Unified approach for spectral properties of weighted adjacency matrices for graphs with degree-based edge-weights*

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Abstract

Let G be a graph and d_i be the degree of a vertex v_i in G. For a symmetric real function $f(x, y)$, one can get an edge-weighted graph in such a way that for each edge v_iv_j of G, the weight of v_iv_j is assigned by $f(d_i, d_j)$. Hence, we have a weighted adjacency matrix $A_f(G)$ of G, in which the ij-entry is equal to $f(d_i, d_j)$ if $v_iv_j \in E(G)$ and 0 otherwise. In this paper, we use a unified approach to deal with the spectral properties of $A_f(G)$ for $f(x, y)$ to be the functions of graphical or topological function-indices. Firstly, we obtain uniform interlacing inequalities for the weighted adjacency eigenvalues. For the edge-weight functions defined by almost a half of popularly used topological indices, it can be shown that our inequalities cannot be improved. Secondly, we establish a uniform equivalent condition for a connected graph G to have m distinct weighted adjacency eigenvalues. As an application, a combinatorial characterization for a graph to have two and three distinct weighted adjacency eigenvalues are presented, respectively. Moreover, bipartite graphs and unicyclic graphs with three distinct weighted adjacency eigenvalues are characterized. This paper attempts to unify the spectral study for weighted adjacency matrices of graphs with degree-based edge-weights.

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. For notation and terminology not defined here, we refer to [8]. Let $G = (V(G), E(G))$ be a graph with vertex-set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge-set $E(G)$. We use $G + e$ to denote the graph obtained from G by adding an edge e. Denote by d_i the degree of a vertex v_i in G. $N_G(v_i)$ denotes the set of neighbors of a vertex v_i in G, and let $N_G[v_i] = N_G(v_i) \cup \{v_i\}$. If the vertex-set $V(G)$ of G admits a partition into two classes and the two ends of its every edge are in different classes, then G is a bipartite graph. The complete bipartite graph denoted by $K_{s,t}$, where $s + t = n$. A unicyclic graph is a connected graph such that the number of its vertices is equal to the number of its edges. As usual, we denote by K_n, P_n, C_n and S_n, respectively, the complete graph, the path, the cycle and the star of order n.

In molecular graph theory, graphical or topological indices are used to represent the structural properties of molecular graphs. The general form of these indices is $\sum\limits_{v_iv_j \in E(G)} f(d_i, d_j)$, where $f(x, y)$ is a symmetric real function, called the edge-weight function, and $f(d_i, d_j)$ is the edge-weight of an edge v_iv_j of G. Gutman [12] collected many significant degree-based indices; see them in Table 1. Rada [23] gave the exponentials of the well-studied degree-based indices to study the discrimination property; see them in Table 2. If we sum up the edge-weights in a molecular graph with edge-weights defined by the function $f(x, y)$, then each index maps a molecular graph into a single number.

In spectral graph theory, there are many researches on matrices associated with a graph G. Because matrices are connected with the structure of a molecular graph with edge-weights separately on its pairs of adjacent vertices, using a matrix to represent graph keeps much more structural information than just a single number, the value of an index. So, the algebraic properties of these structural matrices need intensively study. In 2015, one of the authors Li first proposed this idea in [17]. Up to now, there are various studies on matrices defined by topological indices from algebraic viewpoint, such as the misbalance degree (Albertson) matrix [10], inverse sum indeg matrix [1], ABC matrix [5], Radić matrix [19], AG matrix [24], Zagreb matrix [22] and GA matrix [25]. From the study of these matrices, a lot of interesting properties of graphs are reflected.

In 2018, Das et al. [9] gave the definition of the weighted adjacency matrix for a graph with degree-based edge-weights formally. Let $A_f(G)$ be the weighted adjacency

Edge-weight function f(x,y)	**The corresponding index**
$x+y$	first Zagreb index
xy	second Zagreb index
$(x+y)^2$	first hyper-Zagreb index
$(xy)^2$	second hyper-Zagreb index
$x^{-3}+y^{-3}$	modified first Zagreb index
$\|x-y\|$	Albertson index
$(x/y+y/x)/2$	extended index
$(x-y)^2$	sigma index
$1/\sqrt{xy}$	Randić index
$\sqrt{xy}$	reciprocal Randić index
$1/\sqrt{x+y}$	sum-connectivity index
$\sqrt{x+y}$	reciprocal sum-connectivity index
$2/(x+y)$	harmonic index
$\sqrt{(x+y-2)/(xy)}$	atom-bond-connectivity (ABC) index
$(xy/(x+y-2))^3$	augmented Zagreb index
x^2+y^2	forgotten index
$x^{-2}+y^{-2}$	inverse degree
$2\sqrt{xy}/(x+y)$	geometric-arithmetic (GA) index
$(x+y)/(2\sqrt{xy})$	arithmetic-geometric (AG) index
$xy/(x+y)$	inverse sum index
$x+y+xy$	first Gourava index
$(x+y)xy$	second Gourava index
$(x+y+xy)^2$	first hyper-Gourava index
$((x+y)xy)^2$	second hyper-Gourava index
$1/\sqrt{x+y+xy}$	sum-connectivity Gourava index
$\sqrt{(x+y)xy}$	product-connectivity Gourava index
$\sqrt{x^2+y^2}$	Sombor index

Table 1: Some well-studied chemical indices

Edge-weight function f(x,y)	**The corresponding index**
e^{x+y}	exponential first Zagreb index
e^{xy}	exponential second Zagreb index
$e^{1/\sqrt{xy}}$	exponential Randić index
$e^{\sqrt{(x+y-2)/(xy)}}$	exponential ABC index
$e^{2\sqrt{xy}/(x+y)}$	exponential GA index
$e^{2/(x+y)}$	exponential harmonic index
$e^{1/\sqrt{x+y}}$	exponential sum-connectivity index
$e^{(xy/(x+y-2))^3}$	exponential augmented Zagreb index

Table 2: Some well-known exponential chemical indices

matrix of a graph G with edge-weight function $f(x, y)$, whose ij-entry is

$$(A_f(G))_{ij} = \begin{cases} f(d_i, d_j) & \text{if } v_iv_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Given an index, we can define the corresponding weighted adjacency matrix of a graph edge-weighted by the edge-weight function $f(x, y)$ of the index. As we can see, although the matrix $A_f(G)$ was proposed in a general setting, matrix for a concrete graphical or topological index with function $f(x, y)$ was studied still one by one separately. This means that the introduce of the general weighted adjacency matrix $A_f(G)$ has no significance. In 2021, Li and Wang [18] studied the extremal spectral radius of the weighted adjacency matrices $A_f(G)$ for a general function $f(x, y)$. The extremal spectral radius of weighted adjacency matrices among trees were obtained, when the edge-weight function $f(x, y)$ has some functional properties. This is a good beginning of the study of spectral properties by function classification, and there are still a lot of properties of $A_f(G)$ waiting to be explored in the future when $f(x, y)$ has some functional properties. This will eventually unify the approaches for spectral properties of the weighted adjacency matrices of a graph edge-weighted by the edge-weight function $f(x, y)$ of graphical or topological indices.

The eigenvalues of $A_f(G)$ are called the *weighted adjacency eigenvalues* of a graph G with edge-weight function $f(x, y)$. Because G is undirected with a symmetric real function $f(x, y)$, $A_f(G)$ is a real symmetric matrix, and therefore all its eigenvalues are real numbers. Let

$$\lambda_{max} = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = \lambda_{min} \qquad (1.1)$$

be the weighted adjacency eigenvalues of G with order n. If $f(x,y) \equiv 1$, then the adjacency matrix of G is simply denoted by $A(G)$. For two graphs G and H, if

$$\lambda_{i+1}(A_f(H)) \leq \lambda_i(A_f(G)) \leq \lambda_i(A_f(H)) \tag{1.2}$$

for all i, then we say that the eigenvalues of $A_f(G)$ *interlace* the eigenvalues of $A_f(H)$. From the interlacing property, it clearly shows that the relation between the eigenvalues of $A_f(G)$ and $A_f(H)$. If

$$\lambda_{i+1}(A_f(H)) \leq \lambda_i(A_f(G)) \leq \lambda_{i-1}(A_f(H)) \tag{1.3}$$

for all i, we say that the eigenvalues of $A_f(G)$ and the eigenvalues of $A_f(H)$ are *compatible*. This means that $max\{\lambda_i(A_f(G)), \lambda_i(A_f(H))\} \leq min\{\lambda_{i-1}(A_f(G)), \lambda_{i-1}(A_f(H))\}$. It is easily seen that the interlacing property implies the compatible property, but the converse is not true in general. As is well-known, the interlacing property and compatible property are very popular and important properties for adjacency matrix $A(G)$. It is worth mentioning that Chudnovsky and Seymour in [7] proved that all the roots of the independence polynomial are real if G is clawfree, by using the compatible property. For a survey of interlacing results, we refer to Haemers [13]. Besides the study of the adjacency matrix $A(G)$, Butler [4] got the interlacing results for weighted graphs. In this paper, we first focus on a very familiar conclusion from [14] about the interlacing of eigenvalues as follows.

Theorem 1.1 *Let G be a graph of order n. The adjacency matrices of G and $G+e$ are denoted by $A(G)$ and $A(G+e)$, respectively. Then*

$$\lambda_{i+1}(A(G)) \leq \lambda_i(A(G+e)) \leq \lambda_{i-1}(A(G)), \qquad (2 \leq i \leq n-1).$$

From this result, we can say that the eigenvalues of $A(G)$ and the eigenvalues of $A(G+e)$ are compatible. A natural question is whether this property keeps to be true in general for weighted adjacency matrices of graphs with degree-based edge-weight function $f(x,y)$. Unfortunately, that is not the case. For example, if $f(x,y)$ is the edge-weight function for the Albertson index (see Table 1), then the weighted adjacency eigenvalues of the graphs H_1 and H_1+e in Figure 1 are $4, 0, 0, 0, -4$ and $3.6334, 1, -0.7685, -1, -2.8649$, respectively. One can easily check that $\lambda_4(A_f(H_1)) = 0 > \lambda_3(A_f(H_1+e)) = -0.7685$. Moreover, if $f(x,y)$ is the edge-weight function for the second hyper-Zagreb index, then the weighted adjacency eigenvalues of $A_f(K_4 - e)$ and $A_f(K_4)$ are $123.1090, 0, -42.1090, -81$ and $243, -81, -81, -81$, respectively. We then have $\lambda_3(A_f(K_4-e)) = -42.1090 > \lambda_2(A_f(K_4)) = -81$. One can also find that for many edge-weight functions $f(x,y)$ in Table 1, there exist graphs such

Figure 1: Graphs H_1 and $H_1 + e$.

that their weighted eigenvalues are not compatible and therefore not interlacing. In the following paragraphs, we study interlacing results for the weighted adjacency eigenvalues from the following three aspects.

(i) For the edge-weight functions $f^*(x, y) = ((x+y-\alpha)/xy)^\beta$ or the corresponding exponential forms, where $2 \leq \alpha \leq 2n-2$ and β is a real number, the interlacing result in Theorem 1.1 for weighted adjacency matrices $A_f(G)$ and $A_f(G+e)$ keeps to be true.

(ii) For all functions $f(x, y)$ and the weighted adjacency matrices $A_f(G)$ and $A_f(G+e)$, we have $\lambda_{i+2}(A_f(G)) \leq \lambda_i(A_f(G+e)) \leq \lambda_{i-2}(A_f(G))$ with $3 \leq i \leq n-2$.

(iii) For functions $f(x, y)$ of almost a half of the topological indices in Tables 1 and 2, we can not improve the gap in (ii).

Another question we are interested in this paper is the weighted adjacency matrix $A_f(G)$ which have fewer distinct eigenvalues. This problem has been extensively considered for the adjacency matrix $A(G)$, for which we refer to [3, 6, 20, 26]. In fact, for individual weighted adjacency matrix with fewer distinct eigenvalues, studies have been done one by one matrix separately, such as the ABC matrix [5], Randić matrix [19], GA matrix [21] and AG matrix [24]. In the following paragraphs, we give a unified approach for the weighted adjacency matrix $A_f(G)$ to have fewer distinct eigenvalues, which covers the weighted adjacency matrices defined by nearly all of the topological indices in Tables 1 and 2.

The structure of this paper is arranged as follows. In the next section, we introduce some necessary notation and terminology and list several previous known results that will be used in the subsequent sections. In Section 3, we first establish some interlacing results for the eigenvalues of the weighted adjacency matrices $A_f(G)$ and $A_f(G+e)$. The interlacing result in Theorem 3.3 is good enough for the weighted adjacency matrices defined by almost a half of the indices listed in Tables 1 and 2. Second, if $f(d_i, d_j) > 0$ (this is almost always the case) for each edge $v_iv_j \in E(G)$, we get an equivalent condition for a connected graph to have m ($2 \leq m \leq n$) distinct weighted adjacency eigenvalues. As an application, we give a uniform combinatorial characterization for a graph to have two and three distinct weighted adjacency eigenvalues, respectively. More generally, bipartite graphs and unicyclic graphs with three distinct weighted adjacency eigenvalues are also characterized, respectively.

2 Preliminaries

First, we have some results that will be used in the sequel. An $n \times n$ complex square matrix M is called *Hermitian* if $M^* = M$, where M^* is the conjugate transpose of M. In 1912, Weyl [27] stated a very useful result.

Lemma 2.1 ([27]) *Let M,N be Hermitian matrices of order n, and let the respective eigenvalues of M, N, and $M+N$ be $\{\rho_i(M)\}_{i=1}^n$, $\{\rho_i(N)\}_{i=1}^n$, and $\{\rho_i(M+N)\}_{i=1}^n$, each algebraically ordered as in (1.1). Then*

$$\rho_i(M+N) \leq \rho_j(M) + \rho_{i-j+1}(N), \qquad (1 \leq j \leq i \leq n). \tag{2.1}$$

Also,

$$\rho_j(M) + \rho_{i-j+n}(N) \leq \rho_i(M+N), \qquad (1 \leq i \leq j \leq n). \tag{2.2}$$

This is the root of a great many inequalities involving the sum of two Hermitian matrices and their eigenvalues, for which we refer to Section 3 of Chapter 4 in [15].

Suppose M is an $n \times n$ real matrix. If $P(M) = 0$, then the polynomial $P(t)$ is said to *annihilate* M. The Hamilton-Caylay theorem guarantees that for a matrix M there is a monic polynomial $P_M(t)$ of degree n (the characteristic polynomial) such that $P_M(t) = 0$. The unique monic polynomial $p(t)$ of minimum degree that annihilates M is called the *minimal polynomial* of M.

Lemma 2.2 ([16]) *Let M be an $n \times n$ real symmetric matrix. Then the minimal polynomial of M can be written as*

$$p(t) = \prod_{i=1}^{m} (t - t_i),$$

where t_1, $t_2, \ldots, t_m$ are the distinct eigenvalues of M.

An $n \times n$ real matrix M is called *nonnegative* if its every entry is nonnegative. We say that M is *irreducible* if it is not the 1×1 matrix $[0]$ and if there does not exist a permutation matrix N (a matrix with 1 in every row and column, and 0 for all other entries) such that

$$NMN^{-1} = \begin{pmatrix} M_{11} & M_{12} \\ \mathbf{0} & M_{22} \end{pmatrix},$$

where $\mathbf{0}$ denotes the zero matrix, M_{11} and M_{22} are square matrices of size greater than zero. We now state the famous Perron–Frobenius theorem.

Lemma 2.3 ([8]) *Let M be a nonnegative irreducible square matrix. Then the largest eigenvalue $\rho_1(M)$ is simple, with a corresponding eigenvector whose entries are all positive.*

Suppose the rows and columns of

$$M = \begin{pmatrix} M_{11} & M_{12} & \dots & M_{1m} \\ M_{21} & M_{22} & \dots & M_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m1} & M_{m2} & \dots & M_{mm} \end{pmatrix}$$

are partitioned according to a partitioning $X_1, X_2, \dots, X_m$ of $\{1, 2, \dots, n\}$. The *quotient matrix* is the matrix B whose entries are the average row sums of the blocks of M. The partition is called equitable if each block M_{ij} of M has a constant row sum.

Lemma 2.4 ([28]) *Let B be the equitable quotient matrix of M. The respective eigenvalues of B and M be $\{\rho_i(B)\}_{i=1}^m$ and $\{\rho_i(M)\}_{i=1}^n$. Then*

$$\{\rho_1(B), \rho_2(B), \dots, \rho_m(B)\} \subseteq \{\rho_1(M), \rho_2(M), \dots, \rho_n(M)\}.$$

Lemma 2.5 ([15]) *Let M, N be nonnegative matrices of order n. If $C = M - N$ is a nonnegative matrix, then $\rho_1(M) \geq \rho_1(N)$.*

3 Main results

In this section, we first investigate the interlacing inequalities for weighted adjacency eigenvalues. Because the eigenvalues of $A(G)$ and the eigenvalues of $A(G + e)$ are compatible, we study if the same property holds for some general weighted adjacency matrix $A_f(G)$.

Nowadays, the study of graphical or topological indices is not only limited to a single index. For instance, to investigate the discrimination property, Rada [23] introduced the exponentials of the best known degree-based topological indices. From one index, one can get some similar indices. In 1998, Bollobás and Erdös [2] generalized the Randić index to general Randić index, which is defined as $R_\alpha = \sum_{v_iv_j \in E(G)} (d_id_j)^\alpha$, where α is a nonzero real number. In 2010, Zhou and Trinajstić [29] gave the general sum-connectivity index $\chi_\alpha = \sum_{v_iv_j \in E(G)} (d_i + d_j)^\alpha$. In the same year, Furtula et al. [11] introduced the generalized ABC index $ABC_\alpha = \sum_{v_iv_j \in E(G)} (\frac{d_i+d_j-2}{d_id_j})^\alpha$. Here we consider edge-weight function $f^*(x, y) = ((x + y - \alpha)/xy)^\beta$, where $2 \leq \alpha \leq 2n - 2$ and β is a real number. One can see that $f^*(x, y)$ has some good functional properties. For example, when $x = \alpha$, we get $f^*(\alpha, y) = (1/\alpha)^\beta$. This means that $f^*(\alpha, y)$ is a constant, no matter what y is. Besides, if $\beta > 0$ and the two numbers x, y satisfy $x + y = \alpha$, then $f^*(x, y) = 0$. Now, the first theorem of this section is given below.

Theorem 3.1 *Let G be a graph of order n and the edge-weight function $f^*(x, y) = ((x + y - \alpha)/xy)^\beta$, where $2 \leq \alpha \leq 2n - 2$ and β is a real number. Adding an edge e between v_1 and v_2 yields a graph $G + e$. For each vertex $v_i \in N_G(v_2)$, the degree of v_i is equal to α. Then*

$$\lambda_{i+1}(A_{f^*}(G)) \leq \lambda_i(A_{f^*}(G + e)) \leq \lambda_{i-1}(A_{f^*}(G)), \qquad (2 \leq i \leq n - 1). \tag{3.1}$$

Proof. If we add an edge e between v_1 and v_2, then the degrees of v_1 and v_2 in the graph G will be changed. By properly labelling the vertices of G, we can get a matrix $B = A_{f^*}(G + e) - A_{f^*}(G)$ to be written as follows:

$$B = \begin{pmatrix} 0 & x_1 & x_2 & \dots & x_{n-1} \\ x_1 & 0 & y_1 & \dots & y_{n-2} \\ x_2 & y_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n-1} & y_{n-2} & 0 & \dots & 0 \end{pmatrix},$$

where x_i and y_j are real numbers, $1 \leq i \leq n-1$ and $1 \leq j \leq n-2$. Since the degrees of the neighbors of v_2 are α, we have $f^*(d_2, d_j) = f^*(d_2 + 1, d_j)$ for each vertex v_j adjacent to v_2. Hence we claim that $y_j = 0$ for $1 \leq j \leq n - 2$. By calculating, we have $det(\rho I - B) = \rho^{n-2}(\rho^2 - (x_1^2 + x_2^2 + \cdots + x_{n-1}^2))$, and so the eigenvalues of B are $\rho_n(B) = -\sqrt{x_1^2 + x_2^2 + \cdots + x_{n-1}^2}$, $\rho_1(B) = \sqrt{x_1^2 + x_2^2 + \cdots + x_{n-1}^2}$ and $\rho_2(B) = \rho_3(B) = \cdots = \rho_{n-1}(B) = 0$.

Because $A_{f^*}(G)$ and B are real symmetric matrices, by Lemma 2.1 we can get

$$\lambda_{i+1}(A_{f^*}(G)) \leq \lambda_i(A_{f^*}(G + e)) \leq \lambda_{i-1}(A_{f^*}(G)).$$

Hence the theorem holds. □

It is not difficult to see that Theorem 3.1 is suitable for weighted adjacency matrices $A_f(G)$ defined by the edge-weight functions $f(x,y)$ from the ABC index, augmented Zagreb index, exponential ABC index, exponential augmented Zagreb index and generalized ABC index.

We already knew that, for two graphs G and H, if the eigenvalues of $A_f(G)$ interlace the eigenvalues of $A_f(H)$, then the eigenvalues of $A_f(G)$ and $A_f(H)$ are compatible. But the converse may not be true. For adjacency matrix, the eigenvalues of $A(G)$ can not interlace the eigenvalues of $A(G+e)$, that is, Inequality (1.2) does not hold for all i, because there is no graph G such that $A(G)$ and $A(G+e)$ with the same eigenvalues. However, for the weighted adjacency matrices, we can get a theorem as below.

Theorem 3.2 *Let G be a graph of order n and the edge-weight function $f^*(x,y) = ((x+y-\alpha)/xy)^\beta$, where $2 \le \alpha \le 2n-2$ and β is a real number. Adding an edge e between v_1 and v_2 yields graph $G+e$. If the following statements hold:*

(i) $\beta > 0$;

(ii) $d_1 + d_2 = \alpha - 2$;

(iii) $\forall\ v_i \in N_G(v_1) \cup N_G(v_2),\ d_i = \alpha,$

then

$$\lambda_{i+1}(A_{f^*}(G)) \le \lambda_i(A_{f^*}(G+e)) \le \lambda_i(A_{f^*}(G)), \qquad (1 \le i \le n-1). \tag{3.2}$$

Moreover, the eigenvalues of $A_{f^*}(G)$ are the same as the eigenvalues of $A_{f^*}(G+e)$.

Proof. From the proof of Theorem 3.1, we can conclude that x_i and y_j are equal to zero for $2 \le i \le n-1$ and $1 \le j \le n-2$, since the degrees of the neighbors of v_1 and v_2 are α. In addition, when $\beta > 0$ and $d_1+d_2+2 = \alpha$, we have $x_1 = f^*(d_1+1, d_2+1) = 0$. Hence, matrix $B = A_{f^*}(G+e) - A_{f^*}(G) = \mathbf{0}$. The required result is thus obtained. □

Similarly, Theorem 3.2 is true when the edge-weight function $f(x,y)$ is the ABC index, augmented Zagreb index or generalized ABC index.

From Theorems 3.1 and 3.2, the degree of the vertex $v_i \in N_G[v_1] \cup N_G[v_2]$ has some restrictions. In other words, when adding an edge e to a graph G, we need to consider where to insert it to get the desired result. If we do not care about where to add an edge e, and what the edge-weight function is, then the following conclusion can be obtained.

Theorem 3.3 *Let G be a graph of order n. Then for any symmetric real function $f(x,y)$,*

$$\lambda_{i+2}(A_f(G)) \leq \lambda_i(A_f(G+e)) \leq \lambda_{i-2}(A_f(G)), \qquad (3 \leq i \leq n-2). \tag{3.3}$$

Proof. Let $B = A_f(G+e) - A_f(G)$ be the matrix written as in the proof of Theorem 3.1. We assume that $B = B_1 + B_2$, where

$$B_1 = \begin{pmatrix} 0 & x_1 & x_2 & \dots & x_{n-1} \\ x_1 & 0 & 0 & \dots & 0 \\ x_2 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n-1} & 0 & 0 & \dots & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & y_1 & \dots & y_{n-2} \\ 0 & y_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & y_{n-2} & 0 & \dots & 0 \end{pmatrix}.$$

By calculating, we have $det(\rho I - B_1) = \rho^{n-2}(\rho^2 - (x_1^2 + x_2^2 + \cdots + x_{n-1}^2))$, and so the eigenvalues of B_1 are $\rho_n(B_1) = -\sqrt{x_1^2 + x_2^2 + \cdots + x_{n-1}^2}$, $\rho_1(B_1) = \sqrt{x_1^2 + x_2^2 + \cdots + x_{n-1}^2}$ and $\rho_2(B_1) = \rho_3(B_1) = \cdots = \rho_{n-1}(B_1) = 0$. In the same way, we can get that the eigenvalues of B_2 are $\rho_n(B_2) = -\sqrt{y_1^2 + y_2^2 + \cdots + y_{n-2}^2}$, $\rho_1(B_2) = \sqrt{y_1^2 + y_2^2 + \cdots + y_{n-2}^2}$ and $\rho_2(B_2) = \rho_3(B_2) = \cdots = \rho_{n-1}(B_2) = 0$.

Since $A_f(G)$ and B are real symmetric matrices, using Lemma 2.1 we can obtain that

$$\lambda_{i+2}(A_f(G)) \leq \lambda_{i+1}(A_f(G)+B_1) \leq \lambda_i(A_f(G+e)) \leq \lambda_{i-1}(A_f(G)+B_1) \leq \lambda_{i-2}(A_f(G)).$$

The proof is thus complete. □

Theorem 3.3 holds for all weighted adjacency matrices $A_f(G)$ and $A_f(G+e)$. In addition, this result shows that in the interval $[\lambda_{i+2}(A_f(G)), \lambda_{i-2}(A_f(G))]$ we can find the eigenvalue $\lambda_i(A_f(G+e))$, that is, the change between $\lambda_i(A_f(G+e))$ and $\lambda_i(A_f(G))$ may be at a small value when an edge e is added in a graph G. A vertex of degree 0 is called *isolated*. Using Theorem 3.3, the following result can be obtained directly.

Corollary 3.4 *Let G be a graph of order n, and v_2 be an isolated vertex of G. Adding an edge e between v_1 and v_2 yields the graph $G+e$. Then for any symmetric real function $f(x,y)$,*

$$\lambda_{i+1}(A_f(G)) \leq \lambda_i(A_f(G+e)) \leq \lambda_{i-1}(A_f(G)), \qquad (2 \leq i \leq n-1).$$

For the weighted adjacency matrices $A_f(G)$ and $A_f(G+e)$, Corollary 3.4 is always true. However, in Theorem 3.3 we are unable to improve the gap for many degree-based edge-weighted functions $f(x,y)$ when we add the edge e to G arbitrarily. The following result is given to support our point of view.

Theorem 3.5 *Let G be a graph of order n and the edge-weighted function $f(x,y)$ be a symmetric polynomial with nonnegative coefficients and zero constant term. Then there exist graphs G and $G+e$ that do not have the property that*

$$\lambda_{i+1}(A_f(G)) \leq \lambda_i(A_f(G+e)) \leq \lambda_{i-1}(A_f(G)), \qquad (2 \leq i \leq n-1).$$

Proof. Firstly, we assume that $G+e = K_n$. Since K_n is a regular graph, it is easy to see that $A_f(K_n) = f(n-1,n-1)A(K_n)$, and hence $\lambda_i(A_f(K_n)) = f(n-1,n-1)\lambda_i(A(K_n))$ for $1 \leq i \leq n$. That is, $\lambda_1(A_f(K_n)) = f(n-1,n-1)(n-1)$, $\lambda_2(A_f(K_n)) = \lambda_3(A_f(K_n)) = \cdots = \lambda_n(A_f(K_n)) = -f(n-1,n-1)$. Now, we consider the eigenvalues of $A_f(G)$, where $G = K_n - e$. By properly labelling the vertices of G, we can get

$$A_f(G) = \begin{pmatrix} 0 & 0 & f(n-1,n-2) & \dots & f(n-1,n-2) \\ 0 & 0 & f(n-1,n-2) & \dots & f(n-1,n-2) \\ f(n-1,n-2) & f(n-1,n-2) & 0 & \dots & f(n-1,n-1) \\ f(n-1,n-2) & f(n-1,n-2) & f(n-1,n-1) & \dots & f(n-1,n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f(n-1,n-2) & f(n-1,n-2) & f(n-1,n-1) & \dots & 0 \end{pmatrix}.$$

For $1 \leq i \leq n-3$, let $x_i = (0,0,1,x_{i4},x_{i5},\dots,x_{in})^T$ be the vector such that $x_{ij} = -1$ if $j-i = 3$ and 0 otherwise. Suppose $x_1, x_2, \dots, x_{n-3}$ are linearly dependent vectors. Then there exist real numbers $c_1, c_2, \dots, c_{n-3}$ not all zero, such that

$$c_1x_1 + c_2x_2 + \cdots + c_{n-3}x_{n-3} = o,$$

where o denotes the zero vector. This implies that

$$(0,0,\sum_{i=1}^{n-3} c_i, -c_1, -c_2, \cdots, -c_{n-3}) = o,$$

and it follows that $c_1 = c_2 = \cdots = c_{n-3} = 0$. Therefore, the vectors $x_1, x_2, \dots, x_{n-3}$ can not be linearly dependent. It is not difficult to verify that

$$A_f(G)x_i = -f(n-1,n-1)x_i, \qquad (1 \leq i \leq n-3).$$

So, $x_1, x_2, \dots, x_{n-3}$ are the eigenvectors of $A_f(G)$ corresponding to the eigenvalue $-f(n-1,n-1)$. From a partition of $V(G)$, the remaining eigenvalues of $A_f(G)$ can be obtained. Let $X_1 = \{v_1\}$, $X_2 = \{v_2\}$ and $X_3 = \{v_3, v_4, \dots, v_n\}$. Then the quotient matrix B of the matrix $A_f(G)$ is

$$B = \begin{pmatrix} 0 & 0 & (n-2)f(n-1,n-2) \\ 0 & 0 & (n-2)f(n-1,n-2) \\ f(n-1,n-2) & f(n-1,n-2) & (n-3)f(n-1,n-1) \end{pmatrix}.$$

Because each block of $A_f(G)$ has a constant row sum, this partition is equitable. By Lemma 2.4, the eigenvalues of B are the eigenvalues of $A_f(G)$. By calculating, we have $det(\rho I - B) = \rho(\rho^2 - (n-3)f(n-1,n-1)\rho - 2f^2(n-1,n-2)(n-2))$, and so the eigenvalues of B are $\frac{(n-3)f(n-1,n-1)+((n-3)^2f^2(n-1,n-1)+8f^2(n-1,n-2)(n-2))^{1/2}}{2}$, 0 and $\frac{(n-3)f(n-1,n-1)-((n-3)^2f^2(n-1,n-1)+8f^2(n-1,n-2)(n-2))^{1/2}}{2}$.

Recalling that the edge-weighted function $f(x,y)$ is a symmetric polynomial, each term of $f(x,y)$ has three different types: $(xy)^\alpha$, $x^\beta + y^\beta$ and $(xy)^\gamma(x^\xi + y^\xi)$, where α, β, γ and ξ are nonnegative integers.

Now, we show that the following inequality

$$\frac{(n_0-3)f(n_0-1,n_0-1) - ((n_0-3)^2f^2(n_0-1,n_0-1) + 8f^2(n_0-1,n_0-2)(n_0-2))^{1/2}}{2}$$

$$> -f(n_0-1,n_0-1)$$

holds for some of the integers n_0, if the polynomial $f(x,y)$ does not contain the term $x+y$. Actually, by proper simplification, we only need to show that $\frac{f^2(n_0-1,n_0-1)}{f^2(n_0-1,n_0-2)} > 2$, for some of the integers n_0.

Since

$$\frac{((n_0-1)(n_0-1))^\alpha}{((n_0-1)(n_0-2))^\alpha} = \frac{(n_0-1)^\alpha}{(n_0-2)^\alpha} = (1 + \frac{1}{n_0-2})^\alpha,$$

we can get $(1 + \frac{1}{n_0-2})^{2\alpha} > 2$, when $n_0 < \frac{1}{2^{\frac{1}{2\alpha}}-1} + 2$. Besides

$$\frac{(n_0-1)^\beta + (n_0-1)^\beta}{(n_0-1)^\beta + (n_0-2)^\beta} = \frac{2}{1 + (\frac{n_0-2}{n_0-1})^\beta},$$

it is not difficult to obtain $(\frac{2}{1+(\frac{n_0-2}{n_0-1})^\beta})^2 > 2$, when $n_0 < \frac{1}{1-(\sqrt{2}-1)^{\frac{1}{\beta}}} + 1$. In particular, we assume $\beta > 1$. Let us consider the term of third type: $(xy)^\gamma(x^\xi + y^\xi)$ with $\gamma > 1$ and $\xi > 1$. Because $\frac{n_0-2}{n_0-1} < 1$, we have $(\frac{2}{1+(\frac{n_0-2}{n_0-1})^\beta})^2 > 1$. We can see that inequality

$$\left(\frac{((n_0-1)(n_0-1))^\alpha((n_0-1)^\beta + (n_0-1)^\beta)}{((n_0-1)(n_0-2))^\alpha((n_0-1)^\beta + (n_0-2)^\beta)}\right)^2$$

$$= \frac{(n_0-1)^{2\alpha}}{(n_0-2)^{2\alpha}} \left(\frac{2}{1 + (\frac{n_0-2}{n_0-1})^\beta}\right)^2 > 2$$

is true for $n_0 < \frac{1}{2^{\frac{1}{2\alpha}}-1} + 2$.

It is important to note that if we let $f_1(x,y)$ and $f_2(x,y)$ be two symmetric polynomials, n_1 and n_2 be two integers, such that $\frac{f_1^2(n_1-1,n_1-1)}{f_1^2(n_1-1,n_1-2)} > 2$ and $\frac{f_2^2(n_2-1,n_2-1)}{f_2^2(n_2-1,n_2-2)} >$

2, then we can claim that

$$\frac{(f_1(n_0-1,n_0-1)+f_2(n_0-1,n_0-1))^2}{(f_1(n_0-1,n_0-2)+f_2(n_0-1,n_0-2))^2}$$

$$> \frac{(\sqrt{2}f_1(n_0-1,n_0-2)+\sqrt{2}f_2(n_0-1,n_0-2))^2}{(f_1(n_0-1,n_0-2)+f_2(n_0-1,n_0-2))^2} = 2,$$

for some of the integers n_0.

Up to now, the proof of the inequality

$$\frac{(n_0-3)f(n_0-1,n_0-1)-((n_0-3)^2f^2(n_0-1,n_0-1)+8f^2(n_0-1,n_0-2)(n_0-2))^{1/2}}{2}$$

$$> -f(n_0-1,n_0-1)$$

is finished for certain integer n_0, when the polynomial function $f(x,y)$ does not contain the term $x+y$. This means that

$$\lambda_3(A_f(G)) > \lambda_2(A_f(G+e)).$$

Thus the eigenvalues of $A_f(G)$ and $A_f(G+e)$ can not have the property that

$$\lambda_{i+1}(A_f(G)) \le \lambda_i(A_f(G+e)) \le \lambda_{i-1}(A_f(G))$$

for $2 \le i \le n-1$.

Next, we can also find graphs G and $G+e$, such that the eigenvalues of $A_f(G)$ and $A_f(G+e)$ are not compatible when the edge-weighted function $f(x,y)$ must contain the term $x+y$. First of all, we get that the inequality

$$\frac{(2\times 2)^{\alpha}+2+2+(2\times 2)^{\gamma}(2^{\xi}+2^{\xi})}{2^{\alpha}+2+1+2^{\gamma}(1+2^{\xi})} \ge \frac{4}{3}$$

holds when α, γ and ξ are nonnegative integers, since $2^{\alpha}\times(3\times 2^{\alpha}-4)+2^{\gamma}\times(3\times 2^{\gamma}\times 2^{\xi}-4)+2^{\gamma}\times 2^{\xi}\times(3\times 2^{\gamma}-4)\ge 0$.

Now, set the graphs $G = P_4 \cup P_4$ and $G+e = P_8$, and let $H = P_4$. According to the matrices $A_f(H)$ and $A_f(G+e)$, we have $det(\lambda I - A_f(H)) = \lambda^4 - (2f^2(2,1) + f^2(2,2))\lambda^2 + f^4(2,1)$ and $det(\lambda I - A_f(G+e)) = \lambda^8 - (2f^2(2,1) + 5f^2(2,2))\lambda^6 + (f^4(2,1) + 8f^2(2,1)f^2(2,2) + 6f^4(2,2))\lambda^4 - (3f^4(2,1)f^2(2,2) + 6f^2(2,1)f^4(2,2) + f^6(2,2))\lambda^2 + f^4(2,1)f^4(2,2)$.

On the one hand, from Theorem 3.3 we know that

$$\lambda_1(A_f(H)) = \lambda_1(A_f(G)) = \frac{f(2,2)+(4f^2(2,1)+f^2(2,2))^{1/2}}{2} \ge \lambda_3(A_f(G+e)).$$

On the other hand, since $A_f(G)$, $A_f(G+e)$ and $A_f(G+e)-A_f(G)$ are nonnegative matrices, using Lemma 2.5 we can get

$$\lambda_1(A_f(G+e)) \geq \lambda_1(A_f(G)).$$

For convenience, let $f(2,1)=a_1$, $f(2,2)=b_1$ and $g(\lambda)=\lambda^8-(2a_1^2+5b_1^2)\lambda^6+(a_1^4+8a_1^2b_1^2+6b_1^4)\lambda^4-(3a_1^4b_1^2+6a_1^2b_1^4+b_1^6)\lambda^2+a_1^4b_1^4$. Hence, for the value $g(\lambda_1(A_f(G)))=g(\frac{b_1+(4a_1^2+b_1^2)^{1/2}}{2})$, there are three cases.

Case 1. If $\lambda_1(A_f(G))=\lambda_i(A_f(G+e))$ with $i=1,2$ or 3, then

$$g\left(\frac{b_1+(4a_1^2+b_1^2)^{1/2}}{2}\right)=0.$$

Case 2. If $\lambda_1(A_f(G+e))>\lambda_1(A_f(G))>\lambda_2(A_f(G+e))$, then

$$g\left(\frac{b_1+(4a_1^2+b_1^2)^{1/2}}{2}\right)<0.$$

Case 3. If $\lambda_2(A_f(G+e))>\lambda_1(A_f(G))>\lambda_3(A_f(G+e))$, then

$$g\left(\frac{b_1+(4a_1^2+b_1^2)^{1/2}}{2}\right)>0.$$

We claim that $g(\frac{b_1+(4a_1^2+b_1^2)^{1/2}}{2})>0$. Because, by calculating, we have

$$g(\frac{b_1+(4a_1^2+b_1^2)^{1/2}}{2})=(\frac{b_1+(4a_1^2+b_1^2)^{1/2}}{2})^8-(2a_1^2+5b_1^2)(\frac{b_1+(4a_1^2+b_1^2)^{1/2}}{2})^6$$

$$+(a_1^4+8a_1^2b_1^2+6b_1^4)(\frac{b_1+(4a_1^2+b_1^2)^{1/2}}{2})^4-(3a_1^4b_1^2+6a_1^2b_1^4+b_1^6)(\frac{b_1+(4a_1^2+b_1^2)^{1/2}}{2})^2+a_1^4b_1^4$$

$$=\frac{b_1^8}{2}+\frac{b_1^7(4a_1^2+b_1^2)^{1/2}}{2}-a_1^2b_1^5(4a_1^2+b_1^2)^{1/2}-\frac{5a_1^4b_1^4}{2}+\frac{a_1^4b_1^3(4a_1^2+b_1^2)^{1/2}}{2}+a_1^6b_1^2$$

$$=\frac{a_1^6b_1^2}{2}\left((4+(\frac{b_1}{a_1})^2)^{1/2}((\frac{b_1}{a_1})^5+\frac{b_1}{a_1}-2(\frac{b_1}{a_1})^3)+(\frac{b_1}{a_1})^6+2-5(\frac{b_1}{a_1})^2\right).$$

Let $\frac{b_1}{a_1}=t$ and $h(t)=(4+t^2)^{1/2}(t^5+t-2t^3)+t^6-5t^2+2$. Then the first-order derivative $h'(t)=(4+t^2)^{1/2}(5t^4+1-6t^2)+\frac{t}{(4+t^2)^{1/2}}(t^5+t-2t^3)+6t^5-10t=(4+t^2)^{1/2}(5t^2-1)(t^2-1)+\frac{t^2}{(4+t^2)^{1/2}}(t^2-1)^2+t(6t^4-10)$. Recall that $\frac{b_1}{a_1}=\frac{f(2,2)}{f(2,1)}\geq\frac{4}{3}$. Hence, $h'(t)>0$ and $h(t)$ is an increasing function. It is easy to get $h(\frac{4}{3})\approx 0.6686>0$.

So, we finally get that $g(\frac{b_1+(4a_1^2+b_1^2)^{1/2}}{2}) > 0$ when the symmetric polynomial function $f(x, y)$ has the term $x + y$. This means that,

$$\lambda_2(A_f(G + e)) > \lambda_1(A_f(G)).$$

Hence the eigenvalues of $A_f(G)$ and $A_f(G + e)$ do not satisfy the property that

$$\lambda_{i+1}(A_f(G)) \leq \lambda_i(A_f(G + e)) \leq \lambda_{i-1}(A_f(G))$$

for $2 \leq i \leq n - 1$. Our proof is thus complete. □

This theorem covers the edge-weight functions $f(x, y)$ of nine indices from Table 1, including the first Zagreb index, second Zagreb index, first hyper-Zagreb index, second hyper-Zagreb index, forgotten index, first Gourava index, second Gourava index, first hyper-Gourava index and second hyper-Gourava index.

In fact, from the proof of Theorem 3.5, one can see that the edge-weight functions of some other indices listed in Tables 1 and 2 can also be covered. Let us consider the following two situations.

(i) If there exists an n_0 such that $\frac{f^2(n_0-1,n_0-1)}{f^2(n_0-1,n_0-2)} > 2$, we can deduce that the complete graph K_{n_0} is a counterexample. By calculating, we can find that this is also the case for some other indices such as the product-connectivity Gourava index, augmented Zagreb index, inverse sum index, exponential first Zagreb index, exponential second Zagreb index and exponential augmented Zagreb index.

(ii) Moreover, assume that $\frac{f(2,2)}{f(2,1)} \geq \frac{4}{3}$. Then the eigenvalues of $A_f(P_4 \cup P_4)$ and $A_f(P_8)$ are not compatible. It is not difficult to verify that this also holds for the reciprocal Randić index, inverse sum index, exponential first Zagreb index and exponential second Zagreb index.

Denote by I the identity matrix and J the square matrix with all the entries being ones for appropriate sizes. Any column vector $x = (x_1, \ x_2, \ldots, x_n)^T$ can be regarded as a function defined on $V(G)$ which relates every v_i to x_i, that is, $x(v_i) = x_i$ for all $1 \leq i \leq n$.

Next, a uniform equivalent condition for a connected graph to have m $(2 \leq m \leq n)$ distinct weighted adjacency eigenvalues is established. As an application, we get a uniform combinatorial characterization for a graph to have two and three distinct weighted adjacency eigenvalues, respectively. Note that in the remainder of this paper, we always assume that the edge-weight $f(d_i, d_j) > 0$ for any edge $v_iv_j \in E(G)$.

Theorem 3.6 *Let G be a connected graph of order $n \geq 2$ and the edge-weight $f(d_i, d_j) > 0$ for any edge $v_iv_j \in E(G)$. Let λ_1 be the largest eigenvalue of $A_f(G)$ and x be a corresponding unit eigenvector. Then $A_f(G)$ has m $(2 \leq m \leq n)$ distinct eigenvalues if and only if there exist $m-1$ real numbers $\lambda_2,\ \lambda_3, \ldots, \lambda_m$ such that*

$$\prod_{i=2}^{m}(A_f(G) - \lambda_i I) = \prod_{i=2}^{m}(\lambda_1 - \lambda_i)xx^T. \tag{3.4}$$

Moreover, $\lambda_1 > \lambda_2 > \cdots > \lambda_m$ are exactly the m distinct eigenvalues of $A_f(G)$.

Proof. We first prove the sufficiency. Multiplying $A_f(G) - \lambda_1 I$ for both sides of Equation (3.4) on the left, we get that

$$(A_f(G) - \lambda_1 I)\prod_{i=2}^{m}(A_f(G) - \lambda_i I) = \prod_{i=2}^{m}(\lambda_1 - \lambda_i)(A_f(G) - \lambda_1 I)xx^T = \mathbf{0}.$$

Hence, the polynomial $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_m)$ annihilates $A_f(G)$. However, by Formula (3.4) the polynomial $P(\lambda)/(\lambda - \lambda_1)$ can not annihilate $A_f(G)$. From the definition of minimal polynomial of a matrix, we get that the minimal polynomial of $A_f(G)$ is

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_m).$$

Since $A_f(G)$ is a real symmetric matrix, using Lemma 2.2 we get that $A_f(G)$ has m distinct eigenvalues $\lambda_1,\ \lambda_2, \ldots, \lambda_m$.

For the necessity, let $\lambda_1,\ \lambda_2, ..., \lambda_m$ be m distinct eigenvalues of $A_f(G)$. Because $A_f(G)$ is a real symmetric matrix, from Lemma 2.2 we can get the minimal polynomial $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_m)$ directly. The minimal polynomial $p(\lambda)$ can annihilate $A_f(G)$. This means that

$$(A_f(G) - \lambda_1 I)(A_f(G) - \lambda_2 I)\cdots(A_f(G) - \lambda_m I) = \mathbf{0},$$

that is,

$$(A_f(G) - \lambda_1 I)\prod_{i=2}^{m}(A_f(G) - \lambda_i I) = \mathbf{0}.$$

Recall that G is connected, and for each $v_iv_j \in E(G)$ we have $f(d_i, d_j) > 0$. So, $A_f(G)$ is nonnegative and irreducible. From Lemma 2.3, we get that λ_1 is simple. Let $x = (x_1, x_2, \ldots, x_n)^T$ be a corresponding unit eigenvector. Then

$$\prod_{i=2}^{m}(A_f(G) - \lambda_i I) = x(c_1, c_2, \ldots, c_n),$$

where $c_1, c_2, \ldots, c_n$ are nonzero real numbers. Multiplying x^T to both sides of the above equality on the left, we get

$$\prod_{i=2}^{m}(\lambda_1 - \lambda_i)x^T = (c_1, c_2, \ldots, c_n).$$

For $1 \le i \le n$, it is easy to see that

$$c_i = \prod_{i=2}^{m}(\lambda_1 - \lambda_i)x_i.$$

We obtain the required result. □

Since G is connected, and $f(d_i, d_j) \neq 0$ for $v_iv_j \in E(G)$, Theorem 3.6 is suitable for the edge-weight functions $f(x, y)$ from nearly all of the indices in Tables 1 and 2, apart from the ABC index(when $n = 2$), Albertson index and sigma index. From the proof of Theorem 3.6, the result is true for any nonnegative irreducible symmetric matrix M, indexed by the vertices of a graph G, in which the ij entry is greater than zero if and only if $v_iv_j \in E(G)$. So, from our result, the results for the ABC matrix [5], Randić matrix [19], GA matrix [21] and AG matrix [24] follow directly. There is no need to prove them one by one separately. The following result is an immediate consequence of Theorem 3.6.

Corollary 3.7 *Let G be a connected graph of order $n \ge 2$ and the edge-weight $f(d_i, d_j) > 0$ for any edge $v_iv_j \in E(G)$. Then the weighted adjacency matrix $A_f(G)$ has two distinct eigenvalues if and only if G is the complete graph.*

Proof. By Theorem 3.6, $A_f(G)$ has two distinct eigenvalues if and only if there are two real numbers $\lambda_1,\ \lambda_2$, such that

$$A_f(G) - \lambda_2 I = (\lambda_1 - \lambda_2)xx^T,$$

that is

$$A_f(G) = (\lambda_1 - \lambda_2)xx^T + \lambda_2 I.$$

Since $A_f(G)$ is nonnegative and irreducible, from Lemma 2.3 we get that all the entries of x are not equal to 0, which means that each nondiagonal entry of $A_f(G)$ is not equal to 0. So, the graph G is the complete graph. □

Apart from the ABC index(when $n = 2$), Albertson index and sigma index, Corollary 3.7 holds for the weighted adjacency matrices from all of the indices in Tables 1 and 2. There is no doubt that we can uniformly get the results for the ABC matrix [5], Randić matrix [19], GA matrix [21] and AG matrix [24]. Another immediate consequence is the next one.

Corollary 3.8 *Let G be a connected graph of order $n \geq 3$ and the edge-weight $f(d_i, d_j) > 0$ for any edge $v_iv_j \in E(G)$. Let λ_1 be the largest eigenvalue of $A_f(G)$ and $x = (x_1, x_2, \ldots, x_n)^T$ be the corresponding unit eigenvector. Then the weighted adjacency matrix $A_f(G)$ has three distinct eigenvalues if and only if the following three properties hold:*

(i) For any vertex v_i,

$$\sum_{v_iv_j \in E(G)} f^2(d_i, d_j) = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)x_i^2 - \lambda_2\lambda_3;$$

(ii) For any two adjacent vertices v_i and v_j,

$$\sum_{\substack{v_iv_k \in E(G) \\ v_jv_k \in E(G)}} f(d_i, d_k)f(d_j, d_k) = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)x_ix_j + f(d_i, d_j)(\lambda_2 + \lambda_3);$$

(iii) For any two nonadjacent vertices v_i and v_j,

$$\sum_{\substack{v_iv_k \in E(G) \\ v_jv_k \in E(G)}} f(d_i, d_k)f(d_j, d_k) = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)x_ix_j.$$

Proof. From Theorem 3.6, $A_f(G)$ has three distinct eigenvalues if and only if there are three real numbers $\lambda_1,\ \lambda_2,\ \lambda_3$, such that

$$(A_f(G) - \lambda_2 I)(A_f(G) - \lambda_3 I) = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)xx^T.$$

Considering the diagonal entries and off-diagonal entries for both sides of the above equality, the result then immediately follows. □

Corollary 3.8 holds for the weighted adjacency matrices from nearly all of the indices in Tables 1 and 2 except for the Albertson index and sigma index. This result can directly imply the results for the ABC matrix [5], Randić matrix [19], GA matrix [21] and AG matrix [24].

The *distance* between two vertices v_i and v_j in a graph G is the length of a shortest v_iv_j-path in G. The *diameter* of graph G is the maximum distance between any pair of vertices of G. Let G be a connected graph with three distinct weighted adjacency eigenvalues. By Corollary 3.8 (iii), any two nonadjacent vertices of G have at least one common neighbor. Immediately, we have a relation between the eigenvalues of $A_f(G)$ and the diameter of a graph G.

Corollary 3.9 *Let G be a connected graph of order $n \geq 3$ and the edge-weight $f(d_i, d_j) > 0$ for any edge $v_iv_j \in E(G)$. If the weighted adjacency matrix $A_f(G)$ has three distinct eigenvalues, then the diameter of G is two.*

Furthermore, we can prove the following more general result.

Corollary 3.10 *Let G be a connected graph of order $n \geq 3$ and the edge-weight $f(d_i, d_j) > 0$ for any edge $v_iv_j \in E(G)$. If the weighted adjacency matrix $A_f(G)$ has m distinct eigenvalues, then the diameter of G is less than m.*

Proof. First, it is not difficult for us to have

$$\prod_{i=2}^{m}(A_f(G)-\lambda_i I) = A_f^{m-1}(G)-(\lambda_2+\lambda_3+\cdots+\lambda_m)A_f^{m-2}(G)+\cdots+(-1)^{m-1}\lambda_2\lambda_3\cdots\lambda_m I.$$

By Theorem 3.6, $A_f(G)$ has m $(2 \leq m \leq n)$ distinct eigenvalues if and only if there exist $m-1$ real numbers $\lambda_2,\ \lambda_3, \ldots, \lambda_m$ such that

$$\prod_{i=2}^{m}(A_f(G) - \lambda_i I) = \prod_{i=2}^{m}(\lambda_1 - \lambda_i)xx^T.$$

From Lemma 2.3, we can have a positive eigenvector x about λ_1, thus every entry in $\prod_{i=2}^{m}(A_f(G) - \lambda_i I)$ has a positive number. This means that for $i \neq j$, there is a positive integer r with $1 \leq r \leq m-1$ such that $(A_f^r(G))_{ij} > 0$. By calculating, we get that

$$(A_f^r(G))_{ij} = \sum f(d_i, d_{i_1})f(d_{i_1}, d_{i_2})\cdots f(d_{i_{r-1}}, d_j).$$

Because the edge-weight $f(d_i, d_j) > 0$ for any edge $v_iv_j \in E(G)$, it follows that there exists a path between v_i and v_j of length r. Then the diameter of G is at most $m-1$. The proof is thus complete. □

Except for the Albertson index and sigma index, Corollaries 3.9 and 3.10 hold true for the weighted adjacency matrices from all of the indices in Tables 1 and 2. The results for the ABC matrix [5], Randić matrix [19], GA matrix [21] and AG matrix [24] can be obtained directly. Therefore, to unify the spectral study of weighted adjacency matrices saves us lots of energy. Now we apply Corollary 3.9 to bipartite graphs and unicyclic graphs.

Corollary 3.11 *Let G be a bipartite graph of order $n \geq 3$ and the edge-weight $f(d_i, d_j) > 0$ for any edge $v_iv_j \in E(G)$. Then the weighted adjacency matrix $A_f(G)$ has three distinct eigenvalues if and only if G is the complete bipartite graph $K_{s,t}$.*

Proof. If G is a complete bipartite graph $K_{s,t}$, where $s+t=n$, then we can easily get $A_f(K_{s,t}) = f(s,t)A(K_{s,t})$. Hence, the distinct eigenvalues of $A_f(K_{s,t})$ are 0, $f(s,t)\sqrt{st}$ and $-f(s,t)\sqrt{st}$.

Conversely, assume that G is a bipartite graph of diameter two having three distinct weighted adjacency eigenvalues. Then, any two nonadjacent vertices must have the same neighbour set. If $v_iv_j \notin E(G)$ and v_i has a neighbor not adjacent to v_j, then this neighbor along with v_i, v_j path induce P_4 as subgraph. This contradicts the fact that the diameter of G is two. It follows that G is a complete bipartite graph. □

Corollary 3.11 holds for the weighted adjacency matrices from nearly all of the indices in Tables 1 and 2 except for that Albertson index and sigma index. This result can deduce the results for the ABC matrix [5], Randić matrix [19], GA matrix [21] and AG matrix [24].

Corollary 3.12 *Let G be a unicyclic graph of order $n \geq 3$ and the edge-weight $f(d_i, d_j) > 0$ for any edge $v_iv_j \in E(G)$. Then the weighted adjacency matrix $A_f(G)$ has three distinct eigenvalues if and only if G is either C_4 or C_5.*

Proof. Suppose G is either C_4 or C_5. It is easy to see that $A_f(G) = f(2,2)A(G)$. So, the distinct weighted adjacency eigenvalues of C_4 are $2f(2,2)$, 0 and $-2f(2,2)$, and the distinct weighted adjacency eigenvalues of C_5 are $2f(2,2)$, $2f(2,2)\cos\frac{2\pi}{5}$ and $2f(2,2)\cos\frac{4\pi}{5}$.

Conversely, if G is a unicyclic graph with diameter two, then G must be one of the graphs: C_4, C_5, S_n+e. We show that the graph S_n+e has more than three distinct weighted adjacency eigenvalues. For the convenience of discussion, we index the vertices in the graph S_n+e as shown in Figure 2. If we let the vector

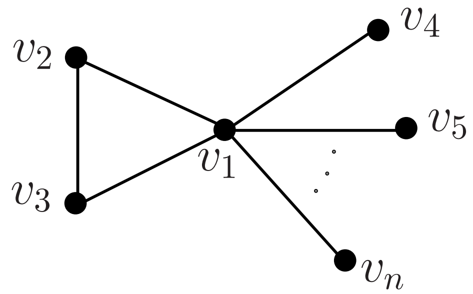

Figure 2: The graph S_n+e.

$x = (0, x_2, -x_2, 0, 0, 0, \ldots, 0)^T$, then it is not difficult to get that $A_f(G)x = -f(2,2)x$. So, $-f(2,2)$ is a weighted adjacency eigenvalue of the graph S_n+e. If we let the vector

$y = (0, 0, 0, y_4, -y_4, 0, \ldots, 0)^T$, then we have $A_f(G)y = 0y = o$. Thus 0 is a weighted adjacency eigenvalue of the graph $S_n + e$. Next, we give a partition $\{v_1, v_2, \ldots, v_n\} = X_1 \cup X_2 \cup X_3$, where $X_1 = \{v_1\}$, $X_2 = \{v_2, v_3\}$, and $X_3 = \{v_4, v_5, \ldots v_n\}$. Then the quotient matrix B of matrix $A_f(G)$ is

$$B = \begin{pmatrix} 0 & 2f(n-1,2) & (n-3)f(n-1,1) \\ f(n-1,2) & f(2,2) & 0 \\ f(n-1,1) & 0 & 0 \end{pmatrix}.$$

It is not difficult to check that this partition is equitable. From Lemma 2.4, each the eigenvalue of B is the eigenvalue of $A_f(G)$.

Let $f(n-1,2) = a$, $f(n-1,1) = b$ and $f(2,2) = c$, we can get

$$f(\rho) = det(\rho I - B) = \rho^3 - c\rho^2 - ((n-3)b^2 + 2a^2)\rho + (n-3)b^2c.$$

It is clear that $f(0) \neq 0$, because $b > 0$ and $c > 0$. For the value $f(-c)$, there are two cases as follows:

Case 1. If $f(-c) \neq 0$, then $f(\rho)$ has at least two distinct eigenvalues. Suppose the polynomial $f(\rho)$ has exactly one eigenvalue ρ_1. Then we have $f(\rho) = (\rho - \rho_1)^3 = \rho^3 - 3\rho_1\rho^2 + 3\rho_1^2\rho - \rho_1^3 = \rho^3 - c\rho^2 - ((n-3)b^2 + 2a^2)\rho + (n-3)b^2c$. This means that $\rho_1 = \frac{c}{3}$. But $3\rho_1^2 = \frac{c^2}{3} > 0$, $-((n-3)b^2 + 2a^2) < 0$, which is a contradiction, and hence $S_n + e$ has at least four distinct weighted adjacency eigenvalues.

Case 2. If $f(-c) = 0$, that is $f(-c) = -2c^3 + 2(n-3)b^2c + 2a^2c = 2c(a^2 + (n-3)b^2 - c^2) = 0$, then by calculating $\frac{f(\rho)}{\rho+c}$, we get $h(\rho) = \rho^2 - 2c\rho + 2c^2 - (n-3)b^2 - 2a^2 = \rho^2 - 2c\rho + (n-3)b^2$. Because $(-2c)^2 - 4 \times 1 \times (n-3)b^2 = 4a^2 > 0$, we know that $h(\rho)$ has two distinct eigenvalues. In addition, $h(-c) = c^2 + 2c^2 + (n-3)b^2 = 3c^2 + (n-3)b^2 > 0$, and thus $-c$ is not an eigenvalue of $h(\rho)$. This means that $f(\rho)$ has two distinct eigenvalues apart from $-c$. Hence, we can also conclude that $S_n + e$ has at least four distinct weighted adjacency eigenvalues. Our proof is thus complete. □

Corollary 3.12 is suitable for the weighted adjacency matrices from nearly all of the indices in Tables 1 and 2 apart from the Albertson index and sigma index. Using this result, we can directly get the results for the ABC matrix [5], Randić matrix [19], GA matrix [21] and AG matrix [24].

4 Concluding remarks

In this paper we are trying to uniformly study the spectral results for the weighted adjacency matrices of graphs with edge-weight function $f(x, y)$, but not separately one by one for concrete topological indices. As one can see, many of them share the same conclusions. For those spectral properties not shared by all the weighted adjacency matrices, one may consider them by groups to deal with. This will save a lot of energy for the study of the spectral properties of the weighted adjacency matrices defined by all kinds of existing degree-based topological function-indices and also for future invented indices that are useful in practical real world, especially in chemistry and biology. This work is aiming to throw a stone in order to induce jades. In the future, we will try to further study what kinds of spectral properties are shared by all or many of them, and which property is unique for some single weighted adjacency matrix defined by a specific index.

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