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Gallai-Ramsey multiplicity for rainbow small trees

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Abstract Let G, H be two non-empty graphs and k be a positive integer. The Gallai-Ramsey number $\mathrm{gr}_k(G : H)$ is defined as the minimum positive integer N such that for all $n \geq N$, every k-edge-coloring of K_n contains either a rainbow subgraph G or a monochromatic subgraph H. The Gallai-Ramsey multiplicity $\mathrm{GM}_k(G : H)$ is defined as the minimum total number of rainbow subgraphs G and monochromatic subgraphs H for all k-edge-colored $K_{\mathrm{gr}_k(G:H)}$. In this paper, we get some exact values of the Gallai-Ramsey multiplicity for rainbow small trees versus general monochromatic graphs under a sufficiently large number of colors. We also study the bipartite Gallai-Ramsey multiplicity.

Keywords Coloring · Ramsey theory · Gallai-Ramsey number · Gallai-Ramsey multiplicity

Mathematics Subject Classification (2020) 05C15 · 05C30 · 05C55

1 Introduction

In this paper, the graphs we consider are finite, undirected, simple and without isolated vertices. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph G, respectively. An *edge-coloring* of G is a function $c : E(G) \to \{1, 2, \ldots, k\}$, where $\{1, 2, \ldots, k\}$ is called the set of colors. We can also use red, blue or other specific names to represent these colors. An edge-colored

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graph is called *rainbow* if all its edges have distinct colors and *monochromatic* if all its edges have the same color. A k-edge-coloring of a graph is *exact* if all the k colors are used at least once. In this paper, we only consider exact edge-colorings of graphs.

The *union* $G \cup H$ of two graphs G and H is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The *degree*, $\deg_G(v)$ or $\deg(v)$ for short, of a vertex v of G is the number of edges incident to v in G. We usually say that a vertex with degree 1 is a *leaf vertex*, and an edge incident to a leaf vertex is called a *pendent edge*. A path with n vertices from v_1 to v_n is denoted as $P_n = v_1v_2 \dots v_n$ or $P_n = e_1e_2 \dots e_{n-1}$, which is a vertex-edge alternative sequence $v_1, e_1, v_2, e_2, \dots, v_{n-1}, e_{n-1}, v_n$ such that $v_1, v_2, \dots, v_n$ are distinct vertices, $e_1, e_2, \dots, e_{n-1}$ are distinct edges and $v_iv_{i+1} = e_i$ for each $i \in \{1, 2, \dots, n-1\}$. S_3^+ is a the graph consisting of a triangle with one pendent edge and P_4^+ is a the graph consisting of P_4 with one pendent edge incident with an inner vertex of P_4. P_4^+ can also be seen as the graph by adding an extra pendent edge to a leaf vertex of $K_{1,3}$. For convenience, we call the newly added pendent edge at $K_{1,3}$ the *tail edge* of P_4^+.

The *automorphism group* of a graph G is denoted as $\mathrm{Aut}(G)$. For the automorphism groups of some special graphs, we have the following conclusions: $\mathrm{Aut}(K_n) \cong S_n, \mathrm{Aut}(K_{m,n}) \cong S_m \times S_n$ for $m \neq n$, and $\mathrm{Aut}(P_n) \cong S_2$, where S_n is the n-order symmetric group. For more notation and terminology not defined here, we refer to [2].

We also define some notations to replace some text descriptions in this paper. We use $e_1 \sim e_2$ to denote two adjacent edges e_1 and e_2; similarly, we use $e_1 \nsim e_2$ to denote two non-adjacent edges e_1 and e_2. Given a subgraph H of a graph G, we define $\mathrm{Num}_G(H)$ to be the number of different copies of H in G. Furthermore, if $e_1, e_2, \dots, e_i$ are edges of graph G, we use $\mathrm{Num}_G(H|e_1, e_2, \dots, e_i)$ to denote the number of different copies of H that contain edges $e_1, e_2, \dots, e_i$ in G. For the balanced complete $(k-1)$-part graph $K_{\underbrace{2, 2, \dots, 2}_{k-1}}$, we re-write it as $K_{(k-1)\times 2}$.

1.1 Gallai-Ramsey number and multiplicity

In 1930, Ramsey problems were first studied by Ramsey in [24]. Given two graphs G and H, the *Ramsey number* $r(G, H)$ is defined as the minimum positive integer n such that every red/blue-edge-coloring of K_n contains either a red subgraph G or a blue subgraph H. If $G = H$, then we simply denote $r(G, H)$ as $r(G)$. More generally, the definition of Ramsey number has been extended to multicolor and hypergraphs, and there are currently many research results available. Determining the exact value of the Ramsey numbers or improving the known upper or lower bounds on the number has always been a hot research topic in graph theory. For more results on Ramsey numbers, we refer to [23].

In 2010, Faudree, Gould, Jacobson and Magnant in [4] provided a definition of a rainbow version of the Ramsey number, called the *Gallai-Ramsey number*.

Definition 1 [4] Given two non-empty graphs G, H and a positive integer k, define the Gallai-Ramsey number $\mathrm{gr}_k(G : H)$ to be the minimum integer N such that for all $n \geq N$, every k-edge-coloring of K_n contains either a rainbow subgraph G or a monochromatic subgraph H.

Considering the Gallai-Ramsey number on k-edge-colored balanced complete bipartite graph $K_{n,n}$ is another research direction. In 2019, Li, Wang and Liu in [17] gave the definition of *bipartite Gallai-Ramsey number*.

Definition 2 [17] Given two non-empty bipartite graphs G, H and a positive integer k, define the bipartite Gallai-Ramsey number $\mathrm{bgr}_k(G : H)$ to be the minimum integer N such that for all $n \geq N$, every k-edge-coloring of $K_{n,n}$ contains either a rainbow subgraph G or a monochromatic subgraph H.

In the past decade, there has been a wealth of research on the Gallai-Ramsey numbers. In terms of current research progress, six types of rainbow graphs have been studied, which are $K_3, S_3^+, K_{1,3}, P_4, P_5$ and P_4^+. An edge-coloring of a complete graph without rainbow triangles is called a Gallai coloring. As early as 2010, Gyárfás, Sárközy, Sebő and Selkow studied the Ramsey problem of complete graphs under Gallai coloring in [10], although they did not use the definition of Gallai-Ramsey formally. The Gallai-Ramsey number involving rainbow triangle has received widespread attention. One of the most important conjectures was proposed by Fox, Grinshpun and Pach in their 2015 paper [6].

Conjecture 1 [6] For integers $k \geq 1$ and $n \geq 3$,

$$\mathrm{gr}_k(K_3 : K_n) = \begin{cases} (r(K_n) - 1)^{k/2} + 1, & \text{if } k \text{ is even,} \\ (n-1)(r(K_n) - 1)^{(k-1)/2} + 1, & \text{if } k \text{ is odd.} \end{cases}$$

Conjecture 1 has been solved in some special cases. When $n = 3$, Gyárfás, Sárközy, Sebő and Selkow gave a simple proof in [10]. In [18], Liu, Magnant, Saito, Schiermeyer and Shi solved the conjecture when $n = 4$. In [20], Magnant and Schiermeyer studied the case of $n = 5$ and concluded that only one of the conjectures of Conjecture 1 and $r(K_5) = 43$ [22] could be true, while the other was false. In [13], Li, Broersma and Wang studied other extremal problems related to Gallai coloring.

For the research on Gallai-Ramsey number involving rainbow S_3^+, we refer to [7, 14, 16]. For rainbow subgraphs $K_{1,3}$ and P_4^+, we refer to [1, 3], and for rainbow subgraphs P_4 and P_5, we refer to [11, 17, 27, 28]. For more results about Gallai-Ramsey numbers, we refer to the monograph [19] and the survey paper [8].

Recently, a counting problem related to the Gallai-Ramsey number has been studied. Li, Broersma and Wang in [12] studied the minimum number of subgraphs H in all k-edge-colored complete graphs without rainbow triangles (also known as Gallai coloring). Later, Mao in [21] proposed the definition of *Gallai-Ramsey multiplicity*.

Definition 3 [21] Given two non-empty graphs G, H and a positive integer k, define the Gallai-Ramsey multiplicity $\mathrm{GM}_k(G : H)$ to be the minimum total number of rainbow subgraphs G and monochromatic subgraphs H for all k-edge-colored $K_{\mathrm{gr}_k(G:H)}$.

Based on the definitions of bipartite Gallai-Ramsey number and Gallai-Ramsey multiplicity, we give the definition of *bipartite Gallai Ramsey multiplicity.*

Definition 4 Given two non-empty bipartite graphs G, H and a positive integer k, define the bipartite Gallai-Ramsey multiplicity bi-$\mathrm{GM}_k(G : H)$ to be the minimum total number of rainbow subgraphs G and monochromatic subgraphs H for all k-edge-colored $K_{\mathrm{bgr}_k(G:H),\mathrm{bgr}_k(G:H)}$.

1.2 Structural theorems under rainbow-tree-free colorings

The five k-edge-colored structures of complete graphs or complete bipartite graphs given below is for convenience of describing several structural theorems.
Colored Structure 1: Let $(V_1, V_2, \ldots, V_k)$ be a partition of $V(K_n)$ such that for each i, all the edges connecting two vertices in V_i are colored by either 1 or i and all the edges between V_i and V_j with $i \neq j$ are colored by 1.
Colored Structure 2: Let K_n be a k-edge-colored complete graph such that $K_n - v$ is monochromatic for some vertex v.
Colored Structure 3: Let (U, V) be the bipartition of complete bipartite graph $K_{n,n}$. U can be partitioned into k non-empty parts $U_1, U_2, \ldots, U_k$ such that all the edges between U_i and V have color i for $i \in \{1, 2, \ldots, k\}$.
Colored Structure 4: Let (U, V) be the bipartition of complete bipartite graph $K_{n,n}$. U can be partitioned into two parts U_1 and U_2 with $|U_1| \geq 1, |U_2| \geq 0$, and V can be partitioned into k parts $V_1, V_2, \ldots, V_k$ with $|V_1| \geq 0$ and $|V_j| \geq 1$, $j \in \{2, 3, \ldots, k\}$, such that all the edges between V_i and U_1 have color i and all the edges between V_i and U_2 have color 1 for $i \in \{1, 2, \ldots, k\}$.
Colored Structure 5: Let (U, V) be the bipartition of complete bipartite graph $K_{n,n}$. U can be partitioned into k parts $U_1, U_2, \ldots, U_k$ with $|U_1| \geq 0, |U_j| \geq 1$ and V can be partitioned into k parts $V_1, V_2, \ldots, V_k$ with $|V_1| \geq 0, |V_j| \geq 1$, $j \in \{2, 3, \ldots, k\}$, such that only colors 1 and i can be used on the edges between U_i and V_i for $i \in \{1, 2, \ldots, k\}$, and all the other edges have color 1.

Thomason and Wagner in [26] obtained the following results.

Theorem 1 [26] *For an integer $n \geq 4$, let K_n be an edge-colored complete graph with at least three colors so that it contains no rainbow P_4 if and only if $n = 4$ and three colors are used, each color forming a perfect matching.*

Theorem 2 [26] *For integers $k \geq 5$ and $n \geq 5$, let K_n be a k-edge-colored complete graph so that it contains no rainbow P_5 if and only if Colored Structure 1 or Colored Structure 2 occurs.*

Bass, Magnant, Ozeki and Pyron in [1] obtained the following results. The study of this structural theorem can be traced back to the study of local k-coloring Ramsey numbers by Gyárfás, Lehel, Schelp and Tuza in [9].

Theorem 3 [1,9] *For integers $k \geq 4$ and $n \geq 4$, let K_n be a k-edge-colored complete graph so that it contains no rainbow $K_{1,3}$ if and only if Colored Structure 1 occurs.*

Bass, Magnant, Ozeki and Pyron in [1] described the colored structure of complete graphs without rainbow P_4^+. Also, Schlage-Puchta and Wagner proved this structural theorem in [25].

Theorem 4 [1,25] *For integers $k \geq 5$ and $n \geq 5$, let K_n be a k-edge-colored complete graph so that it contains no rainbow P_4^+ if and only if Colored Structure 1 occurs.*

In terms of edge-colorings of complete bipartite graphs, Li, Wang and Liu in [17] first obtained the following results.

Theorem 5 [17] *Let (U, V) be the bipartition of a complete bipartite graph. For integers $k \geq 3$ and $n \geq 2$, let $K_{n,n}$ be a k-edge-colored complete bipartite graph so that it contains no rainbow P_4 if and only if Colored Structure 3 occurs.*

Theorem 6 [17] *Let (U, V) be the bipartition of a complete bipartite graph. For integers $k \geq 5$ and $n \geq 3$, let $K_{n,n}$ be a k-edge-colored complete bipartite graph so that it contains no rainbow P_5 if and only if Colored Structure 4 or Colored Structure 5 occurs.*

Li and Wang in [15] first described the colored structure of complete bipartite graphs without rainbow $K_{1,3}$. Recently, Chen, Ji, Mao and Wei in [3] repeoved this structural theorem on balanced complete bipartite graphs.

Theorem 7 [3,15] *Let (U, V) be the bipartition of a complete bipartite graph. For integers $k \geq 5$ and $n \geq 3$, let $K_{n,n}$ be a k-edge-colored complete bipartite graph so that it contains no rainbow $K_{1,3}$ if and only if Colored Structure 5 occurs.*

It should be noted that the structural theorems cited above are only a part of what is needed in this paper. For a complete survey of the structural theorems, we refer to the original references.

1.3 Main results

Due to the fact that the research in this paper is based on the exact k-edge-colorings, it is natural to require that the number of edges in the graph is at least the number of colors that we consider. From the above structural theorems, it can be seen that if there are no rainbow subgraphs $G \in$

$\{P_4, P_5, K_{1,3}, P_4^+\}$ for k-edge-colored complete graphs or complete bipartite graphs, then there must be some monochromatic graphs under this colored structure. For example, a k-edge-colored complete graph without rainbow $K_{1,3}$ must contain a monochromatic $K_{(k-1)\times 2}$; a k-edge-colored balanced complete bipartite graph without rainbow P_4 must contain a monochromatic $K_{1,k}$; a k-edge-colored balanced complete bipartite graph without rainbow P_5 or $K_{1,3}$ must contain a monochromatic $K_{1,\lceil \frac{k-1}{2} \rceil}$. Based on these properties, we know that when the number of colors k is sufficiently large with respect to the subgraph H, the Gallai-Ramsey number $\mathrm{gr}_k(G:H)$ (also bipartite Gallai-Ramsey number $\mathrm{bgr}_k(G:H)$) does not depend on the subgraph H, but only on k.

In the third section of this paper, we give some exact values of the Gallai-Ramsey number and multiplicity when the number of colors $k = \binom{t}{2}$ is sufficiently large with respect to the subgraph H. The main results for the Gallai-Ramsey multiplicity are shown in the Table 1.

Table 1 Main results for Gallai-Ramsey multiplicity. The graph H is a subgraph of some graphs related to k, see Theorems 9, 10, 11, 12, 13, 14, 16 and 17 for details.

	$\mathrm{GM}_{k-1}(G:H)$	$\mathrm{GM}_{k-2}(G:H)$
$G = K_{1,3}$	$3\ (t=4)$ $18\ (t=5)$ $(t-1)\binom{t-1}{3} + \binom{t-3}{3} + 2\binom{t-3}{2}\ (t \geq 6)$	$1\ (t=4)$ $14\ (t=5)$ $(t-3)\binom{t-1}{3} + 3\binom{t-3}{3} + 6\binom{t-3}{2}\ (t \geq 6)$
$G = P_4^+$	$60\binom{t}{5} - 5(t-3)(t-4)\ (t \geq 5)$	$60\binom{t}{5} - 15(t-3)(t-4)\ (t \geq 5)$
$G = P_4$	$8\ (t=4)$ $12\binom{t}{4} - 2(t-3)\ (t \geq 5)$	$4\ (t=4)$ $12\binom{t}{4} - 6(t-3)\ (t \geq 5)$
$G = P_5$	$60\binom{t}{5} - 12(t-4)\ (5 \leq t \leq 6)$ $60\binom{t}{5} - 3(t-3)(t-4)\ (t \geq 7)$	$38\ (t=5)$ $288\ (t=6)$ $60\binom{t}{5} - 9(t-3)(t-4)\ (t \geq 7)$

In the fourth section of this paper, similarly, we give some exact values of the bipartite Gallai-Ramsey number and multiplicity when the number of colors $k = t^2$ is sufficiently large with respect to the subgraph H. The main results for the bipartite Gallai-Ramsey multiplicity are shown in the Table 2.

2 Preliminaries

Some propositions and lemmas presented in section are very helpful for the proof in the third and fourth sections of this paper. In 2008, Fox in [5] provided a result on the total number of different subgraphs G in K_n.

Proposition 1 [5] *If G is a subgraph of K_n, then* $\mathrm{Num}_{K_n}(G) = \frac{|V(G)|!\binom{n}{|V(G)|}}{|\mathrm{Aut}(G)|}$.

Li, Wang and Liu in [17] determined the sharp bound of k such that any k-edge-colored K_n always has a rainbow subgraph P_5. Bass, Magnant, Ozeki

Table 2 Main results for bipartite Gallai-Ramsey multiplicity. The graph H is a subgraph of some graphs related to k, see Theorems 19, 20, 22, 23, 24 and 25 for details.

	bi-GM$_{k-1}(G:H)$	bi-GM$_{k-2}(G:H)$
$G=K_{1,3}$	$5\ (t=3)$ $30\ (t=4)$ $(2t-1)\binom{t}{3}+\binom{t-2}{3}+2\binom{t-2}{2}\ (t\geq 5)$	$4\ (t=3)$ $28\ (t=4)$ $93\ (t=5)$ $(2t-1)\binom{t}{3}+\binom{t-3}{3}+3\binom{t-3}{2}\ (t\geq 6)$
$G=P_4$	$t^2(t-1)^2-2(t-1)\ (t\geq 2)$	$t^2(t-1)^2-6(t-1)\ (t\geq 2)$
$G=P_5$	$t^2(t-1)^2(t-2)-3(t-1)(t-2)\ (t\geq 3)$	$t^2(t-1)^2(t-2)-9(t-1)(t-2)\ (t\geq 3)$

and Pyron in [1] obtained the sharp bound of k such that any k-edge-colored K_n always has a rainbow subgraph $K_{1,3}$ by studying anti-Ramsey numbers.

Proposition 2 [17] *For integers $n\geq 5$ and k with $n+1\leq k\leq\binom{n}{2}$, there is always a rainbow subgraph P_5 under any k-edge-colored K_n.*

Proposition 3 [1] *For integers $n\geq 4$ and k with $\lceil\frac{n+3}{2}\rceil\leq k\leq\binom{n}{2}$, there is always a rainbow subgraph $K_{1,3}$ under any k-edge-colored K_n.*

Also, Li, Wang and Liu in [17] determined the sharp bound of k such that any k-edge-coloring of $K_{n,n}$ always has a rainbow subgraph P_4 or P_5.

Proposition 4 [17] *For integers $n\geq 2$ and k with $n+1\leq k\leq n^2$, there is always a rainbow subgraph P_4 under any k-edge-colored $K_{n,n}$.*

Proposition 5 [17] *For integers $n\geq 3$ and k with $n+2\leq k\leq n^2$, there is always a rainbow subgraph P_5 under any k-edge-colored $K_{n,n}$.*

Similarly, we can directly obtain the following result through Theorem 7.

Proposition 6 *For integers $n\geq 3$ and k with $n+2\leq k\leq n^2$, there is always a rainbow subgraph $K_{1,3}$ under any k-edge-colored $K_{n,n}$.*

The following lemmas are very useful in the proofs of the third section.

Lemma 1 *Let $t\geq 4$ be an integer and e_1,e_2 be two edges in K_t. Then*

$$\mathrm{Num}_{K_t}(P_4|e_1,e_2)=\begin{cases}4, & \text{if } e_1\nsim e_2;\\ 2(t-3), & \text{if } e_1\sim e_2.\end{cases}$$

Proof Assume that $e_1\nsim e_2$, and let $e_1=v_1v_2$, $e_2=v_3v_4$. Since P_4 is a connected graph, there is an edge that connects e_1 and e_2. In this case, there are four different P_4, which are $v_1v_2v_3v_4$, $v_1v_2v_4v_3$, $v_2v_1v_3v_4$ and $v_2v_1v_4v_3$.

Assume that $e_1\sim e_2$. From the structure of P_4, it can be seen that e_1 and e_2 cannot be both pendent edges of P_4, and one of e_1 and e_2 must be the pendent edge of P_4. If e_1 is the pendent edge of P_4, then there are $t-3$ different P_4. By symmetry, if e_2 is the pendent edge of P_4, then there are $t-3$ different P_4. Therefore, in this case, there are $2(t-3)$ different P_4.

Lemma 2 *Let $t \geq 5$ be an integer and e_1, e_2 be two edges in K_t. Then*

$$\mathrm{Num}_{K_t}(P_5|e_1, e_2) = \begin{cases} 12(t-4), & \textit{if } e_1 \nsim e_2; \\ 3(t-3)(t-4), & \textit{if } e_1 \sim e_2. \end{cases}$$

Proof Assume that $e_1 \nsim e_2$. From the structure of P_5, it can be seen that one of e_1 and e_2 must be the pendent edge of P_5. If e_1 is the pendent edge of P_5 but e_2 is not the pendent edge of P_5, then there are $4(t-4)$ different P_5. By symmetry, if e_2 is the pendent edge of P_5 but e_1 is not the pendent edge of P_5, then there are $4(t-4)$ different P_5. If the e_1 and e_2 are both pendent edges of P_5, then there are $4(t-4)$ different P_5. Therefore, in this case, there are $12(t-4)$ different P_5.

Assume that $e_1 \sim e_2$. From the structure of P_5, it can be seen that e_1 and e_2 cannot be both pendent edges of P_5. If e_1 is the pendent edge of P_5 but e_2 is not the pendent edge of P_5, then there are $(t-3)(t-4)$ different P_5. By symmetry, if e_2 is the pendent edge of P_5 but e_1 is not the pendent edge of P_5, then there are $(t-3)(t-4)$ different P_5. If neither e_1 nor e_2 are the pendent edges of P_5, then there are $(t-3)(t-4)$ different P_5. Therefore, in this case, there are $3(t-3)(t-4)$ different P_5.

Lemma 3 *Let $t \geq 5$ be an integer and e_1, e_2 be two edges in K_t. Then*

$$\mathrm{Num}_{K_t}(P_4^+|e_1, e_2) = \begin{cases} 8(t-4), & \textit{if } e_1 \nsim e_2; \\ 5(t-3)(t-4), & \textit{if } e_1 \sim e_2. \end{cases}$$

Proof Assume that $e_1 \nsim e_2$, and let $e_1 = v_1v_2$, $e_2 = v_3v_4$. Consider P_4^+ with edges e_1 and e_2. From the proof of Lemma 1, we know that there are four different P_4, which are $v_1v_2v_3v_4$, $v_1v_2v_4v_3$, $v_2v_1v_3v_4$ and $v_2v_1v_4v_3$. Since P_4^+ is a graph by adding an extra pendent edge to an inner vertex of P_4, it follows that there are $4 \cdot 2(t-4) = 8(t-4)$ different P_4^+.

Assume that $e_1 \sim e_2$. If e_1 is the tail edge of P_4^+, then there are $(t-3)(t-4)$ different P_4^+. By symmetry, if e_2 is the tail edge of P_4^+, then there are $(t-3)(t-4)$ different P_4^+. If neither e_1 nor e_2 are the tail edges of P_4^+, then there are $3(t-3)(t-4)$ different P_4^+. Therefore, in this case, there are $5(t-3)(t-4)$ different P_4^+.

Also, the following lemmas are very useful in the proofs of the fourth section.

Lemma 4 *Let $t \geq 3$ be an integer and e_1, e_2 be two edges in $K_{t,t}$. Then*

$$\mathrm{Num}_{K_{t,t}}(P_4|e_1, e_2) = \begin{cases} 2, & \textit{if } e_1 \nsim e_2; \\ 2(t-1), & \textit{if } e_1 \sim e_2. \end{cases}$$

Proof Assume that $e_1 \nsim e_2$. Let (X, Y) be the bipartition of $K_{t,t}$, $e_1 = v_1v_2$, $e_2 = v_3v_4$ and $v_1, v_3 \in X$, $v_2, v_4 \in Y$. Since P_4 is a connected graph, there is an edge that connects e_1 and e_2. In this case, there are two different P_4, which are $v_1v_2v_3v_4$ and $v_2v_1v_4v_3$.

Assume that $e_1 \sim e_2$. From the structure of P_4, it can be seen that e_1 and e_2 cannot be both pendent edges of P_4, and one of e_1 and e_2 must be the pendent edge of P_4. If e_1 is the pendent edge of P_4, then there are $t-1$ different P_4. By symmetry, if e_2 is the pendent edge of P_4, then there are $t-1$ different P_4. Therefore, in this case, there are $2(t-1)$ different P_4.

Lemma 5 *Let $t \geq 3$ be an integer and e_1, e_2 be two edges in $K_{t,t}$. Then*

$$\mathrm{Num}_{K_{t,t}}(P_5|e_1, e_2) = \begin{cases} 6(t-2), & \text{if } e_1 \nsim e_2; \\ 3(t-1)(t-2), & \text{if } e_1 \sim e_2. \end{cases}$$

Proof Assume that $e_1 \nsim e_2$. From the structure of P_5, it can be seen that one of e_1 and e_2 must be the pendent edge of P_5. If e_1 is the pendent edge of P_5 but e_2 is not the pendent edge of P_5, then there are $2(t-2)$ different P_5. By symmetry, if e_2 is the pendent edge of P_5 but e_1 is not the pendent edge of P_5, then there are $2(t-2)$ different P_5. If e_1 and e_2 are both pendent edges of P_5, then there are $2(t-2)$ different P_5. Therefore, in this case, there are $6(t-2)$ different P_5.

Assume that $e_1 \sim e_2$. From the structure of P_5, it can be seen that e_1 and e_2 cannot be both pendent edges of P_5. If e_1 is the pendent edge of P_5 but e_2 is not the pendent edge of P_5, then there are $(t-1)(t-2)$ different P_5. By symmetry, if e_2 is the pendent edge of P_5 but e_1 is not the pendent edge of P_5, then there are $(t-1)(t-2)$ different P_5. If neither e_1 nor e_2 are the pendent edges of P_5, then there are $(t-1)(t-2)$ different P_5. Therefore, in this case, there are $3(t-1)(t-2)$ different P_5.

Recall that the k-edge-colorings studied in this paper are exact, meaning that each color is used at least once. Based on this, a basic principle is that the number of colors does not exceed the total number of edges in an edge-colored graph. So by solving the equations

$$k \leq \binom{n}{2} = |E(K_n)| \text{ and } k \leq n^2 = |E(K_{n,n})|,$$

we obtain $n \geq \frac{1+\sqrt{1+8k}}{2}$ and $n \geq \sqrt{k}$, respectively. Therefore, we directly get the following basic lower bound lemma.

Lemma 6 *For integer $k \geq 4$, $G \in \{P_4, K_{1,3}\}$ and any graph H, we have*

$$\mathrm{gr}_k(G : H) \geq \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil.$$

For integer $k \geq 5$, $G \in \{P_4, P_5, K_{1,3}\}$ and any bipartite graph H, we have

$$\mathrm{bgr}_k(G : H) \geq \left\lceil \sqrt{k} \right\rceil,$$

in particular, this lower bound also holds when $3 \leq k \leq 4$ and $G = P_4$.

It is worth noting that the idea of Lemma 6 originated from the paper of Zou, Wang, Lai and Mao in [28].

3 Results for Gallai-Ramsey multiplicity

We consider four kinds of rainbow graphs $K_{1,3}$, P_4^+, P_4, P_5, respectively, in the following four subsections.

3.1 Rainbow $K_{1,3}$

Theorem 8 *Let integer $k \geq 4$. If H is a subgraph of the balanced complete $(k-1)$-partite graph $K_{(k-1)\times 2}$. Then*

$$\mathrm{gr}_k(K_{1,3} : H) = \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil.$$

Proof The lower bound follows from Lemma 6. Let $N_k = \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil$. For the upper bound, we consider an arbitrary k-edge-coloring of K_N $(N \geq N_k)$. Noticing that $N_k < 2k-2$ for $k \geq 4$. If $N_k \leq N \leq 2k-3$, then it follows from Proposition 3 that there is always a rainbow $K_{1,3}$, the result thus follows. Next we assume $N \geq 2k-2$. Suppose to the contrary that K_N contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph H. It follows from Theorem 3 that the Colored Structure 1 occurs. From exact k-edge-coloring, we have $|V_i| \geq 2$ for each $i \in \{2, 3, \ldots, k\}$. Since H is a subgraph of the balanced complete $(k-1)$-partite graph $K_{(k-1)\times 2}$, it follows that there is a monochromatic H, a contradiction. The result thus follows.

According to Theorem 8 and Proposition 1, the following corollary can be directly deduced.

Corollary 1 *For integers k and t satisfying $k = \binom{t}{2} \geq 6$, and a subgraph H of the balanced complete $(k-1)$-partite graph $K_{(k-1)\times 2}$ with $|E(H)| \geq 2$, we have*

$$\mathrm{GM}_k(K_{1,3} : H) = \frac{4!\binom{t}{4}}{|\,\mathrm{Aut}(K_{1,3})|} = t\binom{t-1}{3}.$$

Theorem 9 *For integers k and t satisfying $k = \binom{t}{2} \geq 6$, and a subgraph H of the balanced complete $(k-2)$-partite graph $K_{(k-2)\times 2}$ with $|E(H)| \geq 3$, we have*

$$\mathrm{GM}_{k-1}(K_{1,3} : H) = \begin{cases} 3, & t = 4; \\ 18, & t = 5; \\ (t-1)\binom{t-1}{3} + \binom{t-3}{3} + 2\binom{t-3}{2}, & t \geq 6. \end{cases}$$

Proof It follows from Theorem 8 that $\mathrm{gr}_{k-1}(K_{1,3} : H) = t$. Consider any $(k-1)$-edge-coloring of K_t. Since $|E(K_t)| = \binom{t}{2}$ and each color is used at least once, it follows that there are only two edges with the same color in K_t. Without loss of generality, we assume that there are two red edges e_1 and e_2. Since $|E(H)| \geq 3$, it follows that we do not need to consider the number of monochromatic H in K_t. If $e_1 \nsim e_2$, then this case is equivalent to Corollary 1. Therefore,

$$\mathrm{GM}_{k-1}(K_{1,3} : H) \leq t\binom{t-1}{3}.$$

If $e_1 \sim e_2$, then e_1 and e_2 form a red P_3. Let vertex v be incident to the edges e_1 and e_2. We first investigate the number of rainbow copies of $K_{1,3}$ with center v for $t \geq 6$. Noticing that $\deg(v) = t-1$, the number of rainbow copies of $K_{1,3}$ with center v and without red edges is $\binom{t-3}{3}$, and number of rainbow copies of $K_{1,3}$ with center v and with a red edge is $2\binom{t-3}{2}$. In K_t, there are $(t-1)\binom{t-1}{3}$ rainbow copies of $K_{1,3}$ with center in $V(K_t) \setminus \{v\}$. Therefore,

$$\mathrm{GM}_{k-1}(K_{1,3} : H) \leq (t-1)\binom{t-1}{3} + \binom{t-3}{3} + 2\binom{t-3}{2}.$$

It is easy to verify that when $t \geq 6$,

$$\min\left\{t\binom{t-1}{3}, (t-1)\binom{t-1}{3} + \binom{t-3}{3} + 2\binom{t-3}{2}\right\} = (t-1)\binom{t-1}{3}+\binom{t-3}{3}+2\binom{t-3}{2}.$$

When $t = 4$, there is no rainbow $K_{1,3}$ with center v. In K_4, each other vertex has one rainbow $K_{1,3}$. So there are three rainbow copies of $K_{1,3}$ in K_4. Since $3 < 4\binom{4-1}{3}$, it follows that $\mathrm{GM}_5(K_{1,3} : H) = 3$.

When $t = 5$, the number of rainbow copies of $K_{1,3}$ with center v and without red edges is 0, and number of rainbow copies of $K_{1,3}$ with center v and with a red edge is 2. In K_5, there are $4\binom{4}{3} = 16$ rainbow copies of $K_{1,3}$ with center in $V(K_5) \setminus \{v\}$. Therefore, $\mathrm{GM}_9(K_{1,3} : H) \leq 18$. Since $18 < 5\binom{5-1}{3}$. It follows that $\mathrm{GM}_9(K_{1,3} : H) = 18$.

Theorem 10 *For integers k and t satisfying $k = \binom{t}{2} \geq 6$, and a subgraph H of the balanced complete $(k-3)$-partite graph $K_{(k-3)\times 2}$ with $|E(H)| \geq 4$, we have*

$$\mathrm{GM}_{k-2}(K_{1,3} : H) = \begin{cases} 1, & t = 4; \\ 14, & t = 5; \\ (t-3)\binom{t-1}{3} + 3\binom{t-3}{3} + 6\binom{t-3}{2}, & t \geq 6. \end{cases}$$

Proof It follows from Theorem 8 that $\mathrm{gr}_{k-2}(K_{1,3} : H) = t$. Consider a $(k-2)$-edge-coloring of K_t. Since $|E(H)| \geq 4$, it follows that we do not need to consider the number of monochromatic H in K_t. Noticing that each color needs to be used at least once. We first color any $k-2$ edges in K_t with $k-2$ colors, and the remaining two edges are temporarily not colored, denoted as e_1 and e_2. Next, we discuss the edges e_1 and e_2 in two cases.

Case 1 The edges e_1 and e_2 have the same color.

Without loss of generality, we assume that these two edges are red. According to the structure of K_t, it is easy to calculate that if the red edges form a $3P_2$, then there are

$$f_1(t) = t\binom{t-1}{3}$$

rainbow copies of $K_{1,3}$ in K_t; if the red edges form a $P_3 \cup P_2$, then there are

$$f_2(t) = (t-1)\binom{t-1}{3} + \binom{t-3}{3} + 2\binom{t-3}{2}$$

rainbow copies of $K_{1,3}$ in K_t; if the red edges form a P_4, then there are

$$f_3(t) = (t-2)\binom{t-1}{3} + 2\binom{t-3}{3} + 4\binom{t-3}{2}$$

rainbow copies of $K_{1,3}$ in K_t; if the red edges form a K_3, then there are

$$f_4(t) = (t-3)\binom{t-1}{3} + 3\binom{t-3}{3} + 6\binom{t-3}{2}$$

rainbow copies of $K_{1,3}$ in K_t; if the red edges form a $K_{1,3}$, then there are

$$f_5(t) = (t-1)\binom{t-1}{3} + \binom{t-4}{3} + 3\binom{t-4}{2}$$

rainbow copies of $K_{1,3}$ in K_t.

Case 2 The edges e_1 and e_2 have different colors.

When the edges e_1 and e_2 form a P_3 in K_t, without loss of generality, we assume that e_1 is red and e_2 is blue. Let $V(P_3) = \{u, v, w\}$ and vertex v is incident to edges e_1 and e_2. According to the structure of K_t, it is easy to calculate that if the other red edge is not incident to vertex u or v, and the other blue edge is not incident to vertex v or w, then there are

$$f_1(t) = t\binom{t-1}{3}$$

rainbow copies of $K_{1,3}$ in K_t; if the other red edge is incident to vertex u or v, and the other blue edge is not incident to vertex v or w, then there are

$$f_2(t) = (t-1)\binom{t-1}{3} + \binom{t-3}{3} + 2\binom{t-3}{2}$$

rainbow copies of $K_{1,3}$ in K_t; if the other red edge is incident to vertex u, and the other blue edge is incident to vertex w, then there are

$$f_3(t) = (t-2)\binom{t-1}{3} + 2\binom{t-3}{3} + 4\binom{t-3}{2}$$

rainbow copies of $K_{1,3}$ in K_t; if the other red edge is incident to vertex v, and the other blue edge is also incident to vertex v, then there are

$$f_6(t) = (t-1)\binom{t-1}{3} + \binom{t-5}{3} + 4\binom{t-5}{2} + 4(t-5)$$

rainbow copies of $K_{1,3}$ in K_t.

When the edges e_1 and e_2 form a $2P_2$ in K_t, without loss of generality, we assume that e_1 is red and e_2 is blue. According to the structure of K_t, it is easy to calculate that if the other red edge is not adjacent to e_1, and the other blue edge is not adjacent to e_2, then there are

$$f_1(t) = t\binom{t-1}{3}$$

rainbow copies of $K_{1,3}$ in K_t; if the other red edge is adjacent to e_1, and the other blue edge is not adjacent to e_2, then there are

$$f_2(t) = (t-1)\binom{t-1}{3} + \binom{t-3}{3} + 2\binom{t-3}{2}$$

rainbow copies of $K_{1,3}$ in K_t; if the other red edge is adjacent to e_1, and the other blue edge is adjacent to e_2, then there are

$$f_3(t) = (t-2)\binom{t-1}{3} + 2\binom{t-3}{3} + 4\binom{t-3}{2}$$

rainbow copies of $K_{1,3}$ in K_t.

Next, we compare the sizes of $f_1(t), f_2(t), f_3(t), f_4(t), f_5(t)$ and $f_6(t)$. Based on the practical significance of counting in this paper, we know that the count of rainbow copies of $K_{1,3}$ cannot be negative. For example, for $f_6(t)$, the expression is not applicable when $4 \leq t \leq 6$. In other words, for $4 \leq t \leq 6$, $f_6(t)$ can be written as a piecewise expression. But for the convenience of calculation, we only define in the operations of expressions for t in $f_1(t), f_2(t), f_3(t), f_4(t), f_5(t)$ and $f_6(t)$ that when integers $a < b$, we have $\binom{a}{b} \equiv 0$ and $a - b \equiv 0$.

For $t = 4$, we have

$$f_1(4) = 4, f_2(4) = 3, f_3(4) = 2, f_4(4) = 1, f_5(4) = 3, f_6(4) = 3.$$

Thus, $\min\{f_1(4), f_2(4), f_3(4), f_4(4), f_5(4), f_6(4)\} = 1$.

For $t = 5$, we have

$$f_1(5) = 20, f_2(5) = 18, f_3(5) = 16, f_4(5) = 14, f_5(5) = 16, f_6(5) = 16.$$

Thus, $\min\{f_1(5), f_2(5), f_3(5), f_4(5), f_5(5), f_6(5)\} = 14$.

For $t = 6$, we have

$$f_1(6) = 60, f_2(6) = 57, f_3(6) = 54, f_4(6) = 51, f_5(6) = 53, f_6(6) = 54.$$

Thus, $\min\{f_1(6), f_2(6), f_3(6), f_4(6), f_5(6), f_6(6)\} = 51$.

For $t = 7$, we have

$$f_1(7) = 140, f_2(7) = 136, f_3(7) = 132, f_4(7) = 128, f_5(7) = 130, f_6(7) = 132.$$

Thus, $\min\{f_1(7), f_2(7), f_3(7), f_4(7), f_5(7), f_6(7)\} = 128$.

For $t \geq 8$ and $1 \leq i \leq 6$, let $f_{ii}(t) = f_i(t) - (t-3)\binom{t-1}{3} - \frac{1}{2}t^3 + 3t^2 - \frac{5}{2}t$, then

$$f_{11}(t) = 3t - 3, f_{22}(t) = 2t, f_{33}(t) = t + 3, f_{44}(t) = 6, f_{55}(t) = 8, f_{66}(t) = t + 3.$$

Therefore, for $t \geq 8$,

$$\min\{f_{11}(t), f_{22}(t), f_{33}(t), f_{44}(t), f_{55}(t), f_{66}(t)\} = f_{44}(t) = 6,$$

and thus,

$$\min\{f_1(t), f_2(t), f_3(t), f_4(t), f_5(t), f_6(t)\} = f_4(t) = (t-3)\binom{t-1}{3} + 3\binom{t-3}{3} + 6\binom{t-3}{2}.$$

Based on the above discussion, we have

$$\min\{f_1(t), f_2(t), f_3(t), f_4(t), f_5(t), f_6(t)\} = \begin{cases} (t-3)\binom{t-1}{3}, & t = 4; \\ (t-3)\binom{t-1}{3} + 6\binom{t-3}{2}, & t = 5; \\ (t-3)\binom{t-1}{3} + 3\binom{t-3}{3} + 6\binom{t-3}{2}, & t \geq 6. \end{cases}$$

The result thus follows.

3.2 Rainbow P_4^+

According to Theorems 3 and 4, we directly give the following observation.

Observation 1 *For integers $k \geq 5$, if $\mathrm{gr}_k(K_{1,3} : H) \geq 5$, then*

$$\mathrm{gr}_k(P_4^+ : H) = \mathrm{gr}_k(K_{1,3} : H).$$

From Observation 1 and Theorem 8, the following corollary can be directly deduced.

Corollary 2 *Let integer $k \geq 5$. If H is a subgraph of the balanced complete $(k-1)$-partite graph $K_{(k-1)\times 2}$, then*

$$\mathrm{gr}_k(P_4^+ : H) = \begin{cases} 5, & 5 \leq k \leq 6; \\ \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil, & k \geq 7. \end{cases}$$

Noticing that P_4^+ is obtained by adding a pendent edge to a leaf vertex at $K_{1,3}$, there are $t\binom{t-1}{3} \cdot 3(t-4)$ different P_4^+ in K_t. We can also calculate $|\,\mathrm{Aut}(P_4^+)| = 2$ from Proposition 1. Therefore, we directly provide the following corollary.

Corollary 3 *For integers k and t satisfying $k = \binom{t}{2} \geq 10$, and a subgraph H of the balanced complete $(k-1)$-partite graph $K_{(k-1)\times 2}$ with $|E(H)| \geq 2$, we have*

$$\mathrm{GM}_k(P_4^+ : H) = \frac{5!\binom{t}{5}}{|\,\mathrm{Aut}(P_4^+)|} = 60\binom{t}{5}.$$

Theorem 11 *For integers k and t satisfying $k = \binom{t}{2} \geq 10$, and a subgraph H of the balanced complete $(k-2)$-partite graph $K_{(k-2)\times 2}$ with $|E(H)| \geq 3$, we have*

$$\mathrm{GM}_{k-1}(P_4^+ : H) = 60\binom{t}{5} - 5(t-3)(t-4).$$

Proof It follows from Corollary 2 that $\mathrm{gr}_{k-1}(P_4^+ : H) = t$. Consider any $(k-1)$-edge-coloring of K_t. Since $|E(K_t)| = \binom{t}{2}$ and each color is used at least once, it follows that there are only two edges, say e_1 and e_2, with the same color in K_t. Since $|E(H)| \geq 3$, it follows that we do not need to consider the number of monochromatic H in K_t. According to Corollary 3, there are $60\binom{t}{5}$ different P_4^+ in K_t, and we only need to find the number of different P_4^+ containing the edges e_1 and e_2. This is because only P_4^+ containing edges e_1 and e_2 are not rainbow, and all other P_4^+ are rainbow. If $e_1 \nsim e_2$, then according to Lemma 3 that there are $8(t-4)$ different P_4^+ in K_t that contain edges e_1 and e_2. If $e_1 \sim e_2$, then according to Lemma 3 that there are $5(t-3)(t-4)$ different P_4^+ in K_t that contain edges e_1 and e_2. Noticing that $8(t-4) < 5(t-3)(t-4)$ for $t \geq 5$, the result thus follows.

Theorem 12 *For integers k and t satisfy $k = \binom{t}{2} \geq 10$, if H is a subgraph of the balanced complete $(k-3)$-partite graph $K_{(k-3)\times 2}$ with $|E(H)| \geq 4$, then we have*

$$\mathrm{GM}_{k-2}(P_4^+ : H) = 60\binom{t}{5} - 15(t-3)(t-4).$$

Proof It follows from Corollary 2 that $\mathrm{gr}_{k-2}(P_4^+ : H) = t$. Consider $(k-2)$-edge-coloring of K_t. Since $|E(H)| \geq 4$, it follows that we do not need to consider the number of monochromatic H in K_t. Since each color is used at least once, there are only the following two cases. Due to the arbitrariness of colors, we can describe them using specific color names such as red and blue. Next, we calculate the number of different P_4^+ containing two or more edges with the same color. The following counting bases are all based on Lemma 3.

Case 1 There are three red edges e_1, e_2 and e_3. The remaining edges are not red and the colors of any two remaining edges are not the same.

Assume that the edges e_1, e_2 and e_3 form a red $3P_2$. In this subcase, there are $24(t-4)$ different P_4^+ containing two red edges.

Assume that the edges e_1, e_2 and e_3 form a red $P_3 \cup P_2$. In this subcase, there are $5(t-3)(t-4)$ different P_4^+ containing red P_3 and $2(8(t-4)-2)$ different P_4^+ without red P_3. So there are a total of $(5t+1)(t-4)-4$ different P_4^+ containing two or more red edges.

Assume that the edges e_1, e_2 and e_3 form a red $K_{1,3}$. In this subcase, there are $3(2(t-3)(t-4)+3(t-4)^2)$ different P_4^+ containing two red edges and $3(t-4)$ different P_4^+ containing three red edges. So there are a total of $3(5t-17)(t-4)$ different P_4^+ containing two or more red edges.

Assume that the edges e_1, e_2 and e_3 form a red K_3. In this subcase, there are $15(t-3)(t-4)$ different P_4^+ containing two red edges.

Assume that the edges e_1, e_2 and e_3 form a red P_4. In this subcase, there are $2(5(t-3)(t-4)-2(t-4))+3\cdot 2(t-4) = 2(5t-14)(t-4)$ different P_4^+ containing two red edges and $2(t-4)$ different P_4^+ containing three red edges. So there are a total of $2(5t-13)(t-4)$ different P_4^+ containing two or more red edges.

Case 2 There are two red edges e_1, e_2 and two blue edges e_3, e_4. The remaining edges are not red or blue and the colors of any two remaining edges are not the same.

If $e_1 \nsim e_2$ and $e_3 \nsim e_4$, then there are $16(t-4)$ different P_4^+ containing two edges with the same color; if $e_1 \sim e_2$ and $e_3 \nsim e_4$, then there are at most $8(t-4)+5(t-3)(t-4) = (5t-7)(t-4)$ different P_4^+ containing two edges with the same color; if $e_1 \sim e_2$ and $e_3 \sim e_4$, then there are at most $10(t-3)(t-4)$ different P_4^+ containing two edges with the same color.

Let

$$f_1(t) = 24(t-4), f_2(t) = (5t+1)(t-4) - 4,$$

$$f_3(t) = 15(t-3)(t-4), f_4(t) = 2(5t-13)(t-4), f_5(t) = (5t-7)(t-4).$$

Based on the data calculated from the eight subcases above, we need to compare the sizes of $f_1(t), f_2(t), f_3(t), f_4(t)$ and $f_5(t)$.

For $1 \leq i \leq 5$, let $f_{ii}(t) = \frac{f_i(t)}{t-4}$. Then

$$f_{11}(t) = 24, f_{22}(t) = 5t+1-\frac{4}{t-4}, f_{33}(t) = 15(t-3), f_{44}(t) = 2(5t-13), f_{55}(t) = 5t-7.$$

Therefore, for $t \geq 5$

$$\max\{f_{11}(t), f_{22}(t), f_{33}(t), f_{44}(t), f_{55}(t)\} = f_{33}(t) = 15(t-3),$$

and thus,

$$\max\{f_1(t), f_2(t), f_3(t), f_4(t), f_5(t)\} = f_3(t) = 15(t-3)(t-4).$$

The result thus follows.

3.3 Rainbow P_4

From Theorem 1, we directly obtain the following corollary.

Corollary 4 *For a graph H and integer $k \geq 4$, we have*

$$\mathrm{gr}_k(P_4 : H) = \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil.$$

According to Corollary 4 and Proposition 1, the following corollary can be directly deduced.

Corollary 5 *For a graph H with $|E(H)| \geq 2$ and integers k and t satisfying $k = \binom{t}{2} \geq 6$, we have*

$$\mathrm{GM}_k(P_4 : H) = \frac{4!\binom{t}{4}}{|\,\mathrm{Aut}(P_4)|} = 12\binom{t}{4}.$$

Theorem 13 *For a graph H with $|E(H)| \geq 3$ and integers k and t satisfying $k = \binom{t}{2} \geq 6$, we have*

$$\mathrm{GM}_{k-1}(P_4 : H) = \begin{cases} 8, & t = 4; \\ 12\binom{t}{4} - 2(t-3), & t \geq 5. \end{cases}$$

Proof It follows from Corollary 4 that $\mathrm{gr}_{k-1}(P_4 : H) = t$. Consider any $(k-1)$-edge-coloring of K_t. Since $|E(K_t)| = \binom{t}{2}$ and each color is used at least once, it follows that there are only two edges, say e_1 and e_2, with the same color in K_t. Since $|E(H)| \geq 3$, it follows that we do not need to consider the number of monochromatic H in K_t. According to Corollary 5, there are $12\binom{t}{4}$ different P_4 in K_t, and we only need to find the number of different P_4 containing the edges e_1 and e_2. This is because only P_4 containing edges e_1 and e_2 are not rainbow, and all other P_4 are rainbow. If $e_1 \nsim e_2$, then according to Lemma 1 that there are 4 different P_4 in K_t that contain edges e_1 and e_2. If $e_1 \sim e_2$, then according to Lemma 1 that there are $2(t-3)$ different P_4 in K_t that contain edges e_1 and e_2. Noticing that $4 > 2(t-3)$ for $t = 4$ and $4 \leq 2(t-3)$ for $t \geq 5$, the result thus follows.

Theorem 14 *For a graph H with $|E(H)| \geq 4$ and integers k and t satisfying $k = \binom{t}{2} \geq 6$, we have*

$$\mathrm{GM}_{k-2}(P_4 : H) = \begin{cases} 4, & t = 4; \\ 12\binom{t}{4} - 6(t-3), & t \geq 5. \end{cases}$$

Proof It follows from Corollary 4 that $\mathrm{gr}_{k-2}(P_4 : H) = t$. Consider $(k-2)$-edge-coloring of K_t. Since $|E(H)| \geq 4$, it follows that we do not need to consider the number of monochromatic H in K_t. Since each color is used at least once, there are only the following two cases. Due to the arbitrariness of colors, we can describe them using specific color names such as red and blue. Next, we calculate the number of different P_4 containing two or more edges with the same color. The following counting bases are all based on Lemma 1.

Case 1 There are three red edges e_1, e_2 and e_3. The remaining edges are not red and the colors of any two remaining edges are not the same.

Assume that the edges e_1, e_2 and e_3 form a red $3P_2$. In this subcase, there are 12 different P_4 containing two red edges.

Assume that the edges e_1, e_2 and e_3 form a red $P_3 \cup P_2$. In this subcase, there are $8 + 2(t-3) = 2(t+1)$ different P_4 containing two red edges.

Assume that the edges e_1, e_2 and e_3 form a red $K_{1,3}$. In this subcase, there are $6(t-3)$ different P_4 containing two red edges.

Assume that the edges e_1, e_2 and e_3 form a red K_3. In this subcase, there are $6(t-3)$ different P_4 containing two red edges.

Assume that the edges e_1, e_2 and e_3 form a red P_4. In this subcase, there are $4 + 2(t-3) + 2(t-4) = 2(2t-5)$ different P_4 containing two or more red edges.

Case 2 There are two red edges e_1, e_2 and two blue edges e_3, e_4. The remaining edges are not red or blue and the colors of any two remaining edges are not the same.

If $e_1 \nsim e_2$ and $e_3 \nsim e_4$, then there are 8 different P_4 containing two edges with the same color; if $e_1 \sim e_2$ and $e_3 \nsim e_4$, then there are $4+2(t-3) = 2(t-1)$ different P_4 containing two edges with the same color; if $e_1 \sim e_2$ and $e_3 \sim e_4$, then there are at most $4(t-3)$ different P_4 containing two edges with the same color.

We first consider the result when $t = 4$. Noticing that there are no red $3P_2$ or $P_3 \cup P_2$ in a 4-edge-colored K_4. Thus there are at most 8 different P_4 in a 4-edge-colored K_4 that contain two or more edges of the same color.

Let $f_1(t) = 12, f_2(t) = 2(t+1), f_3(t) = 6(t-3), f_4(t) = 2(2t-5)$. Based on the data calculated from the eight subcases above, we need to compare the sizes of $f_1(t), f_2(t), f_3(t)$ and $f_4(t)$.

For $t \geq 5$, we have

$$\max\{f_1(t), f_2(t), f_3(t), f_4(t)\} = f_3(t) = 6(t-3).$$

The result thus follows.

3.4 Rainbow P_5

In 2023, Zou, Wang, Lai and Mao in [28] provided results on the Gallai-Ramsey number for rainbow P_5.

Theorem 15 [28] *For a graph H and an integer $k \geq 5$, we have*

$$\mathrm{gr}_k(P_5 : H) = \begin{cases} \max\left\{\left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil, 5\right\}, & k \geq |V(H)|+1; \\ |V(H)|+1, & k = |V(H)| \text{ and } H \text{ is not a complete graph;} \\ (|V(H)|-1)^2+1, & k = |V(H)| \text{ and } H \text{ is a complete graph.} \end{cases}$$

According to Theorem 15 and Proposition 1, the following corollary can be directly deduced.

Corollary 6 *For a graph H with $|E(H)| \geq 2$ and integers k and t satisfying $k = \binom{t}{2} \geq \max\{|V(H)|+1, 10\}$, we have*

$$\mathrm{GM}_k(P_5 : H) = \frac{5!\binom{t}{5}}{|\operatorname{Aut}(P_5)|} = 60\binom{t}{5}.$$

Theorem 16 *For a graph H with $|E(H)| \geq 3$ and integers k and t satisfying $k = \binom{t}{2} \geq \max\{|V(H)|+2, 10\}$, we have*

$$\mathrm{GM}_{k-1}(P_5 : H) = \begin{cases} 60\binom{t}{5} - 12(t-4), & 5 \leq t \leq 6; \\ 60\binom{t}{5} - 3(t-3)(t-4), & t \geq 7. \end{cases}$$

Proof It follows from Theorem 15 that $\mathrm{gr}_{k-1}(P_5 : H) = t$. Consider any $(k-1)$-edge-coloring of K_t. Since $|E(K_t)| = \binom{t}{2}$ and each color is used at least once, it follows that there are only two edges, say e_1 and e_2, with the same color in K_t. Since $|E(H)| \geq 3$, it follows that we do not need to consider the number of monochromatic H in K_t. According to Corollary 6, there are $60\binom{t}{5}$ different P_5 in K_t, and we only need to find the number of different P_5 containing the edges e_1 and e_2. This is because only P_5 containing edges e_1 and e_2 are not rainbow, and all other P_5 are rainbow. If $e_1 \nsim e_2$, then according to Lemma 2 that there are $12(t-4)$ different P_5 in K_t that contain edges e_1 and e_2. If $e_1 \sim e_2$, then according to Lemma 2 that there are $3(t-3)(t-4)$ different P_5 in K_t that contain edges e_1 and e_2. Noticing that $12(t-4) > 3(t-3)(t-4)$ for $5 \leq t \leq 6$ and $12(t-4) \leq 3(t-3)(t-4)$ for $t \geq 7$, the result thus follows.

Theorem 17 *For a graph H with $|E(H)| \geq 4$ and integers k and t satisfying $k = \binom{t}{2} \geq \max\{|V(H)|+3, 10\}$, we have*

$$\mathrm{GM}_{k-2}(P_5 : H) = \begin{cases} 38, & t = 5; \\ 288, & t = 6; \\ 60\binom{t}{5} - 9(t-3)(t-4), & t \geq 7. \end{cases}$$

Proof It follows from Theorem 15 that $\mathrm{gr}_{k-2}(P_5 : H) = t$. Consider $(k-2)$-edge-coloring of K_t. Since $|E(H)| \geq 4$, it follows that we do not need to consider the number of monochromatic H in K_t. Since each color is used at least once, there are only the following two cases. Due to the arbitrariness of colors, we can describe them using specific color names such as red and blue. Next, we calculate the number of different P_5 containing two or more edges with the same color. The following counting bases are all based on Lemma 2.

Case 1 There are three red edges e_1, e_2 and e_3. The remaining edges are not red and the colors of any two remaining edges are not the same.

Assume that the edges e_1, e_2 and e_3 form a red $3P_2$. In this subcase, there are $36(t-4)$ different P_5 containing two red edges.

Assume that the edges e_1, e_2 and e_3 form a red $P_3 \cup P_2$. Let $P_3 = e_1e_2$ and $P_2 = e_3$. In this subcase, there are $3(t-3)(t-4)$ different P_5 containing red edges e_1 and e_2, $12(t-4)-4$ different P_5 only containing red edges e_1 and e_3, and symmetrically $12(t-4)-4$ different P_5 only containing red edges e_2 and e_3. So there are a total of $3(t-3)(t-4)+2(12(t-4)-4) = 3(t+5)(t-4)-8$ different P_5 containing two or more red edges.

Assume that the edges e_1, e_2 and e_3 form a red $K_{1,3}$. In this subcase, there are $9(t-3)(t-4)$ different P_5 containing two red edges.

Assume that the edges e_1, e_2 and e_3 form a red K_3. In this subcase, there are $9(t-3)(t-4)$ different P_5 containing two red edges.

Assume that the edges e_1, e_2 and e_3 form a red P_4. In this subcase, there are $12(t-4)$ different P_5 containing red $2P_2$ and $2(3(t-3)(t-4)-2(t-4)) = 2(3t-11)(t-4)$ different P_5 without red $2P_2$. So there are a total of $12(t-4)+2(3t-11)(t-4) = 2(3t-5)(t-4)$ different P_5 containing two or more red edges.

Case 2 There are two red edges e_1, e_2 and two blue edges e_3, e_4. The remaining edges are not red or blue and the colors of any two remaining edges are not the same.

If $e_1 \nsim e_2$ and $e_3 \nsim e_4$, then there are at most $24(t-4)$ different P_5 containing two edges with the same color; if $e_1 \sim e_2$ and $e_3 \nsim e_4$, then there are at most $12(t-4)+3(t-3)(t-4) = 3(t+1)(t-4)$ different P_5 containing two edges with the same color; if $e_1 \sim e_2$ and $e_3 \sim e_4$, then there are at most $6(t-3)(t-4)$ different P_5 containing two edges with the same color.

Let

$$f_1(t) = 36(t-4), f_2(t) = 3(t+5)(t-4) - 8,$$

$$f_3(t) = 9(t-3)(t-4), f_4(t) = 2(3t-5)(t-4), f_5(t) = 3(t+1)(t-4).$$

Based on the data calculated from the eight subcases above, we need to compare the sizes of $f_1(t), f_2(t), f_3(t), f_4(t)$ and $f_5(t)$.

For $1 \leq i \leq 5$, let $f_{ii}(t) = \frac{f_i(t)}{t-4}$. Then

$$f_{11}(t) = 36, f_{22}(t) = 3(t+5) - \frac{8}{t-4}, f_{33}(t) = 9(t-3), f_{44}(t) = 2(3t-5), f_{55} = 3(t+1).$$

For $t = 5$, note that there are no red $3P_2$ in a 8-edge-colored K_5. Thus

$$\max\{f_{22}(t), f_{33}(t), f_{44}(t), f_{55}(t)\} = f_{22}(t) = 3(t+5) - \frac{8}{t-4} = 22,$$

and thus

$$\max\{f_2(t), f_3(t), f_4(t), f_5(t)\} = f_3(t) = 3(t+5)(t-4) - 8 = 22.$$

For $t = 6$, we have

$$\max\{f_{11}(t), f_{22}(t), f_{33}(t), f_{44}(t), f_{55}(t)\} = f_{11}(t) = 36,$$

and thus

$$\max\{f_1(t), f_2(t), f_3(t), f_4(t), f_5(t)\} = f_1(t) = 36(t-4) = 72.$$

For $t \geq 7$, we have

$$\max\{f_{11}(t), f_{22}(t), f_{33}(t), f_{44}(t), f_{55}(t)\} = f_{33}(t) = 9(t-3),$$

and thus

$$\max\{f_1(t), f_2(t), f_3(t), f_4(t), f_5(t)\} = f_3(t) = 9(t-3)(t-4).$$

The result thus follows.

4 Results for bipartite Gallai-Ramsey multiplicity

We consider three kinds of rainbow graphs P_4, P_5 and $K_{1,3}$, respectively, in the following two subsections.

4.1 Rainbow P_4

Theorem 18 *Let integer $k \geq 3$. If H is a subgraph of $K_{1,k}$, then*

$$\mathrm{bgr}_k(P_4 : H) = \left\lceil \sqrt{k} \right\rceil.$$

Proof The lower bound follows from Lemma 6. For the upper bound, we consider an arbitrary k-edge-coloring of $K_{N,N}$ ($N \geq \left\lceil \sqrt{k} \right\rceil$). Let (U, V) be the bipartition of $K_{N,N}$ and suppose to the contrary that $K_{N,N}$ contains neither a rainbow subgraph P_4 nor a monochromatic subgraph H. Noticing that $\left\lceil \sqrt{k} \right\rceil \leq k-1$ for $k \geq 3$. If $\left\lceil \sqrt{k} \right\rceil \leq N \leq k-1$, then it follows from Proposition 4 that there is always a rainbow P_4, and the result thus follows. Next we assume $N \geq k$. It follows from Theorem 5 that the Colored Structure 3 occurs. Thus U can be partitioned into k non-empty parts $U_1, U_2, \ldots, U_k$ such that all the edges between U_i and V have color i, $i \in \{1, 2, \ldots, k\}$. Since H is a subgraph of $K_{1,k}$ and $|V| = N \geq k$, it follows that there is a monochromatic H, a contradiction. The result thus follows.

It is easy to calculate that when $t \geq 2$, there are $t^2(t-1)^2$ different P_4 in $K_{t,t}$. Therefore, the following corollary can be directly derived from Theorem 18.

Corollary 7 *For integers k and t satisfying $k = t^2 \geq 4$, and a subgraph H of $K_{1,k}$ with $|E(H)| \geq 2$, we have*

$$\text{bi-GM}_k(P_4 : H) = t^2(t-1)^2.$$

Theorem 19 *For integers k and t satisfying $k = t^2 \geq 4$, and a subgraph H of $K_{1,k-1}$ with $|E(H)| \geq 3$, we have*

$$\text{bi-GM}_{k-1}(P_4 : H) = t^2(t-1)^2 - 2(t-1).$$

Proof It follows from Theorem 18 that $\mathrm{bgr}_{k-1}(P_4 : H) = t$. Consider any $(k-1)$-edge-coloring of $K_{t,t}$. Since $|E(K_{t,t})| = t^2$ and each color is used at least once, it follows that there are only two edges, say e_1 and e_2, with the same color in $K_{t,t}$. Since $|E(H)| \geq 3$, it follows that we do not need to consider the number of monochromatic H in $K_{t,t}$. According to Corollary 7, there are $t^2(t-1)^2$ different P_4 in $K_{t,t}$, and we only need to find the number of different P_4 containing the edges e_1 and e_2. This is because only P_4 containing edges e_1 and e_2 are not rainbow, and all other P_4 are rainbow. If $e_1 \nsim e_2$, then according to Lemma 4 that there are 2 different P_4 in $K_{t,t}$ that contain edges e_1 and e_2. If $e_1 \sim e_2$, then according to Lemma 4 that there are $2(t-1)$ different P_4 in $K_{t,t}$ that contain edges e_1 and e_2. Noticing that $2 \leq 2(t-1)$ for $t \geq 2$, the result thus follows.

Theorem 20 *For integers k and t satisfying $k = t^2 \geq 9$, and a subgraph H of $K_{1,k-2}$ with $|E(H)| \geq 4$, we have*

$$\text{bi-GM}_{k-2}(P_4 : H) = t^2(t-1)^2 - 6(t-1).$$

Proof It follows from Theorem 18 that $\mathrm{bgr}_{k-2}(P_4 : H) = t$. Consider $(k-2)$-edge-coloring of $K_{t,t}$. Since $|E(H)| \geq 4$, it follows that we do not need to consider the number of monochromatic H in $K_{t,t}$. Since each color is used at least once, there are only the following two cases. Due to the arbitrariness of colors, we can describe them using specific color names such as red and blue. Next, we calculate the number of different P_4 containing two or more edges with the same color. The following counting bases are all based on Lemma 4.

Case 1 There are three red edges e_1, e_2 and e_3. The remaining edges are not red and the colors of any two remaining edges are not the same.

Assume that the edges e_1, e_2 and e_3 form a red $3P_2$. In this subcase, there are 6 different P_4 containing two red edges.

Assume that the edges e_1, e_2 and e_3 form a red $P_3 \cup P_2$. In this subcase, there are $4 + 2(t-1) = 2(t+1)$ different P_4 containing two red edges.

Assume that the edges e_1, e_2 and e_3 form a red $K_{1,3}$. In this subcase, there are $6(t-1)$ different P_4 containing two red edges.

Assume that the edges e_1, e_2 and e_3 form a red P_4. In this subcase, there are $2 + 2(t-1) + 2(t-2) = 4(t-1)$ different P_4 containing two or more red edges.

Case 2 There are two red edges e_1, e_2 and two blue edges e_3, e_4. The remaining edges are not red or blue and the colors of any two remaining edges are not the same.

If $e_1 \nsim e_2$ and $e_3 \nsim e_4$, then there are 4 different P_4 containing two edges with the same color; if $e_1 \sim e_2$ and $e_3 \nsim e_4$, then there are $2 + 2(t-1) = 2t$ different P_4 containing two edges with the same color; if $e_1 \sim e_2$ and $e_3 \sim e_4$, then there are $4(t-1)$ different P_4 containing two edges with the same color.

Let $f_1(t) = 6, f_2(t) = 2(t+1), f_3(t) = 6(t-1)$. Based on the data calculated from the seven subcases above, we need to compare the sizes of $f_1(t), f_2(t)$ and $f_3(t)$.

For $t \geq 3$, we have

$$\max\{f_1(t), f_2(t), f_3(t)\} = f_3(t) = 6(t-1).$$

The result thus follows.

4.2 Rainbow P_5 and $K_{1,3}$

Theorem 21 *Let integer $k \geq 5$. If H is a subgraph of $K_{1,\lceil \frac{k-1}{2} \rceil}$, then*

$$\mathrm{bgr}_k(P_5 : H) = \mathrm{bgr}_k(K_{1,3} : H) = \left\lceil \sqrt{k} \right\rceil.$$

Proof The lower bound follows from Lemma 6. For the upper bound, we consider an arbitrary k-edge-coloring of $K_{N,N}$ $(N \geq \left\lceil \sqrt{k} \right\rceil)$. Let (U, V) be the bipartition of $K_{N,N}$ and suppose to the contrary that $K_{N,N}$ contains neither a rainbow subgraph P_5 nor a monochromatic subgraph H. Noticing that $\left\lceil\sqrt{k}\right\rceil \leq k-2$ for $k \geq 5$. If $\left\lceil\sqrt{k}\right\rceil \leq N \leq k-2$, then it follows from Proposition 5 that there is always a rainbow P_5, and the result thus follows. Next we assume $N \geq k-1$. It follows from Theorems 6 and 7 that either the Colored Structure 4 or Colored Structure 5 occurs. If Colored Structure 4 occurs, then U can be partitioned into two parts U_1 and U_2 with $|U_1| \geq 1, |U_2| \geq 0$, and V can be partitioned into k parts $V_1, V_2, \ldots, V_k$ with $|V_1| \geq 0$ and $|V_j| \geq 1$, $j \in \{2, 3, \ldots, k\}$. Since $N \geq k-1$, it follows from pigeonhole principle that $|U_1| \geq \lceil\frac{k-1}{2}\rceil$ or $|U_2| \geq \lceil\frac{k-1}{2}\rceil$. Without loss of generality, we assume that $|U_1| \geq \lceil\frac{k-1}{2}\rceil$. Noticing that $|V_2| \geq 1$ and all the edges between V_2 and U_1 have color 2, there is a monochromatic H with color 2, a contradiction. If the Colored Structure 5 occurs, then U can be partitioned into k parts $U_1, U_2, \ldots, U_k$ with $|U_1| \geq 0, |U_j| \geq 1$ and V can be partitioned into k parts $V_1, V_2, \ldots, V_k$ with $|V_1| \geq 0, |V_j| \geq 1$, $j \in \{2, 3, \ldots, k\}$. Noticing that $\lceil\frac{k-1}{2}\rceil < k-2$ for $k \geq 5$, $|V_2| \geq 1$ and all the edges between V_2 and $U_3 \cup U_4 \cup \ldots \cup U_k$ have color 1, there is a monochromatic H with color 1, a contradiction. The result thus follows.

It is easy to calculate that when $t \geq 3$, there are $t^2(t-1)^2(t-2)$ different P_5 and $2t\binom{t}{3}$ different $K_{1,3}$ in $K_{t,t}$. Therefore, the following corollary can be directly derived from Theorem 21.

Corollary 8 *For integers k and t satisfying $k = t^2 \geq 9$, and a subgraph H of $K_{1,\lceil\frac{k-1}{2}\rceil}$ with $|E(H)| \geq 2$, we have*

$$\text{bi-GM}_k(G : H) = \begin{cases} t^2(t-1)^2(t-2), & G = P_5; \\ 2t\binom{t}{3}, & G = K_{1,3}. \end{cases}$$

Theorem 22 *For integers k and t satisfying $k = t^2 \geq 9$, and a subgraph H of $K_{1,\lceil\frac{k-2}{2}\rceil}$ with $|E(H)| \geq 3$, we have*

$$\text{bi-GM}_{k-1}(P_5 : H) = t^2(t-1)^2(t-2) - 3(t-1)(t-2).$$

Proof It follows from Theorem 21 that $\text{bgr}_{k-1}(P_5 : H) = t$. Consider any $(k-1)$-edge-coloring of $K_{t,t}$. Since $|E(K_{t,t})| = t^2$ and each color is used at least once, it follows that there are only two edges, say e_1 and e_2, with the same color in $K_{t,t}$. Since $|E(H)| \geq 3$, it follows that we do not need to consider the number of monochromatic H in $K_{t,t}$. According to Corollary 8, there are $t^2(t-1)^2(t-2)$ different P_5 in $K_{t,t}$, and we only need to find the number of different P_5 containing the edges e_1 and e_2. This is because only P_5 containing edges e_1 and e_2 are not rainbow, and all other P_5 are rainbow. If $e_1 \nsim e_2$, then according to Lemma 5 that there are $6(t-2)$ different P_5 in $K_{t,t}$ that contain edges e_1 and e_2. If $e_1 \sim e_2$, then according to Lemma 5 that there are $3(t-1)(t-2)$ different P_5 in $K_{t,t}$ that contain edges e_1 and e_2. Noticing that $6(t-2) \leq 3(t-1)(t-2)$ for $t \geq 3$, the result thus follows.

Theorem 23 *For integers k and t satisfying $k = t^2 \geq 9$, and a subgraph H of $K_{1,\lceil \frac{k-3}{2} \rceil}$ with $|E(H)| \geq 4$, we have*

$$\text{bi-GM}_{k-2}(P_5 : H) = t^2(t-1)^2(t-2) - 9(t-1)(t-2).$$

Proof It follows from Theorem 21 that $\text{bgr}_{k-2}(P_5 : H) = t$. Consider $(k-2)$-edge-coloring of $K_{t,t}$. Since $|E(H)| \geq 4$, it follows that we do not need to consider the number of monochromatic H in $K_{t,t}$. Since each color is used at least once, there are only the following two cases. Due to the arbitrariness of colors, we can describe them using specific color names such as red and blue. Next, we calculate the number of different P_5 containing two or more edges with the same color. The following counting bases are all based on Lemma 5.

Case 1 There are three red edges e_1, e_2 and e_3. The remaining edges are not red and the colors of any two remaining edges are not the same.

Assume that the edges e_1, e_2 and e_3 form a red $3P_2$. In this subcase, there are $18(t-2)$ different P_5 containing two red edges.

Assume that the edges e_1, e_2 and e_3 form a red $P_3 \cup P_2$. Let $P_3 = e_1e_2$ and $P_2 = e_3$. In this subcase, there are $3(t-1)(t-2)$ different P_5 containing red edges e_1 and e_2, $6(t-2)-2$ different P_5 only containing red edges e_1 and e_3, and symmetrically $6(t-2)-2$ different P_5 only containing red edges e_2 and e_3. So there are a total of $3(t-1)(t-2)+2(6(t-2)-2) = 3(t+3)(t-2)-4$ different P_5 containing two or more red edges.

Assume that the edges e_1, e_2 and e_3 form a red $K_{1,3}$. In this subcase, there are $9(t-1)(t-2)$ different P_5 containing two red edges.

Assume that the edges e_1, e_2 and e_3 form a red P_4. In this subcase, there are $6(t-2)$ different P_5 containing red $2P_2$ and $2(3(t-1)(t-2)-2(t-2)) = 2(3t-5)(t-2)$ different P_5 without red $2P_2$. So there are a total of $6(t-2)+2(3t-5)(t-2) = 2(3t-2)(t-2)$ different P_5 containing two or more red edges.

Case 2 There are two red edges e_1, e_2 and two blue edges e_3, e_4. The remaining edges are not red or blue and the colors of any two remaining edges are not the same.

If $e_1 \nsim e_2$ and $e_3 \nsim e_4$, then there are at most $12(t-2)$ different P_5 containing two edges with the same color; if $e_1 \sim e_2$ and $e_3 \nsim e_4$, then there are at most $6(t-2)+3(t-1)(t-2) = 3(t+1)(t-2)$ different P_5 containing two edges with the same color; if $e_1 \sim e_2$ and $e_3 \sim e_4$, then there are at most $6(t-1)(t-2)$ different P_5 containing two edges with the same color.

Let

$$f_1(t) = 18(t-2), f_2(t) = 3(t+3)(t-2) - 4,$$

$$f_3(t) = 9(t-1)(t-2), f_4(t) = 2(3t-2)(t-2), f_5(t) = 3(t+1)(t-2).$$

Based on the data calculated from the seven subcases above, we need to compare the sizes of $f_1(t), f_2(t), f_3(t), f_4(t)$ and $f_5(t)$.

For $1 \le i \le 5$, let $f_{ii}(t) = \frac{f_i(t)}{t-2}$. Then

$$f_{11}(t) = 18, f_{22}(t) = 3(t+3) - \frac{4}{t-2}, f_{33}(t) = 9(t-1), f_{44}(t) = 2(3t-2), f_{55} = 3(t+1).$$

For $t \ge 3$, we have

$$\max\{f_{11}(t), f_{22}(t), f_{33}(t), f_{44}(t), f_{55}(t)\} = f_{33}(t) = 9(t-1),$$

and thus

$$\max\{f_1(t), f_2(t), f_3(t), f_4(t), f_5(t)\} = f_3(t) = 9(t-1)(t-2).$$

The result thus follows.

Theorem 24 *For integers k and t satisfying $k = t^2 \ge 9$, and a subgraph H of $K_{1,\lceil \frac{k-2}{2} \rceil}$ with $|E(H)| \ge 3$, we have*

$$\text{bi-GM}_{k-1}(K_{1,3} : H) = \begin{cases} 5, & t = 3; \\ 30, & t = 4; \\ (2t-1)\binom{t}{3} + \binom{t-2}{3} + 2\binom{t-2}{2}, & t \ge 5. \end{cases}$$

Proof It follows from Theorem 21 that $\text{bgr}_{k-1}(K_{1,3} : H) = t$. Consider any $(k-1)$-edge-coloring of $K_{t,t}$. Since $|E(K_{t,t})| = t^2$ and each color is used at least once, it follows that there are only two edges, say e_1 and e_2, with the same color in $K_{t,t}$. Since $|E(H)| \ge 3$, it follows that we do not need to consider the number of monochromatic H in $K_{t,t}$. Next, we calculate the number of rainbow copies of $K_{1,3}$ in $K_{t,t}$. If $e_1 \nsim e_2$, then this case is equivalent to Corollary 8. Hence there are $2t\binom{t}{3}$ rainbow copies of $K_{1,3}$ in $K_{t,t}$. If $e_1 \sim e_2$, then e_1 and e_2 form a monochromatic P_3. Without loss of generality, we assume that the edges e_1 and e_2 are red and vertex v is incident with the edges e_1 and e_2. We first investigate the number of rainbow copies of $K_{1,3}$ with center v for $t \ge 5$. Noticing that $\deg(v) = t$, the number of rainbow copies of $K_{1,3}$ with center v and without red edges is $\binom{t-2}{3}$, and number of rainbow copies of $K_{1,3}$ with center v and with a red edge is $2\binom{t-2}{2}$. In $K_{t,t}$, there are $(2t-1)\binom{t}{3}$ rainbow copies of $K_{1,3}$ with center in $V(K_{t,t}) \setminus \{v\}$. Therefore, the total number of rainbow copies of $K_{1,3}$ in $K_{t,t}$ is $(2t-1)\binom{t}{3} + \binom{t-2}{3} + 2\binom{t-2}{2}$. It is easy to verify that when $t \ge 5$,

$$\min\left\{2t\binom{t}{3}, (2t-1)\binom{t}{3} + \binom{t-2}{3} + 2\binom{t-2}{2}\right\} = (2t-1)\binom{t}{3} + \binom{t-2}{3} + 2\binom{t-2}{2}.$$

When $t = 3$, the number of rainbow copies of $K_{1,3}$ with center v and without red edges is 0, and number of rainbow copies of $K_{1,3}$ with center v and with a red edge is 0. In $K_{3,3}$, there are $5\binom{3}{3} = 5$ rainbow copies of $K_{1,3}$ with center in $V(K_{3,3}) \setminus \{v\}$, therefore $\text{bi-GM}_8(K_{1,3} : H) \le 5$. Since $5 < 6\binom{5}{3}$, it follows that $\text{bi-GM}_8(K_{1,3} : H) = 5$.

When $t = 4$, the number of rainbow copies of $K_{1,3}$ with center v and without red edges is 0, and number of rainbow copies of $K_{1,3}$ with center v

and with a red edge is 2. In $K_{4,4}$, there are $7\binom{4}{3} = 28$ rainbow copies of $K_{1,3}$ with center in $V(K_{4,4}) \setminus \{v\}$, and therefore bi-GM$_{15}(K_{1,3} : H) \leq 30$. Since $30 < 8\binom{4}{3}$, it follows that bi-GM$_{15}(K_{1,3} : H) = 30$.

Theorem 25 *For integers k and t satisfying $k = t^2 \geq 9$, and a subgraph H of $K_{1,\lceil \frac{k-3}{2} \rceil}$ with $|E(H)| \geq 4$, we have*

$$\text{bi-GM}_{k-2}(K_{1,3} : H) = \begin{cases} 4, & t = 3; \\ 28, & t = 4; \\ 93, & t = 5; \\ (2t-1)\binom{t}{3} + \binom{t-3}{3} + 3\binom{t-3}{2}, & t \geq 6. \end{cases}$$

Proof It follows from Theorem 21 that $\text{bgr}_{k-2}(K_{1,3} : H) = t$. Consider $(k-2)$-edge-coloring of $K_{t,t}$. Since $|E(H)| \geq 4$, it follows that we do not need to consider the number of monochromatic H in $K_{t,t}$. Noticing that each color needs to be used at least once. We first color any $k-2$ edges in $K_{t,t}$ with $k-2$ colors, and the remaining two edges are temporarily not colored, denoted as e_1 and e_2. Next, we discuss the edges e_1 and e_2 in two cases.

Case 1 The edges e_1 and e_2 have the same color.

Without loss of generality, we assume that these two edges are red. According to the structure of $K_{t,t}$, it is easy to calculate that if the red edges form a $3P_2$, then there are

$$f_1(t) = 2t\binom{t}{3}$$

rainbow copies of $K_{1,3}$ in $K_{t,t}$; if the red edges form a $P_3 \cup P_2$, then there are

$$f_2(t) = (2t-1)\binom{t}{3} + \binom{t-2}{3} + 2\binom{t-2}{2}$$

rainbow copies of $K_{1,3}$ in $K_{t,t}$; if the red edges form a P_4, then there are

$$f_3(t) = (2t-2)\binom{t}{3} + 2\binom{t-2}{3} + 4\binom{t-2}{2}$$

rainbow copies of $K_{1,3}$ in $K_{t,t}$; if the red edges form a $K_{1,3}$, then there are

$$f_4(t) = (2t-1)\binom{t}{3} + \binom{t-3}{3} + 3\binom{t-3}{2}$$

rainbow copies of $K_{1,3}$ in $K_{t,t}$.

Case 2 The edges e_1 and e_2 have different colors.

When the edges e_1 and e_2 form a P_3 in $K_{t,t}$, without loss of generality, we assume that e_1 is red and e_2 is blue. Let $V(P_3) = \{u, v, w\}$ and vertex v be incident with the edges e_1 and e_2. According to the structure of $K_{t,t}$, it is easy to calculate that if the other red edge is not incident with vertex u or v, and the other blue edge is not incident with vertex v or w, then there are

$$f_1(t) = 2t\binom{t}{3}$$

rainbow copies of $K_{1,3}$ in $K_{t,t}$; if the other red edge is incident with vertex u or v, and the other blue edge is not incident with vertex v or w, then there are

$$f_2(t) = (2t-1)\binom{t}{3} + \binom{t-2}{3} + 2\binom{t-2}{2}$$

rainbow copies of $K_{1,3}$ in $K_{t,t}$; if the other red edge is incident with vertex u, and the other blue edge is incident with vertex w, then there are

$$f_3(t) = (2t-2)\binom{t}{3} + 2\binom{t-2}{3} + 4\binom{t-2}{2}$$

rainbow copies of $K_{1,3}$ in $K_{t,t}$; if the other red edge is incident with vertex v, and the other blue edge is also incident with vertex v, then there are

$$f_5(t) = (2t-1)\binom{t}{3} + \binom{t-4}{3} + 4\binom{t-4}{2} + 4(t-4)$$

rainbow copies of $K_{1,3}$ in $K_{t,t}$.

When the edges e_1 and e_2 form a $2P_2$ in $K_{t,t}$, without loss of generality, we assume that e_1 is red and e_2 is blue. According to the structure of $K_{t,t}$, it is easy to calculate that if the other red edge is not adjacent to e_1, and the other blue edge is not adjacent to e_2, then there are

$$f_1(t) = 2t\binom{t}{3}$$

rainbow copies of $K_{1,3}$ in $K_{t,t}$; if the other red edge is adjacent to e_1, and the other blue edge is not adjacent to e_2, then there are

$$f_2(t) = (2t-1)\binom{t}{3} + \binom{t-2}{3} + 2\binom{t-2}{2}$$

rainbow copies of $K_{1,3}$ in $K_{t,t}$; if the other red edge is adjacent to e_1, and the other blue edge is adjacent to e_2, then there are

$$f_3(t) = (2t-2)\binom{t}{3} + 2\binom{t-2}{3} + 4\binom{t-2}{2}$$

rainbow copies of $K_{1,3}$ in $K_{t,t}$.

Next, we compare the sizes of $f_1(t), f_2(t), f_3(t), f_4(t)$ and $f_5(t)$. Based on the practical significance of counting in this paper, we only define in the operations of expressions for t in $f_1(t), f_2(t), f_3(t), f_4(t)$ and $f_5(t)$ that when integers $a < b$, we have $\binom{a}{b} \equiv 0$ and $a - b \equiv 0$.

For $t = 3$, we have

$$f_1(3) = 6, f_2(3) = 5, f_3(3) = 4, f_4(3) = 5, f_5(3) = 5.$$

Thus, $\min\{f_1(3), f_2(3), f_3(3), f_4(3), f_5(3)\} = 4$.

For $t = 4$, we have

$$f_1(4) = 32, f_2(4) = 30, f_3(4) = 28, f_4(4) = 28, f_5(4) = 28.$$

Thus, $\min\{f_1(4), f_2(4), f_3(4), f_4(4), f_5(4)\} = 28$.

For $t = 5$, we have

$$f_1(5) = 100, f_2(5) = 97, f_3(5) = 94, f_4(5) = 93, f_5(5) = 94.$$

Thus, $\min\{f_1(5), f_2(5), f_3(5), f_4(5), f_5(5)\} = 93$.

For $t = 6$, we have

$$f_1(6) = 240, f_2(6) = 236, f_3(6) = 232, f_4(6) = 230, f_5(6) = 232.$$

Thus, $\min\{f_1(6), f_2(6), f_3(6), f_4(6), f_5(6)\} = 230$.

For $t \geq 7$ and $1 \leq i \leq 5$, let $f_{ii}(t) = f_i(t) - (2t-2)\binom{t}{3} - \frac{1}{3}t^3 + t^2 + \frac{1}{3}t$, then

$$f_{11}(t) = t, f_{22}(t) = 2, f_{33}(t) = -t + 4, f_{44}(t) = -2t + 8, f_{55}(t) = -t + 4.$$

Therefore, when $t \geq 7$ we have

$$\min\{f_{11}(t), f_{22}(t), f_{33}(t), f_{44}(t), f_{55}(t)\} = f_{44}(t) = -2t + 8,$$

and thus

$$\min\{f_1(t), f_2(t), f_3(t), f_4(t), f_5(t)\} = f_4(t) = (2t-1)\binom{t}{3} + \binom{t-3}{3} + 3\binom{t-3}{2}.$$

Based on the above discussion, we have

$$\min\{f_1(t), f_2(t), f_3(t), f_4(t), f_5(t)\} = \begin{cases} (2t-2)\binom{t}{3}, & t = 3; \\ (2t-1)\binom{t}{3}, & t = 4; \\ (2t-1)\binom{t}{3} + 3\binom{t-3}{2}, & t = 5; \\ (2t-1)\binom{t}{3} + \binom{t-3}{3} + 3\binom{t-3}{2}, & t \geq 6. \end{cases}$$

The result thus follows.

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Conflict of interest

The authors declare that they have no conflict of interest.

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