Sharp upper bounds on the k-independence number in graphs with given minimum and maximum degree

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Abstract

The *k-independence number* of a graph G is the maximum size of a set of vertices at pairwise distance greater than k. In this paper, for each positive integer k, we prove sharp upper bounds for the k-independence number in an n-vertex connected graph with given minimum and maximum degree.

Keywords: k-independence number, independence number, chromatic number, k-distance chromatic number, regular graphs

AMS subject classification 2010: 05C69

1 Introduction

Throughout this paper, all graphs are simple, undirected, and finite. For two vertices u and v in a graph G, we define the distance between u and v, written $d_G(u, v)$ or simply $d(u, v)$, to be the length of the shortest path between u and v. For a nonnegative integer k, a *k-independent set* in a graph G is a vertex set $S \subseteq V(G)$ such that the distance between any two vertices in S is bigger than k. Note that the 0-independent set is $V(G)$ and an 1-independent set is an independent set. The *k-independence number* of a graph G, written $\alpha_k(G)$, is the maximum size of a k-independent set in G.

It is known that $\alpha_1(G) = \alpha(G) \geq \frac{n}{\chi(G)}$, where $\chi(G)$ and $\alpha(G)$ are the chromatic number and independence number of a graph G, repsectively. Similarly, by finding the k-distance

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chromatic number of G, we can find a lower bound for $\alpha_k(G)$. It will be discussed in Section 4. Other graph parameters such as the average distance [4], injective chromatic number [6], packing chromatic number [5], and strong chromatic index [9] are also directly related to the k-independence number. Lower bounds on the corresponding distance or packing chromatic number can be given by finding upper bounds on the k-independence number. Alon and Mohar [2] asked the extremal value for the distance chromatic number in graphs of a given girth and degree.

Firby and Haviland [4] proved an upper bound for $\alpha_k(G)$ in an n-vertex connected graph. We give a proof of the theorem below, because with a similar idea, we prove Theorem 3.2, which is one of the main results in this paper.

Theorem 1.1. ([4]) *For a positive integer k, if G is a non-complete n-vertex connected graph with $diam(G) \geq k+1$, then*

$$2 \leq \alpha_k(G) \leq \begin{cases} \dfrac{2n}{k+2}, & \text{if } k \text{ is even,} \\ \dfrac{2n-2}{k+1}, & \text{if } k \text{ is odd.} \end{cases}$$

Furthermore, bounds are sharp.

Proof. Let S_k be a k-independent set in G. Since $diam(G) \geq k+1$ and G is not a complete graph, there are two vertices $u, v \in V(G)$ such that $d_G(u,v) = k+1$. Thus $u, v \in S_k$, which implies $\alpha_k(G) = max\ |S_k| \geq 2$. The graph $K_1 \vee K_{i_1} \vee \cdots K_{i_k} \vee K_1$ attains equality in the lower bound, where $\sum_{j=1}^{k} i_j = n-2$.
For the upper bounds, we consider two cases depending on the parity of k.
Case 1: k is even. When $k=2$, for any pair of vertices $u, v \in S_2$, we have $d_G(u,v) \geq 3$, which means $N(u) \cap N(v) = \emptyset$. Therefore, we have $|N(S_2)| \geq |S_2|$. Since $|N(S_2)| + |S_2| = n$, we have $|S_2| \leq \frac{n}{2}$. The n-vertex comb H has $\alpha_2(H) = \frac{n}{2}$, where a comb is a graph obtained by joining a single pendant edge to each vertex to a path. For $k \geq 4$ and any pair of vertices $u, v \in S_k$, we have $d_G(u,v) \geq k+1$ and $N(v) \cap N(u) = \emptyset$. Thus we have $|N(S_k)| \geq |S_k|$. Simliarly, for $j = 1, \ldots, \frac{k}{2}$, we have $|N^j(S_k)| \geq |N^{j-1}(S_k)|$. Thus we have $n - |S_k| - \frac{k}{2}|N(S_k)| \geq 0$, which implies $\alpha_k(G) = max\ |S_k| \leq \frac{2n}{k+2}$. The n-vertex graph H_k obtained from a comb H with $\frac{4n}{k+2}$ vertices by replacing each pendant edge of H with a path of length $\frac{k}{2}$ has $\alpha_k(H_k) = \frac{2n}{k+2}$.
Case 2: k is odd. When $k=1$, for any pair of vertices $u, v \in S_1$, we have $d_G(u,v) \geq 2$. Thus we have $n - |S_1| \geq 1$. The star $K_{1,n-1}$ have $\alpha_1(G) = n-1$. For $k \geq 3$, for any pair of vertices $u, v \in S_k$, we have $d_G(u,v) \geq k+1$ and $N(v) \cap N(u) = \emptyset$. Similarly to Case 1, we have $|N(S_k)| \geq |S_k|$ and for $j = 1, \ldots, \frac{k-1}{2}$, we have $|N^j(S_k)| \geq |N^{j-1}(S_k)|$ and $N^{\frac{k+1}{2}}(S_k) \neq \emptyset$. Thus we have $n - |S_k| - \frac{k-1}{2}|N(S_k)| \geq 1$, which implies $\alpha_k(G) = max\ |S_k| \leq \frac{2(n-1)}{k+1}$. The

graph F_k obtained from the star $K_{1,\frac{2(n-1)}{k+1}}$ by replacing each edge with a path of length $\frac{k+1}{2}$ has $\alpha_k(F_k) = \frac{2n-2}{k+1}$. $\square$

For a vertex set $S \subseteq V(G)$, let $N(S)$ be the neiborhood of S, and for an integer $j \geq 2$, let $N^j(S) = N(N^{j-1}(S)) \setminus (N^{j-2}(S) \cup N^{j-1}(S))$, where $N^0(S) = S$ and $N^1(S) = N(S)$. For graphs $G_1, \ldots, G_k$, the graph $G_1 \vee \cdots \vee G_k$ is the one such that $V(G_1 \vee \cdots \vee G_k)$ is the disjoint union of $V(G_1), \ldots, V(G_k)$ and $E(G_1 \vee \cdots \vee G_k) = \{e : e \in E(G_i)$ for some $i \in [k]$ or an unordered pair between $V(G_i)$ and $V(G_{i+1})$ for some $i \in [k-1]\}$.

In 2000, Kong and Zhao [7] showed that for every $k \geq 2$, determining $\alpha_k(G)$ is NP-complete for general graphs. They also showed that this problem remains NP-hard for regular bipartite graphs when $k \in \{2, 3, 4\}$ [8]. It is well-known that for an n-vertex r-regular graph G, we have $\alpha_1(G) \leq \frac{n}{2}$. Also, for $k = 2$, we have $\alpha_2(G) \leq \frac{n}{r+1}$ because for any pair of two vertices u, v in a 2-independent set, we have $N(u) \cap N(v) = \emptyset$, which implies $n \geq |S_2| + |N(S_2)| \geq |S_2| + r|S_2|$. For each fixed integer, $k \geq 2$ and $r \geq 3$, Beis, Duckworth, and Zito [3] proved some upper bounds for $\alpha_k(G)$ in random r-regular graphs.

The remainder of the paper is organized as follows. In Subsection 2, for all positive integers k and $r \geq 3$, we provide infinitely many r-regular graphs with $\alpha_k(G)$ attaining the sharp upper bounds. In Section 3, we prove sharp upper bounds for $\alpha_k(G)$ in an n-vertex connected graph with $diam(G) \geq k+1$ for every positive integer k with given minimum and maximum degree. We conclude this paper with some open questions in Section 4.

For undefined terms, see West [11].

2 Construction

In this section, we construct n-vertex r-regular graphs with the k-independence number achieving equality in the upper bounds in Theorem 3.2. For a vertex $v \in V(G)$, we denote the neighborhood of v by $N(v)$ and $N(v) \cup \{v\}$ by $N[v]$, respectively.

Definition 2.1. For a positive integer ℓ, let $k = 6\ell - 4$. Let $H^1_{r,k}$ be the r-regular graph with the vertex sets $V_1, \ldots, V_{3\ell-1}$ satisfying the following properties:
(i) V_1 is an independent set with r vertices $v_{11}, \cdots, v_{1r}$ such that for each $i \in [r]$, the degree of v_{1i} is r, $N(v_{1i})$ induces a copy of $K_r - K_2$ and $N(v_{1i}) \cap N(v_{1j}) = \emptyset$ for $j \neq i$.
(ii) Let $V_2 = \cup_{j=1}^r N(v_{1j})$ such that for each $i \neq j \in [r]$, there is no edge with endpoints in $N(v_{1i})$ and $N(v_{1j})$, and for each $i \in [r]$, $v^1_{2i}, v^2_{2i} \in N(v_{1i})$, and v^1_{2i} is not adjacent to v^2_{2i}.
(iii) For a positive integer $x \in [\ell - 1]$, let $V_{3x} = \{v_{(3x)1}, \cdots, v_{(3x)r}\}$ such that for each $i \in [r]$, $v_{(3x)i}$ is adjacent to $v^h_{(3x-1)i}$ for $h \in \{1, 2\}$, $N(v_{(3x)i}) \setminus v^h_{(3x-1)i}$ induces a copy of K_{r-2} (in V_{3x+1}), and for each $i \neq j \in [r]$, $N[v_{(3x)i}] \cap N[v_{(3x)j}] = \emptyset$.
(iv) Let $V_{3x+1} = \{N(v_{(3x)1}) \backslash v^h_{(3x-1)1}, \ldots, N(v_{(3x)r}) \backslash v^h_{(3x-1)r}\}$ such that $h \in \{1, 2\}$ and for each

$i \neq j \in [r]$, there is no edge with endpoints in $N(v_{(3x)i}) \setminus v_{(3x-1)i}$ and in $N(v_{(3x)j}) \setminus v_{(3x-1)j}$.
(v) Let $V_{3x+2} = \{v^1_{(3x+2)1}, v^2_{(3x+2)1}, \ldots, v^1_{(3x+2)r}, v^2_{(3x+2)r}\}$ such that for each $i \in [r]$, $v^1_{(3x+2)i}$ is adjacent to $v^2_{(3x+2)i}$, and $v^h_{(3x+2)i}$ is adjacent to all vertices in $N(v_{(3x)i}) \setminus v_{(3x-1)i}$, for each $i \neq j \in [r]$ and $h \in \{1,2\}$, $v^h_{(3x+2)i}$ is not adjacent to $v^h_{(3x+2)j}$ except for $x = \ell - 1$.
Let $G^1_{r,k,t}$ be the disjoint union of t copies of $H^1_{r,k}$ (see Figure 1).

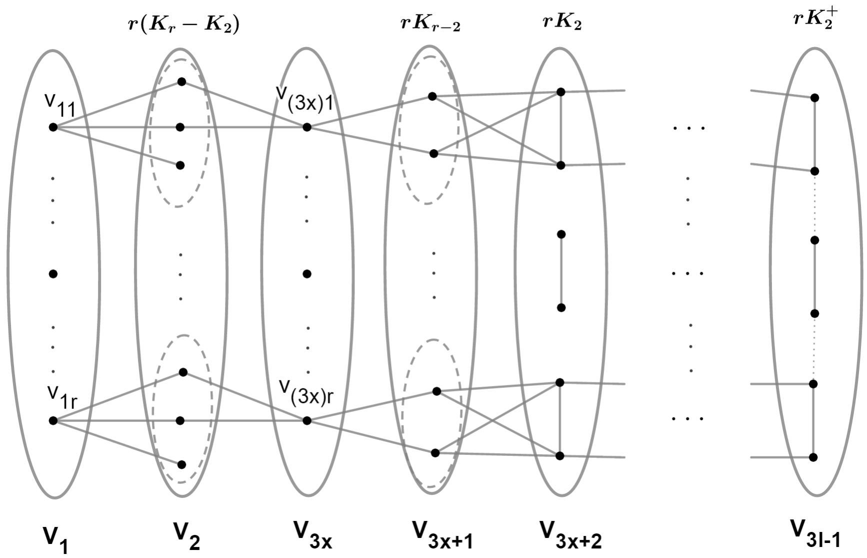

Figure 1: The graph $H^1_{r,k}$

Observation 2.2. *The graph $G^1_{r,k,t}$ in Definition 2.1 is an r-regular graph with $n = t\ell r(r+1)$ vertices and the k-independence number $\frac{n}{\ell(r+1)}$.*

Proof. By the definition of $G^1_{r,k,t}$, every vertex has degree r, V_1 is a k-independent set with size $\frac{n}{\ell(r+1)}$, and for any $u, v \in V_1$ and for any $x, y \in V(G^1_{r,k,t})$, we have $k+1 = d(u,v) \geq d(x,y)$. Also, we have $\sum_{i=1}^{3\ell-1} |V_i| = t\ell r(r+1)$, which gives the desired result. □

For $k = 6\ell - 4$, we can create other r-regular graphs with the k-independence number equal to $\frac{n}{\ell(r+1)}$ (see Figure 2 and 3).

Definition 2.3. For a positive integer ℓ, let $k = 6\ell - 3$. Let $H^4_{r,k}$ be the r-regular graph with the vertex sets $V_1, \ldots, V_{3\ell}$ satisfying the following properties:
(i) For $x \in [\ell - 1]$, follow the definitions of $V_1,\ V_2,\ V_{3x},\ V_{3x+1},\ V_{3x+2}$ in Definition 2.1, and in $V_{3\ell-1}$, for each $i \neq j \in [r]$ and for $h \in \{1,2\}$, $v^h_{(3\ell-1)i}$ is not adjacent to $v^h_{(3\ell-1)j}$. (Note that $V_{3\ell-1}$ in this definition is different from the one in Definition 2.1).
(ii) Let $V_{3\ell} = \{v_{(3\ell)1}, v_{(3\ell)2}\}$ such that for each $i \in [r]$, $v_{(3\ell)1}$ is adjacent to $v^1_{(3\ell-1)i}$ and $v_{(3\ell)2}$

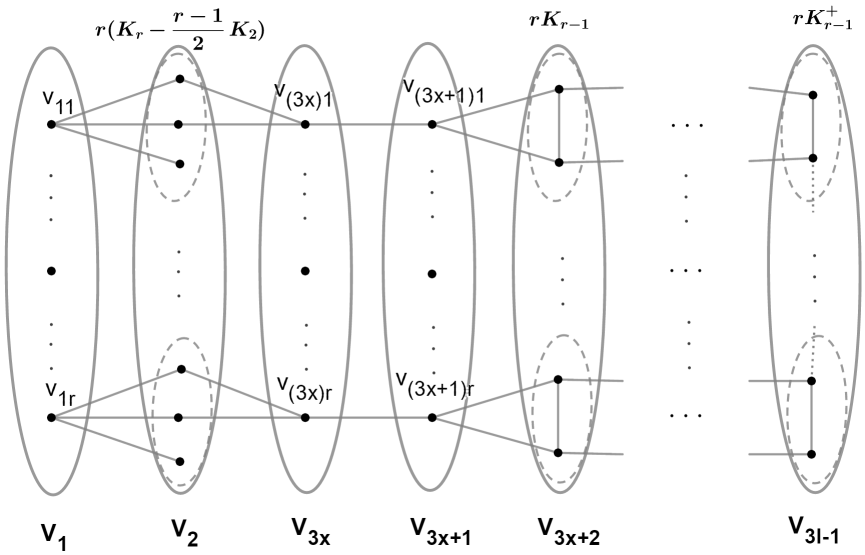

Figure 2: The graph $H^2_{r,k}$

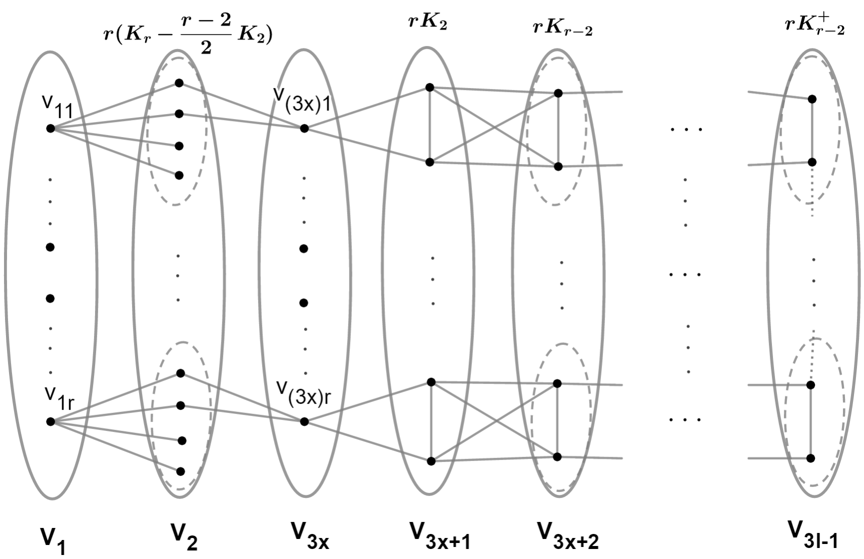

Figure 3: The graph $H^3_{r,k}$

is adjacent to $v^2_{(3\ell-1)i}$.
Let $G^4_{r,k,t}$ be the disjoint union of t copies of $H^4_{r,k}$ (see Figure 4).

Similarly to Observation 2.2, Definition 2.3 guarantees the following observation.

Observation 2.4. *The graph $G^4_{r,k,t}$ in Definition 2.3 is an r-regular graph with $n = t\ell r(r+1) + 2t$ vertices and the k-independence number $\frac{rn}{\ell r(r+1)+2}$.*

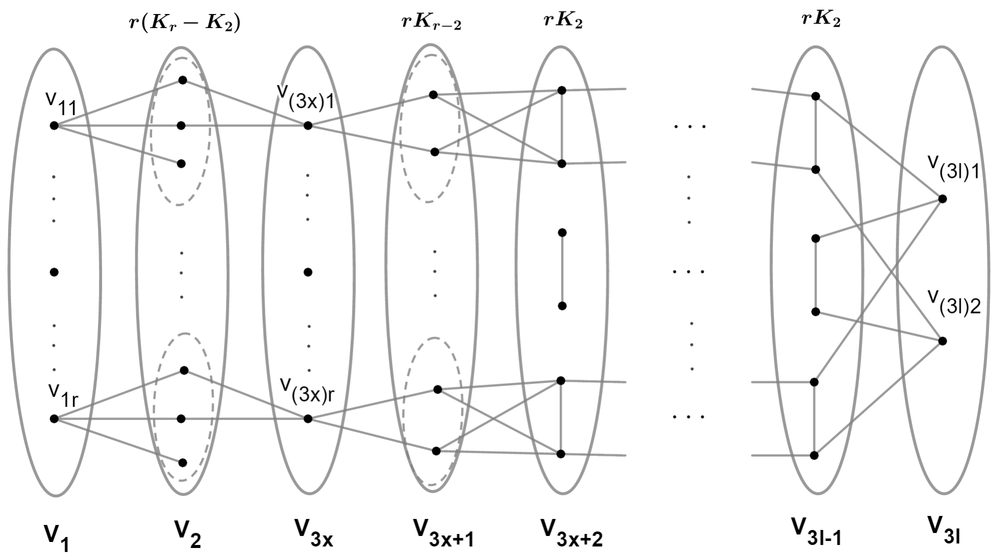

Figure 4: The graph $H^4_{r,k}$

Definition 2.5. For a positive integer ℓ, let $k = 6\ell - 2$. Let $H^5_{r,k}$ be the r-regular graph with the vertex sets $V_1, \ldots, V_{3\ell}$ satisfying the following properties:
(i) For $x \in [\ell - 1]$, follow the definitions of $V_1,\ V_2,\ V_{3x},\ V_{3x+1},\ V_{3x+2}$ in Definition 2.3 except $|V_1| = r - 1$.
(ii) In $V_{3\ell}$, $v_{(3\ell)i}$ is adjacent to $v_{(3\ell)j}$ for $i \neq j \in [r-1]$, i.e., the graph induced by $V_{3\ell}$ is a copy of K_{r-1}.
Let $G^5_{r,k,t}$ be the disjoint union of t copies of $H^5_{r,k}$ (see Figure 5).

Observation 2.6. *The graph $G^5_{r,k,t}$ in Definition 2.5 is an r-regular graph with $n = t\ell(r-1)(r+1) + t(r-1)$ vertices and the k-independence number $\frac{n}{\ell(r+1)+1}$.*

Definition 2.7. Let r be an odd interger at least 3, and for a positive integer ℓ, let $k = 6\ell - 1$. Let $H^6_{r,k}$ be the r-regular graph with the vertex sets $V_1, \ldots, V_{3\ell+1}$ satisfying the following properties:
(i) For $x \in [\ell]$, follow the definitions of $V_1,\ V_2,\ V_{3x},\ V_{3x+1},\ V_{3x+2}$ in graph $H^2_{r,k}$ (see Figure 2), except $V_{3\ell+1}$.(Note that $V_{3\ell-1}$ in this definition is different from the one in $H^2_{r,k}$).
(ii) Let $|V_{3\ell+1}| = 1$ such that all vertices in $V_{3\ell}$ are adjacent to the vertex in $V_{3\ell+1}$.
Let $G^6_{r,k,t}$ be the disjoint union of t copies of $H^6_{r,k}$ (see Figure 6).

Definition 2.8. Let r be an even interger at least 4, and for a positive integer ℓ, let $k = 6\ell - 1$. Let $H^7_{r,k}$ be the r-regular graph with the vertex sets $V_1, \ldots, V_{3\ell+1}$ satisfying the following

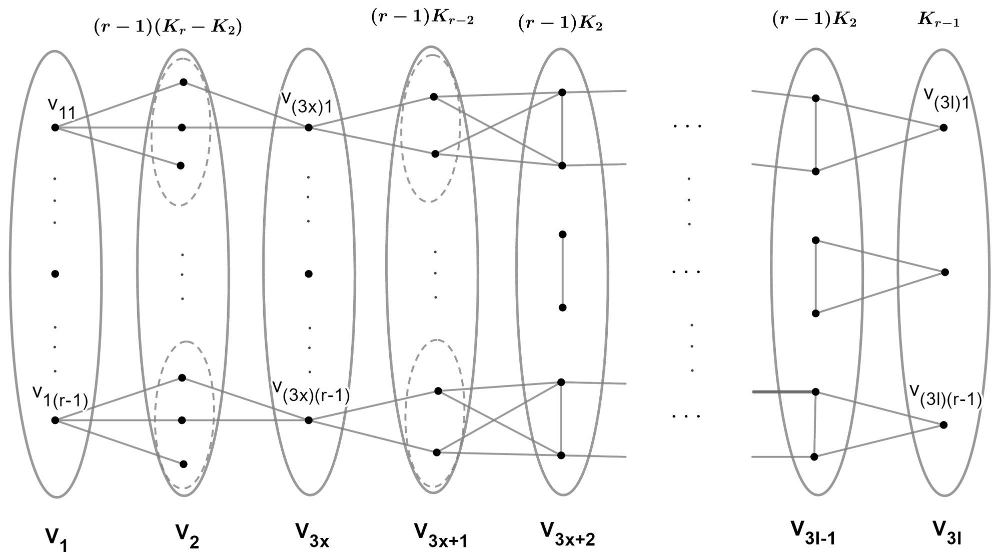

Figure 5: The graph $H^5_{r,k}$

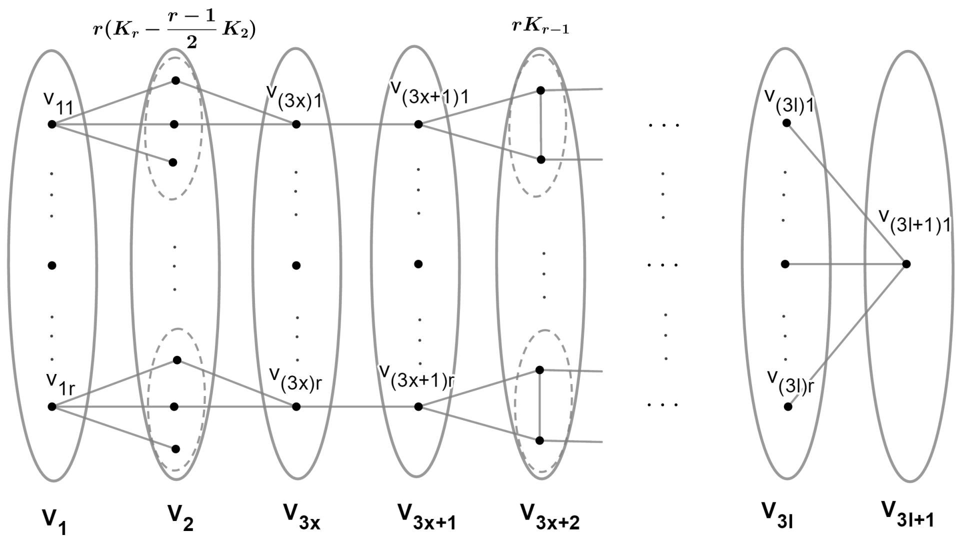

Figure 6: The graph $H^6_{r,k}$

properties:
(i) For $x \in [\ell]$, follow the definitions of $V_1,\ V_2,\ V_{3x},\ V_{3x+1},\ V_{3x+2}$ in graph $H^3_{r,k}$ (see Figure 3), except $V_{3\ell+1}$.(Note that $V_{3\ell-1}$ in this definition is different from the one in $H^3_{r,k}$).
(ii) Let $|V_{3\ell+1}| = 2$ such that all vertices in $V_{3\ell}$ are adjacent to the two vertices in $V_{3\ell+1}$, and $V_{3\ell+1}$ is independent.
Let $G^7_{r,k,t}$ be the disjoint union of t copies of $H^7_{r,k}$ (see Figure 7).

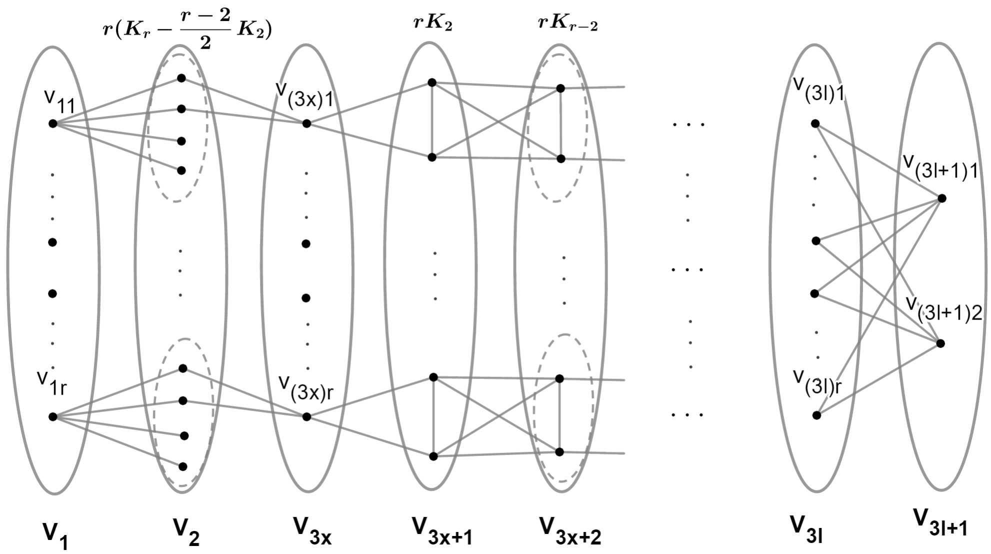

Figure 7: The graph $H^7_{r,k}$

Observation 2.9. *The graph $G^6_{r,k,t}$ in Definition 2.7 is an r-regular graph with $n = t(\ell r + 1)(r+1)$ vertices and the k-independence number $\frac{rn}{(\ell r+1)(r+1)}$.*
Also, the graph $G^7_{r,k,t}$ in Definition 2.8 is an r-regular graph with $n = t(\ell r + 1)(r + 1) + t$ vertices and the k-independence number $\frac{rn}{(\ell r+1)(r+1)+1}$.

Definition 2.10. Let r be an odd integer at least 3, and for a positive integer ℓ, let $k = 6\ell$. Let $H^8_{r,k}$ be the r-regular graph with the vertex sets $V_1, \ldots, V_{3\ell+1}$ satisfying the following properties:
(i) For $x \in [\ell]$, follow the definitions of $V_1,\ V_2,\ V_{3x},\ V_{3x+1},\ V_{3x+2}$ in Definition 2.7.(Note that $V_{3\ell+1}$ in this definition is different from the one in Definition 2.7).
(ii) In $V_{3\ell+1}$, $v_{(3\ell+1)i}$ is adjacent to $v_{(3\ell+1)j}$ for $i \neq j \in [r]$, i.e., the graph induced by $V_{3\ell+1}$ is copy of K_r.
Let $G^8_{r,k,t}$ be the disjoint union of t copies of $H^8_{r,k}$ (see Figure 8).

Definition 2.11. Let r be an even integer at least 4, and for a positive integer ℓ, let $k = 6\ell$. Let $H^9_{r,k}$ be the r-regular graph with the vertex sets $V_1, \ldots, V_{3\ell+1}$ satisfying the following properties:
(i) For $x \in [\ell]$, follow the definitions of $V_1,\ V_2,\ V_{3x},\ V_{3x+1},\ V_{3x+2}$ in Definition 2.8, except $|V_1| = \frac{r}{2}$.(Note that $V_{3\ell+1}$ in this definition is different from the one in Definition 2.8).
(ii) In $V_{3\ell+1}$, $v^h_{(3\ell+1)i}$ is adjacent to $v^h_{(3\ell+1)j}$ for $i \neq j \in [r]$ and $h \in \{1, 2\}$, i.e., the graph induced by $V_{3\ell+1}$ is a copy of K_r.
Let $G^9_{r,k,t}$ be the disjoint union of t copies of $H^9_{r,k}$ (see Figure 9).

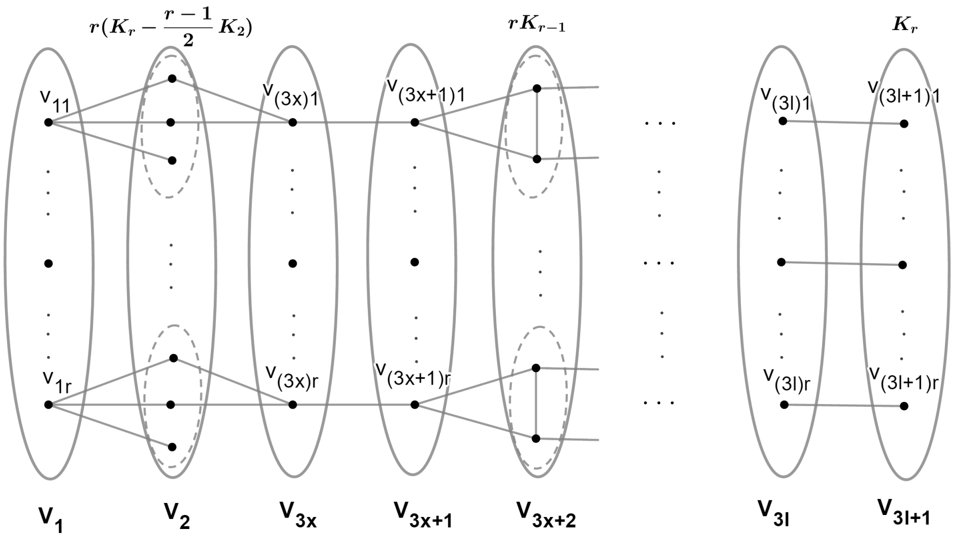

Figure 8: The graph $H^8_{r,k}$

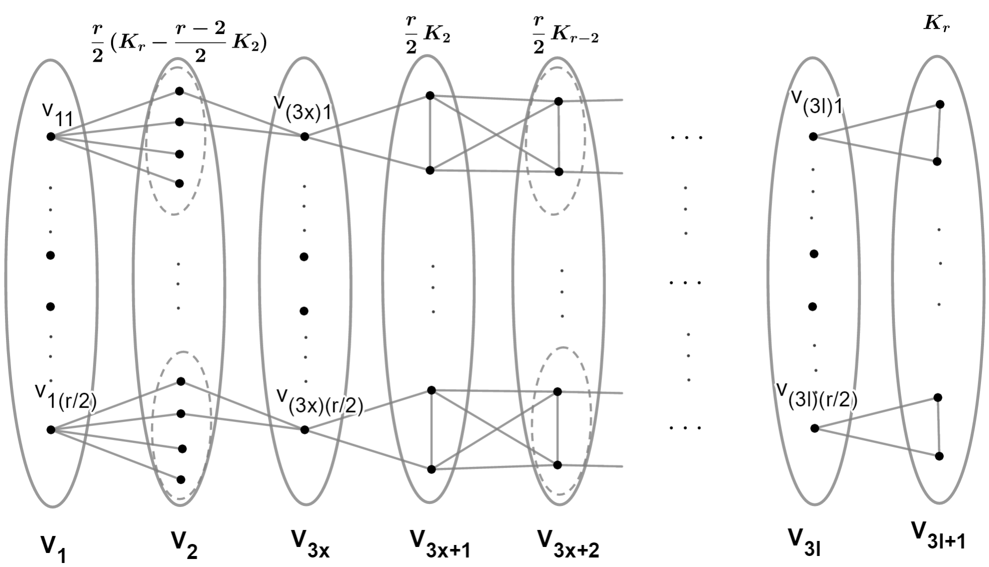

Figure 9: The graph $H^9_{r,k}$

Observation 2.12. *The graph $G^8_{r,k,t}$ in Definition 2.10 is an r-regular graph with $n = t\ell r(r+1) + 2tr$ vertices and the k-independence number $\frac{n}{\ell(r+1)+2}$.*
Also, the graph $G^9_{r,k,t}$ in Definition 2.11 is an r-regular graph with $\frac{t\ell r(r+1)+3tr}{2}$ vertices and the k-independence number $\frac{n}{\ell(r+1)+3}$.

Definition 2.13. Let r be an odd integer at least 3 and for a positive integer ℓ, let $k = 6\ell+1$. Let $H^{10}_{r,k}$ be the r-regular graph with the vertex sets $V_1, \ldots, V_{3\ell+2}$ satisfying the following

properties:
(i) For $x \in [\ell]$, follow the definitions of $V_1,\ V_2,\ V_{3x},\ V_{3x+1},\ V_{3x+2}$ in Definition 2.10, except $V_{3\ell+2}$.(Note that $V_{3\ell+1}$ in this definition is different from the one in Definition 2.10).
(ii) Let $V_{3\ell+2} = \{v_{(3\ell+2)1}, \cdots, v_{(3\ell+2)(r-1)}\}$ such that for each $i \in [r-1]$, $v_{(3\ell+2)i}$ is adjacent to all vertices in $V_{3\ell+1}$.
(iii) $V_{(3\ell+2)}$ is independent.
Let $G^{10}_{r,k,t}$ be the disjoint union of t copies of $H^{10}_{r,k}$ (see Figure 10).

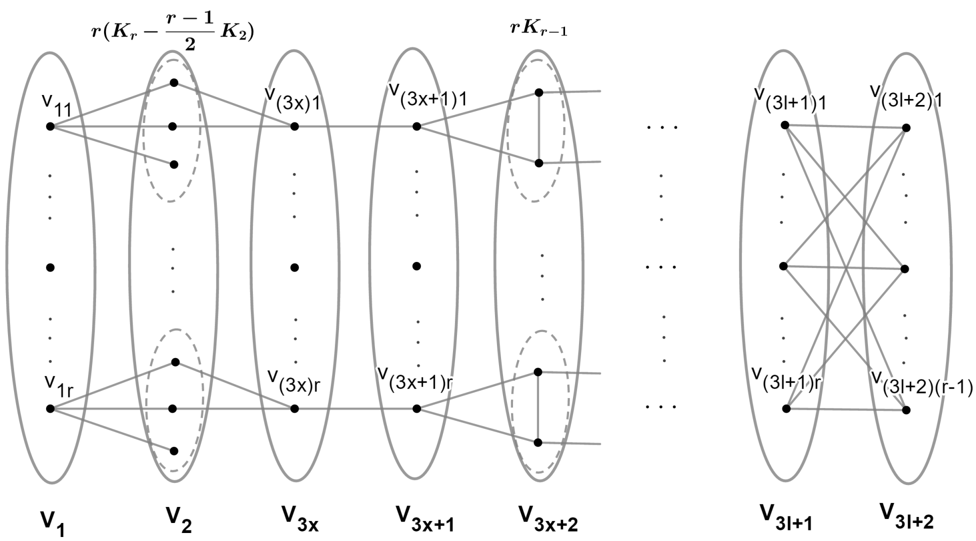

Figure 10: The graph $H^{10}_{r,k}$

Definition 2.14. Let r be an even integer at least 4 and for a positive integer ℓ, let $k = 6\ell+1$. Let $H^{11}_{r,k}$ be the r-regular graph with the vertex sets $V_1, \ldots, V_{3\ell+2}$ satisfying the following properties:
(i) For $x \in [\ell]$, make similiar definitions of $V_1,\ V_2,\ V_{3x},\ V_{3x+1},\ V_{3x+2}$ as $H^2_{r,k}$, but change the positions of V_{3x} and V_{3x+1} and make $V_{3\ell+2}$ and V_2 a little different.
(ii) Let $V_{3\ell+2} = \{v_{(3\ell+2)1}, \cdots, v_{(3\ell+2)(r-2)}\}$ such that for each $i \in [r-2]$, $v_{(3\ell+2)i}$ is adjacent to all vertices in $V_{3\ell+1}$.
(iii) $V_{(3\ell+2)}$ is independent.
Let $G^{11}_{r,k,t}$ be the disjoint union of t copies of $H^{11}_{r,k}$ (see Figure 11).

Observation 2.15. *The graph $G^{10}_{r,k,t}$ in Definition 2.13 is am r-regular graph with $n = t(\ell r+3)(r+1) - 4t$ vertices and the k-independence number $\frac{rn}{(\ell r+3)(r+1)-4}$.*
Also, the graph $G^{11}_{r,k,t}$ in Definition 2.14 is an r-regular graph with $t(\ell r+4)(r+1)-6t$ vertices and the k-independence number $\frac{rn}{(\ell r+4)(r+1)-6}$.

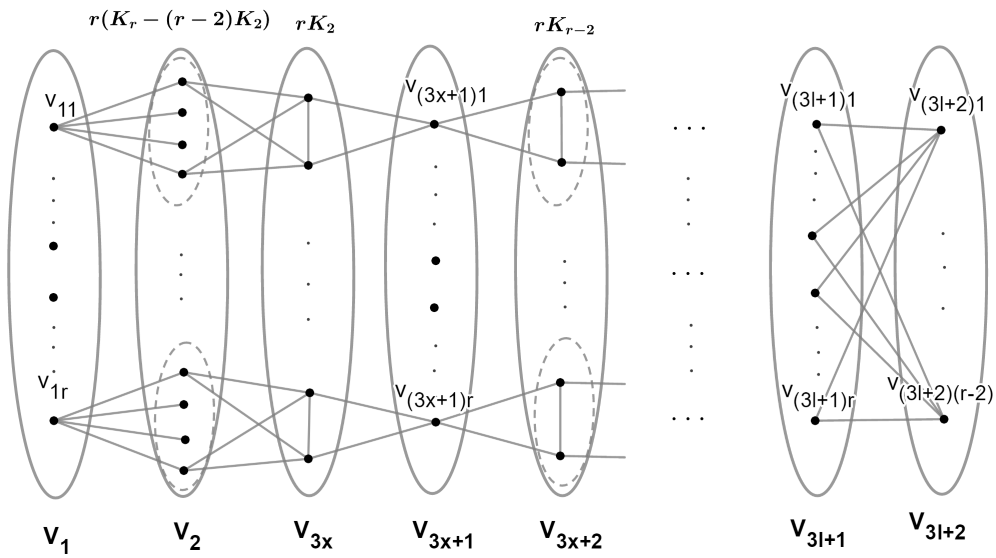

Figure 11: The graph $H^{11}_{r,k}$

3 Sharp Upper Bounds

In this section, for a positive integer k, we prove sharp upper bounds for $\alpha_k(G)$ in an n-vertex connected graph G with $diam(G) \geq k+1$. Before proving the bounds, we investigate the relevant properties of a k-independent set of G.

Now, We recall the definition of $N^i(S)$, which is the subsequent neighborhood of $N^{i-1}(S)$, i.e., $N^i(S) = N(N^{i-1}(S)) \setminus (N^{i-2}(S) \cup N^{i-1}(S))$. Note that $N^0(S) = S$ and $N^1(S) = N(S)$.

For $S \subseteq V(G)$, we denote by $G[S]$ the graph induced by S.

Lemma 3.1. *Let k be a positive integer and let G be an n-vertex connected graph with $diam(G) \geq k+1$. Suppose that S is a k-independent set in G. If $N^{i-1}(S),\ N^i(S),\ N^{i+1}(S)$ are three consecutive sets of G as defined, where $3 \leq i \leq \frac{k}{2}-1$, then we have*

$$|N^{i-1}(S)| + |N^i(S)| + |N^{i+1}(S)| \geq 3|S| \quad \textit{for any} \delta, \tag{1}$$

$$|N^{i-1}(S)| + |N^i(S)| + |N^{i+1}(S)| \geq (\delta+1)|S| \quad \textit{for } \delta \geq 2. \tag{2}$$

Proof. Let $v_j, v_h \in S$. Note that for $i \in \{0, \dots, \lfloor \frac{k}{2} \rfloor\}$, we have $N^i(v_j) \cap N^i(v_h) = \emptyset$ and for any $x \in S$, we have $N^i(x) \neq \emptyset$, which implies that $|N^i(S)| \geq |S|$.

Note that for each $u \in N^i(v_j)$, we have $N[u] \subseteq N^{i-1}(v_j) \cup N^i(v_j) \cup N^{i+1}(v_j)$, which implies $|N^{i-1}(v_j)| + |N^i(v_j)| + |N^{i+1}(v_j)| \geq 3$ for any δ. If $\delta \geq 2$, then we have $|N^{i-1}(v_j)| + |N^i(v_j)| + |N^{i+1}(v_j)| \geq \delta + 1$, which gives the desired result.

□

Lemma 3.1 is used to prove Theorem 3.2, which gives upper bounds for $\alpha_k(G)$ in an n-vertex connected graph with given minimum and maximum degree.

Theorem 3.2. *For positive integers k and ℓ, let δ and Δ be the minimum and maximum degree of G respectively. If G is an n-vertex connected graph with $diam(G) \geq k+1$, then we have*

1. *If $k = 1$, then $\alpha_k(G) \leq \frac{\Delta n}{\Delta+\delta}$.*
2. *If $k \geq 2$ and $\delta \leq 2$, then*
$$\alpha_k(G) \leq \begin{cases} \frac{\Delta n}{\Delta(\delta+\frac{k-1}{2})+1} & \text{if } k \text{ is odd,} \\ \frac{n}{\delta+\frac{k}{2}} & \text{if } k \text{ is even.} \end{cases} \tag{3}$$
3. *If $k = 6\ell-4$ and $\delta \geq 3$, then $\alpha_k(G) \leq \frac{n}{\ell(\delta+1)}$.*
4. *If $k = 6\ell-3$ and $\delta \geq 3$, then*
$$\alpha_k(G) \leq \begin{cases} \frac{\Delta n}{\ell\Delta+\ell\delta\Delta+1} & \text{if } \Delta > \delta, \\ \frac{\Delta n}{\ell\Delta+\ell\delta\Delta+2} & \text{if } \Delta = \delta. \end{cases} \tag{4}$$
5. *If $k = 6\ell-2$ and $\delta \geq 3$, then $\alpha_k(G) \leq \frac{n}{\ell(\delta+1)+1}$.*
6. *If $k = 6\ell-1$ and $\delta \geq 3$, then*
$$\alpha_k(G) \leq \begin{cases} \frac{\Delta n}{\ell\Delta(\delta+1)+\Delta+2} & \text{if } \Delta = \delta \text{ is even,} \\ \frac{\Delta n}{\ell\Delta(\delta+1)+\Delta+1} & \text{otherwise.} \end{cases} \tag{5}$$
7. *If $k = 6\ell$ and $\delta \geq 3$, then*
$$\alpha_k(G) \leq \begin{cases} \frac{n}{\ell(\delta+1)+3} & \text{if } \Delta = \delta \text{ is even,} \\ \frac{n}{\ell(r\delta+1)+2} & \text{otherwise.} \end{cases} \tag{6}$$
8. *If $k = 6\ell+1$ and $\delta \geq 3$, then*
$$\alpha_k(G) \leq \begin{cases} \frac{\Delta n}{\ell\Delta(\delta+1)+2\Delta+\delta-1} & \text{if } \delta \text{ is odd,} \\ \frac{\Delta n}{\ell\Delta(\delta+1)+3\Delta+\delta-2} & \text{if } \delta \text{ is even.} \end{cases} \tag{7}$$

For $i \in \{1, \ldots, 11\}$, $k \geq 2$, and $\delta \geq 3$, equalities hold for the graphs $G^i_{r,k,t}$.

Proof. Let S be a k-independent set of G. Note that $|S| \geq 1$.

Case 1: $k = 1$. Note that $|S_1|\delta \leq |[S,\overline{S_1}]| \leq \Delta(n-|S_1|)$, where $[S,T]$ is the set of edges with endpoints in both S and T. Thus we have $\alpha_1(G) \leq \frac{\Delta n}{\Delta+\delta}$. Equality in the bound requires that G is a (δ,Δ)-biregular, where a graph is (a,b)-biregular if it is bipartite with the vertices of one part all having degree a and the others all having degree b.

Case 2: $k \geq 2$ and $\delta \leq 2$. If k is odd, then we have $|N(S)| \geq \delta|S|$ and $|N^i(S)| \geq |S|$, where $i \in \{2,3,\cdots,t-1\}$ and $t = \frac{k+1}{2}$. Since $N^t(u) \cap N^t(v)$ may not be empty for $u, v \in S$, we have $|N^t(S)| \geq \frac{|S|}{\Delta}$. Thus we have $|S| + \delta|S| + (t-2)|S| + \frac{|S|}{\Delta} \leq n$, which gives the desired result. If $\delta = 1$ and $\Delta = \frac{n-1}{t}$, we have $|S| \leq \frac{2n-1}{k+1}$, which gives the bound in Theorem 1.1. Similiarly to the proof of odd k, for even k, we have $|N(S)| \geq \delta|S|$ and $|N^i(S)| \geq |S|$, where $i \in \{2,3,\cdots,t-1\}$ and $t = \frac{k}{2}$. However, $N^t(u) \cap N^t(v) = \emptyset$ for any $u, v \in S$. Thus we have $|N^t(S)| \geq |S|$. Then we have $|S| + \delta|S| + (t-1)|S| \leq n$, which gives the desired result. If $\delta = 1$, we have $|S| \leq \frac{2n}{2+k}$, which gives the bound in Theorem 1.1.

From Case 3, we assume that $\delta \geq 3$.

Case 3: $k = 6\ell - 4$. For any pair of vertices $u, v \in S$, we have $d(u,v) \geq 6\ell - 3$. Assume that u and v are two distinct vertices in S with $d(u,v) = 6\ell-3$. Then there is a path $P = \{u, x_1, \ldots, x_{3\ell-2}, y_{3\ell-2}, \ldots, y_1, v\}$ with legnth $6\ell - 3$. Note that since $d(u,v) = 6\ell - 3$, which is odd, we have $N^{3\ell-3}(u) \cap N^{3\ell-3}(v) = \emptyset$ and there are edges between $N^{3\ell-2}(u)$ and $N^{3\ell-2}(v)$.

Note that $|N^1(S)| \geq \delta|S|$ and S is k-independent. For a positive integer $h \in [\ell-1]$, by considering $N^{3h-1}(S), N^{3h}(S), N^{3h+1}(S)$ as a unit, we have at least $(\ell-1)$ units since S is a $(6\ell-4)$-independent set.

Thus by Lemma 3.1 (2), we have $|S| + \delta|S| + (\ell-1)(\delta+1)|S| \leq n$, which gives the desired result. Equality holds for the graphs $G^i_{r,k,t}$ for all $i \in \{1,2,3\}$ when $\delta = r$.

Case 4: $k = 6\ell - 3$. The proof is similar to that of Case 3. Since there are two vertices u and v in S such that $d(u,v) = 6\ell-2$, there is a path $P = \{u, x_1, \ldots, x_{3\ell-2}, z, y_{3\ell-2}, \ldots, y_1, v\}$ with legnth $6\ell - 2$. Note that $N^{3\ell-1}(u) \cap N^{3\ell-1}(v)$ can be non-empty.

Since there are $(\ell-1)$ units and $|N^{3\ell-1}(S)| \geq \frac{|S|}{\Delta}$ for $\Delta > \delta$, we have $|S| + \delta|S| + (\ell-1)(\delta+1)|S| + \frac{|S|}{\Delta} \leq n$. Since $|N^{3\ell-1}(S)| \geq \frac{2|S|}{\Delta}$ for $\Delta = \delta$, we have $|S|+\delta|S|+(\ell-1)(\delta+1)|S|+\frac{2|S|}{\Delta} \leq n$, which gives the desired results. Equality holds for the graph $G^4_{r,k,t}$ when $\delta = \Delta = r$.

Case 5: $k = 6\ell - 2$. In this case, we consider $N^{3h}(S), N^{3h+1}(S), N^{3h+2}(S)$ as a unit. Then by Lemma 3.1 (2), we have $|S| + \delta|S| + |S| + (\ell-1)(\delta+1)|S| \leq n$ since $|N^2(S)| \geq |S|$. Equality holds for the graph $G^5_{r,k,t}$ when $\delta = r$.

Case 6: $k = 6\ell - 1$. Similarly to Case 5, we consider $N^{3h}(S), N^{3h+1}(S), N^{3h+2}(S)$ as a unit. Since $N^{3\ell}(u) \cap N^{3\ell}(v)$ can be non-empty for two vertices u and v with $d(u,v) = 6\ell$, we have $|S| + \delta|S| + |S| + (\ell-1)(\delta+1)|S| + \frac{2|S|}{\Delta} \leq n$ for even $\Delta = \delta$ and we have $|S| + \delta|S| + |S| + (\ell-1)(\delta+1)|S| + \frac{|S|}{\Delta} \leq n$ for odd $\Delta = \delta$ or $\Delta > \delta$. Equalities hold for the

graphs $G^6_{r,k,t}$ and $G^7_{r,k,t}$ for $\delta = \Delta = r$ depending on the parity of r.
Case 7: $k = 6\ell$. In this case, we have $N^{3\ell}(u) \cap N^{3\ell}(v) = \emptyset$ for two vertices u and v with $d(u,v) = 6\ell + 1$. Thus for even $\Delta = \delta$, we have $|N^{3\ell}(S)| \geq 2|S|$, which implies $|S| + \delta|S| + |S| + (\ell-1)(\delta+1)|S| + 2|S| \leq n$, and for odd $\Delta = \delta$ or $\Delta > \delta$, we have $|N^{3\ell}(S)| \geq |S|$, which implies $|S| + \delta|S| + |S| + (\ell-1)(\delta+1)|S| + |S| \leq n$. Equalties hold for the graphs $G^8_{r,k,t}$ and $G^9_{r,k,t}$ when $\delta = \Delta = r$ depending on the parity of r.
Case 8: $k = 6\ell + 1$. Like Case 3, we consider $N^{3h-1}(S), N^{3h}(S), N^{3h+1}(S)$ as a unit. Note that $N^{3\ell+1}(u) \cap N^{3\ell+1}(v)$ can be non-empty for two vertices u and v with $d(u,v) = 6\ell + 2$. Thus for odd δ, we have $|N^{3\ell-1}(S)| \geq |S|$, $|N^{3\ell}(S)| \geq |S|$ and $|N^{3\ell+1}(S)| \geq \frac{(\delta-1)|S|}{\Delta}$, which implies $|S| + \delta|S| + (\ell-1)(\delta+1)|S| + |S| + |S| + \frac{(\delta-1)|S|}{\Delta} \leq n$, and for even δ, we have $|N^{3\ell-1}(S)| \geq 2|S|$, $|N^{3\ell}(S)| \geq |S|$ and $|N^{3\ell+1}(S)| \geq \frac{(\delta-2)|S|}{\Delta}$, which implies $|S| + \delta|S| + (\ell-1)(\delta+1)|S| + 2|S| + |S| + \frac{(\delta-2)|S|}{\Delta} \leq n$. Equalties hold for the graphs $G^{10}_{r,k,t}$ and $G^{11}_{r,k,t}$ when $\delta = \Delta = r$ depending on the parity of r. □

4 Questions

Aida, Cioabá, and Tait [1] obtained two spectral upper bounds for the k-independence number of a graph. They constructed graphs that attain equality for their first bound and showed that their second bound compares favorably to previous bounds on the k-independence number. We may ask whether given an independence number, there is an upper or lower bound for the spectral radius (the largest eigenvalue of a graph) in an n-vertex regular graph.

Question 4.1. *Given a positive integer t, what is the best lower bound for the spectral radius in an n-vertex r-regular graph to guarantee that $\alpha_k(G) \geq t + 1$?*

If for $r \geq 3$, G is an n-vertex r-regular graph, which is not a complete graph, then $\alpha_1(G) \geq \frac{n}{\chi(G)} \geq \frac{n}{r}$ by Brooks' Theorem. For $k \geq 2$, it is natural to ask a lower bound for $\alpha_k(G)$ in an n-vertex r-regular graph.

Question 4.2. *For $r \geq 3$, what is the best lower bound for $\alpha_k(G)$ in an n-vertex r-regular graph?*

The k-th power of the graph G, denoted by G^k, is a graph on the same vertex set as G such that two vertices are adjacent in G^k if and only if their distance in G is at most k. The k-distance t-coloring, also called distance (k,t)-coloring, is a k-coloring of the graph G^k (that is, any two vertices within distance k in G receive different colors). The k-distance chromatic number of G, written $\chi_k(G)$, is exactly the chromatic number of G^k. It is easy to see that $\chi(G) = \chi_1(G) \leq \chi_k(G) = \chi(G^k)$.

It was noted by Skupień that the well-known Brooks' theorem can provide the following upper bound:

$$\chi_k(G) \leq 1 + \Delta(G^k) \leq 1 + \Delta \sum_{i=1}^{k} (\Delta - 1)^{k-1} = 1 + \Delta \frac{(\Delta - 1)^k - 1}{\Delta - 2}, \quad (8)$$

for $\Delta \geq 3$. Let $M =: 1 + \Delta \frac{(\Delta-1)^k - 1}{\Delta - 2}$. Consider a $(k, \chi_k(G))$-coloring. Let V_i be the vertex set with the color i for $i \in [\chi_k(G)]$. Then we have $\chi_k(G)\alpha_k(G) \geq n$. Thus for $r \geq 3$, if G is an n-vertex r-regular graph, then we have $\alpha_k(G) \geq \frac{n}{\chi_k(G)} \geq \frac{n}{M}$. Since equality in inequality (8) holds only when G is a Moore graph, the lower bound is not tight. Thus, we might be interested in answering Question 4.2.

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References

[1] Abiad, A., Cioabá, S.M., Tait, M.: Spectral bounds for the k-independence number of a graph. Linear Algebra Appl. 510, 160–170 (2016)

[2] Alon, N., Mohar, B.: The chromatic number of graph powers. Combin. Probab. Comput. 11(1), 1–10 (2002)

[3] Beis, M., Duckworth, W., Zito, M.: Large k-independent sets of regular graphs. Electron. Notes in Discrete Math. 19, 321–327 (2005)

[4] Firby, P., Haviland, J.: Independence and average distance in graphs. Discrete Appl. Math. 75, 27–37 (1997)

[5] Goddard, W., Hedetniemi, S.M., Hedetniemi, S.T., Harris, J.M., Rall, D.F.: Broadcast chromatic numbers of graphs. Ars Combin. 86, 33–49 (2008)

[6] Hahn, G., Kratochvl, J., Širáň, J., Sotteau, D.: On the injective chromatic number of graphs. Discrete Math. 256, 179–192 (2002)

[7] Kong, M.C., Zhao, Y.: On computing maximum k-independent sets. Congr. Numer. 95, 47–60 (1993)

[8] Kong, M.C., Zhao, Y.: Computing k-Independent sets for regular bipartite graphs. Congr. Numer. 143, 65–80 (2000)

[9] Mahdian, M.: The strong chromatic index of graphs. MSc thesis, University of Toronto, (2000)

[10] Z. Skupień, Some maximum multigraphs and edge-vertex distance colourings, *Discuss. Math. Graph Theory*, **15** (1995), 89–106.

[11] D.B. West, Introduction to Graph Theory, Prentice Hall, Inc., Upper Saddle River, NJ, 2001.