Acta Mathematica Sinica, English Series
Sep., 201x, Vol. x, No. x, pp. 1–16
Published online: August 15, 201x
DOI: 0000000000000000
Http://www.ActaMath.com

Spectral properties and energy of weighted adjacency matrix for graphs with a degree-based edge-weight function

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Abstract Let G be a graph and d_i denote the degree of a vertex v_i in G, and let $f(x, y)$ be a real symmetric function. Then one can get an edge-weighted graph in such a way that for each edge v_iv_j of G, the weight of v_iv_j is assigned by the value $f(d_i, d_j)$. Hence, we have a weighted adjacency matrix $\mathcal{A}_f(G)$ of G, in which the ij-entry is equal to $f(d_i, d_j)$ if $v_iv_j \in E(G)$ and 0 otherwise. This paper attempts to unify the study of spectral properties for the weighted adjacency matrix $\mathcal{A}_f(G)$ of graphs with a degree-based edge-weight function $f(x, y)$. Some lower and upper bounds of the largest weighted adjacency eigenvalue λ_1 are given, and the corresponding extremal graphs are characterized. Bounds of the energy $\mathcal{E}_f(G)$ for the weighted adjacency matrix $\mathcal{A}_f(G)$ are also obtained. By virtue of the unified method, this makes many earlier results become special cases of our results.

Keywords degree-based edge-weight function, weighted adjacency matrix, weighted adjacency eigenvalue (energy), topological function-index

MR(2020) Subject Classification 05C50, 05C92, 05C09

1 Introduction

All graphs considered in this paper are simple, finite and undirected. For notation and terminology not defined here, we refer to [5]. Let $G = (V(G), E(G))$ be a graph with vertex-set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge-set $E(G)$. An edge e of G with end vertices v_i and v_j is denoted by v_iv_j. Let d_i be the degree of a vertex v_i in G. The maximum degree and minimum degree of G are denoted by Δ and δ, respectively. If the vertex-set $V(G)$ of G admits a partition into two classes such that the two ends of its every edge are in different classes (or, vertices in the same partition class must be nonadjacent), then G is called a bipartite graph. A bipartite graph in which any two vertices from different partition classes are adjacent is called a complete bipartite graph, denoted by $K_{s,t}$, where $s + t = n$. A complete graph K_n of order n is a graph in which any two vertices are adjacent.

In chemical graph theory, graphical or topological indices in chemistry are used to represent both the structural and chemical properties of molecular graphs. The general form of these indices is

$$TI(G) = \sum_{v_iv_j \in E(G)} f(d_i, d_j),$$

Received March 13, 2023, accepted April 26, 2024
Supported by NSFC (Grant No. 12131013 and 12161141006.)
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where $f(x, y)$ is a real symmetric function, called the edge-weight function, and $f(d_i, d_j)$ is the edge-weight of an edge v_iv_j of G. Each index maps a molecular graph into a single number, obtained by summing up the edge-weights in a molecular graph with edge-weights defined by the function $f(x, y)$.

In spectral graph theory, matrices associated with a graph G play an important role. Thus, using a matrix to represent the structure of a molecular graph with edge-weights separately on its pairs of adjacent vertices, it would much better keep the structural information of the graph. In other words, a matrix keeps much more structural information than just a single number, the value of an index. So, the algebraic properties of these structural matrices are worth thoroughly studying. In 2015, this idea was first proposed by one of the authors Li in [21]. In recent years, various studies on matrices defined by topological indices from algebraic viewpoint were reported, such as the Sombor matrix [11], ABC matrix [4], first(second) Zagreb matrix [30], inverse sum indeg matrix [1], arithmetic-geometric matrix [32] and geometric-arithmetic matrix [33], because many interesting properties of graphs are reflected in the study of these matrices.

In 2018, Das et al. [7] gave the following formal definition of the weighted adjacency matrix for a graph with a degree-based edge-weight function. Let $\mathcal{A}_f(G)$ denote the *weighted adjacency matrix* of a graph G with edge-weight function $f(x, y)$, whose ij-entry is defined as

$$(\mathcal{A}_f(G))_{ij} = \begin{cases} f(d_i, d_j) & \text{if } v_iv_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

That is, for any graphical or topological index, one can define a corresponding weighted adjacency matrix of an edge-weighted graph by the edge-weight function $f(x, y)$ of the index. We will simply call the eigenvalues of the $n \times n$ matrix $\mathcal{A}_f(G)$ as the *weighted adjacency eigenvalues* of a graph G with edge-weight function $f(x, y)$. Since $f(x, y)$ is a real symmetric function and G is an undirected graph. $\mathcal{A}_f(G)$ is a real symmetric matrix, and therefore all its eigenvalues are real numbers. We may adopt the convention that the weighted adjacency eigenvalues λ_i are always arranged in a decreasing order, i.e.,

$$\lambda_{max} = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = \lambda_{min}. \tag{1.1}$$

And $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ are the absolute values of eigenvalues λ_i, $i = 1, 2, \ldots, n$, given in a decreasing order. In [7] the *energy* of the weighted adjacency matrix $\mathcal{A}_f(G)$ was defined as

$$\mathcal{E}_f(G) = \sum_{i=1}^{n} |\lambda_i| = \sum_{i=1}^{n} \gamma_i,$$

then they obtained some lower and upper bounds on $\mathcal{E}_f(G)$. For recent works on investigations of the energy $\mathcal{E}_f(G)$ of the weighted adjacency matrix $\mathcal{A}_f(G)$, we refer to [10, 12, 13, 15, 16, 34].

In fact, although the matrix $\mathcal{A}_f(G)$ was proposed in a general setting, it was studied still one by one separately for a concrete graphical or topological index or function $f(x, y)$, not as a whole for a general function $f(x, y)$. This lost the sense for us to introduce the general weighted adjacency matrix $\mathcal{A}_f(G)$. In 2021, Li and Wang [22] attempted to study the extremal spectral radius of the weighted adjacency matrices in a general setting. They obtained that the tree with the largest spectral radius is star or a double star when $f(x, y)$ is increasing and convex in variable x. In 2023, Zheng et al. [38] added a restriction to $f(x, y)$ and they confirmed that star is the unique tree with the largest spectral radius among trees. In 2022, Li and Yang [23] obtained uniform interlacing inequalities for the weighted adjacency eigenvalues under edge deletion. They also established a uniform equivalent condition for a connected graph G to have m distinct weighted adjacency eigenvalues. In 2023, Li and Yang [24] further discussed interlacing inequalities for the weighted adjacency eigenvalues under some kinds of graph operations including edge subdivision, vertex deletion and vertex contraction. In this paper, we continue the study to consider some eigenvalues properties of the weighted adjacency matrix $\mathcal{A}_f(G)$ and obtain some new upper and lower bounds on the energy $\mathcal{E}_f(G)$.

The structure of this paper is arranged as follows. In the next section, we introduce some necessary notation and terminology and list some previous known results that will be used in the subsequent sections. In Section 3, we first do some investigations on the weighted adjacency matrix $\mathcal{A}_f(G)$. Then, we give some lower and upper bounds for the largest weighted adjacency eigenvalue λ_1 and characterize the corresponding graphs. In Section 4, we obtain some bounds for the energy $\mathcal{E}_f(G)$ of the weighted adjacency matrix $\mathcal{A}_f(G)$. By means of the present approach, many earlier established results can be viewed as special cases of our results.

2 Preliminaries

At the very beginning, we state some fundamental results on matrix theory, which will be used in the sequel. An $n \times n$ real square matrix M is called *symmetric* if $M^T = M$, where M^T is the *transpose* of M. The eigenvalues of M are all real and arranged as

$$\rho_1(M) \geq \rho_2(M) \geq \cdots \geq \rho_n(M).$$

Lemma 2.1 ([3]) *Let M be a symmetric matrix of order n. Then for any nonzero vector $x \in R^n$,*

$$\rho_1(M) \geq \frac{x^T M x}{x^T x},$$

where equality holds if and only if x is an eigenvector of M corresponding to the largest eigenvalue $\rho_1(M)$.

If we delete several rows and the corresponding columns from a real square matrix, the remaining matrix is a *principal submatrix* of the original matrix. Then we have the following result.

Lemma 2.2 ([19]) *Suppose M is a symmetric matrix of order n, partitioned as*

$$M = \begin{bmatrix} B_{m\times m} & C_{m\times(n-m)} \\ (C_{m\times(n-m)})^T & D_{(n-m)\times(n-m)} \end{bmatrix}.$$

Let

$$\rho_1(B) \geq \rho_2(B) \geq \cdots \geq \rho_m(B) \text{ and } \rho_1(M) \geq \rho_2(M) \geq \cdots \geq \rho_n(M)$$

be the eigenvalues of B and M, respectively. Then the inequalities

$$\rho_i(M) \geq \rho_i(B) \geq \rho_{n-m+i}(M),$$

hold for each $i = 1, 2, \ldots, m$.

An $n \times n$ real matrix M is called *nonnegative* if its every entry is nonnegative. We say that M is *irreducible* if it is not the 1×1 matrix [0] and if there does not exist a permutation matrix N (a matrix with 1 in every row and column, and 0 for all other entries) such that

$$NMN^{-1} = \begin{bmatrix} M_{11} & M_{12} \\ \mathbf{0} & M_{22} \end{bmatrix},$$

where M_{11} and M_{22} are square matrices of size greater than zero. We now state the famous Perron–Frobenius theorem.

Lemma 2.3 ([19]) *Let* M *be a nonnegative irreducible square matrix of order* n*. Then the largest eigenvalue* $\rho_1(M)$ *is simple, there exists an eigenvector with all entries positive. Moreover,* $\rho_1(M) \geq |\rho_i(M)|$ *for* $2 \leq i \leq n$*.*

The following two results are from our paper [23] for the weighted adjacency matrix $\mathcal{A}_f(G)$ with few distinct eigenvalues.

Lemma 2.4 ([23]) *Let* $f(x, y)$ *be a real symmetric function, and let* G *be a connected graph of order* $n \geq 2$ *with edge-weight* $f(d_i, d_j) > 0$ *for any edge* $v_iv_j \in E(G)$*. Then the weighted adjacency matrix* $\mathcal{A}_f(G)$ *has two distinct eigenvalues if and only if* G *is the complete graph* K_n*.*

Lemma 2.5 ([23]) *Let* $f(x, y)$ *be a real symmetric function, and let* G *be a bipartite graph of order* $n \geq 3$ *with edge-weight* $f(d_i, d_j) > 0$ *for any edge* $v_iv_j \in E(G)$*. Then the weighted adjacency matrix* $\mathcal{A}_f(G)$ *has three distinct eigenvalues if and only if* G *is the complete bipartite graph* $K_{s,t}$*.*

3 Spectral properties of the weighted adjacency matrix

In this section, we first present two equalities, which show the elementary properties of the weighted adjacency matrix $\mathcal{A}_f(G)$.

$$tr(\mathcal{A}_f(G)) = \lambda_1 + \lambda_2 + \cdots + \lambda_n = 0 \tag{3.1}$$

and

$$tr(\mathcal{A}_f^2(G)) = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2 = \gamma_1^2 + \gamma_2^2 + \cdots + \gamma_n^2 = 2 \sum_{v_iv_j \in E(G)} f^2(d_i, d_j), \tag{3.2}$$

where $tr(\mathcal{A}_f(G))$ and $tr(\mathcal{A}_f^2(G))$ are the *traces* of matrices $\mathcal{A}_f(G)$ and $\mathcal{A}_f^2(G)$, respectively.

Theorem 3.1 *Let* G *be a graph of order* n *and the edge-weight* $f(d_i, d_j) > 0$ *for any edge* $v_iv_j \in E(G)$*. Then the weighted adjacency matrix* $\mathcal{A}_f(G)$ *has exactly one eigenvalue if and only if* G *is an empty graph.*

Proof Since $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 0$ and $\mathcal{A}_f(G)$ has exactly one eigenvalue. We have $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$. Noting that $\mathcal{A}_f(G)$ be a symmetric matrix and the edge-weight $f(d_i, d_j) > 0$ for any edge $v_iv_j \in E(G)$. Thus $\mathcal{A}_f(G) = \mathbf{0}$, and so G is an empty graph.

Conversely, if $G = \overline{K_n}$, then $\mathcal{A}_f(G) = \mathbf{0}$, which yields that all eigenvalues are zero. Hence the proof is complete. $\square$

Theorem 3.2 *Let G be a graph of order n and the edge-weight $f(d_i,d_j) > 0$ for any edge $v_iv_j \in E(G)$. Then the eigenvalues of the weighted adjacency matrix $\mathcal{A}_f(G)$ are symmetric with respect to the origin if and only if G is a bipartite graph.*

Proof If G is a bipartite graph of order n, then the weighted adjacency matrix $\mathcal{A}_f(G)$ has the following form:

$$\mathcal{A}_f(G) = \begin{bmatrix} \mathbf{0} & B \\ B^T & \mathbf{0} \end{bmatrix}.$$

For an eigenvalue λ_i of $\mathcal{A}_f(G)$, suppose $x = (x_1, x_2)^T$ is its corresponding eigenvector. It is clear that $\mathcal{A}_f(G)x = \lambda_i x$, or equivalently, $(Bx_2, B^T x_1) = \lambda_i(x_1, x_2)$. For the vector $x' = (x_1, -x_2)^T$, one can easily see that $\mathcal{A}_f(G)x' = -\lambda_i x'$.

Conversely, if the eigenvalues of $\mathcal{A}_f(G)$ are symmetric with respect to origin, we have $tr(\mathcal{A}_f^{2k+1}(G)) = \sum_{i=1}^{n} \lambda_i^{2k+1} = 0$ for any $k \geq 1$. Since the edge-weight $f(d_i,d_j) > 0$ for any edge $v_iv_j \in E(G)$, this implies that the number of closed walks of odd length $2k+1$ is 0 for any integer k. Therefore, G does not contain any odd cycles, and so it is bipartite. Our proof is thus complete. □

Next, we give two results for the bounds of the largest weighted adjacency eigenvalue λ_1.

Theorem 3.3 *Let G be a graph of order n and λ_1 be the largest eigenvalue of the weighted adjacency matrix $\mathcal{A}_f(G)$. Then*

$$\lambda_1 \geq \frac{2TI(G)}{n},$$

where the equality holds if and only if $\mathcal{A}_f(G)$ has a constant row sum.

Proof Let $x = (x_1, x_2, \cdots, x_n)^T$ be any unit vector in R^n. Then

$$x^T \mathcal{A}_f(G)x = 2 \sum_{v_iv_j \in E(G)} f(d_i,d_j)x_i x_j.$$

If $x_1 = x_2 = \cdots = x_n = \frac{1}{\sqrt{n}}$, we can get

$$\frac{x^T \mathcal{A}_f(G)x}{x^T x} = \frac{2 \sum\limits_{v_iv_j \in E(G)} f(d_i,d_j)}{n} = \frac{2TI(G)}{n}.$$

By Lemma 2.1, we obtain $\lambda_1 \geq \frac{2TI(G)}{n}$, where the equality holds if and only if $x = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \cdots, \frac{1}{\sqrt{n}})^T$ is an eigenvector corresponding to the eigenvalue λ_1. In other words, the equality holds if and only if $\mathcal{A}_f(G)$ has a constant row sum. The result is thus obtained. □

Theorem 3.4 *Let G be a nonempty graph of order $n \geq 2$ and the edge-weight $f(d_i,d_j) > 0$ for any edge $v_iv_j \in E(G)$. Let λ_1 be the largest eigenvalue of the weighted adjacency matrix $\mathcal{A}_f(G)$. Then*

$$\sqrt{\frac{tr(\mathcal{A}_f^2(G))}{n}} \leq \lambda_1 \leq \sqrt{\frac{(n-1)tr(\mathcal{A}_f^2(G))}{n}},$$

where the left equality holds if and only if $G = \frac{n}{2}K_2$ and the right equality holds if and only if $G = K_n$.

Proof From equality (3.2), we have

$$\lambda_1^2 = tr(\mathcal{A}_f^2(G)) - \lambda_2^2 - \lambda_3^2 - \cdots - \lambda_n^2.$$

By the Cauchy-Schwarz inequality and equality (3.1), we get

$$\lambda_1^2 \leq tr(\mathcal{A}_f^2(G)) - \frac{1}{n-1}\left(\sum_{i=2}^{n}\lambda_i\right)^2 = tr(\mathcal{A}_f^2(G)) - \frac{1}{n-1}\lambda_1^2,$$

which implies that

$$\lambda_1 \leq \sqrt{\frac{(n-1)tr(\mathcal{A}_f^2(G))}{n}}.$$

Suppose that the equality on the right-hand side holds. It is not difficult to get $\lambda_2 = \lambda_3 = \cdots = \lambda_n$. If $\lambda_1 = \lambda_2 = \lambda_3 = \cdots = \lambda_n$, from Theorem 3.1, we have $G = \overline{K_n}$, which cannot be true. If $\lambda_1 \neq \lambda_2 = \lambda_3 = \cdots = \lambda_n$, from equality (3.1) we obtain $\lambda_1 > 0$ and $\lambda_2 < 0$. Hence, G is a graph with two distinct nonzero eigenvalues. Next, we claim that G is connected. If G is disconnected, there are two vertices v_i and v_j such that $v_iv_j \notin E(G)$. Therefore, the weighted adjacency matrix $\mathcal{A}_f(G)$ has a principal submatrix B of order 2 in which all the entries are zero. By Lemma 2.2, we have $\lambda_2 \geq \rho_2(B) = 0$, which is a contradiction. Since G is connected, using Lemma 2.4, we thus get that G is the complete graph K_n.

Conversely, if $G = K_n$, the eigenvalues of the weighted adjacency matrix $\mathcal{A}_f(G)$ are $f(n-1, n-1)(n-1)$ and $-f(n-1, n-1)$ with multiplicity $n-1$. It is easy to check that

$$\lambda_1 = \sqrt{\frac{(n-1)tr(\mathcal{A}_f^2(G))}{n}} = \sqrt{\frac{n(n-1)^2 f^2(n-1, n-1)}{n}} = f(n-1, n-1)(n-1).$$

Next, we prove the bound on the left-hand side. Since $tr(\mathcal{A}_f^2(G)) \leq \lambda_1^2 + \lambda_1^2 + \cdots + \lambda_1^2$, which gives

$$\lambda_1 \geq \sqrt{\frac{tr(\mathcal{A}_f^2(G))}{n}},$$

where the equality holds if and only if $|\lambda_1| = |\lambda_2| = \cdots = |\lambda_n|$.

Suppose that the equality on the left-hand side holds. Because G is a nonempty graph of order $n \geq 2$ and the edge-weight $f(d_i, d_j) > 0$ for any edge $v_iv_j \in E(G)$, we have $|\lambda_1| = |\lambda_2| = \cdots = |\lambda_n| \neq 0$. Recall that $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 0$, which means that n is an even number. Furthermore, $\lambda_1 = \lambda_2 = \cdots = \lambda_{\frac{n}{2}}$ and $\lambda_{\frac{n}{2}+1} = \lambda_{\frac{n}{2}+2} = \cdots = \lambda_n$. This means that the eigenvalues of the weighted adjacency matrix $\mathcal{A}_f(G)$ are symmetric with respect to the origin. Using Theorem 3.2, we can get that G is a bipartite graph. If its maximum degree $\Delta = 1$, then $G = \frac{n}{2}K_2$. If its maximum degree $\Delta \geq 2$, then G has an induced subgraph $K_{1,2}$. Let B be a principle submatrix of $\mathcal{A}_f(G)$ corresponding to $K_{1,2}$. From Lemma 2.2, we can directly have $\lambda_2 \geq \rho_2(B) = 0$. Besides, G must contain a connected component H with at least 3 vertices. By Lemma 2.3, we get $\lambda_1(H) > \lambda_2(H) \geq 0$, which is a contradiction.

Conversely, if $G = \frac{n}{2}K_2$, the eigenvalues of the weighted adjacency matrix $\mathcal{A}_f(G)$ are $f(1,1)$ with multiplicity $n/2$ and $-f(1,1)$ with multiplicity $n/2$. It is easy to check that

$$\lambda_1 = \sqrt{\frac{tr(\mathcal{A}_f^2(G))}{n}} = \sqrt{\frac{nf^2(1,1)}{n}} = f(1,1).$$

This completes the proof. □

4 Bounds for the energy of the weighted adjacency matrix

In this section, we establish some upper and lower bounds for the energy $\mathcal{E}_f(G)$ of the weighted adjacency matrix $\mathcal{A}_f(G)$.

Theorem 4.1 *Let G be a graph of order n. For any real number k with the property $\gamma_1 \geq k \geq \gamma_n$, we have*

$$\mathcal{E}_f(G) \leq k + \sqrt{(n-1)(tr(\mathcal{A}_f^2(G)) - k^2)}.$$

Proof Recall that $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ are the absolute values of the weighted adjacency eigenvalues λ_i for $1 \leq i \leq n$, given in a decreasing order. Thus

$$\mathcal{E}_f(G) = |\lambda_1| + |\lambda_2| + \cdots + |\lambda_n| = \gamma_1 + \gamma_2 + \cdots + \gamma_n.$$

By the Cauchy-Schwarz inequality, we have

$$\gamma_2 + \gamma_3 + \cdots + \gamma_n \leq \sqrt{(n-1)(\gamma_2^2 + \gamma_3^2 + \cdots + \gamma_n^2)} = \sqrt{(n-1)(tr(\mathcal{A}_f^2(G)) - \gamma_1^2)}.$$

This means that

$$\mathcal{E}_f(G) \leq \gamma_1 + \sqrt{(n-1)(tr(\mathcal{A}_f^2(G)) - \gamma_1^2)}.$$

Similarly, we can obtain

$$\mathcal{E}_f(G) \leq \gamma_n + \sqrt{(n-1)(tr(\mathcal{A}_f^2(G)) - \gamma_n^2)}.$$

Now, consider the function

$$f(x) = x + \sqrt{(n-1)(tr(\mathcal{A}_f^2(G)) - x^2)}.$$

It is monotonously increasing in the interval $[0, \sqrt{\frac{tr(\mathcal{A}_f^2(G))}{n}})$ and decreasing in the interval $(\sqrt{\frac{tr(\mathcal{A}_f^2(G))}{n}}, \gamma_1)$. Hence, for any real number k such that $\gamma_1 \geq k \geq \gamma_n$, we have

$$\mathcal{E}_f(G) \leq k + \sqrt{(n-1)(tr(\mathcal{A}_f^2(G)) - k^2)}.$$

Our proof is thus complete. □

Theorem 4.1 is consistent with the previous result in [7]. However, we present a different method to prove it. The following result is an immediate consequence of Theorem 4.1.

Corollary 4.2 *Let G be a graph of order n. Then*

$$\mathcal{E}_f(G) \leq min\left\{\gamma_1 + \sqrt{(n-1)(tr(\mathcal{A}_f^2(G)) - \gamma_1^2)}, \gamma_n + \sqrt{(n-1)(tr(\mathcal{A}_f^2(G)) - \gamma_n^2)}\right\}.$$

For $0 \leq x_0 \leq \gamma_1$, we know that $f(x_0) \leq f\left(\sqrt{\frac{tr(\mathcal{A}_f^2(G))}{n}}\right) = \sqrt{n \cdot tr(\mathcal{A}_f^2(G))}$. Hence, the following result can be obtained from Theorem 4.1 directly.

Corollary 4.3 *Let G be a graph of order n. Then*

$$\mathcal{E}_f(G) \leq \sqrt{n \cdot tr(\mathcal{A}_f^2(G))}.$$

Remark 4.4 Using Corollary 4.3, by appropriate choices of $f(d_i, d_j)$ we arrive at the following inequalities for some particular degree-based graph energies of a graph G. It is worth noting that these inequalities are not listed in [7].

- If $f(d_i,d_j)=\sqrt{d_i^2+d_j^2}$, then $\mathcal{A}_f(G)$ is the Sombor matrix $\mathcal{A}_{Som}(G)$ [11]. We have the Sombor energy
$$\mathcal{E}_{Som}(G)\leq\sqrt{2nF(G)},\tag{4.1}$$
where $F(G)$ is the forgotten index of the graph G.

- If $f(d_i,d_j)=\sqrt{d_i+d_j}$, then $\mathcal{A}_f(G)$ is the Nirmala matrix $\mathcal{A}_{Nir}(G)$ [14]. We have the Nirmala energy
$$\mathcal{E}_{Nir}(G)\leq\sqrt{2n\cdot Zag1(G)},\tag{4.2}$$
where $Zag1(G)$ is the first Zagreb index of the graph G.

- If $f(d_i,d_j)=\frac{d_id_j}{d_i+d_j}$, then $\mathcal{A}_f(G)$ is the inverse sum indeg matrix $\mathcal{A}_{isi}(G)$ [37]. We have the inverse sum indeg energy
$$\mathcal{E}_{isi}(G)\leq\sqrt{n\cdot tr(\mathcal{A}_{isi}^2(G))}.\tag{4.3}$$

- If $f(d_i,d_j)=\frac{d_i+d_j}{2\sqrt{d_id_j}}$, then $\mathcal{A}_f(G)$ is the arithmetic-geometric matrix $\mathcal{A}_{ag}(G)$ [39]. We have the arithmetic-geometric energy
$$\mathcal{E}_{ag}(G)\leq\sqrt{\frac{n}{2}\sum_{v_iv_j\in E(G)}\left(\sqrt{\frac{d_i}{d_j}}+\sqrt{\frac{d_j}{d_i}}\right)^2}.\tag{4.4}$$

- If $f(d_i,d_j)=\frac{g(d_i,d_j)}{\sqrt{d_id_j}}$, then $\mathcal{A}_f(G)$ is the Randić degree-based adjacency matrix $\mathcal{A}_{Rdb}(G)$ [40]. We have the Randić degree-based energy
$$\mathcal{E}_{Rdb}(G)\leq\sqrt{n\cdot tr(\mathcal{A}_{Rdb}^2(G))}.\tag{4.5}$$

- If $f(d_i,d_j)=\frac{d_i}{d_j}+\frac{d_j}{d_i}$, then $\mathcal{A}_f(G)$ is the symmetric division deg matrix $\mathcal{A}_{sdd}(G)$ [29]. We have the symmetric division deg energy
$$\mathcal{E}_{sdd}(G)\leq\sqrt{2nP(G)},\tag{4.6}$$
where $P(G)$ is one half of the second spectral moment of $\mathcal{A}_{sdd}(G)$.

- If $f(d_i,d_j)=\frac{(d_id_j)^\alpha}{(d_i+d_j)^\beta}$, then $\mathcal{A}_f(G)$ is the generalized inverse sum indeg matrix $\mathcal{A}_{gisi}(G)$ [18]. We have the generalized inverse sum indeg energy
$$\mathcal{E}_{gisi}(G)\leq\sqrt{2nQ(G)},\tag{4.7}$$
where $Q(G)$ is one half of the second spectral moment of $\mathcal{A}_{gisi}(G)$.

- If $f(d_i,d_j)=\sqrt{d_id_j}+\frac{d_i+d_j}{2}$, then $\mathcal{A}_f(G)$ is the sum geometric arithmetic means matrix $\mathcal{A}_{sgam}(G)$ [27]. We have the sum geometric arithmetic means energy
$$\mathcal{E}_{sgam}(G)\leq\sqrt{n\cdot tr(\mathcal{A}_{sgam}^2(G))}.\tag{4.8}$$

- If $f(d_i,d_j)=\frac{1}{2}(\frac{d_i}{d_j}+\frac{d_j}{d_i})$, then $\mathcal{A}_f(G)$ is the extended adjacency matrix $\mathcal{A}_{ext}(G)$ [36]. We have the extended energy
$$\mathcal{E}_{ext}(G)\leq\sqrt{n\cdot tr(\mathcal{A}_{ext}^2(G))}.\tag{4.9}$$

- If $f(d_i, d_j) = ((d_i)^p + (d_j)^p)^{\frac{1}{p}}$, then $\mathcal{A}_f(G)$ is the p-Sombor matrix $\mathcal{A}_{pSom}(G)$ [26]. We have the p-Sombor energy
$$\mathcal{E}_{pSom}(G) \leq \sqrt{n \cdot N_2(G)}, \tag{4.10}$$
where $N_2(G)$ is the second spectral moment of $\mathcal{A}_{pSom}(G)$.

- If $f(d_i, d_j) = \frac{1}{\sqrt{d_i^2 d_j + d_j^2 d_i}}$, then $\mathcal{A}_f(G)$ is the product of Randić and sum-connectivity adjacency matrix $\mathcal{A}_{prs}(G)$ [28]. We have the product of Randić and sum-connectivity energy
$$\mathcal{E}_{prs}(G) \leq \sqrt{2n \sum_{v_iv_j \in E(G)} \frac{1}{d_i^2 d_j + d_j^2 d_i}}. \tag{4.11}$$

The inequality (4.1) was proved in [11], inequality (4.2) in [14], inequality (4.3) in [25] and [17], inequality (4.4) in [39] and [32], inequality (4.5) in [40], inequality (4.6) in [29], inequality (4.7) in [18], inequality (4.8) in [27], inequality (4.9) in [8], inequality (4.10) in [26] and inequality (4.11) in [28].

Next, we obtain an upper bound for the energy $\mathcal{E}_f(G)$ of the weighted adjacency matrix $\mathcal{A}_f(G)$ in terms of the order n, the topological index $TI(G)$ of a graph G and the trace of $\mathcal{A}_f^2(G)$.

Theorem 4.5 *Let G be a nonempty graph of order $n \geq 2$ and the edge-weight $f(d_i, d_j) > 0$ for any edge $v_iv_j \in E(G)$. If $2TI(G) \geq n\gamma_n$, then*
$$\mathcal{E}_f(G) \leq \frac{2TI(G)}{n} + \sqrt{(n-1)\left(tr(\mathcal{A}_f^2(G)) - \frac{4(TI(G))^2}{n^2}\right)}.$$
Moreover, the equality holds if and only if G is $\frac{n}{2}K_2$, or K_n, or a graph with three distinct eigenvalues $\frac{2TI(G)}{n}$, $\sqrt{\frac{tr(\mathcal{A}_f^2(G)) - \frac{4(TI(G))^2}{n^2}}{n-1}}$ and $-\sqrt{\frac{tr(\mathcal{A}_f^2(G)) - \frac{4(TI(G))^2}{n^2}}{n-1}}$.

Proof From Theorem 3.3, we have
$$\lambda_1 \geq \frac{2TI(G)}{n}.$$
Since $\frac{2TI(G)}{n} \geq \gamma_n$, together with Theorem 4.1, the following inequality can be obtained
$$\mathcal{E}_f(G) \leq \frac{2TI(G)}{n} + \sqrt{(n-1)\left(tr(\mathcal{A}_f^2(G)) - \frac{4(TI(G))^2}{n^2}\right)}.$$
Thus the bound follows.

Now, we consider what happens when the equality holds. If $\lambda_1 = \frac{2TI(G)}{n}$, then $\mathcal{A}_f(G)$ has a constant row sum. Because the equality must also hold in the Cauchy-Schwarz inequality, we have
$$\gamma_i = \sqrt{\frac{tr(\mathcal{A}_f^2(G)) - \frac{4(TI(G))^2}{n^2}}{n-1}},$$
for $2 \leq i \leq n$. Recall that $G \neq \overline{K_n}$ and $f(d_i, d_j) > 0$ for any edge $v_iv_j \in E(G)$, we have $\gamma_1 \neq 0$. Hence, we can reduce it into three possibilities: either G has two eigenvalues with equal absolute values, in which case $G = \frac{n}{2}K_2$; or G has two eigenvalues with distinct absolute values,

in which case $G = K_n$; or graph G with three distinct eigenvalues $\frac{2TI(G)}{n}$, $\sqrt{\frac{tr(\mathcal{A}_f^2(G)) - \frac{4(TI(G))^2}{n^2}}{n-1}}$ and $-\sqrt{\frac{tr(\mathcal{A}_f^2(G)) - \frac{4(TI(G))^2}{n^2}}{n-1}}$.

Conversely, if G consists of $n/2$ copies of K_2, or $G = K_n$ or a graph with three distinct eigenvalues $\frac{2TI(G)}{n}$, $\sqrt{\frac{tr(\mathcal{A}_f^2(G)) - \frac{4(TI(G))^2}{n^2}}{n-1}}$ and $-\sqrt{\frac{tr(\mathcal{A}_f^2(G)) - \frac{4(TI(G))^2}{n^2}}{n-1}}$, then the equality clearly follows. This completes the proof. □

Remark 4.6 From the inequality in Theorem 4.5, by appropriate choices of $f(d_i, d_j)$ we get the following inequalities for some particular degree-based graph energies of a graph G.

- If $f(d_i, d_j) = \sqrt{d_i^2 + d_j^2}$, then $\mathcal{A}_f(G)$ is the Sombor matrix $\mathcal{A}_{Som}(G)$ [11]. We have
$$\mathcal{E}_{Som}(G) \leq \frac{2SO(G)}{n} + \sqrt{(n-1)\Big(2F(G) - (\frac{2SO(G)}{n})^2\Big)}, \tag{4.12}$$
where $SO(G)$ is the Sombor index of the graph G.
- If $f(d_i, d_j) = \sqrt{d_i + d_j}$, then $\mathcal{A}_f(G)$ is the Nirmala matrix $\mathcal{A}_{Nir}(G)$ [14]. We have
$$\mathcal{E}_{Nir}(G) \leq \frac{2N(G)}{n} + \sqrt{2(n-1)\Big(Zag1(G) - (\frac{2N(G)}{n})^2\Big)}, \tag{4.13}$$
where $N(G)$ is the Nirmala index of the graph G.
- If $f(d_i, d_j) = \frac{1}{\sqrt{d_i^2 + d_j^2}}$, then $\mathcal{A}_f(G)$ is the modified Sombor matrix $\mathcal{A}_{mSom}(G)$ [20]. We have the modified Sombor energy
$$\mathcal{E}_{mSom}(G) \leq \frac{2(^mSO(G))}{n} + \sqrt{(n-1)\Big(2B(G) - (\frac{2(^mSO(G))}{n})^2\Big)}, \tag{4.14}$$
where $B(G)$ is one half of the second spectral moment of $\mathcal{A}_{mSom}(G)$ and $^mSO(G)$ is the modified Sombor index of the graph G.
- If $f(d_i, d_j) = \frac{1}{\sqrt{d_i d_j}}$, then $\mathcal{A}_f(G)$ is the Randić matrix $\mathcal{A}_R(G)$ [2]. We have the Randić energy
$$\mathcal{E}_R(G) \leq 1 + \sqrt{(n-1)\Big(2\sum_{v_iv_j \in E(G)} \frac{1}{d_i d_j} - 1\Big)}. \tag{4.15}$$
- If $f(d_i, d_j) = ((d_i)^p + (d_j)^p)^{\frac{1}{p}}$, then $\mathcal{A}_f(G)$ is the p-Sombor matrix $\mathcal{A}_{pSom}(G)$ [26]. We have
$$\mathcal{E}_{pSom}(G) \leq \frac{2^{(1+\frac{1}{p})}\delta m}{n} + \sqrt{(n-1)\Big(2^{(1+\frac{2}{p})}m\Delta^2 - (\frac{2^{(1+\frac{1}{p})}\delta m}{n})^2\Big)}, \tag{4.16}$$
where n and m are the order and size of the graph G, respectively.

The inequality (4.12) was proved in [11], inequality (4.13) was proved in [14], inequality (4.14) was proved in [41], inequality (4.15) was proved in [2] and [9] and inequality (4.16) was proved in [26].

Now we consider the bipartite graph case of Theorem 4.1.

Theorem 4.7 *Let G be a bipartite graph of order n. Suppose the multiplicity of eigenvalue 0 of the weighted adjacency matrix $\mathcal{A}_f(G)$ is not equal to 1. Then, for any real number k with the property $\gamma_1 \geq k \geq \gamma_n$, we have*

$$\mathcal{E}_f(G) \leq 2k + \sqrt{(n-2)\big(tr(\mathcal{A}_f^2(G)) - 2k^2\big)}.$$

Proof If G is a bipartite graph, from Theorem 3.2, we know that the eigenvalues of the weighted adjacency matrix $\mathcal{A}_f(G)$ are symmetric with respect to the origin. Thus $\gamma_1 = \gamma_2$. Using the Cauchy-Schwarz inequality, we have

$$\gamma_3 + \gamma_4 + \cdots + \gamma_n \leq \sqrt{(n-2)(\gamma_3^2 + \gamma_4^2 + \cdots + \gamma_n^2)}.$$

Because $\gamma_1^2 + \gamma_2^2 + \cdots + \gamma_n^2 = tr(\mathcal{A}_f^2(G))$, it follows that

$$\mathcal{E}_f(G) = \gamma_1 + \gamma_2 + \cdots + \gamma_n \leq 2\gamma_1 + \sqrt{(n-2)(tr(\mathcal{A}_f^2(G)) - 2\gamma_1^2)}.$$

Since the multiplicity of eigenvalue 0 of the weighted adjacency matrix $\mathcal{A}_f(G)$ is not equal to 1, we have $\gamma_n = \gamma_{n-1}$. Similarly, we can obtain

$$\mathcal{E}_f(G) \leq 2\gamma_n + \sqrt{(n-2)(tr(\mathcal{A}_f^2(G)) - 2\gamma_n^2)}.$$

Now, consider the function

$$f(x) = 2x + \sqrt{(n-2)(tr(\mathcal{A}_f^2(G)) - 2x^2)}.$$

It is easily seen that $f(x)$ is increasing in the interval $[0, \sqrt{\frac{tr(\mathcal{A}_f^2(G))}{n}})$ and decreasing in the interval $(\sqrt{\frac{tr(\mathcal{A}_f^2(G))}{n}}, \gamma_1)$. Therefore, for any real number k with the property $\gamma_1 \geq k \geq \gamma_n$, we have

$$\mathcal{E}_f(G) \leq 2k + \sqrt{(n-2)\big(tr(\mathcal{A}_f^2(G)) - 2k^2\big)}.$$

This proves the result. □

From the proof of Theorem 4.7, for a bipartite graph G we have in general

$$\mathcal{E}_f(G) \leq 2\gamma_1 + \sqrt{(n-2)(tr(\mathcal{A}_f^2(G)) - 2\gamma_1^2)}.$$

In addition, by using Theorem 4.7 we can obtain the following two results.

Corollary 4.8 *Let G be a bipartite graph of order n. If the multiplicity of eigenvalue 0 of the weighted adjacency matrix $\mathcal{A}_f(G)$ is not equal to 1, then*

$$\mathcal{E}_f(G) \leq min\left\{2\gamma_1 + \sqrt{(n-2)(tr(\mathcal{A}_f^2(G)) - 2\gamma_1^2)}, 2\gamma_n + \sqrt{(n-2)(tr(\mathcal{A}_f^2(G)) - 2\gamma_n^2)}\right\}.$$

Theorem 4.9 *Let G be a nonempty bipartite graph of order $n \geq 2$ and the edge-weight $f(d_i, d_j) > 0$ for any edge $v_iv_j \in E(G)$. Suppose the multiplicity of eigenvalue 0 of the weighted adjacency matrix $\mathcal{A}_f(G)$ is not equal to 1. If $2TI(G) \geq n\gamma_n$, then*

$$\mathcal{E}_f(G) \leq \frac{4TI(G)}{n} + \sqrt{(n-2)\left(tr(\mathcal{A}_f^2(G)) - \frac{8(TI(G))^2}{n^2}\right)}.$$

Moreover, the equality holds if and only if $G = \frac{n}{2}K_2$, or $G = K_{\frac{n}{2},\frac{n}{2}}$, or a graph G with four distinct eigenvalues $\frac{2TI(G)}{n}$, $-\frac{2TI(G)}{n}$, $\sqrt{\frac{tr(\mathcal{A}_f^2(G)) - \frac{8(TI(G))^2}{n^2}}{n-2}}$ and $-\sqrt{\frac{tr(\mathcal{A}_f^2(G)) - \frac{8(TI(G))^2}{n^2}}{n-2}}$.

Proof Firstly, by Theorem 3.3 we have

$$\lambda_1 \geq \frac{2TI(G)}{n}.$$

If $\frac{2TI(G)}{n} \geq \gamma_n$, then from Theorem 4.7 we have

$$\mathcal{E}_f(G) \leq \frac{4TI(G)}{n} + \sqrt{(n-2)\left(tr(\mathcal{A}_f^2(G)) - \frac{8(TI(G))^2}{n^2}\right)}.$$

Hence, the bound follows.

Now, let us consider what happens when the equality holds. If $\lambda_1 = \frac{2TI(G)}{n}$, then the weighted adjacency matrix $\mathcal{A}_f(G)$ has a constant row sum. Since the equality must also hold in the Cauchy-Schwarz inequality, we have

$$\gamma_i = \sqrt{\frac{tr(\mathcal{A}_f^2(G)) - \frac{8(TI(G))^2}{n^2}}{n-2}},$$

for $3 \leq i \leq n$. Noting that $G \neq \overline{K_n}$ and $f(d_i, d_j) > 0$ for any edge $v_iv_j \in E(G)$, we have $\gamma_1 \neq 0$. Thus, we can reduce it into three possibilities: either G has two eigenvalues with equal absolute values, in which case G must be equal to $\frac{n}{2}K_2$; or G has three distinct eigenvalues, that is, $\gamma_i = 0$ for $3 \leq i \leq n$, and so from Lemma 2.5, $G = K_{\frac{n}{2},\frac{n}{2}}$; or G has four distinct eigenvalues $\frac{2TI(G)}{n}$, $-\frac{2TI(G)}{n}$, $\sqrt{\frac{tr(\mathcal{A}_f^2(G))-\frac{8(TI(G))^2}{n^2}}{n-2}}$ and $-\sqrt{\frac{tr(\mathcal{A}_f^2(G))-\frac{8(TI(G))^2}{n^2}}{n-2}}$.

Conversely, it is not difficult to verify that the upper bound is achieved by $\frac{n}{2}K_2$, $K_{\frac{n}{2},\frac{n}{2}}$ and a graph G with four distinct eigenvalues $\sqrt{\frac{tr(\mathcal{A}_f^2(G))-\frac{8(TI(G))^2}{n^2}}{n-2}}$, $-\sqrt{\frac{tr(\mathcal{A}_f^2(G))-\frac{8(TI(G))^2}{n^2}}{n-2}}$, $\frac{2TI(G)}{n}$ and $-\frac{2TI(G)}{n}$. The proof is thus complete. □

Remark 4.10 Using the inequality in Theorem 4.9, by appropriate choices of $f(d_i, d_j)$ we obtain the following inequalities for some particular degree-based graph energies of a bipartite graph G.

- If $f(d_i, d_j) = \sqrt{d_i^2 + d_j^2}$, then $\mathcal{A}_f(G)$ is the Sombor matrix $\mathcal{A}_{Som}(G)$ [11]. We have

$$\mathcal{E}_{Som}(G) \leq \frac{4SO(G)}{n} + \sqrt{(n-2)\left(2F(G) - 2(\frac{2SO(G)}{n})^2\right)}. \quad (4.17)$$

- If $f(d_i, d_j) = \sqrt{d_i + d_j}$, then $\mathcal{A}_f(G)$ is the Nirmala matrix $\mathcal{A}_{Nir}(G)$ [14]. We have

$$\mathcal{E}_{Nir}(G) \leq \frac{4N(G)}{n} + \sqrt{2(n-2)\left(Zag1(G) - 2(\frac{2N(G)}{n})^2\right)}. \quad (4.18)$$

- If $f(d_i, d_j) = \frac{1}{\sqrt{d_id_j}}$, then $\mathcal{A}_f(G)$ is the Randić matrix $\mathcal{A}_R(G)$ [2]. We have

$$\mathcal{E}_R(G) \leq 2 + \sqrt{(n-2)\left(2\sum_{v_iv_j \in E(G)} \frac{1}{d_id_j} - 2\right)}. \quad (4.19)$$

The inequality (4.17) was proved in [11], inequality (4.18) was proved in [14], and inequality (4.19) was proved in [2].

The next result gives a lower bound on the energy $\mathcal{E}_f(G)$ of the weighted adjacency matrix $\mathcal{A}_f(G)$ in terms of the trace of $\mathcal{A}_f^2(G)$ and the least weighted adjacency eigenvalue λ_n.

Theorem 4.11 *Let G be a connected graph of order $n \geq 2$ and the edge-weight $f(d_i, d_j) > 0$ for any edge $v_iv_j \in E(G)$. Then*

$$\mathcal{E}_f(G) \geq |\lambda_n| + \sqrt{2tr(\mathcal{A}_f^2(G)) - 3\lambda_n^2}.$$

The lower bound is achieved by $G = K_{s,t}$.

Proof Recall that $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ are the absolute values of eigenvalues λ_i, $i = 1, 2, \ldots, n$, given in a decreasing order. We assume that $|\lambda_n| = \gamma_{i_0}$ for some i_0 with $2 \leq i_0 \leq n$. Since the trace of the weighted adjacency matrix $\mathcal{A}_f(G)$ is zero, we have

$$\lambda_n^2 = \gamma_{i_0}^2 = \left(\sum_{i=1}^{n-1} \lambda_i\right)^2 = \sum_{i=1}^{n-1} \lambda_i^2 + \sum_{\substack{1 \leq i,j \leq n-1 \\ i \neq j}} \lambda_i \lambda_j.$$

Note that

$$\left(\sum_{i=1, i\neq i_0}^{n} \gamma_i\right)^2 = \sum_{i=1, i\neq i_0}^{n} \gamma_i^2 + \sum_{\substack{1 \leq i,j \leq n \\ i \neq j \neq i_0}} \gamma_i\gamma_j = \sum_{i=1, i\neq i_0}^{n} \gamma_i^2 + \sum_{\substack{1 \leq i,j \leq n-1 \\ i \neq j}} |\lambda_i||\lambda_j|.$$

Now, we obtain

$$\begin{aligned}
(\mathcal{E}_f(G) - |\lambda_n|)^2 = \left(\sum_{i=1, i\neq i_0}^{n} \gamma_i\right)^2 &= \sum_{i=1, i\neq i_0}^{n} \gamma_i^2 + \sum_{\substack{1 \leq i,j \leq n-1 \\ i \neq j}} |\lambda_i||\lambda_j| \\
&\geq \sum_{i=1, i\neq i_0}^{n} \gamma_i^2 + \left|\sum_{\substack{1 \leq i,j \leq n-1 \\ i \neq j}} \lambda_i\lambda_j\right| \\
&= \sum_{i=1, i\neq i_0}^{n} \gamma_i^2 + \left|\lambda_n^2 - \sum_{i=1}^{n-1} \lambda_i^2\right| \\
&= \sum_{i=1}^{n-1} \lambda_i^2 + \left|\lambda_n^2 - \sum_{i=1}^{n-1} \lambda_i^2\right|.
\end{aligned}$$

Since G is a connected graph of order $n \geq 2$ and the edge-weight $f(d_i, d_j) > 0$ for any edge $v_iv_j \in E(G)$, from Lemma 2.3 we have $\lambda_1 \geq |\lambda_n|$, that is, $\lambda_n^2 \leq \sum\limits_{i=1}^{n-1} \lambda_i^2$. By view of this, we have

$$(\mathcal{E}_f(G) - |\lambda_n|)^2 \geq 2\sum_{i=1}^{n} \lambda_i^2 - 3\lambda_n^2.$$

Thus the bound follows.

If $G = K_{s,t}$, then $\mathcal{E}_f(G) = 2\sqrt{st}f(s,t)$. Since $tr(\mathcal{A}_f^2(G)) = 2stf^2(s,t)$ and $|\lambda_n| = \sqrt{st}f(s,t)$, we have $\sqrt{st}f(s,t) + \sqrt{4stf^2(s,t) - 3stf^2(s,t)} = 2\sqrt{st}f(s,t)$. This completes the proof. □

Remark 4.12 From the result in Theorem 4.11, by appropriate choices of $f(d_i, d_j)$ we deduce the following inequalities for some particular degree-based graph energies of a graph G.

- If $f(d_i, d_j) = \frac{1}{\sqrt{d_i^2+d_j^2}}$, then $\mathcal{A}_f(G)$ is the modified Sombor matrix $\mathcal{A}_{mSom}(G)$ [20]. We have

$$\mathcal{E}_{mSom}(G) \geq |\lambda_n| + \sqrt{4B(G) - 3\lambda_n^2}. \tag{4.20}$$

- If $f(d_i, d_j) = \frac{d_i d_j}{d_i+d_j}$, then $\mathcal{A}_f(G)$ is the inverse sum indeg matrix $\mathcal{A}_{isi}(G)$ [37]. We have

$$\mathcal{E}_{isi}(G) \geq |\lambda_n| + \sqrt{m\delta^2 - 3\lambda_n^2}. \tag{4.21}$$

- If $f(d_i, d_j) = d_i + d_j$, then $\mathcal{A}_f(G)$ is the first Zagreb matrix $\mathcal{A}_{Zag1}(G)$ [30]. We have the first Zagreb energy

$$\mathcal{E}_{Zag1}(G) \geq |\lambda_n| + \sqrt{4HM(G) - 3\lambda_n^2}, \tag{4.22}$$

  where $HM(G)$ is one half of the second spectral moment of $\mathcal{A}_{Zag1}(G)$.

- If $f(d_i, d_j) = d_i d_j$, then $\mathcal{A}_f(G)$ is the second Zagreb matrix $\mathcal{A}_{Zag2}(G)$ [30]. We have the second Zagreb energy

$$\mathcal{E}_{Zag2}(G) \geq |\lambda_n| + \sqrt{4R_2(G) - 3\lambda_n^2}, \tag{4.23}$$

  where $R_2(G)$ is one half of the second spectral moment of $\mathcal{A}_{Zag2}(G)$.

- If $f(d_i, d_j) = \frac{1}{2}(\frac{d_i}{d_j} + \frac{d_j}{d_i})$, then $\mathcal{A}_f(G)$ is the extended adjacency matrix $\mathcal{A}_{ext}(G)$ [36]. We have

$$\mathcal{E}_{ext}(G) \geq |\lambda_n| + \sqrt{\sum_{v_i v_j \in E(G)} \left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right)^2 - 3\lambda_n^2}. \tag{4.24}$$

- If $f(d_i, d_j) = \frac{d_i+d_j}{d_i d_j}$, then $\mathcal{A}_f(G)$ is the degree-based matrix $\mathcal{A}_{db}(G)$ [35]. We have the degree based energy

$$\mathcal{E}_{db}(G) \geq |\lambda_n| + \sqrt{4 \sum_{v_i v_j \in E(G)} \left(\frac{d_i + d_j}{d_i d_j}\right)^2 - 3\lambda_n^2}. \tag{4.25}$$

The inequality (4.20) was proved in [41], inequality (4.21) was proved in [25], inequality (4.22) was proved in [31], inequality (4.23) was proved in [31], inequality (4.24) was proved in [6] and inequality (4.25) was proved in [35].

5 Concluding remark

In this paper, we investigate the weighted adjacency matrix $\mathcal{A}_f(G)$ with degree-based edge-weight function $f(x, y)$, where $f(x, y)$ is a real symmetric function. We do it not by one by one concrete function, say Zagreb edge-weight functions $x + y$ and xy, Randić edge-weight function $\sqrt{\frac{1}{xy}}$, ABC edge-weight function $\sqrt{\frac{x+y-2}{xy}}$, etc. On the contrary, by looking at $f(x, y)$ as an arbitrary function we use a unified method to obtain spectral bounds and properties of $\mathcal{A}_f(G)$. In doing so, we give a unified proof for many known spectral bounds and bounds for various types of graph energies.

Conflict of Interest The authors declare no conflict of interest.

Acknowledgements We thank the referees for their time and comments.

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