Vertices of Schubitopes

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Abstract. Schubitopes were introduced by Monical, Tokcan and Yong as a specific family of generalized permutohedra. It was proven by Fink, Mészáros and St. Dizier that Schubitopes are the Newton polytopes of the dual characters of flagged Weyl modules. Important cases of Schubitopes include the Newton polytopes of Schubert polynomials and key polynomials. In this paper, we develop a combinatorial rule to generate the vertices of Schubitopes. As an application, we show that the vertices of the Newton polytope of a key polynomial can be generated by permutations in a lower interval in the Bruhat order, settling a conjecture of Monical, Tokcan and Yong.

1 Introduction

The objective of this paper is to investigate the vertices of Schubitopes introduced by Monical, Tokcan and Yong [18] during their study of Newton polytopes in algebraic combinatorics. Schubitopes are a specific family of generalized permutohedra extensively studied by Postnikov [21]. It was conjectured by Monical, Tokcan and Yong [18] and shown by Fink, Mészáros and St. Dizier [8] that the Newton polytopes of Schubert and key polynomials are Schubitopes. More generally, Fink, Mészáros and St. Dizier [8] showed that Schubitopes are the Newton polytopes of the dual characters of flagged Weyl modules.

We provide a combinatorial algorithm to generate the vertices of Schubitopes. As an application, we prove that the vertices of the Newton polytope of a key polynomial can be generated by permutations in a lower interval in the Bruhat order, thus confirming a conjecture of Monical, Tokcan and Yong [18, Conjecture 3.13]. This also establishes a connection between the Newton polytopes of key polynomials associated to permutations and the Bruhat interval polytopes introduced by Kodama and Williams [13].

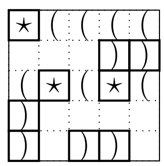
Schibitopes are polytopes associated to diagrams in an $n \times n$ grid. A diagram D is a collection of boxes in an $n \times n$ grid. We adopt the notation $[n] = \{1, 2, \ldots, n\}$. We also abbreviate an $n \times n$ grid to $[n]^2$, and use (i, j) to denote the box in row i and column j. Here the rows (resp., columns) are labeled $1, 2, \ldots, n$ from top to bottom (resp., from left to right). The Schubitope $\mathcal{S}_D$ associated to D can be defined as follows. For $1 \leq j \leq n$ and a subset S of, define a string $\mathrm{word}_{j,S}(D)$ by reading the j-th column of the $n \times n$ grid from top to bottom and recording:

- (if $(i,j) \notin D$ and $i \in S$;
-) if $(i,j) \in D$ and $i \notin S$;
- $\star$ if $(i,j) \in D$ and $i \in S$.

Let

$$\theta_D^j(S) = \#\{\text{paired ()'s in } \mathrm{word}_{j,S}(D)\} + \#\{\star\text{'s in } \mathrm{word}_{j,S}(D)\},$$

where the pairing is by the standard "inside-out" convention. For example, for the following diagram and $S = \{1,3\}$, the strings $\mathrm{word}_{j,S}(D)$ along with the corresponding values $\theta_D^j(S)$ (which are abbreviated as θ^j) are illustrated below.



$$\begin{array}{ll} \mathrm{word}_{1,S}(D) = \star()) & \theta^1 = 2 \\ \mathrm{word}_{2,S}(D) = (\star & \theta^2 = 1 \\ \mathrm{word}_{3,S}(D) = (() & \theta^3 = 1 \\ \mathrm{word}_{4,S}(D) = ()\star) & \theta^4 = 2 \\ \mathrm{word}_{5,S}(D) = ()(& \theta^5 = 1. \end{array}$$

Set

$$\theta_D(S) = \sum_{j=1}^{n} \theta_D^j(S).$$

The Schubitope $\mathcal{S}_D$ is defined by

$$\mathcal{S}_D = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \colon \sum_{i \in [n]} x_i = \#D \ \text{ and } \ \sum_{i \in S} x_i \le \theta_D(S) \ \text{ for } S \subsetneq [n] \right\}.$$

By definition, $\mathcal{S}_D$ is a generalized permutohedren parameterized by $\{\theta_D(S)\}$, see for example Postnikov [21].

In this paper, we characterize the vertices of the Schubitopes $\mathcal{S}_D$ in terms of certain fillings of D. Let S_n denote the set of permutations of $[n]$. Given a permutation $w = w_1 w_2 \cdots w_n \in S_n$, define $\mathcal{F}_w(D)$ to be the filling of D with the entries of w as follows. The filling is described based on an assignment of the entries of w into each column of D, independent of the order of columns. For the j-th column D_j, fill the integers $w_1, \ldots, w_n$ in turn into the empty boxes of D_j as below. From $k = 1$ to $k = n$, put w_k into the first (from top to bottom) empty box whose row index is larger than or equal to w_k. If there are no such empty boxes, then w_k does not appear in the filling and skip to w_{k+1}. For example, Figure 1.1 illustrates the filling $\mathcal{F}_w(D)$ for $w = 315624$.

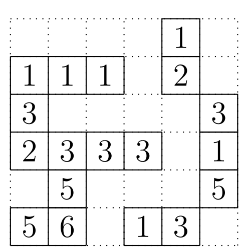


Figure 1.1: The filling $\mathcal{F}_w(D)$ for $w = 315624$.

Theorem 1.1. *Let D be a diagram of $[n]^2$. Then the vertex set of the Schubitope $\mathcal{S}_D$ is*

$$\{x(w) \colon w \in S_n\},$$

where $x(w) = (x_1, x_2, \ldots, x_n)$ is the vector such that x_k $(1 \le k \le n)$ is the number of appearances of k in $\mathcal{F}_w(D)$.

For the running example as displayed in Figure 1.1, we have $x(w) = (6, 2, 6, 0, 3, 1)$.

Let us use an example to demonstrate Theorem 1.1. Let $D = \{(1,1), (3,1), (3,2), (3,3)\}$ be a diagram of $[3]^2$. The fillings $\mathcal{F}_w(D)$ for the six permutations $w = 123$, 132, 213, 231, 312, 321 are listed in Figure 1.2 in turn from left to right. These fillings gener-

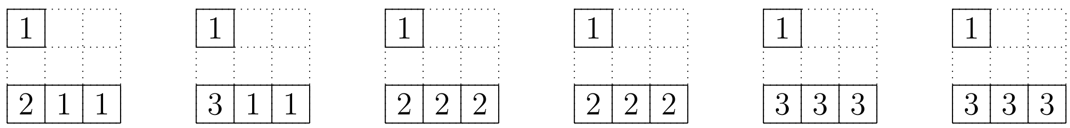


Figure 1.2: The six fillings of D for $D = \{(1,1), (3,1), (3,2), (3,3)\}$.

ate four vertices: $x(123) = (3,1,0)$, $x(132) = (3,0,1)$, $x(213) = x(231) = (1,3,0)$, and $x(312) = x(321) = (1,0,3)$. The corresponding Schubitope $\mathcal{S}_D$ is a trapezoid as displayed in Figure 1.3, where the lattice points in $\mathcal{S}_D$ are signified by bullets.

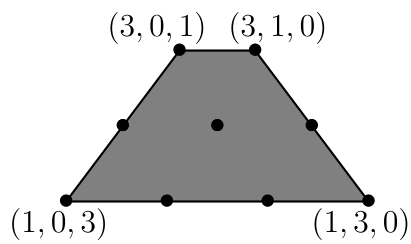


Figure 1.3: The Schubitope $\mathcal{S}_D$ for $D = \{(1,1), (3,1), (3,2), (3,3)\}$.

Given a polynomial $f = \sum_{\alpha \in \mathbb{Z}^n_{\ge 0}} c_\alpha x^\alpha \in \mathbb{R}[x_1, \ldots, x_n]$, the Newton polytope of f is the convex hull of the exponent vectors of f, namely,

$$\mathrm{Newton}(f) = \mathrm{conv}(\{\alpha \colon c_\alpha \neq 0\}).$$

Specifying D to the Rothe diagram $D(w)$ of a permutation w, the Schubitope $\mathcal{S}_{D(w)}$ is the Newton polytope $\mathrm{Newton}(\mathfrak{S}_w)$ of the Schubert polynomial $\mathfrak{S}_w(x)$ [8, 18]. Schubert polynomials were introduced by Lascoux and Schützenberger [15], which represent the cohomology classes of Schubert cycles in flag varieties. Schubert polynomials can be defined in terms of the divided difference operator ∂_i, which sends a polynomial f to

$$\partial_i f = (f - s_i f)/(x_i - x_{i+1}),$$

where $s_i f$ is obtained from f by exchanging x_i and x_{i+1}. For the permutation $w_0 = n\,(n-1) \cdots 1$, set $\mathfrak{S}_{w_0}(x) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$. For $w \neq w_0$, choose a position $1 \le i < n$ such that $w_i < w_{i+1}$. Let w' be the permutation obtained from w by interchanging w_i and w_{i+1}. Set $\mathfrak{S}_w(x) = \partial_i \mathfrak{S}_{w'}(x)$.

The Rothe diagram $D(w)$ of $w \in S_n$ is the diagram obtained from the $n \times n$ grid by deleting the box (i, w_i) as well as the boxes to the right of (i, w_i) or below (i, w_i). Figure 1.4(a) illustrates the Rothe diagram of $w = 1432$. So, when D is the Rothe diagram $D(w)$, Theorem 1.1 gives a characterization of the vertices of Newton$(\mathfrak{S}_w)$.

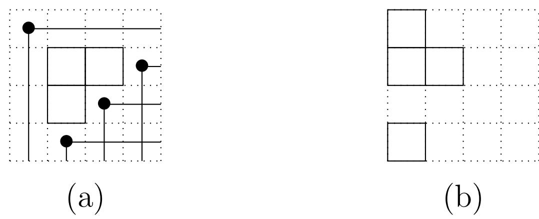


Figure 1.4: (a) $D(w)$ for $w = 1432$, (b) $D(\alpha)$ for $\alpha = (1, 2, 0, 1)$.

When D is restricted to the skyline diagram $D(\alpha)$ of a composition α, the Schubitope $\mathcal{S}_{D(\alpha)}$ is the Newton polytope Newton(κ_α) of the key polynomial $\kappa_\alpha(x)$ [8, 18]. Key polynomials, also called Demazure characters, are characters of the Demazure modules for the general linear groups [5, 6]. Key polynomials can be defined using the Demazure operator $\pi_i = \partial_i x_i$. If α is a partition, then set $\kappa_\alpha(x) = x^\alpha$. Otherwise, choose i such that $\alpha_i < \alpha_{i+1}$. Let α' be the composition obtained from α by interchanging α_i and α_{i+1}. Set $\kappa_\alpha(x) = \pi_i \kappa_{\alpha'}(x)$. It is known that $\kappa_\alpha(x)$ can be realized as a specialization of the nonsymmetric Macdonald polynomial $E_\alpha(x; q, t)$ at $q = t = \infty$, see Ion [12]. It is also worth mentioning that every Schubert polynomial is a positive sum of key polynomials, see for example Assaf [3], Lascoux and Schützenberger [16], or Reiner and Shimozono [22].

The skyline diagram $D(\alpha)$ of a composition α is the diagram consisting of the first α_i boxes in row i, see Figure 1.4(b) for the skyline diagram of $\alpha = (1, 2, 0, 1)$. In this case, Theorem 1.1 can be employed to generate the vertices of Newton(κ_α).

Monical, Tokcan and Yong [18, Conjecture 3.13] conjectured an alternative characterization of the vertices of Newton(κ_α) in terms of the Bruhat order on permutations. Let α be a composition, and $\lambda(\alpha)$ be the partition obtained by resorting the parts of α decreasingly. Write $w(\alpha)$ for the (unique) permutation of shortest length that sends $\lambda(\alpha)$ to α. Here, given a permutation $w = w_1 \cdots w_n \in S_n$ and a vector $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, the (right) action of w on v is defined as

$$v \cdot w = (v_{w_1}, \ldots, v_{w_n}). \tag{1.1}$$

For two compositions α, β, define

$$\beta \leq \alpha \quad \text{if} \quad \lambda(\beta) = \lambda(\alpha) \quad \text{and } w(\beta) \leq w(\alpha) \quad \text{in the Bruhat order.} \tag{1.2}$$

Searles [23] gave an alternative description of the partial order in (1.2). For $i < j$ and $\alpha_i < \alpha_j$, let $t_{i,j}(\alpha)$ be obtained from α by interchanging α_i and α_j. Then $\beta \leq \alpha$ if and only if β can be obtained from α by applying a sequence of $t_{i,j}$ [23, Lemma 3.1].

Based on the decomposition of a key polynomial into Demazure atoms [10, 14, 17], Monical, Tokcan and Yong [18, Theorem 3.12] showed that if $\beta \leq \alpha$, then β is a vertex of Newton(κ_α). They [18, Conjecture 3.13] conjectured that the converse is still true, that is, if β is a vertex of $\mathrm{Newton}(\kappa_\alpha)$, then $\beta \leq \alpha$. Applying Theorem 1.1 together with some analysis on skyline diagrams, we confirm this conjecture.

Theorem 1.2. *Let α be a composition. Then the vertex set of the Newton polytope* $\mathrm{Newton}(\kappa_\alpha)$ *is* $\{\beta\colon \beta \leq \alpha\}$.

For example, let $\alpha = (1, 0, 3)$. The key polynomial corresponding to α is

$$\kappa_\alpha(x) = x_1^3x_2 + x_1^3x_3 + x_1^2x_2^2 + x_1^2x_2x_3 + x_1^2x_3^2 + x_1x_2^3 + x_1x_2^2x_3 + x_1x_2x_3^2 + x_1x_3^3.$$

By Theorem 1.2, it is easily checked that $\mathrm{Newton}(\kappa_\alpha)$ has vertex set

$$\{\beta\colon \beta \leq \alpha\} = \{(3, 1, 0), (3, 0, 1), (1, 3, 0), (1, 0, 3)\}.$$

Notice that the skyline diagram of α is the diagram shown in Figure 1.2. Hence $\mathrm{Newton}(\kappa_\alpha)$ agrees with the Schubitope in Figure 1.3.

When the parts of α are weakly increasing, $\kappa_\alpha(x)$ is the Schur polynomial $s_{\lambda(\alpha)}(x)$ [22]. In this case, Theorem 1.2 implies the classical result that the Newton polytope of a Schur polynomial $s_\lambda(x)$ is $\mathcal{P}_\lambda$, the permutohedron whose vertices are rearrangements of λ.

Remark. The permutations in S_n are usually redundant to generate vertices of a Schubitope, as can be seen in the example illustrated in Figure 1.2. It is natural to ask which permutations are needed to obtain all vertices of a Schubitope. In other words, for two permutations w and w' in S_n, find a characterization to determine whether $x(w) = x(w')$. Propositions 4.4 and 4.5 seem relevant to this question. When D is a skyline diagram, Theorem 1.2 essentially implies that permutations in a lower Bruhat interval are enough to generate the vertices. In the case when D is a Rothe diagram, we still do not know if Theorem 1.1 could be simplified to a version similar to Theorem 1.2 for a skyline diagram.

Theorem 1.2 also establishes a connection between the Newton polytopes of certain key polynomials and Bruhat interval polytopes. For two permutations $u \leq v$ in the Bruhat order, the Bruhat interval polytope $\mathsf{Q}_{u,v}$ is the convex hull of the permutations in the Bruhat interval $[u, v]$. Bruhat interval polytopes were introduced by Kodama and Williams [13] in the context of the Toda lattice and the moment map on the flag variety, and their combinatorial properties were studied by Tsukerman and Williams [24]. The following corollary is a direct consequence of Theorem 1.2.

Corollary 1.3. *Let $w = w_1 \cdots w_n \in S_n$ be a permutation. View w as a composition $(w_1, \ldots, w_n)$. Then the Newton polytope* $\mathrm{Newton}(\kappa_w)$ *of $\kappa_w(x)$ is the Bruhat interval polytope Q_{w,w_0}, where $w_0 = n \cdots 21$ is the largest permutation of S_n in the Bruhat order.*

This paper is structured as follows. In Section 2, we review a result shown in [8] that Schubitopes are Minkowski sums of Schubert matroid polytopes. This implies that the Schubitope $\mathcal{S}_D$ is a base polytope associated to some submodular function. Edmonds [7] found a characterization of vertices of base polytopes for submodular functions. Based on Edmonds's characterization, we prove Theorem 1.1 in Section 3. In the final section, we present a proof of Theorem 1.2.

2 Schubert matroid polytopes

A matroid is a pair $M = (E, \mathcal{I})$ consisting of a finite set E and a collection $\mathcal{I}$ of subsets of E, called independent sets, such that

(i) $\emptyset \in \mathcal{I}$;

(ii) If $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$;

(iii) If $I, J \in \mathcal{I}$ and $|I| < |J|$, then there exists $j \in J \setminus I$ such that $I \cup \{j\} \in \mathcal{I}$.

By (ii), a matroid M is determined by the collection $\mathcal{B}$ of maximal independent sets, called the bases of M. So we can write $M = (E, \mathcal{B})$. Moreover, it follows from (iii) that the bases have the same size. Equivalently, a matroid $M = (E, \mathcal{B})$ can be defined by means of the exchange axiom for bases:

(i') $\mathcal{B} \neq \emptyset$;

(ii') If $A, B \in \mathcal{B}$ and $a \in A \setminus B$, then there exists $b \in B \setminus A$ such that $(A \setminus \{a\}) \cup \{b\} \in \mathcal{B}$.

Let S be a subset of $[n]$. The Schubert matroid $SM_n(S)$ is the matroid with basis

$$\{T \subseteq [n] \colon T \leq S\}.$$

The notation $T \leq S$ means that

(1) $\#T = \#S$;

(2) If we write $T = \{a_1 < a_2 < \cdots < a_k\}$ and $S = \{b_1 < b_2 < \cdots < b_k\}$, then $a_i \leq b_i$ for $1 \leq i \leq k$.

As pointed out by an anonymous referee, Schubert matroids have been rediscovered in different contexts, which have been called freedom matroids, generalized Catalan matroids, PI-matroids, and shifted matroids, among others, see Ardila, Fink and Rincón [2, Example 2.4], or the comments after [1, Theorem 4.1] by Ardila and the comments after [4, Corollary 3.13] by Bonin, de Mier and Noy. It should also be noted that Schubert matroids are specific families of lattice path matroids [4], or more generally transversal matroids [1] and positroids [19, Lemma 23].

Given a matroid $M = (E, \mathcal{B})$ with $E = [n]$, the associated matroid polytope of M is constructed as follows. Let $\{e_i \colon 1 \leq i \leq n\}$ be the standard basis of $\mathbb{R}^n$. For a subset $B = \{b_1, \ldots, b_k\}$ of $[n]$, write

$$e_B = e_{b_1} + \cdots + e_{b_k}.$$

The matroid polytope $P(M)$ is defined by

$$P(M) = \mathrm{conv}\{e_B \colon B \in \mathcal{B}\}.$$

The matroid polytope is a generalized permutohedron parameterized by its rank function $\{r_M(S)\}$, see [8] for a reference. To be specific,

$$P(M) = \left\{ x \in \mathbb{R}^n \colon \sum_{i \in [n]} x_i = r_M([n]) \;\text{ and }\; \sum_{i \in S} x_i \leq r_M(S) \;\text{ for } S \subsetneq [n] \right\}, \tag{2.1}$$

where the rank function r_M of M is a map from the subsets of E to $\mathbb{Z}_{\geq 0}$ defined by

$$r_M(S) = \max\{\#(S \cap B) \colon B \in \mathcal{B}\}, \quad \text{for } S \subseteq E.$$

It turns out that the Schubitope $\mathcal{S}_D$ is the Minkowski sum of Schubert matroid polytopes associated to the columns of D. Let D be a diagram of $[n]^2$. Write $D = (D_1, \ldots, D_n)$, where, for $1 \leq j \leq n$, D_j is the j-th column of D. The column D_j can be viewed as a subset of $[n]$:

$$D_j = \{1 \leq i \leq n \colon (i,j) \in D\}.$$

Then the column D_j defines a Schubert matroid $SM_n(D_j)$. For two polytopes P and Q, the Minkowski sum of P and Q is defined as

$$P + Q = \{u + v \colon u \in P, v \in Q\}.$$

Theorem 2.1 (Fink-Mészáros-St. Dizier [8])**.** *Let $D = (D_1, \ldots, D_n)$ be a diagram of $[n]^2$, and let r_j denote the rank function of $SM_n(D_j)$. Then*

$$\begin{aligned} \mathcal{S}_D &= P(SM_n(D_1)) + \cdots + P(SM_n(D_n)) \\ &= \left\{ x \in \mathbb{R}^n \colon \sum_{i \in [n]} x_i = \#D \;\textit{ and }\; \sum_{i \in S} x_i \leq r_D(S) \;\textit{ for } S \subsetneq [n] \right\}, \end{aligned} \tag{2.2}$$

where

$$r_D(S) = r_1(S) + \cdots + r_n(S). \tag{2.3}$$

3 Proof of Theorem 1.1

In this section, we present a proof of Theorem 1.1. A crucial observation is that the Schubitope $\mathcal{S}_D$ is the base polytope associated to the function r_D. Edmonds [7] obtained a characterization of the vertices of any given base polytope. Based on Edmonds's characterization, we arrive at a proof of Theorem 1.1.

3.1 Schubitopes are base polytopes

Base polytopes are polytopes associated to submodular functions. A function f from the subsets of $[n]$ to $\mathbb{R}$ is called a submodular function, if, for any subsets $S, T \subseteq [n]$,

$$f(S) + f(T) \geq f(S \cup T) + f(S \cap T).$$

To a submodular function f, the associated base polytope B_f is defined by

$$B_f = \left\{ x \in \mathbb{R}^n \colon x_1 + \cdots + x_n = f([n]), \quad \sum_{i \in S} x_i \leq f(S) \ \text{ for } S \subsetneq [n] \right\}.$$

Using the greedy algorithm, Edmonds [7] obtain the following description of the vertices of base polytopes for submodular functions, see also [9, Theorem 3.22].

Theorem 3.1 ([7,9]). *Let $f \colon 2^{[n]} \to \mathbb{R}$ be a submodular function. Then the vertex set of the base polytope B_f is precisely*

$$\{x(w) \colon w \in S_n\},$$

where $x(w) = (x_1, \ldots, x_n)$ is the vector in $\mathbb{R}^n$ defined by

$$x_{w_k} = f(\{w_1, \ldots, w_k\}) - f(\{w_1, \ldots, w_{k-1}\}).$$

A fundamental property of a matroid M is that its rank function r_M is submodular [20]. Hence the function r_D defined in (2.3) is submodular. By Theorem 3.1, we obtain the following characterization of the vertex set of a Schubitope.

Theorem 3.2. *Let D be a diagram of $[n]^2$. Then the vertex set of the Schubitope $\mathcal{S}_D$ is*

$$\{x(w) \colon w \in S_n\},$$

where $x(w) = (x_1, \ldots, x_n)$ is the vector in $\mathbb{R}^n$ defined by

$$x_{w_k} = r_D(\{w_1, \ldots, w_k\}) - r_D(\{w_1, \ldots, w_{k-1}\}).$$

3.2 Rank function of a Schubert matroid

Throughout this subsection, we let C be a column of a diagram of $[n]^2$. Of course, we can regard C itself as a diagram of $[n]^2$ such that the boxes lie in exactly one column. Let $SM_n(C)$ be the Schubert matroid associated to C. We show that the filling $\mathcal{F}_w(C)$ generated by the algorithm in Introduction can be used to compute the rank function r_C of $SM_n(C)$. This, together with Theorem 3.2, leads to a proof of Theorem 1.1.

A filling $\mathcal{F}$ of C is an assignment of positive integers into some of the boxes of C. A box of $\mathcal{F}$ is called empty if it is not assigned any number. A filling $\mathcal{F}$ is called column-strict if the numbers appearing in $\mathcal{F}$ are distinct, and $\mathcal{F}$ is called flagged if for any nonempty box in row i, the number assigned in it does not exceed i. For a subset S of $[n]$, we denote by $\mathcal{F}(C,S)$ the set of column-strict flagged fillings $\mathcal{F}$ of C such that all the integers appearing in $\mathcal{F}$ belong to S. We also denote $\mathcal{F}_{\leq}(C,S)$ to be the subset consisting of the fillings $\mathcal{F} \in \mathcal{F}(C,S)$ such that the numbers in $\mathcal{F}$ are increasing from top to bottom. Let $|\mathcal{F}|$ denote the number of non-empty boxes of $\mathcal{F}$.

For a permutation π of a subset S of $[n]$, we can generate a filling $\mathcal{F}_\pi(C)$ of C by the algorithm given in Introduction. Notice that there may exist empty boxes in $\mathcal{F}_\pi(C)$.

Theorem 3.3. *Let C be a column of a diagram of $[n]^2$, and r_C be the rank function of $SM_n(C)$. For a k-subset S of $[n]$, let $\pi = \pi_1\pi_2\cdots\pi_k$ be any given permutation of elements of S. Then*

$$r_C(S) = |\mathcal{F}_\pi(C)|. \tag{3.1}$$

To prove Theorem 3.3, we need the following characterization of the rank function r_C.

Theorem 3.4. *For any subset S of $[n]$, we have*

$$r_C(S) = \max\{|\mathcal{F}|\colon \mathcal{F} \in \mathcal{F}(C,S)\}. \tag{3.2}$$

To prove Theorem 3.4, we define two operations acting on $\mathcal{F}(C,S)$ and $\mathcal{F}_\leq(C,S)$, respectively. Let $\mathcal{F} \in \mathcal{F}(C,S)$. The first one is the sorting operation, which transforms $\mathcal{F}$ to a filling $\mathrm{sort}(\mathcal{F})$ by keeping the empty boxes of $\mathcal{F}$ unchanged and rearranging the entries of $\mathcal{F}$ increasingly from top to bottom. Figure 3.5 gives an example to illustrate the sorting operation.

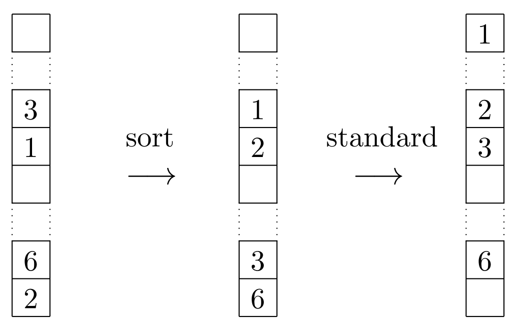


Figure 3.5: The sorting operation and standardization operation.

Proposition 3.5. *For $\mathcal{F} \in \mathcal{F}(C,S)$, the filling* $\mathrm{sort}(\mathcal{F})$ *belongs to $\mathcal{F}_\leq(C,S)$.*

Proof. This is trivially true by treating a Schubert matroid as a transversal matroid, see the proof of [1, Theorem 2.1]. Here, we give a simple verification to make it self-contained. Obviously, $\mathrm{sort}(\mathcal{F})$ is column-strict. We need to verify that $\mathrm{sort}(\mathcal{F})$ is flagged. Let $a_1a_2\cdots a_k$ be the word by reading the numbers of $\mathcal{F}$ from top to bottom. Define the inversion number $\mathrm{inv}(\mathcal{F})$ of $\mathcal{F}$ to be the number of pairs (i,j) such that $a_i > a_j$.

The proof is by induction on $\mathrm{inv}(\mathcal{F})$. If $\mathrm{inv}(\mathcal{F}) = 0$, then $\mathrm{sort}(\mathcal{F}) = \mathcal{F} \in \mathcal{F}_\leq(C,S)$. We now consider the case $\mathrm{inv}(\mathcal{F}) > 0$. Choose $i < j$ such that $a_i > a_j$. Let $\mathcal{F}'$ be the filling obtained from $\mathcal{F}$ by interchanging a_i and a_j. Clearly, $\mathrm{inv}(F') < \mathrm{inv}(F)$. We claim that $\mathcal{F}'$ belongs to $\mathcal{F}(C,S)$. This can be seen as follows. Suppose that a_i lies in the p-th row of $\mathcal{F}$, and a_j lies in the q-th row of $\mathcal{F}$, where $p < q$. Since $\mathcal{F}$ is flagged, we have $a_i \leq p$ and $a_j \leq q$. Combining the facts that $a_i > a_j$ and $p < q$, we reach that $a_i \leq q$ and $a_j \leq p$. This implies that $\mathcal{F}'$ is flagged, concluding the claim. By induction, $\mathrm{sort}(\mathcal{F}')$ belongs to $\mathcal{F}_\leq(C,S)$. Since $\mathrm{sort}(\mathcal{F}) = \mathrm{sort}(\mathcal{F}')$, we complete the proof. ∎

The second operation is the standardization operation acting on $\mathcal{F}_\leq(C,S)$. Let $\mathcal{F} \in \mathcal{F}_\leq(C,S)$. The standardization of $\mathcal{F}$ is the filling $\mathrm{standard}(\mathcal{F})$ obtained by moving upwards the numbers in $\mathcal{F}$ as high as possible subject to the flag condition. More precisely, let $a_1 < a_2 < \cdots < a_k$ be the integers appearing in $\mathcal{F}$. Construct a sequence of fillings $\mathcal{F} = \mathcal{F}^{(0)}, \mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(k)}$ as follows. For $1 \leq t \leq k$, $\mathcal{F}^{(t)}$ is generated from $\mathcal{F}^{(t-1)}$ according to the following two cases.

(1) The row indices of empty boxes in $\mathcal{F}^{(t-1)}$ above a_t are all strictly smaller than a_t. In this case, let $\mathcal{F}^{(t)} = \mathcal{F}^{(t-1)}$;

(2) There exist empty boxes in $\mathcal{F}^{(t-1)}$ above a_t with row indices greater than or equal to a_k. Let i_t be the smallest such row index. Then $\mathcal{F}^{(t)}$ is obtained from $\mathcal{F}^{(t-1)}$ by moving a_t up to the box in row i_t.

Define standard$(\mathcal{F}) = \mathcal{F}^{(k)}$. By construction, it is easily seen that standard$(\mathcal{F})$ belongs to $\mathcal{F}_{\leq}(C, S)$. See Figure 3.5 for an illustration of the standardization operation.

We can now give a proof of Theorem 3.4.

Proof of Theorem 3.4. Let

$$\overline{r}_C(S) = \max\{|\mathcal{F}| \colon \mathcal{F} \in \mathcal{F}(C, S)\}. \tag{3.3}$$

We first show that $\overline{r}_C(S) \leq r_C(S)$. Suppose that $\mathcal{F}_0 \in \mathcal{F}(C, S)$ attains the maximal cardinality among all fillings in $\mathcal{F}(C, S)$, namely, $\overline{r}_C(S) = |\mathcal{F}_0|$. Set

$$\mathcal{F}_0' = \text{standard}(\text{sort}(\mathcal{F}_0)).$$

By Proposition 3.5, $\mathcal{F}_0'$ belongs to $\mathcal{F}_{\leq}(C, S)$. Let $\mathcal{F}_0''$ be the filling of C obtained from $\mathcal{F}_0'$ by assigning each empty box with its row index. By the construction of the standardization operator, it is easily checked that $\mathcal{F}_0''$ is a column-strict flagged filling of C such that the integers in $\mathcal{F}_0''$ are increasing from top to bottom. Hence the set of integers in $\mathcal{F}_0''$ forms a base, say B_0, of the Schubert matroid $SM_n(C)$. Moreover,

$$\#(S \cap B_0) \geq |\mathcal{F}_0|,$$

which implies that

$$\overline{r}_C(S) = |\mathcal{F}_0| \leq \#(S \cap B_0) \leq r_C(S).$$

We now verify the reverse direction $\overline{r}_C(S) \geq r_C(S)$. Let B_0 be a base of the Schubert matroid $SM_n(C)$ such that $S \cap B_0$ has the maximal cardinality, that is, $r_C(S) = \#(S \cap B_0)$. Define a filling $\mathcal{F}_{B_0}$ of C as follows: Assign the elements of B_0 into the boxes of C such that the integers are increasing from top to bottom, and then delete the integers not belonging to S. Since $B_0 \leq C$, $\mathcal{F}_{B_0}$ is a filling in $\mathcal{F}_{\leq}(C, S)$. As $|\mathcal{F}_{B_0}| = \#(S \cap B_0)$, we see that

$$\overline{r}_C(S) \geq |\mathcal{F}_{B_0}| = r_C(S).$$

This completes the proof. ∎

Using Theorem 3.4, we can finish the proof of Theorem 3.3.

Proof of Theorem 3.3. We make induction on the cardinality of $S = \{\pi_1, \ldots, \pi_k\}$. Consider the initial case $k = 1$. It is obvious that $r_C(S) = 1$ or 0, depending on whether C has a box with row index greater than or equal to π_1. So the equality (3.1) holds.

Assume now that $k \geq 2$ and (3.1) is true for $k-1$. Let

$$S' = S \setminus \{\pi_k\} = \{\pi_1, \ldots, \pi_{k-1}\}$$

and $\pi' = \pi_1\pi_2\cdots\pi_{k-1}$. Recall that

$$r_C(S') = \max\{\#(S' \cap B)\colon B \in \mathcal{B}\} \quad \text{and} \quad r_C(S) = \max\{\#(S \cap B)\colon B \in \mathcal{B}\},$$

where $\mathcal{B}$ is the basis of the Schubert matroid $SM_n(C)$. So we see that

$$r_C(S) = r_C(S') \quad \text{or} \quad r_C(S) = r_C(S') + 1. \tag{3.4}$$

Keep in mind that $\mathcal{F}_\pi(C)$ is obtained from $\mathcal{F}_{\pi'}(C)$ by putting π_k into the topmost empty box of $\mathcal{F}_{\pi'}(C)$ subject to the flag condition. We conclude the proof by considering the following cases.

Case 1. $\mathcal{F}_\pi(C) \neq \mathcal{F}_{\pi'}(C)$. In this case, $|\mathcal{F}_\pi(C)| = |\mathcal{F}_{\pi'}(C)| + 1$. Since $\mathcal{F}_\pi(C) \in \mathcal{F}(C,S)$, it follows from Theorem 3.4 that

$$r_C(S) \geq |\mathcal{F}_\pi(C)| = |\mathcal{F}_{\pi'}(C)| + 1.$$

By induction, $r_C(S') = |\mathcal{F}_{\pi'}(C)|$. So $r_C(S) \geq r_C(S') + 1$. In view of (3.4), we have

$$r_C(S) = r_C(S') + 1 = |\mathcal{F}_\pi(C)|,$$

as desired.

Case 2. $\mathcal{F}_\pi(C) = \mathcal{F}_{\pi'}(C)$. In this case, there are no allowable empty boxes in $\mathcal{F}_{\pi'}(C)$ to place π_k. There are two subcases.

Case I. There are no empty boxes in $\mathcal{F}_{\pi'}(C)$. By induction, we have $r_C(S') = |\mathcal{F}_{\pi'}(C)| = \#C$. Since $r_C(S) \leq \#C$, it follows from (3.4) that $r_C(S) = r_C(S') = \#C$, and hence $r_C(S) = |\mathcal{F}_\pi(C)|$.

Case II. There exist empty boxes in $\mathcal{F}_{\pi'}(C)$, but we cannot put π_k into any of these empty boxes. Suppose that l is the largest row index of the empty boxes. Assume that there are b boxes of C lying strictly below row l. By the construction of $\mathcal{F}_{\pi'}(C)$, each integer filled in those b boxes below row l is strictly larger than l. As the box in row l is empty, by the construction of $\mathcal{F}_\pi(C)$, we have $\pi_k > l$.

Assume that $r_C(S') = m$. Let $\pi_{i_1}, \ldots, \pi_{i_m}$ be the elements of S' that are filled in $\mathcal{F}_{\pi'}(C)$. Again, as the box in row l is empty, by the construction of $\mathcal{F}_{\pi'}(C)$, it is clear that each integer in the set $S' \setminus \{\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_m}\}$ is strictly larger than l.

We aim to show that $r_C(S) = m$. Suppose to the contrary that $r_C(S) \neq m$. By (3.4), we have $r_C(S) = m+1$. By Theorem 3.4, there is a filling $\mathcal{F} \in \mathcal{F}(C,S)$ such that $|\mathcal{F}| = m+1$. Notice that π_k must belong to $\mathcal{F}$, since otherwise $\mathcal{F}$ is a filling in $\mathcal{F}(C,S')$ which, together with Theorem 3.4, would imply that $r_C(S') \geq m+1$, leading to a contradiction.

Assume that $\pi_{j_1}, \ldots, \pi_{j_m}, \pi_k \in S$ are the integers filled in $\mathcal{F}$. Notice that each integer in the set

$$\{\pi_{j_1}, \ldots, \pi_{j_m}\} \setminus \{\pi_{i_1}, \ldots, \pi_{i_m}\}$$

is strictly larger than l. Recall that the integers filled in those b boxes of $\mathcal{F}_{\pi'}(C)$ below row l are strictly larger than l. So $\{\pi_{i_1},\ldots,\pi_{i_m}\}$ contains exactly b integers strictly larger than l. Thus $\{\pi_{j_1},\ldots,\pi_{j_m}\}$ contains at least b integers strictly larger than l. Combining the fact that $\pi_k > l$, the set $\{\pi_{j_1},\ldots,\pi_{j_m},\pi_k\}$ contains at least $b+1$ integers strictly larger than l. However, there are exactly b boxes of C with row indices strictly larger than l. This means the $m+1$ integers $\pi_{j_1},\ldots,\pi_{j_m},\pi_k$ cannot be filled into the boxes of C to form a flagged filling, leading to a contradiction. Thus the assumption that $r_C(S) = m+1$ is false. So we have $r_C(S) = m = |\mathcal{F}_\pi(C)|$. This finishes the proof. ∎

3.3 Proof of Theorem 1.1

Using Theorem 3.2 and Theorem 3.3, we can now present a proof of Theorem 1.1, which we restate below.

Theorem 1.1. *Let D be a diagram of $[n]^2$. Then the vertex set of the Schubitope $\mathcal{S}_D$ is*

$$\{x(w) \colon w \in S_n\},$$

where $x(w) = (x_1, x_2, \ldots, x_n)$ is the vector such that x_k $(1 \le k \le n)$ is the number of appearances of k in $\mathcal{F}_w(D)$.

Proof. By Theorem 3.2 and Theorem 3.3, we find that

$$\begin{aligned}
x_{w_k} &= r_D(\{w_1,\ldots,w_k\}) - r_D(\{w_1,\ldots,w_{k-1}\})\\
&= \sum_{j=1}^{n} r_j(\{w_1,\ldots,w_k\}) - \sum_{j=1}^{n} r_j(\{w_1,\ldots,w_{k-1}\})\\
&= \sum_{j=1}^{n} |\mathcal{F}_{w_1\cdots w_k}(D_j)| - \sum_{j=1}^{n} |\mathcal{F}_{w_1\cdots w_{k-1}}(D_j)|\\
&= |\mathcal{F}_{w_1\cdots w_k}(D)| - |\mathcal{F}_{w_1\cdots w_{k-1}}(D)|.
\end{aligned}$$

Thus x_{w_k} is equal to the number of appearances of w_k in $\mathcal{F}_{w_1\cdots w_k}(D)$. It is obvious that the numbers of appearances of w_k in $\mathcal{F}_{w_1\cdots w_k}(D)$ and in $\mathcal{F}_w(D)$ are the same, and so x_{w_k} is equal to the number of appearances of w_k in $\mathcal{F}_w(D)$. This completes the proof. ∎

4 Proof of Theorem 1.2

Let us begin by reviewing the Bruhat order. We view a permutation $w = w_1w_2\cdots w_n \in S_n$ as a bijection on $[n]$, that is, w maps i to $w(i) = w_i$. As usual, for $1 \le i \le n-1$, let $s_i = (i, i+1)$ denote the adjacent transposition. So ws_i is the permutation obtained from w by interchanging w_i and w_{i+1}, while s_iw is obtained by interchanging the values i and $i+1$. For example, for $w = 2143$, we have $ws_2 = 2413$ but $s_2w = 3142$.

Each permutation can be written as a product of adjacent transpositions. The length $\ell(w)$ of a permutation w is the minimum k such that $w = s_{i_1}s_{i_2}\cdots s_{i_k}$, and in this case,

$s_{i_1}s_{i_2}\cdots s_{i_k}$ is called a reduced expression of w. The (strong) Bruhat order $\leq$ on S_n is the closure of the following covering relation. For $w, w' \in S_n$, we say that w covers w' if there exists a transposition $t_{ij} = (i, j)$ such that $w = w'\, t_{ij}$ and $\ell(w) = \ell(w') + 1$. The Bruhat order can also be characterized by the Subword Property, see for example [11].

Theorem 4.1 (Subword Property)**.** *Let $s_{i_1}s_{i_2}\cdots s_{i_k}$ be any given reduced expression of a permutation w. Then $w' \leq w$ in the Bruhat order if and only if there exists a subexpression of $s_{i_1}s_{i_2}\cdots s_{i_k}$ that is a reduced expression of w'.*

4.1 A decomposition of the set $\{\beta\colon \beta \leq \alpha\}$

Recall that for a composition $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\lambda(\alpha)$ is the partition obtained by resorting the parts of α decreasingly, and $w(\alpha)$ is the shortest length permutation such that

$$\lambda(\alpha) \cdot w(\alpha) = \alpha,$$

where the action of a permutation on a vector is as defined in (1.1). The permutation $w(\alpha)$ can be read off directly from α as follows. Let t_1 be the largest part of α appearing in α at positions $l_1 < l_2 < \cdots < l_{a_1}$ from left to right. Then put $1, 2, \ldots, a_1$ in increasing order at the positions $l_1, l_2, \ldots, l_{a_1}$. Let t_2 be the second largest part of α, and t_2 appears in α at positions $l_1' < l_2' < \cdots < l_{a_2}'$. Then put $a_1 + 1, a_1 + 2, \ldots, a_1 + a_2$ in increasing order at the positions $l_1', l_2', \ldots, l_{a_2}'$. Repeat the same process for the third largest part of α, etc. For example, for $\alpha = (2, 0, 1, 3, 2, 0, 1)$, we have $w(\alpha) = 2641375$. We can also construct $w(\alpha)$ by a recursive procedure. If α is a partition, then $w(\alpha)$ is the identity permutation. Otherwise, choose a position r such that $\alpha_r < \alpha_{r+1}$. Let $\alpha' = \alpha \cdot s_r$. Then

$$w(\alpha) = w(\alpha')\, s_r.$$

The above recursive construction eventually leads to a reduced expression of $w(\alpha)$.

Let $V(\alpha)$ denote the set appearing in Theorem 1.2:

$$V(\alpha) = \{\beta\colon \beta \leq \alpha\}.$$

Lemma 4.2. *For any composition α, we have*

$$V(\alpha) = \{\lambda(\alpha) \cdot \sigma\colon \sigma \leq w(\alpha)\}.$$

Proof. By definition (1.2), it is clear that $\{\beta\colon \beta \leq \alpha\} \subseteq \{\lambda(\alpha) \cdot \sigma\colon \sigma \leq w(\alpha)\}$. We next verify the reverse inclusion. Assume that $\sigma \leq w(\alpha)$ and $\beta = \lambda(\alpha) \cdot \sigma$. We aim to show that $\beta \leq \alpha$. In other words, we need to verify $w(\beta) \leq w(\alpha)$.

Let us first give a description of $w(\beta)$. Suppose that α has m distinct parts, and that for $1 \leq i \leq m$, the number of appearances of the i-th largest part is equal to a_i. Set $b_0 = 0$, and $b_i = a_1 + \cdots + a_i$ for $1 \leq i \leq m$. It is easy to check that $w(\beta)$ can be obtained from σ by rearranging the integers in the interval $[b_i + 1, b_{i+1}]$ $(0 \leq i \leq m - 1)$ increasingly from left to right.

The above description of $w(\beta)$ leads to an equivalent characterization of $w(\beta)$. It is well known that S_n is the Coxeter group of type A_{n-1}, where $n = b_m$, with generating set $\{s_1, s_2, \ldots, s_{n-1}\}$. Let

$$J = \{s_1, s_2, \ldots, s_{n-1}\} \setminus \{s_{b_1}, s_{b_2}, \ldots, s_{b_m}\}.$$

Let $(S_n)_J$ denote the parabolic subgroup of S_n generated by J, and let $(S_n)_J\,\sigma$ be the right coset of $(S_n)_J$ with respect to σ. Then $w(\beta)$ is the (unique) minimal coset representative of $(S_n)_J\,\sigma$, that is, $\ell(s_j\,w(\beta)) > \ell(w(\beta))$ for any $s_j \in J$. Hence there is a unique $\tau \in (S_n)_J$ satisfying that $\sigma = \tau\,w(\beta)$ and $\ell(\sigma) = \ell(\tau) + \ell(w(\beta))$ [11, Chapter 1.10]. This implies that the concatenation of any two reduced expressions of τ and $w(\beta)$ is a reduced expression of σ, which, combined with the Subword Property in Theorem 4.1, yields that $w(\beta) \leq \sigma$. Since $\sigma \leq w(\alpha)$, we have $w(\beta) \leq w(\alpha)$. This completes the proof. ∎

By Lemma 4.2, we obtain the following decomposition of $V(\alpha)$.

Proposition 4.3. *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a composition. Assume that there exists $1 \leq r \leq n-1$ such that $\alpha_r < \alpha_{r+1}$. Let $\alpha' = \alpha \cdot s_r$. Then*

$$V(\alpha) = V(\alpha') \cup \{v \cdot s_r \colon v \in V(\alpha')\}. \tag{4.1}$$

Proof. To conclude (4.1), by Lemma 4.2 it suffices to show that

$$\{\sigma\colon \sigma \leq w(\alpha)\} = \{\tau\colon \tau \leq w(\alpha')\} \cup \{\tau s_r\colon \tau \leq w(\alpha')\}.$$

This can be easily deduced from the Subword Property. Since $\alpha_r < \alpha_{r+1}$, from the arguments above Lemma 4.2 it follows that that $w(\alpha) = w(\alpha')s_r$ and $\ell(w(\alpha)) = \ell(w(\alpha'))+1$. Let $s_{i_1}\cdots s_{i_k}$ be a reduced expression of $w(\alpha')$. Then $s_{i_1}\cdots s_{i_k}s_r$ is a reduced expression of $w(\alpha)$.

We first show that

$$\{\sigma\colon \sigma \leq w(\alpha)\} \subseteq \{\tau\colon \tau \leq w(\alpha')\} \cup \{\tau s_r\colon \tau \leq w(\alpha')\}. \tag{4.2}$$

There are two cases.

Case 1. s_r is not a (right) descent of σ, that is, $\ell(\sigma) = \ell(\sigma s_r) - 1$. In this case, any reduced expression of σ does not end with s_r. This means that we can choose a subexpression from $s_{i_1}\cdots s_{i_k}$ to from a reduced expression of σ, which, by the Subword Property, implies $\sigma \leq w(\alpha')$.

Case 2. s_r is a (right) descent of σ, that is, $\ell(\sigma) = \ell(\sigma s_r) + 1$. Then s_r is not a (right) descent of σs_r. As $\sigma s_r \leq \sigma \leq w(\alpha)$, it follows from Case 1 that $\sigma s_r \leq w(\alpha')$. Since $\sigma = (\sigma s_r)s_r$, we have $\sigma \in \{\tau s_r\colon \tau \leq w(\alpha')\}$. This verifies (4.2).

The reverse set inclusion can be checked in a similar manner, and thus is omitted. ∎

4.2 Properties on vertices of $\mathrm{Newton}(\kappa_\alpha)$

In this subsection, we use Theorem 1.1 to give two relationships on the vertices of $\mathrm{Newton}(\kappa_\alpha)$, which will be used in the proof of Theorem 1.2.

Proposition 4.4. *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a composition. Assume that there exists $1 \leq r \leq n-1$ such that $\alpha_r < \alpha_{r+1}$, and that w is a permutation in S_n such that r appears before $r+1$ in w. Then*

$$x(w) = x(s_r w) \cdot s_r. \tag{4.3}$$

Proof. Write $x(w) = (x_1, \ldots, x_n)$. By Theorem 1.1, x_k is the number of appearances of k in $\mathcal{F}_w(D(\alpha))$. Let D_j be the j-th column of $D(\alpha)$, which is here viewed as a subset $\{i\colon (i,j) \in D_j\}$ of $[n]$. It suffices to prove the following claim.

Claim. The numbers of appearances of r and $r+1$ in $\mathcal{F}_w(D_j)$ and $\mathcal{F}_{s_r w}(D_j)$ are exchanged, while, for $k \neq r, r+1$, the number of appearances of k in $\mathcal{F}_w(D_j)$ is the same as the the number of appearances of k in $\mathcal{F}_{s_r w}(D_j)$.

For ease of description, for any filling $\mathcal{F}$, we use $i \in \mathcal{F}$ to mean that the integer i appears in $\mathcal{F}$. To verify the Claim, since $\alpha_r < \alpha_{r+1}$, we have the following three cases.

Case 1. $r \notin D_j$ and $r+1 \notin D_j$. In this case, it is easy to check that $\mathcal{F}_{s_r w}(D_j)$ is obtained from $\mathcal{F}_w(D_j)$ by replacing r (if any) by $r+1$, and replacing $r+1$ (if any) by r.

Case 2. $r \notin D_j$ and $r+1 \in D_j$. This case is essentially the same as Case 1.

Case 3. $r \in D_j$ and $r+1 \in D_j$. This case is divided into the following subcases.

Subcase I. $r \notin \mathcal{F}_w(D_j)$ and $r+1 \notin \mathcal{F}_w(D_j)$. It is easy to check that $\mathcal{F}_w(D_j) = \mathcal{F}_{s_r w}(D_j)$.

Subcase II. $r \notin \mathcal{F}_w(D_j)$ and $r+1 \in \mathcal{F}_w(D_j)$. Since r appears before $r+1$ in w, this case is impossible to occur.

Subcase III. $r \in \mathcal{F}_w(D_j)$ and $r+1 \notin \mathcal{F}_w(D_j)$. In this case, we still have two situations to consider.

(1) r is not filled in the box (r,j). In this case, it easy to check that $\mathcal{F}_{s_r w}(D_j)$ is obtained from $\mathcal{F}_w(D_j)$ by replacing r with $r+1$.

(2) r is filled in the box (r,j). Since $r+1$ does not appear in $\mathcal{F}_w(D_j)$, the box $(r+1,j)$ is filled with an integer, say w_i, which is smaller than r. By the construction of $\mathcal{F}_w(D_j)$, w_i must appear after r, but before $r+1$. Hence, when we construct $\mathcal{F}_{s_r w}(D_j)$, the box $(r+1,j)$ is occupied by $r+1$, the box (r,j) is occupied by w_i, and each box other than (r,j) and $(r+1,j)$ is filled with the same integer as $\mathcal{F}_w(D_j)$. This implies that $\mathcal{F}_{s_r w}(D_j)$ is obtained from $\mathcal{F}_w(D_j)$ by replacing r with $r+1$, and then exchanging the values $r+1$ and w_i. The above arguments are best understood by an example as given in Figure 4.6, where $w = 324615$, $r = 4$ and $s_r w = 325614$.

Subcase IV. $r \in \mathcal{F}_w(D_j)$ and $r+1 \in \mathcal{F}_w(D_j)$. This case is similar to Subcase III.

(1) r is not filled in the box (r,j). In this case, it easy to check that $\mathcal{F}_{s_r w}(D_j)$ is obtained from $\mathcal{F}_w(D_j)$ by interchanging r and $r+1$.

(2) r is filled in the box (r,j), and $r+1$ is filled in the box $(r+1,j)$. In this case, it is easy to check that $\mathcal{F}_{s_r w}(D_j) = \mathcal{F}_w(D_j)$.

$$\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline 1\\\hline 6\\\hline\end{array}\qquad\qquad\begin{array}{|c|}\hline 2\\\hline 3\\\hline 1\\\hline 5\\\hline 6\\\hline\end{array}$$

$\mathcal{F}_{324615}(D_j)$ $\qquad$ $\mathcal{F}_{325614}(D_j)$

Figure 4.6: An illustration of the proof of Subcase (III)(2).

(3) r is filled in the box (r, j), but $r + 1$ is filled in a box below $(r + 1, j)$. Since $r + 1$ is filled in a box below $(r + 1, j)$, the box $(r + 1, j)$ is filled with an integer, say w_i, which is smaller than r. By the same arguments as those in Subcase III(2), we see that $\mathcal{F}_{s_r w}(D_j)$ is obtained from $\mathcal{F}_w(D_j)$ by interchanging r with $r + 1$, and then interchanging $r + 1$ and w_i.

The above analysis allows us to conclude the Claim, and so the proof is complete. ∎

Proposition 4.5. *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a composition. Assume that there exists $1 \leq r \leq n - 1$ such that $\alpha_r < \alpha_{r+1}$, and that w is a permutation in S_n such that r appears before $r+1$ in w. Let $\alpha' = \alpha \cdot s_r$, and let $x'(w)$ denote the vertex of* $\mathrm{Newton}(\kappa_{\alpha'})$ *labeled by w. Then*

$$x(w) = x'(w). \tag{4.4}$$

Proof. Let D_j be the j-th column of $D(\alpha)$. Write $D' = D(\alpha \cdot s_r)$, and let D'_j be the j-th column of $D(\alpha')$. If $D_j = D'_j$, it is clear that $\mathcal{F}_w(D_j) = \mathcal{F}_w(D'_j)$. If $D_j \neq D'_j$, since $\alpha_r < \alpha_{r+1}$, we must have $(r, j) \notin D_j$, $(r + 1, j) \in D_j$ and $(r, j) \in D'_j, (r + 1, j) \notin D'_j$. Keeping in mind that r appears before $r + 1$ in w, it is readily checked that $\mathcal{F}_w(D'_j)$ is obtained from $\mathcal{F}_w(D_j)$ by moving the box $(r + 1, j)$, together with the integer filled in the box, up to row r. This, along with Theorem 1.1, completes the proof. ∎

4.3 Proof of Theorem 1.2

Based on Propositions 4.3, 4.4 and 4.5, we can now provide a proof of Theorem 1.2.

Theorem 1.2. *Let α be a composition. Then the vertex set of the Newton polytope* $\mathrm{Newton}(\kappa_\alpha)$ *is $\{\beta \colon \beta \leq \alpha\}$.*

Proof. Denote by $U(\alpha)$ the vertex set of $\mathrm{Newton}(\kappa_\alpha)$. By Theorem 1.1,

$$U(\alpha) = \{x(w) \colon w \in S_n\}. \tag{4.5}$$

As mentioned in Introduction, Monical, Tokcan and Yong [18, Theorem 3.12] showed that $V(\alpha) \subseteq U(\alpha)$. We finish the proof of Theorem 1.2 by proving

$$U(\alpha) \subseteq V(\alpha). \tag{4.6}$$

The proof is by induction on the "reverse" inversion number of α:

$$\mathrm{rinv}(\alpha) = \#\{1 \le i < j \le n \colon \alpha_i < \alpha_j\}.$$

We first verify (4.6) for the case $\mathrm{rinv}(\alpha) = 0$. In this case, α is a partition. So $w(\alpha)$ is the identity permutation, and thus $V(\alpha) = \{\alpha\}$. On the other hand, it is easy to see that for any permutation $w \in S_n$, $\mathcal{F}_w(D)$ is the filling with boxes in row k $(1 \le k \le n)$ filled with k. By Theorem 1.1, we have $U(\alpha) = \{\alpha\}$. This verifies (4.6) for the case $\mathrm{rinv}(\alpha) = 0$.

We next consider the case $\mathrm{rinv}(\alpha) > 0$. Assume that r is a row index such that $\alpha_r < \alpha_{r+1}$. Let $\alpha' = \alpha \cdot s_r$. It is obvious that $\mathrm{rinv}(\alpha') = \mathrm{rinv}(\alpha) - 1$. Let $S_n^<$ denote the subset consisting of the permutations w of S_n such that r appears before $r+1$. Let $S_n^>$ denote the complement of $S_n^<$, namely,

$$S_n^> = \{s_r w \colon w \in S_n^<\}.$$

Write

$$U^<(\alpha) = \{x(w) \colon w \in S_n^<\} \quad \text{and} \quad U^>(\alpha) = \{x(w) \colon w \in S_n^>\}.$$

Then

$$U(\alpha) = U^<(\alpha) \cup U^>(\alpha).$$

By Proposition 4.4, we have

$$U^>(\alpha) = \{v \cdot s_r \colon v \in U^<(\alpha)\}. \tag{4.7}$$

By Proposition 4.5, we have

$$U^<(\alpha) = U^<(\alpha') \subseteq U(\alpha'). \tag{4.8}$$

Therefore,

$$\begin{aligned}
U(\alpha) &= U^<(\alpha) \cup U^>(\alpha) \\
&= U^<(\alpha) \cup \{v \cdot s_r \colon v \in U^<(\alpha)\} && \text{(by (4.7))} \\
&\subseteq U(\alpha') \cup \{v \cdot s_r \colon v \in U(\alpha')\} && \text{(by (4.8))} \\
&\subseteq V(\alpha') \cup \{v \cdot s_r \colon v \in V(\alpha')\} && \text{(by induction)} \\
&= V(\alpha), && \text{(by Proposition 4.3)}
\end{aligned}$$

which proves (4.6), as desired. ∎

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