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Energies of Complements of Borderenergetic Graphs*

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Abstract

A graph G of order n is said to be *(L)-borderenergetic* if G has the same *(Laplacian) energy* as the complete graph K_n, i.e., $\mathcal{E}(G) = 2(n-1)$. In this paper, by using a few new *Nordhaus-Gaddum-type* results on the (Laplacian) energies of graphs, we obtain some upper bounds of the energy of the complement $\overline{G}$ of an *(L)-borderenergetic* graph G. Then, we show that $\varepsilon(G) + \varepsilon(\overline{G}) < O(n)$, which means that there could be graphs G for which both G and $\overline{G}$ are borderenergetic. As a result, we obtain that for any graph G, except for three graphs (one of order 9 and two of order 11), at most one of G and its complement $\overline{G}$ can be a borderenergetic, and there is a unique self-complementary graph, of order 9.

1 Introduction

All graphs considered in this paper are simple and undirected. Let G be a graph with order n and size m. The maximum degree of G is denoted by $\Delta(G)$. The complete graph

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of order n is denoted by K_n. The complement of a graph G is denoted by $\overline{G}$, whose order and size are n and $\overline{m} = \binom{n}{2} - m$, respectively. Denote the degrees of G and $\overline{G}$ by $d_1 \geq d_2 \geq \cdots \geq d_n$ and $\overline{d_1} \geq \overline{d_2} \cdots \geq \overline{d_n}$, respectively. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\overline{\lambda_1} \geq \overline{\lambda_2} \geq \cdots \geq \overline{\lambda_n}$ be the eigenvalues of the adjacency matrices of G and $\overline{G}$, respectively. For terminology and notation not given here, we refer to [1,3].

The energy of a graph G [14,15], denoted by $\mathcal{E}(G)$, is defined as

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i| .$$

For more information on the graph energy and its applications in chemistry, we refer to [13,16,20].

In 2015, Gong et al. [11] proposed the concept of *borderenergetic* graphs, i.e., graphs of order n satisfying $\mathcal{E}(G) = 2(n-1)$. More results on the borderenergetic graphs can be seen in [4,7,10,17,21,22,25].

Analogously, the concept of *Laplacian borderenergetic*, i.e., *L-borderenergetic* graphs was proposed in [28], that is, a graph G of order n is called L-borderenergetic if $\mathcal{LE}(G) = \mathcal{LE}(K_n) = 2(n-1)$, where $\mathcal{LE}(G)$ is the Laplacian energy of G; see [12]. Results on the L-borderenergetic graphs can be referred to [5,6,8,9,23,26,28,29].

In order to investigate borderenergetic graphs further with a new way different from those before, in this paper we study the energy for the complements of borderenergetic graphs. By using a few new Nordhaus-Gaddum-type results on the (Laplacian) energies of graphs, some upper bounds of the energies for the complements of (L-)borderenergetic graphs are established. Interestingly, we get that, except for three graphs of orders 9 and 11, there is no graph G for which both G and $\overline{G}$ are borderenergetic, and and there is a unique self-complementary graph. .

2 Complements of Borderenergetic Graphs

In [30], a Nordhaus-Gaddum-type result on energies of graphs was presented. See the following.

Theorem 1. *[30] Let G be a graph with n vertices. Then*

$$\varepsilon(G) + \varepsilon(\overline{G}) < \sqrt{2}n + (n-1)\sqrt{n-1}.$$

From this we can immediately get

Corollary 2. *If G is a borderenergetic graph with n vertices, then*

$$\varepsilon(\overline{G}) < \sqrt{2}n + (n-1)(\sqrt{n-1} - 2).$$

Proof. If G is a borderenergetic graph with n vertices, then $\varepsilon(G) = 2(n-1)$. From Theorem 1, the result follows directly. ■

Next a better upper bound (see Corollary 4) of the energy for the complement of any borderenergetic graph is obtained by the following result. Let ω and $\overline{\omega}$ be the clique numbers of G and $\overline{G}$, respectively.

Theorem 3. *[30] Let G be a graph with n vertices. Then*

$$\varepsilon(G) + \varepsilon(\overline{G}) < \sqrt{(2 - \frac{1}{\omega} - \frac{1}{\overline{\omega}})n(n-1)} + (n-1)\sqrt{n-1}\ .$$

With the same reason as Corollary 2, we immediately obtain

Corollary 4. *If G is a borderenergetic graph with n vertices, then*

$$\varepsilon(\overline{G}) < \sqrt{\left(2 - \frac{1}{\omega} - \frac{1}{\overline{\omega}}\right) n(n-1)} + (n-1)(\sqrt{n-1} - 2)\ .$$

In the case of regular graphs, we have

Lemma 5. *[30] Let G be a k-regular graph with n vertices. Then*

$$\begin{aligned} \varepsilon(G) + \varepsilon(\overline{G}) &\leq n - 1 + \left[\sqrt{k(n-k)} + \sqrt{(k+1)(n-k-1)}\right] \\ &< (n-1)\left(\sqrt{n+1} + 1\right)\ . \end{aligned}$$

From Lemma 5, one can immediately arrive at

Theorem 6. *If G is a borderenergetic regular graph with n vertices, then*

$$\varepsilon(\overline{G}) < (n-1)\left(\sqrt{n+1} - 1\right)\ . \tag{1}$$

In fact, the above upper bound in (1) can be improved by using Lemma 7.

Lemma 7. *[2] Let G be a k-regular graph with spectrum*

$$Spec(G) = \begin{pmatrix} k & \lambda_2 & \cdots & \lambda_t \\ 1 & m_2 & \cdots & m_t \end{pmatrix}.$$

Then $\overline{G}$ is an $(n-1-k)$-regular graph with spectrum

$$Spec(\overline{G}) = \begin{pmatrix} n-1-k & -\lambda_2 - 1 & \cdots & -\lambda_t - 1 \\ 1 & m_2 & \cdots & m_t \end{pmatrix}.$$

Theorem 8. *If G is a k-regular borderenergetic graph with n vertices, then*

$$\varepsilon(\overline{G}) \leq 4(n-1) - 2k. \tag{2}$$

Proof. If G is a borderenergetic graph with n vertices, then $\varepsilon(G) = 2(n-1)$. Since G is regular, its spectrum has a form as follows:

$$Spec(G) = \begin{pmatrix} k & \lambda_2 & \cdots & \lambda_t \\ 1 & m_2 & \cdots & m_t \end{pmatrix}.$$

Then $\varepsilon(G) = 2(n-1) = k + \sum_{i=2}^{n} m_i|\lambda_i|$. By Lemma 7, we get

$$\begin{aligned} \varepsilon(\overline{G}) &= n-1-k+\sum_{i=2}^{n} m_i|-\lambda_i-1| \\ &\leq n-1-k+\sum_{i=2}^{n} m_i(|\lambda_i|+1) \\ &= 4(n-1)-2k\ . \end{aligned}$$

The above equality holds if and only if $|-\lambda_i - 1| = |\lambda_i| + 1$ is satisfied for each i with $2 \leq i \leq n$. ■

For the spectral radii of a graph G and its complement $\overline{G}$, Nikiforov [24] showed

$$\lambda_1(G) + \lambda_1(\overline{G}) \geq n - 1 + \sqrt{2}\frac{s^2(G)}{n^3}, \tag{3}$$

where $s(G) = \sum_{1\leq i\leq n} |d_i - \frac{2m}{n}|$.

By the lemma below, another lower bound on $\lambda_1(G) + \lambda_1(\overline{G})$ can be obtained. For a real number $\alpha \in [0,1)$, the A_α-matrix of a graph G is defined as

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)\,,$$

where $A(G)$ and $D(G)$ are the adjacency matrix and diagonal degree matrix of G, respectively. The A_α-spectral radius of G is denoted by $\rho_\alpha(G)$.

Lemma 9. *[18] Let G be a graph of order n and size m. Then*

$$\rho_\alpha(G) + \rho_\alpha(\overline{G}) \geq (n-1) + \frac{s^2(G)}{n^2}\frac{\sqrt{2}}{\sqrt{\alpha^2\frac{n}{2}(n-1)^2 + (1-\alpha)^2 n(n-1)}}\ . \tag{4}$$

When $\alpha = 0$, we see that $\rho_0(G) = \lambda_1$, $\rho_0(\overline{G}) = \overline{\lambda_1}$, and a direct corollary of Lemma 9 is as follows.

Corollary 10. *Let G be a graph of order n. Then*

$$\lambda_1(G) + \lambda_1(\overline{G}) \geq n - 1 + \frac{s^2(G)}{n^2} \frac{\sqrt{2}}{\sqrt{n(n-1)}} . \tag{5}$$

Lemma 11. *[27] Let G be a graph of order n. Then*

$$\lambda_1(G) + \lambda_1(\overline{G}) < \frac{4}{3}n - 1. \tag{6}$$

So, the result in Theorem 1 can be improved as follows.

Theorem 12. *Let G be a graph of order n. Then*

$$\varepsilon(G) + \varepsilon(\overline{G}) < \frac{4}{3}n - 1 + \sqrt{2n(n-1)^2 - (n-1)\left[n - 1 + \frac{\sqrt{2}s^2(G)}{n^2\sqrt{n(n-1)}}\right]^2} . \tag{7}$$

Proof. In [19], it was shown that for a graph G with n vertices and m edges,

$$\varepsilon(G) \leq \lambda_1 + \sqrt{(n-1)(2m - \lambda_1^2)}. \tag{8}$$

From the inequalities (5) , (6) and (8), we have

$$\begin{aligned}
\varepsilon(G) + \varepsilon(\overline{G}) &\leq \lambda_1 + \overline{\lambda_1} + \sqrt{(n-1)(2m - \lambda_1^2)} + \sqrt{(n-1)\left(2\overline{m} - \overline{\lambda_1}^2\right)} \\
&\leq \lambda_1 + \overline{\lambda_1} + \sqrt{2(n-1)\left[2m + 2\overline{m} - \left(\lambda_1^2 + \overline{\lambda_1}^2\right)\right]} \\
&\leq \lambda_1 + \overline{\lambda_1} + \sqrt{2(n-1)\left[n(n-1) - \frac{1}{2}\left(\lambda_1 + \overline{\lambda_1}\right)^2\right]} \\
&< \frac{4}{3}n - 1 + \sqrt{2n(n-1)^2 - (n-1)\left[n - 1 + \frac{\sqrt{2}s^2(G)}{n^2\sqrt{n(n-1)}}\right]^2} .
\end{aligned}$$

■

Observing the following two types of strongly regular graphs constructed in [30], it is easy to check that the upper bound (7) is asymptotically tight.

Type 1. If G is a strongly regular graph with parameters $(n,\ (n+\sqrt{n})/2,\ (n+2\sqrt{n})/4,\ (n + 2\sqrt{n})/4)$, then

$$\varepsilon(G) + \varepsilon(\overline{G}) = (n-1)(\sqrt{n} + 1) - 1.$$

Type 2. For a Paley graph H, which is a strongly regular graph with parameters (n, $(n-1)/2$, $(n-5)/4$, $(n-1)/4$),

$$\varepsilon(H) + \varepsilon(\overline{H}) = (n-1)(\sqrt{n}+1) \ .$$

If G is borderenergetic, by Theorem 12 we get

Theorem 13. *Let G be a borderenergetic graph. Then*

$$\varepsilon(\overline{G}) \ < \ \sqrt{2n(n-1)^2 - (n-1)\left[n-1+\frac{\sqrt{2}s^2(G)}{n^2\sqrt{n(n-1)}}\right]^2 - \frac{2n}{3}+1} \ . \tag{9}$$

From the above inequality (9), as n is large enough, it yields that

$$\varepsilon(\overline{G}) \ < \ O(n^{3/2}). \tag{10}$$

In practical, the inequality (10) can be better improved as follows.

Theorem 14. *Let G be a borderenergetic graph of order n. Then*

$$\varepsilon(\overline{G}) \ = \ O(1), \tag{11}$$

as n is large enough.

Proof. In [4], an asymptotically tight bound on the size of a borderenergetic graph was given as follows:

$$m \geq \left\lceil \frac{\left[2(n-1) - \sqrt{\frac{1}{n}\sum_{i=1}^{n} d_i^2}\right]^2}{2(n-1)} + \frac{\sum_{i=1}^{n} d_i^2}{2n} \right\rceil . \tag{12}$$

If G is $(n-3)$-regular, then the bound in (12) is asymptotically tight.

As n is large enough, by (12) we can see that

$$m \geq O(n^2). \tag{13}$$

As $m \leq \frac{n(n-1)}{2}$, we get $\overline{m} = O(1)$. From the fact that

$$2\sqrt{\overline{m}} \leq \varepsilon(\overline{G}) \leq 2\overline{m}, \tag{14}$$

we obtain $\varepsilon(\overline{G}) = O(1)$. ■

What is more important, by Theorem 14 we see that, in the case of borderenergetic graphs, the upper bound behaves much better for the Nordhaus-Gaddum-type result (i.e., much better than Theorem 1). That is

Theorem 15. *Let G be a borderenergetic graph with n vertices. Then*

$$\varepsilon(G) + \varepsilon(\overline{G}) < O(n),$$

as n is large enough.

An interesting question is whether there is a graph G such that both G and its complement $\overline{G}$ are borderenergetic ? In fact, if it is yes, then one will have

$$\varepsilon(G) + \varepsilon(\overline{G}) = 4(n-1),$$

which is in linear of n. Theorem 15 tells us that such graphs could exist. As a result of our investigation, we find that there are exactly three such graphs, one with order 9 and two with order 11; see Figure 1.

In fact, borderenergetic graphs of order n with $1 \leq n \leq 11$ were found by using computers in [11, 21, 25], and there is no borderenergetic graph of order less than 7. One can check that among them, only three graphs have the property that both the graph and its complement are borderenergetic. The adjacency spectra of the three graphs are given as follows:

$$Sp_A(G_9^1) = \{4, 1, 1, 1, 1, -2, -2, -2, -2\};$$
$$Sp_A(G_{11}^2) = \{5, 1, 1, 1, 1, 1, -2, -2, -2, -2, -2\};$$
$$Sp_A(G_{11}^3) = \{6, 1, 1, 1, 1, 0, -2, -2, -2, -2, -2\}.$$

One can see that $\overline{G_9^1} \cong G_9^1$ and $\overline{G_{11}^2} \cong G_{11}^3$, and especially, the graph G_9^1 is self-complementary.

So, we are left to deal with the case of $n \geq 12$ only.

Theorem 16. *Except for the three graphs G_9^1, G_{11}^2 and G_{11}^3, for any graph G at most one of G and its complement $\overline{G}$ can be a borderenergetic graph.*

Proof. For the case that $1 \leq n \leq 11$, we are done in the above discussion. Next we consider the case of $n \geq 12$. Let G be a borderenergetic graph with order $n \geq 12$. By contradiction, suppose $\overline{G}$ is also borderenergetic. Then $\varepsilon(\overline{G}) = 2(n-1)$. Due to (14), we have

$$\overline{m} \geq n - 1. \tag{15}$$

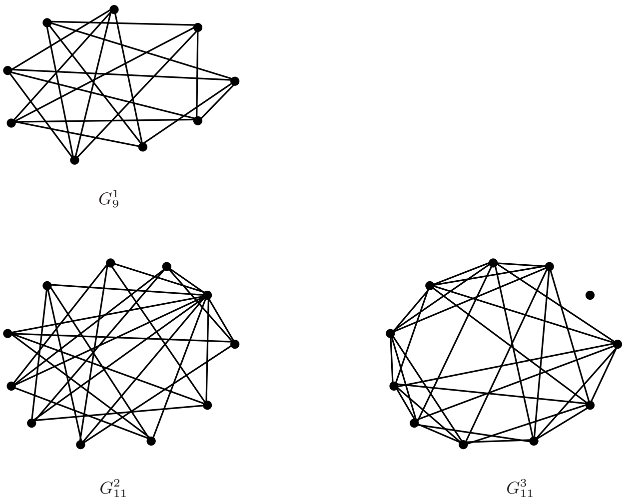

Figure 1. Three borderenergetic graphs: G_9^1, G_{11}^2 and G_{11}^3.

Let $f(x)$ be a function on $x > 0$, where

$$f(x) = \frac{\left[2(n-1) - \sqrt{\frac{x}{n}}\,\right]^2}{2(n-1)} + \frac{x}{2n}.$$

Then the derivative of $f(x)$ is

$$f^{'}(x) = \frac{n\sqrt{\frac{x}{n}} - 2n + 2}{2n(n-1)\sqrt{\frac{x}{n}}}.$$

It can be seen that $f(x)$ is increasing as $x > \frac{4(n-1)^2}{n}$. Assume $x = \sum_{i=1}^{n} d_i^2$. One can check that

$$n\sum_{i=1}^{n} d_i^2 \geq \left(\sum_{i=1}^{n} d_i\right)^2 = 4m^2 > 4(n-1)^2 \ .$$

The second inequality in the above holds when a connected borderenergetic graph is not a tree [4]. Then we have

$$\sum_{i=1}^{n} d_i^2 > \frac{4(n-1)^2}{n}.$$

By (12), we get

$$\overline{m} \ \leq \ \frac{(n-1)n}{2} - \left[\frac{\left[2(n-1) - \sqrt{\frac{1}{n}\sum_{i=1}^{n} d_i^2}\,\right]^2}{2(n-1)} + \frac{\sum_{i=1}^{n} d_i^2}{2n}\right]$$

$$\leq \frac{(n-1)n}{2} - \left(\frac{\left[2(n-1) - \sqrt{\frac{1}{n}\sum_{i=1}^{n} d_i^2}\,\right]^2}{2(n-1)} + \frac{\sum_{i=1}^{n} d_i^2}{2n} \right)$$

Combining (15), we have

$$\begin{aligned} n-1 &\leq \frac{(n-1)n}{2} - \left(\frac{\left[2(n-1) - \sqrt{\frac{1}{n}\sum_{i=1}^{n} d_i^2}\,\right]^2}{2(n-1)} + \frac{\sum_{i=1}^{n} d_i^2}{2n} \right) \\ &= \frac{(n-1)n}{2} - \left(\frac{\left[2(n-1) - \sqrt{\frac{1}{n}x}\,\right]^2}{2(n-1)} + \frac{x}{2n} \right). \qquad (16) \end{aligned}$$

By direct computation, we can see that the above inequality (16) holds if $x_1 \leq x \leq x_2$, where

$$\begin{aligned} x_1 &= \frac{n^4 - 8n^3 + 21n^2 - 22n + 8 - 4\sqrt{n^6 - 10n^5 + 34n^4 - 56n^3 + 49n^2 - 22n + 4}}{n}, \\ x_2 &= \frac{n^4 - 8n^3 + 21n^2 - 22n + 8 + 4\sqrt{n^6 - 10n^5 + 34n^4 - 56n^3 + 49n^2 - 22n + 4}}{n}. \end{aligned}$$

Now we distinguish the following two cases by considering the maximum degree $\Delta(G)$.

Case 1. $\Delta(G) \leq n-6$.

Then $\sum_{i=1}^{n} d_i^2 \leq n(n-6)^2$. By computation, we get that

$$\begin{aligned} & x_1 - n(n-6)^2 \\ &= \frac{n^4 - 8n^3 + 21n^2 - 22n + 8 - 4\sqrt{n^6 - 10n^5 + 34n^4 - 56n^3 + 49n^2 - 22n + 4}}{n} \\ & \quad -n(n-6)^2 \\ &= \frac{4n^3 - 15n^2 - 22n + 8 - 4\sqrt{n^6 - 10n^5 + 34n^4 - 56n^3 + 49n^2 - 22n + 4}}{n} > 0 \,. \end{aligned}$$

That is, $x_1 > n(n-6)^2 \geq x$, which means that the inequality (16) does not hold and this is a contradiction.

Case 2. $\Delta(G) > n-6$.

Then $\Delta(G) \in \{n-5, n-4, n-3, n-2, n-1\}$. From (16) and that $f(x)$ is increasing for $n \geq 12$, we have

$$n-1 \leq \frac{(n-1)n}{2} - \left(\frac{\left[2(n-1) - \sqrt{\frac{1}{n}\sum_{i=1}^{n} d_i^2}\,\right]^2}{2(n-1)} + \frac{\sum_{i=1}^{n} d_i^2}{2n} \right)$$

$$\leq \frac{(n-1)n}{2} - \left(\frac{\left[2(n-1) - \sqrt{\frac{\Delta^2(G)}{n}}\right]^2}{2(n-1)} + \frac{\Delta^2(G)}{2n} \right). \tag{17}$$

Then (17) holds if $x_1 \leq \Delta^2(G) \leq x_2$. But in fact we can check that $x_1 > \Delta^2(G)$ as

$$x_1 - \Delta^2(G) > x_1 - n(n-6)^2 > 0,$$

for each $\Delta(G) \in \{n-5, n-4, n-3, n-2, n-1\}$, which implies that the inequality (17) does not hold and this is a contradiction. ■

From the above result, one can immediately derive the following corollaries.

Corollary 17. *There is a unique self-complementary borderenergetic graph, which is the graph G_9^1 on 9 vertices.*

Corollary 18. *There is a unique regular borderenergetic graph for which both the graph and its complement are borderenergetic, which is the graph G_9^1 on 9 vertices.*

3 Complements of Laplacian borderenergetic graphs

In this section, two upper bounds of the Laplacian energies for the complements of L-borderenergetic graphs are established. The auxiliary quantity $M(G)$ of G [12] is defined as

$$M(G) = m + \frac{1}{2}\sum_{i=1}^{n}\left(d_i - \frac{2m}{n}\right)^2 .$$

Similarly,

$$M(\overline{G}) = \overline{m} + \frac{1}{2}\sum_{i=1}^{n}\left(\overline{d_i} - \frac{2\overline{m}}{n}\right)^2 .$$

These two kinds of inequalities below can be found in [12].

The Koolen–Moulton type of inequality on the Laplacian energy is

$$\mathcal{LE}(G) \leq \frac{2m}{n} + \sqrt{(n-1)\left[2M - \left(\frac{2m}{n}\right)^2\right]} . \tag{18}$$

The McClelland type of inequality on the Laplacian energy is

$$\mathcal{LE}(G) \leq \sqrt{2nM}. \tag{19}$$

Denote the maximum degrees of G and $\overline{G}$ by Δ and $\overline{\Delta}$, respectively. Suppose $\Delta_0 = max\{\Delta, \overline{\Delta}\}$. We get a Nordhaus-Gaddum-Type bound for the Laplacian energy as follows.

Theorem 19. *Let G be a graph of order n. Then*

$$\mathcal{LE}(G) + \mathcal{LE}(\overline{G}) \;<\; n-1+2(n-1)\sqrt{\frac{n}{2}(\Delta_0+1)-1-\frac{1}{n}}. \tag{20}$$

Proof. By surveying the quality $M(G)+M(\overline{G})$, we have

$$\begin{aligned}
M(G)+M(\overline{G}) &= m+\overline{m}+\frac{1}{2}\sum_{i=1}^{n}\left(d_i-\frac{2m}{n}\right)^2+\frac{1}{2}\sum_{i=1}^{n}\left(\overline{d_i}-\frac{2\overline{m}}{n}\right)^2 \\
&= \frac{1}{2}n(n-1)+\frac{1}{2}\left(\sum_{i=1}^{n}d_i^2-\frac{4m^2}{n}\right)+\frac{1}{2}\left(\sum_{i=1}^{n}\overline{d_i}^2-\frac{4\overline{m}^2}{n}\right) \\
&= \frac{1}{2}n(n-1)+\frac{1}{2}\sum_{i=1}^{n}d_i^2-\frac{2m^2}{n}+\frac{1}{2}\sum_{i=1}^{n}\overline{d_i}^2-\frac{2\overline{m}^2}{n} \\
&\le \frac{1}{2}n(n-1)+m\Delta+\overline{m}\overline{\Delta}-\frac{2}{n}\left(m^2+\overline{m}^2\right) \\
&< \frac{1}{2}n(n-1)+(m+\overline{m})\Delta_0-\frac{2}{n}(m+\overline{m}) \\
&= \frac{1}{2}n(n-1)\left(\Delta_0+1-\frac{2}{n}\right)
\end{aligned}$$

Then by the Koolen-Moulton type of inequality, i.e., (18), we obtain

$$\begin{aligned}
\mathcal{LE}(G)+\mathcal{LE}(\overline{G}) &\le \frac{2m}{n}+\sqrt{(n-1)\left[2M-\frac{4m^2}{n^2}\right]}+\frac{2\overline{m}}{n}+\sqrt{(n-1)\left[2\overline{M}-\frac{4\overline{m}^2}{n^2}\right]} \\
&= n-1+\sqrt{n-1}\left(\sqrt{2M-\frac{4m^2}{n^2}}+\sqrt{2\overline{M}-\frac{4\overline{m}^2}{n^2}}\right) \\
&\le n-1+2\sqrt{n-1}\sqrt{M+\overline{M}-\frac{2(m^2+\overline{m}^2)}{n^2}} \\
&\le n-1+2\sqrt{n-1}\sqrt{M+\overline{M}-\frac{2(m+\overline{m})}{n^2}} \\
&= n-1+2\sqrt{n-1}\sqrt{M+\overline{M}-\frac{n-1}{n}}
\end{aligned}$$

$$< \quad n-1+2\sqrt{n-1}\sqrt{\frac{1}{2}n(n-1)\left(\Delta_0+1-\frac{2}{n}\right)-\frac{n-1}{n}}$$

$$= \quad n-1+2(n-1)\sqrt{\frac{n}{2}(\Delta_0+1)-1-\frac{1}{n}}.$$

■

By Theorem 19, it easy to see that

Corollary 20. *Let G be an L-borderenergetic graph of order n. Then*

$$\mathcal{LE}(\overline{G}) \quad < \quad 2(n-1)\sqrt{\frac{n}{2}(\Delta_0+1)-1-\frac{1}{n}}-n+1. \tag{21}$$

On the other hand, from the McClelland type of inequality on the Laplacian energy, a better bound on $\mathcal{LE}(G)+\mathcal{LE}(\overline{G})$ is presented.

Theorem 21. *Let G be a graph of order n. Then*

$$\mathcal{LE}(G)+\mathcal{LE}(\overline{G}) \quad < \quad n\sqrt{2(n-1)\left(\Delta_0+1-\frac{2}{n}\right)}\ . \tag{22}$$

Proof. Note that

$$M(G)+M(\overline{G}) \quad < \quad \frac{1}{2}n(n-1)\left(\Delta_0+1-\frac{2}{n}\right)\ .$$

Then by applying the McClelland type of inequality on the Laplacian energy, i.e., (19), we have

$$\begin{aligned}
\mathcal{LE}(G)+\mathcal{LE}(\overline{G}) &\leq \sqrt{2nM}+\sqrt{2n\overline{M}} \\
&= \sqrt{2n}\left(\sqrt{M}+\sqrt{\overline{M}}\right) \\
&\leq \sqrt{2n}\sqrt{2(M+\overline{M})} \\
&= \sqrt{4n\left(M+\overline{M}\right)} \\
&< n\sqrt{2(n-1)\left(\Delta_0+1-\frac{2}{n}\right)}\ .
\end{aligned}$$

■

From Theorem 21, it directly yields that

Theorem 22. *Let G be an L-borderenergetic graph with n vertices. Then*

$$\mathcal{LE}(\overline{G}) \quad \leq \quad n\sqrt{2(n-1)\left(\Delta_0+1-\frac{2}{n}\right)}-2(n-1)\ . \tag{23}$$

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