c^3+214545c^2+97693c+20416\right),$ $\Delta_7 = 393216(c+1)^4\left(85c^2+35c+64\right),$ $\Delta_8 = 786432(c+1)^4\left(85c^2+35c+64\right).$
$(0,6)$	$2cz^8+(26c+6)z^7+(32c+12)z^6$ $+(52c+48)z^5+(50c+50)z^4+(18c+38)z^3$ $+(12c+32)z^2+4z+2$	$\Delta_1 = 26c+6,\ \Delta_2 = 8\left(91c^2+51c+9\right),$ $\Delta_3 = 24\left(208c^3+373c^2+266c+69\right),$ $\Delta_4 = 48\left(2144c^4+7296c^3+9657c^2+5802c+1329\right),$ $\Delta_5 = 4608(c+1)^2\left(64c^3+169c^2+160c+53\right),$ $\Delta_6 = 9216(c+1)^2\left(384c^4+1408c^3+2005c^2\right.$ $\left.+1304c+325\right),$ $\Delta_7 = 3538944(c+1)^6,\ \Delta_8 = 7077888(c+1)^6.$
$(1,2)$	$4cz^7+6cz^6+(38c+18)z^5$ $+(48c+26)z^4+(48c+52)z^3+(38c+52)z^2$ $+(6c+26)z+(4c+18)$	$\Delta_1 = 6c,\ \Delta_2 = 4c(9c+1),$ $\Delta_3 = 8c\left(114c^2+63c+13\right),$ $\Delta_4 = 128c\left(103c^3+135c^2+69c+13\right),$ $\Delta_5 = 256c\left(1570c^4+3195c^3+2751c^2+1153c+195\right),$ $\Delta_6 = 196608c(c+1)^5,\ \Delta_7 = 393216c(c+1)^5(2c+9).$
$(1,3)$	$4cz^7+(12c+2)z^6+(38c+18)z^5$ $+(50c+32)z^4+(48c+52)z^3+(32c+50)z^2$ $+(6c+26)z+(2c+12)$	$\Delta_1 = 12c+2,\ \Delta_2 = 4\left(64c^2+41c+9\right),$ $\Delta_3 = 8\left(928c^3+1157c^2+595c+118\right),$ $\Delta_4 = 128\left(928c^4+1831c^3+1623c^2+745c+145\right),$ $\Delta_5 = 256\left(10208c^5+27053c^4+32859c^3\right.$ $\left.+23009c^2+9293c+1698\right),$ $\Delta_6 = 196608(c+1)^4\left(32c^2+19c+17\right),$ $\Delta_7 = 393216(c+1)^4\left(32c^3+211c^2+131c+102\right).$
$(1,5)$	$4cz^7+(18c+4)z^6+(38c+18)z^5$ $+(52c+38)z^4+(48c+52)z^3+(26c+48)z^2$ $+(6c+26)z+6$	$\Delta_1 = 18c+4,\ \Delta_2 = 476c^2+324c+72,$ $\Delta_3 = 8\left(1384c^3+1881c^2+1071c+238\right),$ $\Delta_4 = 128\left(1497c^4+3031c^3+2961c^2+1557c+346\right),$ $\Delta_5 = 768\left(3456c^5+9015c^4+11417c^3+9135c^2\right.$ $\left.+4491c+998\right),$ $\Delta_6 = 5308416(c+1)^4\left(3c^2+c+2\right),$ $\Delta_7 = 31850496(c+1)^4\left(3c^2+c+2\right).$
$(1,6)$	$4cz^7+(26c+6)z^6+(38c+18)z^5$ $+(52c+48)z^4+(48c+52)z^3+(18c+38)z^2$ $+(6c+26)z+4$	$\Delta_1 = 26c+6,\ \Delta_2 = 12\left(65c^2+42c+9\right),$ $\Delta_3 = 48\left(208c^3+373c^2+266c+69\right),$ $\Delta_4 = 4608(c+1)^2\left(33c^2+56c+25\right),$ $\Delta_5 = 18432(c+1)^2\left(64c^3+169c^2+160c+53\right),$ $\Delta_6 = 7077888(c+1)^6,\ \Delta_7 = 28311552(c+1)^6.$
$(2,3)$	$6cz^7+(12c+2)z^6+(48c+26)z^5$ $+(50c+32)z^4+(38c+52)z^3+(32c+50)z^2$ $+(4c+18)z+(2c+12)$	$\Delta_1 = 12c+2,\ \Delta_2 = 4\left(69c^2+54c+13\right),$ $\Delta_3 = 8\left(1329c^3+1788c^2+933c+182\right),$ $\Delta_4 = 768\left(53c^4+102c^3+98c^2+54c+13\right),$ $\Delta_5 = 1536\left(325c^5+492c^4+394c^3+474c^2\right.$ $\left.+421c+134\right),$ $\Delta_6 = 589824(c-1)^2(c+1)^4,$ $\Delta_7 = 1179648(c-1)^2(c+1)^4(c+6).$
$(2,5)$	$6cz^7+(18c+4)z^6+(48c+26)z^5$ $+(52c+38)z^4+(38c+52)z^3+(26c+48)z^2$ $+(4c+18)z+6$	$\Delta_1 = 18c+4,\ \Delta_2 = 8\left(69c^2+54c+13\right),$ $\Delta_3 = 48\left(400c^3+561c^2+310c+65\right),$ $\Delta_4 = 3072\left(53c^4+102c^3+98c^2+54c+13\right),$ $\Delta_5 = 18432\left(64c^5+69c^4+14c^3+54c^2+86c+33\right),$ $\Delta_6 = 4718592(c-1)^2(c+1)^4,$ $\Delta_7 = 28311552(c-1)^2(c+1)^4.$

(i,j)	$\tilde{H}_{i,j}(z)$	$\Delta_k(\tilde{H}_{i,j}(z))$
$(2,6)$	$6cz^7+(26c+6)z^6+(48c+26)z^5$ $+(52c+48)z^4+(38c+52)z^3+(18c+38)z^2$ $+(4c+18)z+4$	$\Delta_1=26c+6,\ \Delta_2=52\left(18c^2+13c+3\right),$ $\Delta_3=8\left(3224c^3+4956c^2+3042c+702\right),$ $\Delta_4=128\left(2842c^4+7605c^3+9009c^2+5239c+1209\right),$ $\Delta_5=512\left(5184c^5+17990c^4+31707c^3+32895c^2\right.$ $\left.+18473c+4263\right),$ $\Delta_6=5308416(c+1)^4\left(2c^2+c+3\right),$ $\Delta_7=21233664(c+1)^4\left(2c^2+c+3\right).$
$(3,5)$	$(12c+2)z^7+(18c+4)z^6+(50c+32)z^5$ $+(52c+38)z^4+(32c+50)z^3+(26c+48)z^2$ $+(2c+12)z+6$	$\Delta_1=18c+4,\ \Delta_2=4\left(69c^2+54c+13\right),$ $\Delta_3=24\left(400c^3+561c^2+310c+65\right),$ $\Delta_4=768\left(53c^4+102c^3+98c^2+54c+13\right),$ $\Delta_5=4608\left(64c^5+69c^4+14c^3+54c^2+86c+33\right),$ $\Delta_6=589824(c-1)^2(c+1)^4,\ \Delta_7=3538944(c-1)^2(c+1)^4.$
$(3,6)$	$(12c+2)z^7+(26c+6)z^6+(50c+32)z^5$ $+(52c+48)z^4+(32c+50)z^3+(18c+38)z^2$ $+(2c+12)z+4$	$\Delta_1=26c+6,\ \Delta_2=676c^2+452c+96,$ $\Delta_3=16\left(1196c^3+1641c^2+929c+204\right),$ $\Delta_4=128\left(1345c^4+3481c^3+4071c^2+2359c+544\right),$ $\Delta_5=512\left(3264c^5+12179c^4+22091c^3+22581c^2\right.$ $\left.+12245c+2720\right),$ $\Delta_6=196608(c+1)^4\left(17c^2+19c+32\right),$ $\Delta_7=786432(c+1)^4\left(17c^2+19c+32\right).$
$(5,6)$	$(18c+4)z^7+(26c+6)z^6+(52c+38)z^5$ $+(52c+48)z^4+(26c+48)z^3+(18c+38)z^2$ $+6z+4$	$\Delta_1=26c+6,\ \Delta_2=4\left(104c^2+57c+9\right),$ $\Delta_3=16\left(780c^3+804c^2+337c+57\right),$ $\Delta_4=128\left(384c^4+973c^3+1029c^2+519c+103\right),$ $\Delta_5=512\left(1728c^5+6720c^4+11147c^3+9651c^2\right.$ $\left.+4305c+785\right),$ $\Delta_6=196608(c+1)^5,\ \Delta_7=786432(c+1)^5.$

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