

Monochromatic disconnection of graphs¹

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Abstract

For an edge-colored graph G , we call an edge-cut M of G monochromatic if the edges of M are colored with the same color. The graph G is called monochromatically disconnected if any two distinct vertices of G are separated by a monochromatic edge-cut. For a connected graph G , the monochromatic disconnection number of G , denoted by $md(G)$, is the maximum number of colors that are needed in order to make G monochromatically disconnected. We show that almost all graphs have monochromatic disconnection numbers equal to 1. We also obtain the Nordhaus-Gaddum-type results for $md(G)$.

Keywords: monochromatic edge-cut, monochromatic disconnection number, Nordhaus-Gaddum-type results.

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1 Introduction

Let G be a graph and let $V(G)$, $E(G)$ denote the vertex set and the edge set of G , respectively. Let $|G|$ (also $v(G)$) denote the number of vertices of G , called the order of G , and let $||G||$ (also $e(G)$) denote the number of edges of G , called the size of G . If there is no confusion, we use n and m to denote, respectively, the number of vertices and the number of edges of a graph, throughout this paper. For $v \in V(G)$, let $d_G(v)$ denote the degree of v . We call a vertex v a t -degree vertex of G if $d_G(v) = t$. Let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of G , respectively. Sometimes, we also use Δ to denote a triangle. We use \overline{G} to denote the complement graph of G . Let S and F be a vertex subset and an edge subset of G , respectively. $G - S$ is a graph obtained from G by deleting the vertices of S together with the edges incident with the vertices of S . $G - F$ is a graph whose vertex set is $V(G)$ and edge set is

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$E(G) - F$. Let $G[S]$ and $G[F]$ be the vertex-induced and edge-induced subgraphs of G , by S and F , respectively. The distance of u, v in G is denoted by $d_G(u, v)$. For all other terminology and notation not defined here we follow Bondy and Murty [3].

Throughout this paper, let K_n^- be the graph obtained from K_n by deleting an arbitrary edge. K_3 is also called a *triangle*. We call a cycle C a t -*cycle* if $|C| = t$. If r is a positive integers, we use $[r]$ to denote the set $\{1, 2, \dots, r\}$ of integers. If $r = 0$, $[r]$ is an empty set.

For a graph G , let $\Gamma : E(G) \rightarrow [r]$ be an *edge-coloring* of G that allows the same color to be assigned to adjacent edges. For an edge e of G , we use $\Gamma(e)$ to denote the color of e . If H is a subgraph of G , we also use $\Gamma(H)$ to denote the set of colors on the edges of H and use $|\Gamma(H)|$ to denote the number of colors in $\Gamma(H)$. An edge-coloring Γ of G is *trivial* if $|\Gamma(G)| = 1$, otherwise, it is *nontrivial*.

The notion *rainbow connection coloring* was introduced by Chartrand et al. in [7]. A rainbow connection coloring of a graph G is an edge-coloring of G such that any two distinct vertices are connected by a rainbow path (a path of G whose edges are colored pairwise differently.) The notion *rainbow disconnection coloring* was introduced by Chartrand et al. in [6]. A rainbow disconnection coloring of a graph G is an edge-coloring of G such that any two distinct vertices are separated by a rainbow cut (a cut of G whose edges are colored pairwise differently.)

Contrary to the concepts for rainbow connection and disconnection, monochromatic versions of these concepts naturally appeared, as the other extremal. A *monochromatic connection coloring* of a graph G , which was introduced by Caro and Yuster in [4], is an edge-coloring of G such that any two vertices are connected by a monochromatic path (a path of G whose edges are colored the same.)

As a counterpart of the rainbow disconnection coloring and a similar object of the monochromatic connection coloring, we now introduce the notion of monochromatic disconnection coloring of a graph. For an edge-colored graph G , we call an edge-cut M a *monochromatic edge-cut* if the edges of M are colored with the same color. For two distinct vertices u, v of G , a *monochromatic uv -cut* is a monochromatic edge-cut that separates u and v . An edge-colored graph G is *monochromatically disconnected* if any two distinct vertices of G has a monochromatic cut separating them. An edge-coloring of G is a *monochromatic disconnection coloring* (*MD-coloring* for short) if it makes G monochromatically disconnected. For a connected graph G , the *monochromatic disconnection number* of G , denoted by $md(G)$, is the maximum number of colors that are needed in order to make G monochromatically disconnected. An *extremal MD-coloring* of G is an *MD-coloring* that uses $md(G)$ colors. If H is a subgraph of G and Γ is an edge-coloring of G , we call Γ an edge-coloring *restricted* on H .

As we know that there are two ways to study the connectivity of a graph, one way is by using paths and the other is by using cuts. Both rainbow connection and monochromatic connection provide ways to study the colored connectivity of a graph by

colored paths. However, both rainbow disconnection and monochromatic disconnection can provide ways to study the colored connectivity of a graph by colored cuts. All these parameters or numbers coming from studying the colored connectivity of a graph should be regarded as some kinds of chromatic numbers. However, they are different from classic chromatic numbers. These kinds of chromatic numbers come from colorings by keeping some global structural properties of a graph, say connectivity; whereas the classic chromatic numbers come from colorings by keeping some local structural properties of a graph, say adjacent vertices or edges. So, the employed methods to study them appear quite different sometimes. Of course, local structural properties may yield global structural properties, and vice versa. But this is not always the case, say, local connectedness of a graph cannot guarantee connectedness of the entire graph. So, many colored versions of connectivity parameters appeared in recent years, and we refer the reader to [2, 10, 11, 12, 14, 15, 16, 17] for surveys.

Let G be a graph that may have parallel edges but no loops. By deleting all parallel edges but one of them, we obtain a simple spanning subgraph of G , and call it the *underling graph* of G . If there are some parallel edges of an edge $e = ab$, then any monochromatic ab -cut contains e and its parallel edges. Therefore, the following result is obvious, which means that we only need to think about simple graphs in the sequel.

Proposition 1.1. *Let G' be the underling graph of a graph G . Then $md(G) = md(G')$.*

The following result means that we only need to consider connected graphs in the sequel.

Proposition 1.2. *If a simple graph G has t components D_1, \dots, D_t , then $md(G) = \sum_{i \in [t]} md(D_i)$.*

Let G and H be two graphs. The *union* of G and H is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If G and H are vertex-disjoint, then let $G \vee H$ denote the *join* of G and H , which is obtained from G and H by adding an edge between each vertex of G and every vertex of H .

A block of a graph G is *trivial* if it is a cut-edge of G . If $e = uv$ is an edge of G with $d_G(v) = 1$, we call e a *pendent edge* of G and v a *pendent vertex* of G .

2 Some basic results

Let G be a graph with at least two blocks. An edge-coloring of G is an MD -coloring if and only if it is also an MD -coloring restricted on each block of G . Therefore, the following result is obvious.

Remark 2.1. *If a connected graph G has r blocks B_1, B_2, \dots, B_r , then $md(G) = \sum_{i \in [r]} md(B_i)$.*

From the above remark, if G is a tree, then $md(G) = n - 1$.

Proposition 2.2. *If G is a cycle, then $md(G) = \lfloor \frac{|G|}{2} \rfloor$. Furthermore, if G is a unicycle graph with cycle C , then $md(G) = n - \lfloor \frac{|C|}{2} \rfloor$.*

Proof. From Remark 2.1, we only need to prove that $md(G) = \lfloor \frac{|G|}{2} \rfloor$ if G is a cycle.

Let $G = C$ be a cycle. Suppose $C = v_1e_1v_2e_2 \cdots v_{n-1}e_{n-1}v_n e_n v_1$. Let $r = \lfloor \frac{n}{2} \rfloor$. For $i \in [r]$ and $j \in [n]$, if $j \equiv i \pmod{r}$, then color e_j by $i + 1$. It is easy to verify that the edge-coloring of C is an MD -coloring, and so $md(C) \geq r$.

For a contradiction, let Γ be an MD -coloring of C such that $|\Gamma(C)| \geq r + 1$. Then there exists a color i of Γ that colors only one edge e of C , say $e = ab$. Since the monochromatic ab -cut must contain e and some other edges of $C - e$, a contradiction. ■

Since an MD -coloring of G separates any two vertices by a monochromatic cut, it also separates any two vertices of a subgraph of G . So the following result is obvious.

Proposition 2.3. *Let D be a subgraph of a graph G . If Γ is an MD -coloring of G , then Γ is also an MD -coloring restricted on D .*

Lemma 2.4. *Let H be the union of graphs H_1, \dots, H_r . If $\bigcap_{i \in [r]} E(H_i) \neq \emptyset$ and $md(H_i) = 1$ for each $i \in [r]$, then $md(H) = 1$.*

Proof. We prove it by contradiction. Suppose Γ is an MD -coloring of H with $|\Gamma(H)| \geq 2$. Then there are two edges e_1, e_2 of H such that $\Gamma(e_1) = 1$ and $\Gamma(e_2) = 2$. W.l.o.g., let $e_1 \in E(H_1)$ and $e_2 \in E(H_2)$. Since Γ is an MD -coloring restricted on H_1 (also H_2) and $md(H_1) = md(H_2) = 1$, all edges of H_1 are colored by 1 and all edges of H_2 are colored by 2 under Γ , which contradicts that $E(H_1) \cap E(H_2) \neq \emptyset$. ■

Lemma 2.5. *If H is a connected spanning subgraph of G , then $md(H) \geq md(G)$.*

Proof. Let H' be a graph obtained from G by deleting an edge $e = ab$, where e is in a cycle of G . If $md(H') \leq md(G) - 1$, let Γ be an extremal MD -coloring of G . Then Γ is an MD -coloring that is restricted on H' , and this implies that e is the only edge of G colored by $\Gamma(e)$. However, e is in a cycle of G , and the monochromatic ab -cut has at least 2 edges, a contradiction. Therefore, $md(H') \geq md(G)$.

If H is a connected proper spanning subgraph of G , H can be obtained from G by deleting some edges in cycles one by one, consecutively. Therefore, the lemma is true. ■

Corollary 2.6. *For any connected graph G of order n , $md(G) \leq n - 1$, and the equality holds if and only if G is a tree.*

Proof. Since each connected graph has a spanning tree T , by Lemma 2.5 we have that $md(G) \leq md(T) = n - 1$ if G is connected. On the other hand, if G is a connected graph with $md(G) = n - 1$ but G is not a tree, then G has a connected unicycle spanning subgraph G' . By Lemma 2.5 and Proposition 2.2, $md(G) \leq md(G') < n - 1$, a contradiction. ■

Claim 2.7. *Let G be a connected graph and $v \in V(G)$ is neither a pendent vertex nor a cut-vertex of G . Then for any MD-coloring Γ' of G , $\Gamma'(G) - \Gamma'(G - v) = \emptyset$.*

Proof. We proceed by contradiction. Let $e = vu$ be an edge of $E(G) - E(G - v)$ and $\Gamma'(e) \notin \Gamma'(G - v)$. Since $d_G(v) \geq 2$, there is another edge incident with v , say $f = vw$. Since v is not a cut-vertex, there is a cycle C of G with $E(C) - E(G - v) = \{e, f\}$. Since Γ' is an MD-coloring restricted on C , there are at least two edges in the monochromatic uv -cut of C and the monochromatic uv -cut contains e . Since $\Gamma'(G) - \Gamma'(G - v) \neq \emptyset$, f is in the monochromatic uv -cut, i.e., $\Gamma'(e) = \Gamma'(f)$. Then, there is no monochromatic uw -cut in C , a contradiction. ■

The following is a direct consequence of Claim 2.7.

Lemma 2.8. *Let G be a connected graph and $v \in V(G)$. If v is neither a pendent vertex nor a cut-vertex of G , then $md(G) \leq md(G - v)$.*

Theorem 2.9. *If G is a 2-connected graph of order n , then $md(G) \leq \lfloor \frac{n}{2} \rfloor$.*

Proof. Let $F = \{C, P_1, \dots, P_t\}$ be an ear-decomposition of G , where C is a cycle and P_i is a path for $i \in [t]$. The proof proceeds by induction on $|F|$. If $|F| = 1$, then G is a cycle, the theorem holds. If $|F| = t + 1 \geq 2$, let Γ be an extremal MD-coloring of G . Then Γ is an MD-coloring restricted on G' , where G' is a graph obtained from G by deleting $E(P_t)$ and the internal vertices of P_t . By induction, we have

$$|\Gamma(G')| \leq md(G') \leq \left\lfloor \frac{|G'|}{2} \right\rfloor = \left\lfloor \frac{n - |P_t| + 2}{2} \right\rfloor.$$

Suppose that the ends of P_t are a, b and L is an ab -path of G' . Then $C' = L \cup P_t$ is a cycle of G . Since Γ is an MD-coloring restricted on C' , the monochromatic ab -cut contains at least one edge of L and at least one edge of P_t , say e . Therefore, there are at most $|P_t| - 2$ edges colored by colors of $\Gamma(G) - \Gamma(G')$. Since each color of $\Gamma(G) - \Gamma(G')$ colors at least two edges of $P_t - e$, $|\Gamma(G) - \Gamma(G')| \leq \left\lfloor \frac{|P_t| - 2}{2} \right\rfloor$. So,

$$md(G) = |\Gamma(G)| = |\Gamma(G')| + |\Gamma(G) - \Gamma(G')| \leq \left\lfloor \frac{n - |P_t| + 2}{2} \right\rfloor + \left\lfloor \frac{|P_t| - 2}{2} \right\rfloor \leq \left\lfloor \frac{n}{2} \right\rfloor. \quad \blacksquare$$

3 Graphs with monochromatic disconnection number one

In this section we consider the monochromatic disconnection numbers for some special graphs, such as triangular graphs (i.e., graphs with each of its edges in a triangle), complete multipartite graphs, chordal graphs, square graphs and line graphs (the definitions of the last four graphs are well-known, we omit them). We denote the square graph and the line graph of a graph G by G^2 and $L(G)$, respectively.

For a graph G , we define a *relation* θ on the edge set $E(G)$ as follows: for two edges e and e' of G , we say that $e\theta e'$ if there exists a sequence of subgraphs G_1, \dots, G_k of G with $md(G_i) = 1$ for any $i \in [k]$, such that $e \in G_1$ and $e' \in G_k$, and $|V(G_i) \cap V(G_{i+1})| \geq 2$ for $i \in [k-1]$. It is easy to check that θ is *symmetric*, *reflexive* and *transitive* and therefore an equivalent relation on $E(G)$. We call a graph G a *closure* if $e\theta e'$ for any two edges e, e' of $E(G)$.

Lemma 3.1. *If a graph G is a closure, then $md(G) = 1$.*

Proof. Suppose that $md(G) \geq 2$ and Γ is an extremal MD -coloring of G . Then there exist two edges, say e and e' , of G , such that $\Gamma(e) \neq \Gamma(e')$. Since G is a closure, there is a sequence of subgraphs G_1, \dots, G_k with $md(G_i) = 1$ for any $i \in [k]$, such that $e \in G_1$ and $e' \in G_k$, and G_i and G_{i+1} have at least two common vertices, say a_i, b_i , for $i \in [k-1]$. Since all edges of each G_i must be colored with the same color under Γ , $\Gamma(G_i) = \Gamma(G_{i-1})$. Otherwise there is no monochromatic $a_i b_i$ -cut. Therefore, $\Gamma(e) = \Gamma(G_1) = \Gamma(G_2) = \dots = \Gamma(G_k) = \Gamma(e')$, a contradiction. So, $md(G) = 1$. ■

Theorem 3.2. *If G is one of the following graphs, then $md(G) = 1$.*

1. $G = H \vee v$, where H is a connected graph and v is an additional vertex;
2. G is a complete multipartite graph other than $K_{1,n-1}$ and $K_{2,2}$;
3. G is a 2-connected chordal graph;
4. $G = H^2$, where H is a connected graph other than K_1 ;
5. $G = L(H)$, where H is a connected triangular graph.

Proof. (1) If $H = K_1$, then the result holds. If $|H| \geq 2$, then let T be a spanning tree of H and u be a leaf of T . By induction, $md((T - u) \vee v) = 1$. Since $V(T \vee v) - V((T - u) \vee v) = \{u\}$ and u is neither a pendent vertex nor a cut vertex of $T \vee v$, $md(T \vee v) \leq md((T - u) \vee v)$ by Lemma 2.8. Since $T \vee v$ is a connected spanning subgraph of $H \vee v$, $md(H \vee v) \leq md(T \vee v)$ by Lemma 2.5. Therefore, $md(H \vee v) = 1$.

(2) We prove $md(K_{2,3}) = 1$. Any MD -coloring of C_4 can have only two cases, one is trivial and the other is to assign colors 1, 2 to the four edges of C_4 alternately. Let

$H = K_{2,3}$ and the bipartition of H be $A = \{a, c\}$ and $B = \{b, d, u\}$. If $md(H) \geq 2$, then there is an MD -coloring Γ of $K_{2,3}$ with $|\Gamma(H)| \geq 2$. Therefore, at least one of the three 4-cycles of H has a nontrivial MD -coloring. Let the three 4-cycles of H be $H_1 = H[a, b, c, d]$, $H_2 = H[a, b, c, u]$ and $H_3 = H[a, d, c, u]$. By symmetry, suppose that H_1 is colored nontrivially, say $\Gamma(ad) = \Gamma(bc) = 1$ and $\Gamma(ab) = \Gamma(cd) = 2$. Then Γ is a nontrivial MD -coloring restricted on H_2 with $\Gamma(au) = 1$ and $\Gamma(cu) = 2$. It is obvious that Γ is not an MD -coloring restricted on H_3 , which contradicts that Γ is an MD -coloring of G . Therefore, $md(H) = 1$.

Let G be a complete bipartite graph other than $K_{1,n-1}$ and $K_{2,2}$. Suppose that A, B are the bipartition of G with $A = \{u, v, a_1, \dots, a_s\}$ and $B = \{u', v', b_1, \dots, b_t\}$. Then at least one of s, t is not zero. Let $G_i = G[u, u', v, v', a_i]$ and $G'_j = G[u, u', v, v', b_j]$ for $i \in [s]$ and $j \in [t]$. Since each G_i or G'_j is a $K_{2,3}$ and $\bigcap_{i \in [s]} E(G_i) \cap \bigcap_{j \in [t]} E(G'_j) = E(G[u, u', v, v'])$, by Lemma 2.4 we have $md(\bigcup_{i \in [s]} G_i \cup \bigcup_{j \in [t]} G'_j) = 1$. Since $\bigcup_{i \in [s]} G_i \cup \bigcup_{j \in [t]} G'_j$ is a connected spanning subgraph of G , $md(G) = 1$.

Let $G = G_1$ be a complete r -partite graph with $r \geq 3$ and let $V = \{v_1, \dots, v_t\}$ be one part of G . Let $G_i = G - \{v_1, \dots, v_{i-1}\}$ for $i \in \{2, \dots, t\}$. Then each v_i is neither a pendent vertex nor a cut vertex of G_i , and so $md(G_1) \leq md(G_2) \leq \dots \leq md(G_t)$ by Lemma 2.8. However, $G_t = (G - V) \vee v_t$ and $G - V$ is a connected graph, which implies $md(G_t) = 1$. Therefore, $md(G) = 1$.

(3) A *simplicial order* of a graph H is an enumeration v_1, \dots, v_n of its vertices such that the neighbors of v_i in $H[\{v_i, \dots, v_n\}]$ induce a complete graph. A graph is chordal if and only if it has a simplicial order (see Corollary 9.22 on page 273 of [3]). Suppose that a simplicial order of G is u_1, \dots, u_n and $G_i = G[\{u_i, \dots, u_n\}]$ for $i \in [n-2]$ (then $G = G_1$). Let $H_i = G_i[N_{G_i}(u_i)]$. Since G is 2-connected, each H_i is a complete graph other than K_1 and thus $G_{n-1} = G[u_{n-1}, u_n]$ is a K_2 . Therefore, u_i is neither a pendent vertex nor a cut vertex of G_i for $i \in [n-2]$, and hence by Lemma 2.8, $md(G_i) \leq md(G_{i+1})$. So, $md(G) \leq md(G_{n-1}) = 1$.

(4) The result holds for $G = K_2$. We prove it by induction on $|G|$. If $|G| \geq 3$, let T be a spanning tree of G and v be a leaf of T . Then $T^2 - v = (T - v)^2$. Since v is neither a pendent vertex nor a cut vertex of T^2 , $md(T^2) \leq md((T - v)^2) = 1$. Since T^2 is a spanning subgraph of G^2 , $md(G^2) \leq md(T^2)$. Therefore, $md(G^2) = 1$.

(5) Let A, B be two edge-induced subgraphs of G . We define

$$d_G(A, B) = \min\{d_G(u, v) : u \in V(A), v \in V(B)\}.$$

It is sufficient to show that $L(G)$ is a closure, i.e., we need to show that for every two edges l_1, l_2 of $L(G)$, $l_1 \theta l_2$. For each edge e_i of G , we denote the corresponding vertex of $L(G)$ by u_i . We proceed by induction on $d_{L(G)}(l_1, l_2)$.

If $d_{L(G)}(l_1, l_2) = 0$, then l_1 and l_2 has a common vertex. Let $l_1 = u_1 u_2$ and $l_2 = u_2 u_3$. If $G[e_1, e_2, e_3]$ is a triangle (denote it by Δ) of G , then $L(\Delta)$ is a triangle of $L(G)$ containing l_1, l_2 , and so $l_1 \theta l_2$. If just two edges of e_1, e_2, e_3 are in a triangle Δ of G ,

suppose $\Delta = G[e_1, e_2, e_4]$. Then $G[e_2, e_3, e_4]$ is a star (call the star S). Since $L(\Delta)$ and $L(S)$ are two triangles of $L(G)$ and they have a common edge u_2u_4 , and since $L(\Delta)$ contains l_1 and $L(S)$ contains $l_2, l_1\theta l_2$. If none of the triangles of G contains at least two of e_1, e_2, e_3 , suppose that $\Delta = G[e_4, e_2, e_5]$ is a triangle of G , where e_4 is adjacent to e_3 and e_5 is adjacent to e_1 . Then $S_1 = G[e_1, e_2, e_5]$ and $S_2 = G[e_3, e_2, e_4]$ are two stars of G . Therefore, $L(S_1), L(\Delta)$ and $L(S_2)$ are three triangles of $L(G)$ such that $L(S_1), L(\Delta)$ have a common edge u_2u_5 and $L(S_2), L(\Delta)$ have a common edge u_2u_4 . So, $l_1\theta l_2$.

If $d_{L(G)}(l_1, l_2) = r > 0$, let $l_1 = u_1u_2$ and $l_2 = u_3u_4$. Suppose that P is a shortest path of $L(G)$ connecting l_1 and l_2 . Then $\|P\| = r$. W.l.o.g., suppose $l_3 = u_3u_5$ is a pendent edge of P . Then $d_{L(G)}(l_1, l_3) = r - 1$ and $d_{L(G)}(l_2, l_3) = 0$. By induction, $l_1\theta l_3$ and $l_2\theta l_3$. Therefore, $l_1\theta l_2$. \blacksquare

From Theorem 3.2 (2), $md(K_n) = 1$ for $n \geq 2$. Let v be a minimum degree vertex of K_n^- ($n \geq 4$). Then $K_n^- - v = K_{n-1}$. Since v is neither a pendent vertex nor a cut vertex of K_n^- , $md(K_n^-) \leq md(K_{n-1}) = 1$, i.e., $md(K_n^-) = 1$ for $n \geq 4$.

As we have seen, a lot of graphs have the monochromatic disconnection number equal to 1, and this suggest us to show that following result.

Theorem 3.3. *For almost all graphs G , $md(G) = 1$ holds.*

Proof. Let $G \sim \mathcal{G}_{n, \frac{1}{2}}$, that is, G is a random graph on n vertices chosen by picking each pair of vertices as an edge randomly and independently with probability $\frac{1}{2}$. Let \mathcal{A}_{uv} be the set of events that u and v have at most 2 common neighbors and $\mathcal{A} = \bigcup_{u,v \in V(G)} \mathcal{A}_{uv}$. Let \mathcal{A}_{uv}^i be the set of events that u, v have exactly i common neighbors. Then $\mathcal{A}_{uv} = \bigcup_{i=0}^2 \mathcal{A}_{uv}^i$. For a vertex w of $V(G) - \{u, v\}$, since

$$Pr[w \text{ is a common neighbor of } u \text{ and } v] = \frac{1}{4}$$

and

$$Pr[w \text{ is not a common neighbor of } u \text{ and } v] = \frac{3}{4},$$

we get

$$Pr[\mathcal{A}_{uv}^i] = \binom{n-2}{i} \left(\frac{1}{4}\right)^i \left(\frac{3}{4}\right)^{n-i-2}.$$

Therefore, $Pr[\mathcal{A}_{uv}] < 3n^2\left(\frac{3}{4}\right)^{n-4}$ when n is large enough. Since

$$Pr[\mathcal{A}] \leq \binom{n}{2} Pr[\mathcal{A}_{uv}] < 3n^4 \left(\frac{3}{4}\right)^{n-2} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

it follows that almost all graphs have the property that any two vertices have at least 3 common neighbors. We will complete the proof by showing that $md(G) = 1$ if a graph G has the property that every two vertices of G have at least three common neighbors.

For any two edges $e = ab$ and $f = uv$ of G , there is a path P of G such that the pendent edges of P are e and f . Let $e_1 = x_1x_2$ and $e_2 = x_2x_3$ be two adjacent edges of P . Then x_1 and x_3 have three common neighbors (x_2 is one of them) and thus e_1 and e_2 are in a $K_{2,3}$ of G . This implies $e_1\theta e_2$. By transitivity, $e\theta f$. Therefore, G is a closure, and so $md(G) = 1$ by Lemma 3.1. ■

4 Nordhaus-Gaddum-type results

For a graph parameter, it is always interesting to get the Nordhaus-Gaddum-type results, see [1] and [5, 8, 9, 13, 18, 19, 20] for more such results on various kinds of graph parameters. This section is devoted to get the Nordhaus-Gaddum-type results for our parameter $md(G)$.

For a connected graph G , a vertex v is *deletable* if $G - v$ is connected. Let \mathcal{B} be the set of blocks of G and S be the set of cut-vertices of G . A *block-tree* of G is a bipartite graph $B(G)$ with bipartition \mathcal{B} and S , and a block B of G has an edge with a cut vertex v in $B(G)$ if and only if B contains v . Therefore, every pendent vertex of $B(G)$ is a block (call it a *leaf-block*).

Since $B(G)$ is a tree, there are at least two leaves in $B(G)$ if G has more than one block. For a leaf-block B of G , there are $|B| - 1$ deletable vertices in the block. This implies that every graph has at least two deletable vertices.

Remark 4.1. *If G is a connected simple graph with $|G| \geq 2$, then G has at least two deletable vertices. Furthermore, G has exactly two deletable vertices if and only if G is a path.*

Proof. We only need to deal with the case that G is not a path. If $B(G)$ has at least three leaves, or $B(G)$ has two leaves with one being nontrivial, then G has at least three deletable vertices. If $B(G)$ has exactly two trivial leaf blocks, since G is not a path, there is a nontrivial block B and B has exactly two cut vertices, then B has at least $|B| - 2 \geq 1$ deletable vertices. Therefore, G has at least three deletable vertices. ■

Since a complete graph K_n can be decomposed into two connected graphs if and only if $n \geq 4$, in this section we always assume $n \geq 4$.

Lemma 4.2. *Suppose that G and \overline{G} are connected spanning subgraphs of K_n with $n \geq 5$. Then there is a vertex v of K_n such that v is deletable for both G and \overline{G} .*

Proof. If both G and \overline{G} are 2-connected spanning subgraphs of K_n , then every vertex is deletable for both G and \overline{G} . So, we assume that at least one of $G_1 = G$ and $G_2 = \overline{G}$ has cut vertices. Let v be a cut vertex of G_1 and let S_1, \dots, S_r be the components of $G_1 - v$. Then $F_i = G_1[v \cup S_i]$ is a connected graph. It is obvious that $G_2 - v$ contains

a complete r -partite spanning subgraph, denote it by H . W.l.o.g., let $e = vu_1$ be an edge of G_2 and $u_1 \in S_1$. We distinguish the following cases to discuss.

Case 1. $r = 2$ and $|S_2| \geq 2$, or $r \geq 3$.

By Remark 4.1, there is a deletable vertex u_2 of F_2 and $u_2 \neq v$. Then $G_1 - u_2$ is connected. If $r = 2$ and $|S_2| \geq 2$, then $G_2 - u_2$ is connected, since H is a complete bipartite graph with $|S_2| \geq 2$ and vu_1 is an edge of G_2 with $u_1 \in S_1$. If $r \geq 3$, then $G_2 - u_2$ is also connected. Therefore, u_2 is deletable for both G and \overline{G} .

Case 2. $r = 2$ and $|S_2| = 1$.

Let $S_2 = \{u_2\}$. If F_1 is not a path, by Remark 4.1 F_1 has a deletable vertex w different from v and u_1 . Then $G_1 - w$ is connected. Since u_2 connects to all vertices of S_1 and vu_1 is not affected in $G_2 - w$, $G_2 - w$ is also connected. Therefore, w is deletable for both G and \overline{G} .

If F_1 is a path, then let y be a leaf of F_1 other than v and let x be adjacent to y in F_1 . Since $n \geq 5$, v is not connected to both x and y in F_1 . Therefore, vx and vy are edges of G_2 , both $G_1 - y$ and $G_2 - y$ are connected. Therefore, y is deletable for both G and \overline{G} . ■

Theorem 4.3. *Suppose that G and \overline{G} are connected spanning subgraphs of K_n . Then $md(G) + md(\overline{G}) \leq n + 1$ for $n \geq 5$, and $md(G) + md(\overline{G}) \geq 2$ for $n \geq 8$. Furthermore, the upper bound and the lower bound are sharp.*

Proof. Since both G and \overline{G} are nonempty graphs, $md(G) + md(\overline{G}) \geq 2$ is obvious for $n \geq 8$. So, we need to show that $md(G) + md(\overline{G}) \leq n + 1$ for $n \geq 5$.

If $n = 5$, there are five cases to consider for the graphs G and \overline{G} , and all of the five cases imply that $md(G) + md(\overline{G}) \leq 6 = n + 1$ (see Figure 2).

We proceed by induction on n . The theorem holds for $n = 5$. If $n > 5$, by Lemma 4.2 there is a deletable vertex v for both G and \overline{G} . Let $G' = G - v$. Then G' and \overline{G}' are connected subgraphs of K_{n-1} . By induction, $md(G') + md(\overline{G}') \leq n$. Let Γ be an extremal MD-coloring of G .

Since $n > 5$, at least one of $d_G(v)$ and $d_{\overline{G}}(v)$ is greater than 1 (say $d_G(v) = r \geq 2$). Then v is neither a pendent vertex nor a cut vertex of G , and so by Lemma 2.8, $md(G) \leq md(G')$. If $d_{\overline{G}}(v) \geq 2$, we also have $md(\overline{G}) \leq md(\overline{G}')$. If $d_{\overline{G}}(v) = 1$, then $md(\overline{G}) = md(\overline{G}') + 1$. Therefore, $md(G) + md(\overline{G}) \leq md(G') + md(\overline{G}') + 1 \leq n + 1$.

Now we show that the upper bound is sharp for $n \geq 5$. Let B_n be a tree with $|B_n| = n$ and $\Delta(B_n) = n - 2$. Then $\overline{B_n}$ is a graph obtained by joining a pendent edge to one of the vertices of K_{n-1}^- with minimum degree. Since B_n and $\overline{B_n}$ are connected graphs and $md(B_n) = n - 1$, $md(\overline{B_n}) = 2$, $md(B_n) + md(\overline{B_n}) = n + 1$.

We prove that the lower bound is sharp for $n \geq 8$. Let $V(K_n) = A \cup B \cup \{a, b, u, v\}$, where both $|A|$ and $|B|$ are greater than 1. Let J be a complete bipartite graph with bipartition $A \cup \{a, u\}$ and $B \cup \{b, v\}$. Then $C = J[a, b, u, v]$ is a C_4 . Let G be a graph

obtained from J by deleting the edges of C . Let $G_a = J - \{b, u, v\}$, $G_b = J - \{a, u, v\}$, $G_u = J - \{a, b, v\}$ and $G_v = J - \{a, b, u\}$. Then G is the union of G_a, G_b, G_u and G_v . Since G_a, G_b, G_u and G_v are complete bipartite graphs other than $K_{2,2}$ and stars, by Theorem 3.2 (2) we have $md(G_a) = md(G_b) = md(G_u) = md(G_v) = 1$. Thus, $md(G) = 1$ (see Figure 1) by Lemma 2.4. Since $H_1 = \overline{G}[A \cup a \cup u]$, $H_2 = \overline{G}[a, b, u, v]$ and $H_3 = \overline{G}[B \cup b \cup v]$ are complete graphs, and $E(H_i) \cap E(H_{i+1}) \neq \emptyset$ for $i = 1, 2$, $md(\overline{G}) = 1$ (see Figure 1). Therefore, the lower bound is sharp for $n \geq 8$. ■

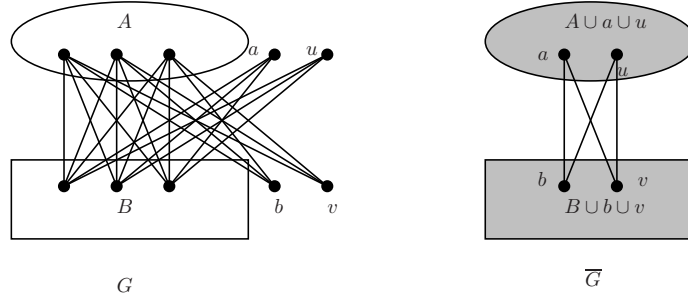


Figure 1: Extremal graphs for $md(G) + md(\overline{G}) = 2$ when $|G| \geq 5$.

In Theorem 4.3, the lower bound of $md(G) + md(\overline{G})$ for $4 \leq n \leq 7$ and the upper bound of $md(G) + md(\overline{G})$ for $n = 4$ are not considered. We discuss them below.

(I) For $n = 4$, since K_4 can only be decomposed into two Hamiltonian paths, $md(G) + md(\overline{G}) = 6 = n + 2$.

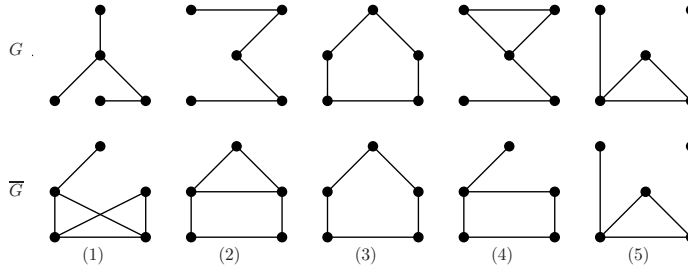


Figure 2: The five cases of G and \overline{G} when $n = 5$.

(II) For $n = 5$, there are ten cases for G and \overline{G} to deal with. However, by symmetry we only need to discuss the five cases depicted in Figure 2. Among all the five cases, (3) implies that the lower bound of $md(G) + md(\overline{G})$ is 4.

(III) For $n = 6$, $e(K_6) = 15$. Since G and \overline{G} are connected spanning subgraphs of K_6 , both $e(G)$ and $e(\overline{G})$ are greater than or equal to 5.

If $e(G) = 5$ and $e(\overline{G}) = 10$, then $md(G) + md(\overline{G}) \geq 6$.

If $e(G) = 6$ and $e(\overline{G}) = 9$, then G is a unicycle graph and the length of the cycle is at most 6. By Proposition 2.2, we have $md(G) \geq 3$. So, $md(G) + md(\overline{G}) \geq 4$.

If $e(G) = 7$ and $e(\overline{G}) = 8$, we assume that G has t blocks. If $t \geq 3$, by Remark 2.1 we have $md(G) \geq 3$. Thus, $md(G) + md(\overline{G}) \geq 4$. If $t = 2$, G is isomorphic to

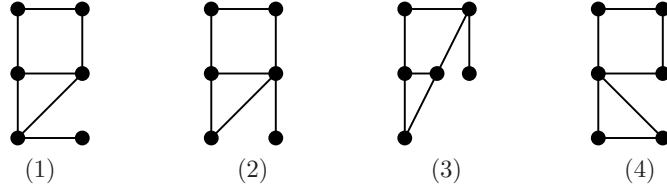


Figure 3: The four cases of graph G when $t = 2$.

one of the four graphs in Figure 3. Since every graph F in Figure 3 has $md(F) = 3$, $md(G) + md(\overline{G}) \geq 4$. If $t = 1$, there are three cases to consider. In Figure 4, we give an extremal MD -coloring for each graph. Since the last two cases of Figure 4 imply that $md(G) + md(\overline{G}) = 4$, the lower bound of $md(G) + md(\overline{G})$ is 4.

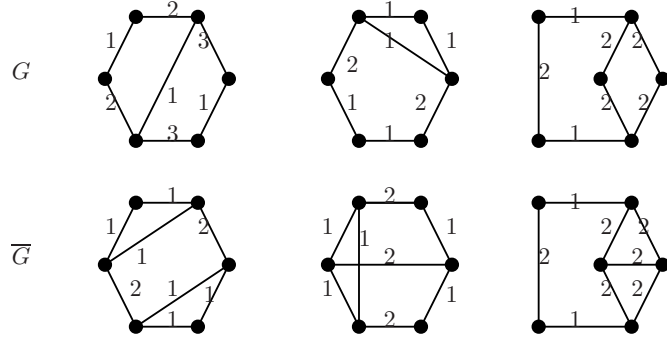


Figure 4: The three cases of G and \overline{G} when $t = 1$.

(IV) For $n = 7$, the lower bound of $md(G) + md(\overline{G})$ is 2. In fact, we only need to construct a graph G (see in Figure 5 (1)) and \overline{G} (see in Figure 5 (2)) such that $md(G) = md(\overline{G}) = 1$.

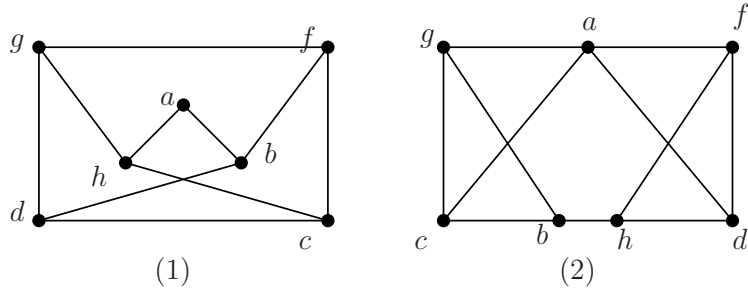


Figure 5: Extremal graphs for $n = 7$.

Let Γ be an extremal MD -coloring of G . Since $G_1 = G[g, f, c, d, h]$ and $G_2 = G[g, f, c, d, b]$ are isomorphic to $K_{2,3}$, and since G_1 and G_2 have common edges, by Lemma 2.4 we have $md(G_1 \cup G_2) = 1$. Since a is neither a cut vertex nor a pendent vertex of G , $md(G) \leq md(G - a) = 1$.

Since $\overline{G}[g, b, c, a]$ and $\overline{G}[a, f, h, d]$ are isomorphic to K_4^- , all the edges of $\overline{G}[g, b, c, a]$ ($\overline{G}[a, f, h, d]$) are colored the same under any MD -coloring of \overline{G} . Then, the 5-cycle $\overline{G}[g, a, f, h, b]$ just has one trivial MD -coloring. Therefore, $md(\overline{G}) = 1$.

For ease of reading, the lower bounds and upper bounds of $md(G) + md(\overline{G})$ for $n \geq 4$ are summarized in the following table.

	$n = 4$	$n = 5$	$n = 6$	$n \geq 7$
Lower bound	6	4	4	2
Upper bound	6	6	7	$n + 1$

Table 1: The bounds of $md(G) + md(\overline{G})$.

Theorem 4.4. *If both G and \overline{G} are connected and $|G| = n \geq 4$, then $md(G) \cdot md(\overline{G}) = 9$ for $n = 4$; $4 \leq md(G) \cdot md(\overline{G}) \leq 9$ for $n = 5$; $3 \leq md(G) \cdot md(\overline{G}) = 10$ for $n = 6$ and $1 \leq md(G) \cdot md(\overline{G}) \leq 2(n - 1)$ for $n \geq 7$. Furthermore, the bounds are sharp.*

Proof. We first consider the upper bounds.

If $n = 4$, then $G = \overline{G} = P_3$, and so $md(G) \cdot md(\overline{G}) = 9$. If $n = 5$, then since $md(G) + md(\overline{G}) = 6$, we have $md(G) \cdot md(\overline{G}) \leq 9$. The graphs G and \overline{G} are shown in Figure 2 (4), implying that $md(G) \cdot md(\overline{G}) = 9$.

For upper bounds, whenever $n \geq 6$, we proceed by induction on n . We prove the inductive base $n = 6$ and the inductive step $n > 6$ simultaneously. Let G and \overline{G} be connected graphs with $n \geq 6$. By Lemma 4.2, there is a vertex v such that both $G - v$ and $\overline{G} - v$ are connected.

Case 1. $d_G(v) \geq 2$ and $d_{\overline{G}}(v) \geq 2$.

Then, v is neither a pendent vertex nor a cut vertex of G and \overline{G} . By Lemma 2.8, $md(G) \leq md(G - v)$ and $md(\overline{G}) \leq md(\overline{G} - v)$. Therefore, $md(G) \cdot md(\overline{G}) \leq md(G - v) \cdot md(\overline{G} - v)$. If $n = 6$, then $md(G) \cdot md(\overline{G}) \leq 9 < 2(n - 1)$. If $n > 6$, then by induction on n , $md(G) \cdot md(\overline{G}) \leq md(G - v) \cdot md(\overline{G} - v) \leq 2(n - 2) < 2(n - 1)$.

Case 2. $d_G(v) = 1$ and $d_{\overline{G}}(v) = n - 2$.

Let u be the neighbor of v in G . Then, v connects every vertex of $V(\overline{G}) - \{u, v\}$ in \overline{G} .

If u is not a cut vertex of $\overline{G} - v$, then $\overline{G} - u = v \vee (\overline{G} - \{u, v\})$ and thus $md(\overline{G} - u) = 1$. Therefore, $md(\overline{G}) \leq 2$.

If $\overline{G} - \{u, v\}$ has two components S_1 and S_2 , then u is a cut vertex of $\overline{G} - \{v\}$ and $\overline{G} - u = (v \vee S_1) \cup (v \vee S_2)$. Since $md(v \vee S_1) = md(v \vee S_2) = 1$, $md(\overline{G} - u) = 2$. Since u is neither a pendent vertex nor a cut vertex of \overline{G} , $md(\overline{G}) \leq md(\overline{G} - u) \leq 2$ by Lemma 2.8.

If $\overline{G} - \{u, v\}$ has components S_1, \dots, S_k , where $k \geq 3$, let w_i be a vertex connecting u in S_i for $i \in [k]$. Then $md(v \vee S_i) = 1$ for $i \in [k]$. If $md(\overline{G}) \neq 1$, then there is

an MD -coloring Γ of \overline{G} with $|\Gamma(\overline{G})| \geq 2$. Since u is neither a pendent vertex nor a cut vertex of \overline{G} , $\Gamma(\overline{G} - u) = \Gamma(\overline{G})$ by Claim 2.7. Then there are two edges e_1 and e_2 of $\overline{G} - u$ such that $\Gamma(e_1) \neq \Gamma(e_2)$. Since $md(v \vee S_i) = 1$ for $i \in [k]$, w.l.o.g., let $e_1 = vw_1$ and $e_2 = vw_2$, then $\overline{G}[u, v, w_1, w_2, w_3] \cong K_{2,3}$. This contradicts that Γ is an MD -coloring restricted on the subgraph $\overline{G}[u, v, w_1, w_2, w_3]$. Therefore, $md(\overline{G}) = 1$. Since $md(G) \leq n - 1$, $md(G) \cdot md(\overline{G}) \leq 2(n - 1)$ for $n \geq 6$.

The graphs B_n and $\overline{B_n}$ defined in the proof of Theorem 4.3 show that $md(B_n) \cdot md(\overline{B_n}) = 2(n - 1)$. So, the upper bound is sharp for $n \geq 6$.

Now we show the lower bounds. If $n = 4$, $md(G) \cdot md(\overline{G}) = 9$. If $n \geq 7$, since there are graphs G and \overline{G} such that $md(G) + md(\overline{G}) = 2$, $md(G) \cdot md(\overline{G}) = 1$, i.e., the lower bound is sharp. If $n = 5$, $md(G) \cdot md(\overline{G})$ is minimum when G and \overline{G} are graphs shown in Figure 2 (3), which implies that $md(G) \cdot md(\overline{G}) = 4$. If $n = 6$, since $md(G) + md(\overline{G}) \geq 4$, $md(G) \cdot md(\overline{G}) \geq 3$. Let G be a graph obtained by connecting an additional vertex w to a vertex u of a 5-cycle (which implies $md(G) = 3$). Then \overline{G} is a graph obtained by connecting w to every vertex of $\overline{C_5}$ except u . Then u is neither a pendent vertex nor a cut vertex of \overline{G} , $md(\overline{G}) \leq md(\overline{G} - u)$. Since $\overline{G} - \{w, u\}$ is a path and $\overline{G} - u = v \vee (\overline{G} - \{w, u\})$, $md(\overline{G} - u) = 1$. Therefore, $md(\overline{G}) = 1$, the lower bound is sharp for $n = 6$. ■

For ease of reading, the lower bounds and upper bounds of $md(G) \cdot md(\overline{G})$ for $n \geq 4$ are summarized in the following table.

	$n = 4$	$n = 5$	$n = 6$	$n \geq 7$
Lower bound	9	4	3	1
Upper bound	9	9	10	$2(n - 1)$

Table 2: The bounds of $md(G) \cdot md(\overline{G})$.

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