Nowhere-Zero 3-Flows in Signed Planar Graphs

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Abstract

Extending Grötzsch's 3-coloring theorem in the flow setting, Steinberg and Younger in 1989 proved that every 4-edge-connected planar or projective planar graph admits a nowhere-zero 3-flow (3-NZF for short), while Tutte's 3-flow conjecture asserts all 4-edge-connected graphs admit 3-NZFs. In this paper, we generalize Grötzsch's theorem to signed planar graphs by showing that *every* 4*-edge-connected signed planar graph with two negative edges admits a* 3*-NZF*. On the other hand, a result from Máčajová and Škoviera implies that there exist infinitely many 4-edge-connected signed planar graphs with three negative edges admitting no 3-NZFs but permitting 4-NZFs. Our proof employs the flow extension ideas from Steinberg-Younger and Thomassen, as well as refined exploration of the location of negative edges and elaborated discharging arguments in signed planar graphs.

Mathematics Subject Classifications: 05C10, 05C21, 05C22

1 Introduction

Besides the Four Color Theorem, in the field of planar graph coloring, there is another influential result which is called Grötzsch's 3-coloring theorem. It states that every triangle-free planar graph has a proper 3-coloring. As a generalization of the dual concept of graph coloring, Tutte [22, 23] initiated the study of flow theory and observed that a plane graph admits a nowhere-zero k-flow if and only if its dual graph has a proper k-coloring. Here,

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a nowhere-zero k-flow (k-NZF for short) of a graph G is an orientation together with an edge-mapping $f : E(G) \mapsto \{\pm 1, \pm 2, \dots, \pm(k-1)\}$ such that the sum of the incoming flow is equal to the sum of the outgoing flow at each vertex. Motivated by the 3-coloring theorem of Grötzsch, Tutte proposed his 3-flow conjecture as follows.

Conjecture 1. *(Tutte's 3-flow conjecture, 1972)* Every 4-edge-connected graph admits a nowhere-zero 3-flow.

As a major open problem in the flow theory, Tutte's 3-flow conjecture is still open as of today but many progresses have been made [5, 13, 14, 18, 21], and we refer readers to a recent survey [10].

The concepts of flows are naturally generalized to signed graphs, motivated from the study of graphs embedding on non-orientable surfaces. Flows of signed graphs are defined similarly, except that the orientation of each negative edge is directed as both away from or both towards its ends (see Section 2 for more details). It is natural to consider the analogue of Tutte's 3-flow conjecture for signed graphs. However, it fails in general as to be explained below. Youngs [29] constructed infinite families of triangle-free projective plane graphs with chromatic number 4, and he also proved that a quadrangulation of the projective plane graph has chromatic number either 2 or 4, but not 3. Thus by Bouchet's flow-coloring duality theorem [2], there exist infinitely many 4-edge-connected signed projective plane graphs admitting no 3-NZFs. Moreover, Máčajová and Škoviera [15] showed that a signed eulerian graph has a 3-NZF if and only if it can be edge-decomposed into three eulerian subgraphs sharing a common vertex and the number of negative edges in each of them is odd. This particularly implies the following proposition.

Proposition 2. ([15]) *Every* 4*-regular signed graph with odd negative edges does not admit* 3*-NZFs. In particular, there exist infinitely many* 4*-edge-connected signed planar graphs with three negative edges that do not admit* 3*-NZFs, while they admit* 4*-NZFs.*

For more illustrated examples, an infinite family of 4-edge-connected signed planar graphs with three negative edges is shown in Figure 1, and a directed proof of this simple fact is presented in Proposition 6 in Section 2.

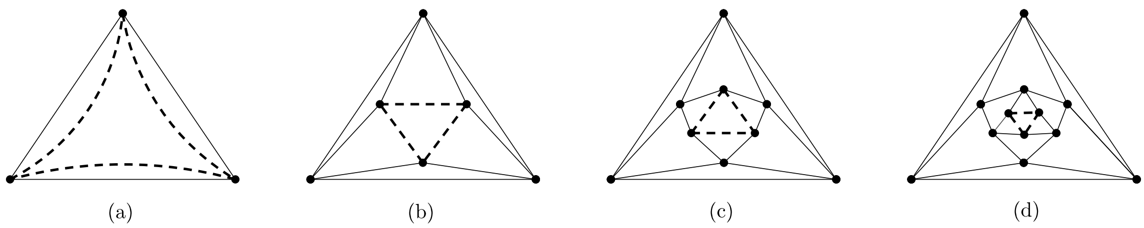


Figure 1: An infinite family of 4-edge-connected signed planar graphs without 3-NZFs.

This leaves a possible case on signed graphs with 2 negative edges (as each signed graph containing exactly 1 negative edge is not flow-admissible). We observe that this case is indeed equivalent to Tutte's 3-flow conjecture for ordinary graphs.

Proposition 3. *The following are equivalent.*
(a)(Tutte's 3-flow conjecture) *Every* 4*-edge-connected graph admits a nowhere-zero* 3*-flow.*
(b)(Signed Version with two negative edges) *Every* 4*-edge-connected signed graph with two negative edges admits a nowhere-zero* 3*-flow.*

It is clear in Proposition 3 that (b) implies (a): By arbitrarily adding two parallel negative edges to a 4-edge-connected ordinary graph G, we apply (b) to obtain a 3-NZF of the signed graph with two parallel negative edges, which still provides a 3-NZF when restricted to G. The reverse statement is proved in Proposition 7 in Section 2. Here, we construct a (potentially nonplanar) ordinary graph from two copies of a signed graph and execute the reduction by utilizing statement (a) on the constructed graph. Consequently, the reduction in the proof unavoidably requires nonplanar graphs if the two negative edges are not incident with the same face in the planar embedding.

Motivated by the above facts on 3-NZFs and some recent developments of flows in signed graphs [3, 17, 25, 26], we shall confirm the planar version of Proposition 3(b) (without using Tutte's 3-flow conjecture), which, in some sense, generalizes Grötzsch's theorem and supports Tutte's 3-flow conjecture.

Theorem 4. *Every* 4*-edge-connected signed planar graph with two negative edges admits a nowhere-zero* 3*-flow.*

The flow extension ideas employed in the proof of Theorem 4 are from Steinberg and Younger [18] in their proof of flow version of Grötzsch's theorem and from Thomassen [19, 20, 21] in his proofs of the 3-colorability of graphs with small genus and the weak 3-flow conjecture. Moreover, due to the existence of negative edges, we need not only to handle the small cuts in potential counterexamples but also to treat the negative edges in various location for different cases. After some prior reductions on the structure of potential counterexamples, in several different situations, we can always find Grötzsch Configurations to perform a desired reduction and complete the proof of Theorem 4.

After this paper has been completed, together with C.-Q. Zhang, we further develop the methods used in this paper to verify Tutte's 3-flow conjecture for all toroidal graphs [13].

2 Prerequisites

We consider finite graphs throughout this paper, both parallel edges and loops are allowed. We refer readers to [1, 31] for undefined notation and terminology.

Let k be a positive integer and $G = (V, E)$ be a graph. For convenience, we simply write $[k]$ for $\{1, 2, \ldots, k\}$. The degree of $x \in V(G)$ is denoted by $d_G(x)$. If x satisfies $d_G(x) = k$ ($d_G(x) \leqslant k$, respectively), then we call it a *k-vertex* (*k^--vertex*, respectively). Set $V_k(G) = \{x : d_G(x) = k\}$ and $V_{k^-}(G) = \{x : d_G(x) \leqslant k\}$. The set of all edges containing x as an endpoint is denoted by $E_G(x)$. Denote $\mu_G(x, y) = |E_G(x) \cap E_G(y)|$ for any two vertices $x, y \in V(G)$, and set $\mu(G) = \max\{\mu_G(x, y) : x, y \in V(G)\}$. Moreover, we use $N_G(x) = \{y : \mu_G(x, y) \geqslant 1, y \neq x\}$ to denote the neighborhood of x, and denote

$N_G[x] = N_G(x) \cup \{x\}$. For any two sets $T, U \subseteq V(G)$, $[T, U]_G$ denotes the set of edges with exact one endpoint in each of T and U. Furthermore, we simply write $[x, U]_G$ for $[T, U]_G$ when $T = \{x\}$, and write $[T, T^c]_G$ for $[T, U]_G$ when $U = V(G) \setminus T$. If $[T, T^c]_G$ is not empty, then it is called an *edge cut* of G. Let $d_G(T) = |[T, T^c]_G|$ and denote by $G[T]$ the subgraph of G induced by T.

An edge cut is called a *cut* if it is minimal which is also known as a bond in [1]. A cut $[T, T^c]_G$ is defined as a *t-cut* (*t^--cut*, respectively) if $d_G(T) = t$ ($d_G(T) \leqslant t$, respectively). Therefore t-cut is a shorten of t-edge-cut in this paper, and generally no vertex cut would be involved in context below. In a connected graph H, we use $\kappa'(H)$ to denote the *edge connectivity* of H. A cut $[T, T^c]_H$ is called *essential* if $\min\{|T|, |T^c|\} \geqslant 2$. If H does not have any essential $(t-1)^-$-cut, then we call it *essentially t-edge-connected.* The size of the minimum essential edge cut of H is called the *essential edge connectivity* and denoted by $\kappa'_e(H)$. Besides, if a graph G does not have $(2l-1)$-cut for any $0 < l \leqslant \lfloor \frac{t}{2} \rfloor$, then it is called *odd-t-edge-connected.* We use $\kappa'_o(G)$ to denote the *odd edge connectivity* of G which equals to the number of edges in the minimum odd edge cut.

An ordinary graph G with a *signature* σ, assigning each edge in $E(G)$ a signature of $\{1, -1\}$, is called a *signed graph* and denoted by (G, σ). For an edge $e \in E(G)$, we call it *negative* if $\sigma(e) = -1$ and *positive* otherwise. The edge set of (G, σ) can be decomposed into two subsets, that one consists of all negative edges and the other consists of all positive edges, where the former is denoted by $E^-_\sigma(G)$ and the latter is denoted by $E^+_\sigma(G)$. If $E^-_\sigma(G) = \emptyset$, then (G, σ) is called *all-positive* and can be viewed as an *ordinary graph*; if $E^+_\sigma(G) = \emptyset$, then (G, σ) is called *all-negative.* For two signed graphs (G, σ) and (G_1, σ_1), if G_1 is a subgraph of G and σ_1 is the restriction of σ on $E(G_1)$, then (G_1, σ_1) is a signed subgraph of (G, σ). Every edge e in (G, σ) consists of two half-edges h_x and h_y, where both x and y are endpoints of e, and h_x (h_y, respectively) is the half-edge of e incident with x (y, respectively). For each $x \in V(G)$, the set of all half-edges incident with x is denoted by $H_G(x)$. Set $H(G) = \bigcup_{x \in V(G)} H_G(x)$. An orientation of (G, σ) assigns a direction to each half-edge of (G, σ) as follows: every positive edge $e = xy$ is either directed out of x and directed into y or directed out of y and directed into x; every negative edge $e = xy$ is either directed into both x and y or directed out of both x and y. In particular, if a negative edge $e = xy$ is directed into (out of, respectively) both x and y, then it is called a *source edge* (a *sink edge*, respectively).

Let $\tau = \tau(G, \sigma)$ be an orientation of (G, σ) and $x \in V(G)$. Define $\tau(h_x) = 1$ if $h_x \in H_G(x)$ is directed out of x and $\tau(h_x) = -1$ if $h_x \in H_G(x)$ is directed into x. By definition of orientations, we have $\tau(h_x)\tau(h_y) = -\sigma(xy)$ for each edge $e = xy \in E(G)$. We use $E^-_\tau(x)$ and $E^+_\tau(x)$ to denote the set of all half-edges, incident with x, directed into x and directed out of x, respectively. Moreover, set $d^-_\tau(x) = |E^-_\tau(x)|$ and $d^+_\tau(x) = |E^+_\tau(x)|$. If the orientation τ satisfies that $d^+_\tau(x) - d^-_\tau(x) \equiv 0 \pmod 3$ for every vertex x of (G, σ), then it is called a *modulo 3-orientation* of (G, σ). For an Abelian group A, an orientation τ of (G, σ), together with a mapping f: $E(G) \mapsto A$, is called an *A-flow* of (G, σ) and denoted by (τ, f) if each vertex $x \in V(G)$ is balanced, that is

$$\partial f(x) = \sum_{h \in H_G(x)} \tau(h) f(e_h) = 0,$$

where "$\sum$" refers to the addition in A and e_h is the edge containing h as a half-edge. If an A-flow (τ, f) satisfies $f(e) \in A - \{0\}$ for each edge $e \in E(G)$, then we call it *nowhere-zero*. As $2 = -1$ in $\mathbb{Z}_3$, it is clear that in (signed) graphs, the existence of a modulo 3-orientation and the existence of a nowhere-zero $\mathbb{Z}_3$-flow are equivalent. Let k be a positive integer. For a nowhere-zero $\mathbb{Z}$-flow (τ, f), if $0 < |f(e)| < k$ for each $e \in E(G)$, then it is called a *nowhere-zero k-flow* (*k-NZF* for short). A signed graph is *flow-admissible* if it admits a k-NZF for some positive integer k. In [23], Tutte obtained a theorem that an ordinary graph admits a nowhere-zero $\mathbb{Z}_k$-flow if and only if it admits a k-NZF. While Tutte's theorem fails for signed graphs in general, the following lemma of Xu and Zhang [27] extends Tutte's work to signed graphs for 3-NZFs, which is a tool that will be used frequently in later proofs.

Lemma 5. (Xu and Zhang [27]) *A 2-edge-connected signed graph has a 3-NZF if and only if it has a modulo 3-orientation.*

By Lemma 5, to prove the existence of 3-NZFs of (signed) graphs, we shall study the existence of modulo 3-orientations in context, which allows us to only consider the orientations and ignore the edge-mapping f. Note that in the notation of a signed graph (G, σ), the signature σ is sometimes omitted if there is no chance for confusion. For example, we shall use G, $E^-(G)$ and $E^+(G)$ for (G, σ), $E^-_\sigma(G)$ and $E^+_\sigma(G)$, respectively. As a warmup, we start with the proofs of Propositions 2 and 3 using modulo 3-orientations.

Proposition 6. *Let G_1 be the signed graph on 3 vertices consisting of an all-positive triangle and an all-negative triangle, as shown in Figure 1(a). For each $i \geqslant 1$, construct a new signed graph G_{i+1} from G_i as follows: replacing every negative edge of G_i by a positive path of length two, and then adding an all-negative triangle such that each vertex is of degree 4. See Figure 1 for their constructions.*

Then for each $i \geqslant 1$, G_i is a signed planar graph with $\kappa'(G_i) = 4$ and $|E^-(G_i)| = 3$ that admits no 3-NZFs, but admits a 4-NZF.

Proof. By definition, in every modulo 3-orientation of a signed graph, the difference between the number of sink edges and the number of source edges is a multiple of 3. Thus, G_1 admits no modulo 3-orientation, which follows from the fact that all those three negative edges must be oriented as sink edges or all as source edges in such an orientation. Similarly, by the same fact, if G_{i+1} admits a modulo 3-orientation, then all those three negative edges must receive the same orientation. Thus, for any vertex $v \in V(G_{i+1})$ incident with negative edges, the two positive edges incident with v must either all be oriented into or all out of v. This would provide a modulo 3-orientation of G_i, which leads to a contradiction. Thus, G_i admits no 3-NZFs for each $i \geqslant 1$. Raspaud and Zhu [16] proved that every flow-admissible 4-edge-connected signed graph admits a 4-NZF. So for each $i \geqslant 1$, G_i admits a 4-NZF. □

For Proposition 3, as discussed above, it is clear that if every signed graph G with $\kappa'(G) \geqslant 4$ and $|E^-(G)| = 2$ admits a 3-NZF, then Tutte's 3-flow conjecture holds (by the trick of adding two parallel negative edges to an ordinary graph H with $\kappa'(H) \geqslant 4$). It remains to show the reverse as below.

Proposition 7. *If Tutte's* 3*-flow conjecture holds, then every signed graph* G *with* $\kappa'(G) \geqslant 4$ *and* $|E^-(G)| = 2$ *admits a* 3*-NZF.*

Proof. Assume that G is a signed graph with $\kappa'(G) \geqslant 4$ and $E^-(G) = \{ab, cd\}$. Take two copies G_1, G_2 of G with negative edges a_1b_1, c_1d_1 in G_1 and a_2b_2, c_2d_2 in G_2, delete all those negative edges and add a new vertex x incident with vertices a_1, b_1, a_2, b_2 and add a new vertex y incident with vertices c_1, d_1, c_2, d_2, x. The new obtained ordinary graph is denoted by H; see Figure 2. Then $\kappa'(H) \geqslant 4$ by the above construction of H. If Tutte's 3-flow conjecture holds, then H admits a modulo 3-orientation τ. Consider the edges xa_1 and xb_1. If they admit the same orientation (both oriented in or both out at x), then we obtain that the orientations of yc_1, yd_1 are also both in or both out, since the orientation in the 4-cut $\{xa_1, xb_1, yc_1, yd_1\}$ is balanced modulo 3. This results in a modulo 3-orientation of G_1 by retrieving the two negative edges. Otherwise, we assume that xa_1 and xb_1 receive opposite directions. As x is a 5-vertex, the three edges xy, xa_2, xb_2 all admit the same direction. Similarly, yc_2 and yd_2 also have the same direction as the orientation in the 4-cut $\{xa_2, xb_2, yc_2, yd_2\}$ is balanced modulo 3. Hence it results in a modulo 3-orientation of G_2. In any case, we get a modulo 3-orientation of G, and so G admits a 3-NZF. $\square$

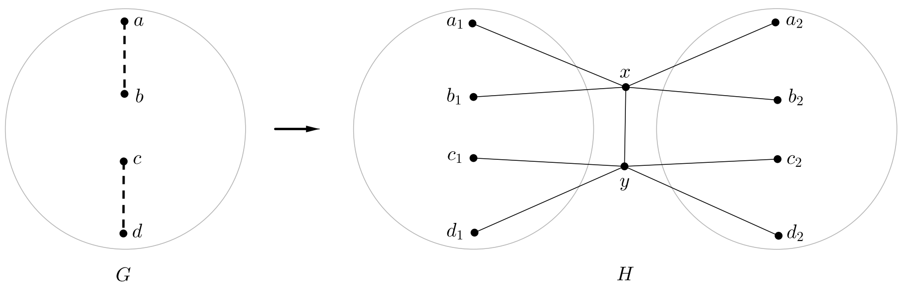


Figure 2: A construction for proving Proposition 7.

Next, let us recall the Steinberg-Younger proof [18] of the flow version of Grötzsch's theorem. For a vertex x of an ordinary graph G, if any pre-orientation τ_0 at $E_G(x)$ satisfying $d^+_{\tau_0}(x) - d^-_{\tau_0}(x) \equiv 0 \pmod 3$ can be extended to a modulo 3-orientation of G, then we call that G is $\mathcal{M}_3$*-extendable* at x. If $d_G(x) = 5$, then we have $\{d^+_{\tau_0}(x), d^-_{\tau_0}(x)\} = \{1, 4\}$; the incident edge opposing the other four in direction is called the *minority edge*.

Theorem 8. (Steinberg and Younger [18]) *Let* G *be an ordinary graph with no* 1*-cut that is either*
(i) planar and has at most three 3*-cuts; or*
(ii) projective planar and has at most one 3*-cut.*
Then G *has a modulo* 3*-orientation.*
Moreover, if G *is planar and has at most one* 3*-cut, then for any* 4*-vertex or* 5*-vertex* u*,* G *is* $\mathcal{M}_3$*-extendable at* u*, provided that* u *is not a cut-vertex and when* u *is a* 5*-vertex the minority edge at* u *does not lie in a* 3*-cut.*

The idea behind Theorem 8 is to find some simple reducible configurations or a Grötzsch Configuration (to be defined below) for reductions. But Theorem 8 alone is not enough to prove Theorem 4 on signed graphs, especially when there exist certain small cuts. Following Theorem 8, we shall provide a similar result below for our purpose of proving Theorem 4, which is also inspired by Thomassen's work [19, 21].

Theorem 9. *Let G be an ordinary planar graph with $u \in V(G)$.*
(i) If $\kappa'(G) \geqslant 5$ and $d_G(u) \leqslant 7$, then G is $\mathcal{M}_3$-extendable at u.
(ii) If $\kappa'_o(G) \geqslant 5$, $d_G(u) \leqslant 5$ and u is not a cut-vertex, then G is $\mathcal{M}_3$-extendable at u.

Note that Theorem 9(ii) is easily implied by Theorem 9(i) (Theorem 9(ii) is also a consequence of Theorem 8), while it seems that Theorem 9(i) cannot be derived from Theorem 8 directly. Actually, Theorem 9(i) is a corollary of Lemma 3.1 of [13], which is proved by authors of this paper together with C.-Q. Zhang. For an ordinary graph G, if there exists an edge e of G whose deletion results in a planar graph, then e is called a *handle-edge* and G is called *nearly-planar*.

Theorem 10. (Lemma 3.1 of [13]) *Let G be a nearly-planar ordinary graph with $u \in V(G)$. If $\kappa'(G) \geqslant 5$, $d_G(u) \leqslant 7$ and u is incident with a handle-edge of G, then G is $\mathcal{M}_3$-extendable at u.*

The usage of Theorem 9 is to eliminate certain small essential cuts in potential counterexamples of Theorem 4. Actually, using Theorem 9, we shall prove a stronger result than Theorem 4 as follows, which may contain 2-cuts.

Theorem 11. *Every connected signed planar graph G with $\kappa'_o(G) \geqslant 5$ and $|E^-(G)| = 2$ admits a modulo 3-orientation.*

Now, let us introduce the *splitting* operation and the *contracting* operation for signed graphs. Both of them are useful tools for later proofs.

Assume uu_1 and uu_2 are two edges of (G, σ). We say that the signed graph, denoted by $(G_{(uu_1,uu_2)}, \sigma')$, is obtained from (G, σ) by *splitting* uu_1 and uu_2 if it is got from (G, σ) by deleting uu_1 and uu_2, and adding a new edge $e' = u_1u_2$ joining u_1, u_2 with a signature σ' as follows: $\sigma'(e) = \sigma(uu_1)\sigma(uu_2)$ if $e = e' = u_1u_2$, and $\sigma'(e) = \sigma(e)$ if $e \in E(G) \setminus \{uu_1, uu_2\}$. Note that splitting can be properly performed even if vv_1 or vv_2 is a (negative) loop. The following lemma about the splitting operation is easy to verify from the definition, which is widely used in the literature [12, 26].

Lemma 12. *Let uu_1 and uu_2 be two edges of a signed graph (G, σ). If $(G_{(uu_1,uu_2)}, \sigma')$ admits a modulo 3-orientation, then so does (G, σ).*

Next, let us give the definition of *contracting* operation. Let $e = uv$ be an edge of a signed graph G. Contracting e means to identify vertices u and v, and delete the resulting positive loops but keep the resulting negative loops. The resulting signed graph is denoted by G/e. Moreover, we simply write G/H for $G/E(H)$ if H is a signed subgraph of G. From the definition of the contracting operation, we know the edge connectivity of (signed) graphs is preserved after contracting.

Observation 13. *Let (G,σ) be a signed graph of order at least* 3.

(i) Assume $e \in E(G)$. If G contains no k-cut, then so does G/e. Furthermore, if $\kappa'(G) \geqslant k$, then $\kappa'(G/e) \geqslant k$ as well.

(ii) Assume $uu_1, uu_2 \in E(G)$. If (G,σ) has exactly 2 *negative edges, then $(G_{(uu_1,uu_2)}, \sigma')$ has* 0 *or* 2 *negative edges.*

We also need some more observations below in later proofs.

Observation 14. *Let G be an ordinary graph with $\kappa'(G) \geqslant k \geqslant 1$ and let $S \subset V(G)$ be a nonempty set. If $d_G(S) \leqslant 2k-1$, then $G[S]$ is connected.*

A planar graph together with a planar embedding is called a plane graph. For a cycle C of a (signed) plane graph G, denote by $O_G(C)$ ($I_G(C)$, respectively) the set of vertices located in the exterior (interior, respectively) of C in G. If $|V(C)| = k$ and $\min\{|O_G(C)|, |I_G(C)|\} \geqslant 1$, then C is called a *separating k-cycle.* For any two adjacent edges of G, we call them consecutive if they are adjacent in the boundary of a face of G.

Lemma 15. *Every signed plane graph G with $\kappa'(G) \geqslant 4$ and $\kappa'_e(G) \geqslant 5$ contains no separating* 3*-cycle $C = xyzx$ with $\max\{d_G(x), d_G(y), d_G(z)\} \leqslant 5$, $O_G(C) \cap V_5 \neq \emptyset$ and $I_G(C) \cap V_5 \neq \emptyset$.*

Proof. By contradiction, assume that G contains a separating 3-cycle $C = xyzx$ with $\max\{d_G(x), d_G(y), d_G(z)\} \leqslant 5$, $O_G(C) \cap V_5 \neq \emptyset$ and $I_G(C) \cap V_5 \neq \emptyset$. Clearly, we have $(I_G(C) \cup O_G(C)) \cap V(C) = \emptyset$ and $d_G(I_G(C)) + d_G(O_G(C)) = d_G(V(C)) \leqslant 9$. Hence $\min\{d_G(I_G(C)), d_G(O_G(C))\} \leqslant 4$. Without loss of generality, say $d_G(I_G(C)) \leqslant 4$. Since $\kappa'(G) \geqslant 4$ and $\kappa'_e(G) \geqslant 5$, we must have that $|I_G(C)| = 1$ and the unique vertex in $I_G(C)$ is not a 5-vertex, a contradiction. □

Another important tool in flow theory is group connectivity. It is well-known that A-connected graphs are contractible configurations for nowhere-zero A-flow problems [6], where A is an Abelian group. Here we only introduce the $\mathbb{Z}_3$-connectivity of ordinary graphs (which is sufficient for our proofs). Note that there is a slightly different concept of $\mathbb{Z}_3$-connectivity for unbalanced signed graphs introduced in [12]. Let H be an ordinary graph. For a function β mapping from the vertex set of H to $\mathbb{Z}_3$, if it satisfies $\sum_{x \in V(H)} \beta(x) \equiv 0 \pmod 3$, then we call it a *$\mathbb{Z}_3$-boundary* of H. Meanwhile, an orientation τ of H, satisfying $d^+_\tau(x) - d^-_\tau(x) \equiv \beta(x) \pmod 3$ for each $x \in V(H)$, is called a *β-orientation.* If H has a β-orientation for each $\mathbb{Z}_3$-boundary β of H, then it is *$\mathbb{Z}_3$-connected.*

Lemma 16. (see [12]) *Let H be a signed subgraph of a* 2*-edge-connected signed graph G with $|E^-(H)| = 0$. Then any modulo* 3*-orientation of G/H can be extended to a modulo* 3*-orientation of G if the underlying graph of H is $\mathbb{Z}_3$-connected.*

The graph consisting of two vertices and a pair of parallel edges is denoted by $2K_2$. Besides, denote by W_k the graph obtained by adding a center vertex connecting to all vertices of a k-cycle. Note that $2K_2$ and W_4 are $\mathbb{Z}_3$-connected ordinary graphs, as observed in [6, 9] among others.

Proposition 17. *The ordinary graphs $2K_2$ and W_4 are both $\mathbb{Z}_3$-connected.*

The rest of the paper is organized as follows. In Section 3, we introduce some concepts and properties of Grötzsch Configurations, with the discharging method, we prove the existence of Grötzsch Configurations of a signed plane graph with some restrictions. In Section 4, in order to keep the paper self-contained, a short proof of Theorem 9 is presented, which will also help to overview the main strategy and some methods to prove our main result Theorem 11. We complete the proof of Theorem 11 in Section 5 and discuss some related questions in Section 6.

3 Discharging and Finding a Grötzsch Configuration

For an easier and clearer proof in later sections, we state and perform our reductions related to Grötzsch Configurations, and then we prove the existences of Grötzsch Configurations under various conditions with discharging arguments.

Before introducing the relevant definitions, we want to present the following property of planar graphs, which will be frequently used in the subsequent proof.

Lemma 18. (Euler's formula, see [1]) *Every plane graph G satisfies $|V(G)| - |E(G)| + |F(G)| = 2$, where $F(G)$ is the set of all faces of G.*

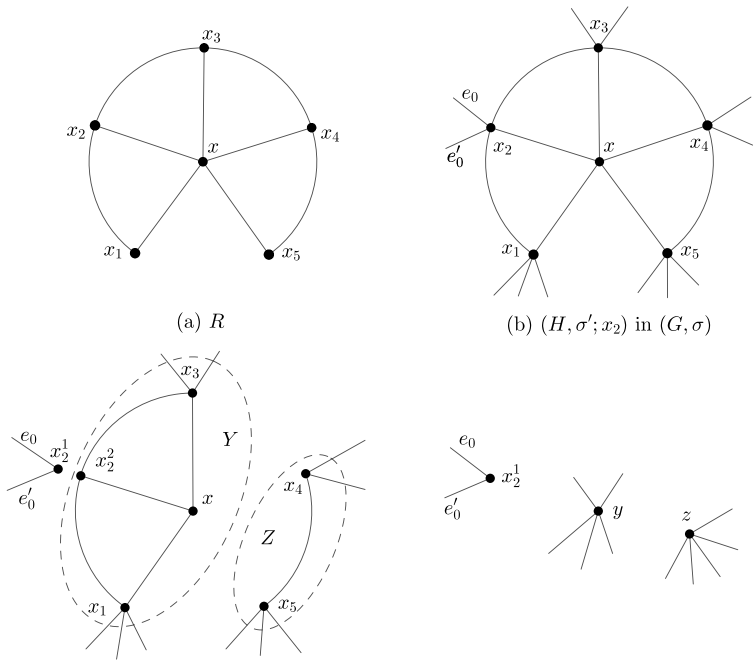
The lemma above implies that every simple plane graph G has at most $3|V(G)| - 6$ edges. Hence, by Handshaking Theorem, which states $\sum_{v \in V(G)} d_G(v) = 2|E(G)|$, G must have a vertex with degree at most 5.

By deleting an edge in W_5 which joins two 3-vertices, we obtain a new graph R whose vertices are labeled as shown in Figure 3(a).

Definition 19. Let (G, σ) be a signed plane graph which contains certain pre-oriented edges. Assume that (G, σ) contains a signed plane subgraph (H, σ') such that H is isomorphic to R, where σ' is the restriction of σ on H. The labels of vertices of R are also applied to H. Then (H, σ') is a *Valid Grötzsch Configuration* (or VGC for short) of (G, σ) if all of the following hold:

(i) all vertices in $V(H)$ are of degree 5 in G;
(ii) no 3-cycle of H is a separating 3-cycle in G;
(iii) for any $uv \in E(H)$, $\mu_G(u, v) = 1$ and uv is not negative or pre-oriented in G;
(iv) at least one vertex in $\{x_2, x_4\}$ is not incident with parallel edges in G.

Similarly, (H, σ') is called a *Nearly Grötzsch Configuration* (or NGC for short) of (G, σ) if it satisfies all conditions in Definition 19 except perhaps (iv). The vertex x in a VGC or NGC is called a *center*. Each vertex $w \in \{x_2, x_4\}$ is called a *corner*. Furthermore, w is called a *good corner* of a VGC (H, σ') if it is not incident with parallel edges in (G, σ). Meanwhile, we use $(H, \sigma'; w)$ to denote a VGC (H, σ') with a good corner w. By symmetry and up to relabeling, we always assume that the vertex x_2 is a good corner of a VGC (H, σ') and label the two edges in $E_G(x_2) \setminus E_H(x_2)$ as e_0 and e'_0 (see Figure 3(b)).



(a) R

(b) $(H, \sigma'; x_2)$ in (G, σ)

(c) The signed graph G_0

(d) The Grötzsch Reduction (G^*, σ^*)

Figure 3: A Grötzsch Configuration and the related Grötzsch Reduction.

Definition 20. With respect to a VGC $(H, \sigma'; x_2)$ of a signed plane graph (G, σ), we construct a *Grötzsch Reduction* $(G^*, \sigma^*) = GR(G, \sigma; H; x_2)$ by the following operations:

(1) In the VGC $(H, \sigma'; x_2)$, split x_2 into a 2-vertex x_2^1 and a 3-vertex x_2^2 such that the edges incident with x_2^1 are e_0 and e_0', and delete all edges in $\{x_3x_4, xx_4, xx_5\}$. Denote by G_0 the resulting signed graph (see Figure 3(c)).

(2) Set $Y = \{x, x_1, x_2^2, x_3\}$ and $Z = \{x_4, x_5\}$. Contract $E(G_0[Y])$ and denote by y the new vertex obtained from contraction. Contract $E(G_0[Z])$ and denote by z the new vertex obtained from contraction. Then we get the desired signed graph (see Figure 3(d)).

The following two useful lemmas are main ingredients of Grötzsch Reductions, which are initially developed (with similar forms) by Steinberg and Younger [18] in ordinary planar graphs. Luckily, they turn out to be effective for signed planar graphs in proving Theorem 11. We provide their proofs here for completeness, which are inspired from the work of Steinberg and Younger [18].

Lemma 21. *Any modulo* 3*-orientation* τ^* *of* $(G^*, \sigma^*) = GR(G, \sigma; H; x_2)$ *can be extended to a modulo* 3*-orientation* τ *of* (G, σ).

Proof. We use τ' to denote the restriction of τ^* on (G, σ). It is clear that every edge in (G, σ) but not in $(H, \sigma'; x_2)$ has been oriented and each $w \in V(G) \setminus V(H)$ satisfies $d^+_{\tau'}(w) - d^-_{\tau'}(w) \equiv 0 \pmod 3$. For each $w' \in V(H)$, set $\beta(w') \equiv d^-_{\tau'}(w') - d^+_{\tau'}(w') \pmod 3$. Clearly, we will obtain a desired orientation of (G, σ) if the VGC $(H, \sigma'; x_2)$ has a β-

orientation. Since τ^* is a modulo 3-orientation, we have

$$\beta(x_2) \equiv \beta(x) \equiv 0 \pmod 3,$$
$$\beta(x_1) + \beta(x_3) \equiv 0 \pmod 3, \text{ and } \beta(x_4) + \beta(x_5) \equiv 0 \pmod 3.$$

Now, let us show that H has a β-orientation. We need to consider nine cases in total according to the different values of $\beta(x_1)$ and $\beta(x_5)$, since $\beta(x_1) \in \{0,1,2\}$ and $\beta(x_5) \in \{0,1,2\}$. For convenience, we provide several orientations of $H - x_4x_5$ in Figure 4 to obtain a desired β-orientation of H, where $\beta(v) \equiv i \pmod 3$ is simply written by $(v : i)$. By assigning the edge x_4x_5 an appropriate direction, the orientation in Figure 4(a) and its reverse provide such β-orientation for $\beta(x_1) = 0$ and $\beta(x_5) \in \{0,1,2\}$. Similarly, we can also obtain such β-orientation for $\beta(x_1) = 2$ and $\beta(x_5) \in \{0,1\}$ and for $\beta(x_1) = 1$ and $\beta(x_5) \in \{0,2\}$ from the orientation in Figure 4(b), and such β-orientation for $\beta(x_1) = 1$ and $\beta(x_5) \in \{0,1\}$ and for $\beta(x_1) = 2$ and $\beta(x_5) \in \{0,2\}$ from the orientation in Figure 4(c). This verifies all cases and completes the proof. $\square$

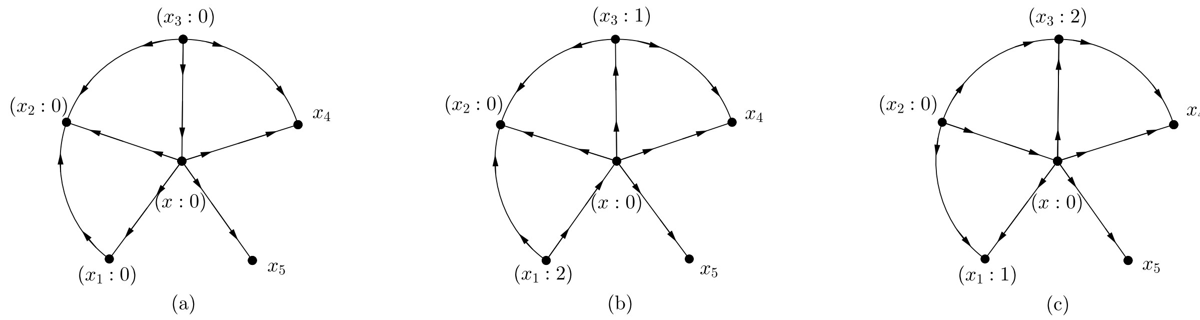


Figure 4: Some orientations of the graph $H - x_4x_5$ in Lemma 21.

Lemma 22. *Let (G, σ) be a k-edge-connected signed plane graph with a VGC $(H, \sigma'; x_2)$, where $k \in \{4, 5\}$. Then the Grötzsch Reduction $(G^*, \sigma^*) = GR(G, \sigma; H; x_2)$ is connected and any $(k-1)^-$-cut separates x_2^1 and z. That is, each $(k-1)^-$-cut $[S, S^c]_{G^*}$ satisfies $|S \cap \{x_2^1, z\}| = 1$ and $|S^c \cap \{x_2^1, z\}| = 1$.*

Proof. We first claim that, if G^* is disconnected, then

any component of G^* must contain either y or both x_2^1 and z.

Let $G^*[B]$ be a component of G^*, where $B \subset V(G^*)$. We observe the following:

- $d_G(B) = 0$ if $|B \cap \{x_2^1, y, z\}| = 0$;
- $d_G(B - \{x_2^1\}) \leqslant |\{e_0, e_0'\}| = 2$ if $B \cap \{x_2^1, y, z\} = \{x_2^1\}$;
- $d_G((B - \{z\}) \cup \{x_4, x_5\}) \leqslant |\{x_3x_4, xx_4, xx_5\}| = 3$ if $B \cap \{x_2^1, y, z\} = \{z\}$.

Since G is k-edge-connected, where $k \in \{4,5\}$, each of the above cases cannot occur. Hence both $G^*[B]$ and $G^*[B^c]$ are connected, and one of them contains y and the other contains $\{x_2^1, z\}$.

Therefore, it suffices to prove that, no matter G^* is connected or not, there is no set $S \subset V(G^*)$ such that $G^*[S]$ and $G^*[S^c]$ are both connected with $y \in S$, $\{x_2^1, z\} \subseteq S^c$, and $d_{G^*}(S) \leqslant k-1$. To the contrary, suppose that such a set S exists. So there is an (x_2^1, z)-path in $G^*[S^c]$, which implies that

$$G[S_1] \text{ contains an } (x_2,x_4)\text{-path or an } (x_2,x_5)\text{-path } P, \tag{1}$$

where $S_1 = (S^c \setminus \{x_2^1, z\}) \cup \{x_2, x_4, x_5\}$. Note that $x \notin S_1$.

Let $S_2 = (S \setminus \{y\}) \cup \{x_1, x_3\}$. Clearly, $S_2 \cap S_1 = \emptyset$, $x \notin S_2$ and $d_G(S_2) = d_{G^*}(S) + |\{x_1x, x_1x_2, x_3x_2, x_3x, x_3x_4\}| \leqslant k+4 \leqslant 9$. We first show that $G[S_2]$ is connected. Otherwise suppose that $G[S_2]$ is disconnected and contains two components, denoted by $G[A]$ and $G[B]$, where $A \subset S_2$ and $B \subset S_2$. Since $d_G(S_2) \leqslant k+4 \leqslant 9$, we have $k=4$ by Observation 14, and so $S_2 = A \cup B$, and $d_G(A) = 4$ and $d_G(B) = 4$. Without loss of generality, assume $x_3 \in A$. By the definition of VGC, we know that $d_G(x_3) = 5$ and so $|A| \geqslant 2$. Denote $A_1 = A \setminus \{x_3\}$. Then $A_1 \neq \emptyset$ and we obtain that $d_G(A_1) \leqslant d_G(A) - 1 = 3$ since $\{x_3x_2, x_3x, x_3x_4\} \subseteq [A, A^c]_G$. This contradicts $k=4$. Thus, $G[S_2]$ is connected and so

$$G[S_2] \text{ contains an } (x_1,x_3)\text{-path } Q. \tag{2}$$

Statements (1) and (2) will lead to a contradiction to planarity. Since $S_1 \cap S_2 = \emptyset$ and $x \notin S_1 \cup S_2$, as shown in Figure 5, we have $V(P) \cap V(Q) = \emptyset$ and $x \notin V(P) \cup V(Q)$. This is a contradiction to that G is a plane graph. $\square$

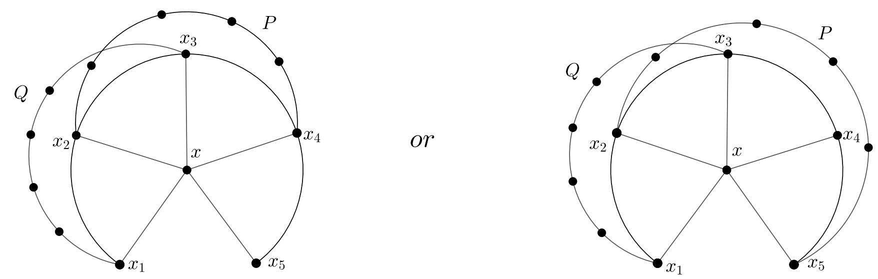


Figure 5: The paths P and Q.

For a 2-connected loopless signed plane graph (G, σ) with $\mu(G) \leqslant 2$ which contains no separating 2-cycles, we set $E_\sigma^-(G) = \{e_1, e_2, \ldots\}$ and denote $e_i = a_ib_i$ for each $e_i \in E_\sigma^-(G)$. If $\mu_G(a_i, b_i) = 2$, then denote the edge parallel to e_i by e_i'. We use $f_l(a_i, b_i)$ and $f_r(a_i, b_i)$ to denote two 3^+-faces (if exist) incident with e_i or e_i'. For each face $f \in \{f_l(a_i,b_i), f_r(a_i,b_i)\}$, denote by $d(f)$ the number of edges in the boundary of f. If $d(f_l(a_i,b_i)) = 3$, then the vertex, incident with $f_l(a_i,b_i)$ other than a_i and b_i, is denoted by c_i. Similarly, if $d(f_r(a_i,b_i)) = 3$, then the vertex, incident with $f_r(a_i,b_i)$ other than a_i and b_i, is denoted by d_i. Moreover, we set $J_i = \{a_i, b_i, c_i, d_i\}$ for each $e_i \in E_\sigma^-(G)$. Note that c_i and d_i

may not exist (so the size of J_i may be less than 4 in this case). For any $w \in V(G)$, let $L_G(w) = \{w' | w' \in N_G(w) \textit{ and } \mu_G(w, w') = 2\}$. More generally, for a set $W \in \{\emptyset, \{w\}\}$ with $w \in V(G)$, we define $L_G(W) = \emptyset$ if $W = \emptyset$, and $L_G(W) = L_G(w)$ if $W = \{w\}$.

Next, we give some sufficient conditions for the existence of a VGC. The statements of Lemma 23 below are rather complicated since we would like to summarize the discharging arguments used in the proofs of Theorems 9 and 11 together.

Lemma 23. *Let (G, σ) be a 2-connected loopless signed plane graph without separating 2-cycles, and u be a vertex of (G, σ). Let $X \in \{\emptyset, \{u\}\}$, and assume that all edges incident with X are pre-oriented. Assume that (G, σ) and X satisfy all of the following conditions:*
(I) $\{a_i, b_i\} \cap \{a_j, b_j\} = \emptyset$ for any $e_i, e_j \in E^-_\sigma(G)$ with $e_i \neq e_j$;
(II) Every essential cut $[S, S^c]_G$ satisfies $d_G(S) \geqslant 6$, and the equality holds if and only if $\mu_G(a_i, b_i) = 2$ and $S = \{a_i, b_i\}$ or $S^c = \{a_i, b_i\}$ for some $e_i \in E^-_\sigma(G)$;
(III) $d_G(v) = 5$ for each $v \in V(G) \setminus X$, and $4 \leqslant d_G(u) \leqslant 7$ if $X = \{u\}$;
(IV) $\mu(G) \leqslant 2$, and if $\mu_G(v, v') = 2$ for $v, v' \in V(G)$, then either $\{v, v'\} \cap X \neq \emptyset$ or $\{v, v'\} = \{a_i, b_i\}$ for some $e_i \in E^-_\sigma(G)$;
(V) G contains no ordinary graph W_4 as a subgraph if $X = \emptyset$, and the center of any ordinary subgraph W_4 (if exists) of G belongs to $N_G[u]$ if $X = \{u\}$.

Then all of the following hold.
(a) If $X = \{u\}$ and $E^-_\sigma(G) = \emptyset$, then (G, σ) has a VGC.
(b) If $X = \emptyset$ and $E^-_\sigma(G) = \{e_1, e_2\}$, then (G, σ) has a VGC.
(c) If $X = \{u\}$, $E^-_\sigma(G) = \{e_1\}$, $d_G(u) \leqslant 6$ and $(\{u\} \cup L_G(u)) \cap \{a_1, b_1\} = \emptyset$, then (G, σ) has an NGC, and furthermore, (G, σ) has a VGC if, additionally, $|L_G(u)| \leqslant 2$.

Proof. With conditions (II), (III) and (IV), we obtain $|V(G)| \geqslant 3$ and $\kappa'(G) \geqslant 4$ since G is loopless. To complete the proof, we start with Euler's formula:

$$|V(G)| - |E(G)| + |F(G)| = 2,$$

and Handshaking Theorem:

$$\sum_{v \in V(G)} d_G(v) = 2|E(G)|.$$

Following Lebesgue [11], we assign each $v \in V(G)$ a weight:

$$w(v) = \left(\sum_{f \in F_G(v)} \frac{1}{d(f)} \right) - \frac{d_G(v) - 2}{2},$$

where $F_G(v)$ is the set of faces incident with v and $d(f)$ is the number of edges in the boundary of a face f. For convenience, we define $w(X) = 0$ if $X = \emptyset$, and $w(X) = w(u)$ if $X = \{u\}$. Combining Euler's formula and Handshaking Theorem, we have

$$\sum_{v \in V(G)} w(v) = |F(G)| - |E(G)| + |V(G)| = 2.$$

With the condition (III), we have

$$w(X)+\sum_{v\in V_5-X} w(v)=\sum_{v'\in V(G)} w(v')=2. \tag{3}$$

Except the vertex in X (if it exists), there are at most $|L_G(X)|+2|E_\sigma^-(G)|$ vertices incident with parallel edges. By definition, an NGC (H,σ') is not a VGC if and only if the vertices x_2 and x_4 of H are both incident with parallel edges in G. Note that any vertex in X can not be x_2 or x_4 of an NGC. For any vertex v incident with parallel edges in G, there is at most one vertex in its neighborhood which can be a center of an NGC with v as a corner. So we conclude that there are at most $\lfloor\frac{|L_G(X)|+2|E_\sigma^-(G)|}{2}\rfloor$ NGCs with distinct centers which are not VGCs. Hence for parts (a) and (b), it suffices to prove that there are at least $\lfloor\frac{|L_G(X)|+2|E_\sigma^-(G)|}{2}\rfloor+1$ NGCs with distinct centers.

Denote by T^* the set consisting of all distinct centers of NGCs in (G,σ) and let $T=N_G[X]\cup(\bigcup_{e_i\in E_\sigma^-(G)} J_i)$, where $N_G[X]=\emptyset$ if $X=\emptyset$, and $N_G[X]=N_G[u]$ if $X=\{u\}$. With the condition (IV), it is clear that any vertex incident with parallel edges is in T. Hence, each $v\in V(G)\setminus\{T^*\cup T\}$ satisfies $d_G(v)=5$ and $|N_G(v)|=5$, and has at most three incident 3-faces, so it contributes at most $w(v)\leqslant\frac{1}{3}\times 3+\frac{1}{4}\times 2-\frac{3}{2}=0$ to the sum in Equation (3). Thus, we have $w(T^*)+w(T)\geqslant 2$, where $w(T^*)=\sum_{v^*\in T^*} w(v^*)$ and $w(T)=\sum_{v\in T} w(v)$. Since the center of any NGC contributes at most $\frac{1}{3}\times 5-\frac{3}{2}=\frac{1}{6}$ to the sum, *it suffices to prove that* $w(T)<2$ *for the case* $|L_G(u)|=3$ *in part (c) and* $w(T)<2-\frac{1}{6}\times\lfloor\frac{|L_G(X)|+2|E_\sigma^-(G)|}{2}\rfloor$ *for parts (a) and (b).*

For the case $|L_G(u)|\leqslant 2$ in part (c), we need more details as follows. If there is an NGC H in which the center is incident with five 3-faces, then either there exists a VGC with the same center as H or $L_G(u)=\{x_2,x_4\}$ and $\{x_1,x_5\}=\{a_1,b_1\}$, since $|L_G(X)|+2|E_\sigma^-(G)|\leqslant 4$. Clearly, for the latter case, any NGC other than H must be a VGC and H contributes at most $\frac{1}{3}\times 5-\frac{3}{2}=\frac{1}{6}$ to the sum in Equation (3). So it suffices to prove that $w(T)<2-\frac{1}{6}=\frac{11}{6}$ for this case. If there is no such NGC, then each center of an NGC contributes at most $\frac{1}{3}\times 4+\frac{1}{4}-\frac{3}{2}=\frac{1}{12}$ to the sum in Equation (3). Note that the number of NGCs with distinct centers which are not VGCs is at most 2 for the case $|L_G(u)|=2$ and at most 1 for the case $|L_G(u)|\leqslant 1$, respectively. So it suffices to prove that $w(T)<2-2\times\frac{1}{12}=\frac{11}{6}$ for the case $|L_G(u)|=2$, and $w(T)<2-\frac{1}{12}=\frac{23}{12}$ for the case $|L_G(u)|\leqslant 1$. Hence, *for the case* $|L_G(u)|\leqslant 2$ *in part (c), it suffices to prove that* $w(T)<\frac{11}{6}$ *for the case* $|L_G(u)|=2$, *and* $w(T)<\frac{23}{12}$ *for the case* $|L_G(u)|\leqslant 1$.

(a) We have $T=N_G[u]$ in this case, and it suffices to show that $w(T)<2-\frac{1}{6}\times\lfloor\frac{|L_G(u)|}{2}\rfloor$. Clearly, we have

$$\begin{gathered}
w(u)\leqslant\frac{|L_G(u)|}{2}+\frac{d_G(u)-|L_G(u)|}{3}-\frac{d_G(u)-2}{2},\\
w(v)\leqslant\frac{1}{3}\times 5-\frac{3}{2}=\frac{1}{6}\ \text{ for each } v\in N_G(u)\setminus L_G(u),\ \text{and}\\
w(v)\leqslant\frac{1}{3}\times 4+\frac{1}{2}-\frac{3}{2}=\frac{1}{3}\ \text{ for each } v\in L_G(u).
\end{gathered}$$

Thus,

$$w(T) \leqslant w(u) + \frac{d_G(u) - 2|L_G(u)|}{6} + \frac{|L_G(u)|}{3} = 1 + \frac{|L_G(u)|}{6}.$$

Since $d_G(u) \leqslant 7$, we have $|L_G(u)| \leqslant 3$, and so $w(T) \leqslant 1 + \frac{3}{6} < 2 - \frac{1}{6} \times \lfloor \frac{|L_G(u)|}{2} \rfloor$ as desired.

(b) We have $T = J_1 \cup J_2$ in this case and it suffices to prove that $w(T) < 2 - \frac{2}{6} = \frac{5}{3}$. Clearly, any vertex in $V(G) \setminus \{a_1, a_2, b_1, b_2\}$ is not incident with parallel edges in this case. Let $T_i = \{a_i, b_i, c_i, d_i\} \setminus \{a_j, b_j\}$, where $\{i, j\} = \{1, 2\}$. Since $\{a_1, b_1\} \cap \{a_2, b_2\} = \emptyset$, we have $w(T) \leqslant w(T_1) + w(T_2)$. So we will prove $w(T_i) < \frac{5}{6}$ for each $i \in \{1, 2\}$ below.

Recall that the positive edge parallel to e_i is denoted by e'_i if it exists. First, suppose $e'_i \notin E(G)$. We obtain that $w(v) \leqslant \frac{1}{3} \times 5 - \frac{3}{2} = \frac{1}{6}$ for each $v \in T_i$, and hence $w(T_i) \leqslant \frac{1}{6} \times 4 < \frac{5}{6}$. Next, suppose $e'_i \in E(G)$. By symmetry, the following two cases are considered.

Case I $d(f_r(a_i, b_i)) \geqslant 4$ or $d(f_l(a_i, b_i)) \geqslant 4$.

Without loss of generality, we may assume $d(f_r(a_i, b_i)) \geqslant 4$. Clearly, $T_i \subseteq \{a_i, b_i, c_i\}$. Notice that we have $w(c_i) \leqslant \frac{1}{3} \times 5 - \frac{3}{2} = \frac{1}{6}$ if $c_i \in T_i$ and $w(v) \leqslant \frac{1}{2} + \frac{1}{3} \times 3 + \frac{1}{4} - \frac{3}{2} = \frac{1}{4}$ for each $v \in \{a_i, b_i\}$. So $w(T_i) \leqslant \frac{1}{6} + \frac{1}{4} \times 2 = \frac{2}{3} < \frac{5}{6}$.

Case II $d(f_l(a_i, b_i)) = 3$ and $d(f_r(a_i, b_i)) = 3$.

First, suppose $T_i = \{a_i, b_i, c_i, d_i\}$. Clearly, c_i and d_i are not incident with negative edges. The condition (V) implies that each $v \in \{a_i, b_i\}$ is incident with at least one 4^+-face. If a_i and b_i are incident with the same 4^+-face, then $d_G(c_i) = 2$ or $d_G(d_i) = 2$ since $d_G(a_i) = 5$, a contradiction. Hence, there are at least two 4^+-faces incident with exactly one of a_i and b_i. Furthermore, they are also incident with c_i or d_i. Then, we have $w(v) \leqslant \frac{1}{2} + \frac{1}{3} \times 3 + \frac{1}{4} - \frac{3}{2} = \frac{1}{4}$ for each $v \in \{a_i, b_i\}$, and $w(c_i) + w(d_i) \leqslant \frac{1}{3} \times 8 + \frac{1}{4} \times 2 - \frac{3}{2} \times 2 = \frac{1}{6}$. Thus, $w(T_i) \leqslant \frac{1}{4} \times 2 + \frac{1}{6} = \frac{2}{3} < \frac{5}{6}$. Next, by symmetry, we suppose $T_i \subseteq \{a_i, b_i, c_i\}$. Clearly, we have $w(v) \leqslant \frac{1}{2} + \frac{1}{3} \times 4 - \frac{3}{2} = \frac{1}{3}$ for each $v \in \{a_i, b_i\}$ and $w(c_i) \leqslant \frac{1}{3} \times 5 - \frac{3}{2} = \frac{1}{6}$ if $c_i \in T_i$. Thus, $w(T_i) \leqslant \frac{1}{3} \times 2 + \frac{1}{6} = \frac{5}{6}$. If the equality holds, then c_i is incident with five 3-faces, and a_i and b_i are incident with four 3-faces as shown in Figure 6(a). Since the negative edge other than e_i is not incident with any vertex in $\{a_i, b_i, c_i\}$, one of a_i and b_i is a center of an ordinary graph W_4, which contradicts the condition (V).

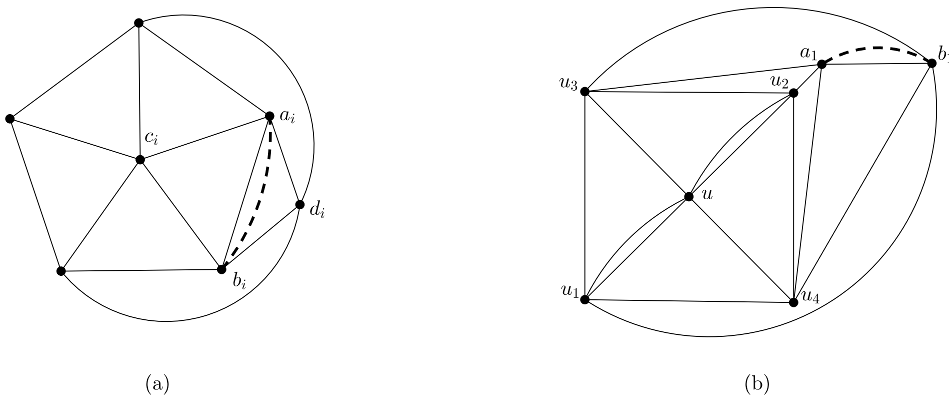


Figure 6: Two special configurations in Lemma 23.

(c) Clearly, we have $T = N_G[u] \cup J_1$ in this case. As discussed above, it suffices to prove that $w(T) < 2$ if $|L_G(u)| = 3$, $w(T) < \frac{11}{6}$ if $|L_G(u)| = 2$, and $w(T) < \frac{23}{12}$ if $|L_G(u)| \leqslant 1$.

First, we claim the following:

$$u \text{ is incident with at least one } 4^+\text{-face if } |L_G(u)| = 2. \tag{4}$$

To the contrary, we suppose that u is incident with $d_G(u) - 2$ 3-faces. With the condition (III), we have $4 \leqslant d_G(u) \leqslant 6$, and $d_G(v) = 5$ for each $v \in V(G) \setminus \{u\}$. By Handshaking Theorem, we obtain that $2|E(G)| = 5|V(G)| + d_G(u) - 5$. Since G is a planar graph with at most three pairs of parallel edges, we obtain $|E(G)| \leqslant 3|V(G)| - 6 + 3$, which implies $|V(G)| \geqslant d_G(u) + 1$. Let $S = V(G) \setminus N_G[u]$. Then $|S| \geqslant (d_G(u) + 1) - (d_G(u) - 1) \geqslant 2$, $|S^c| \geqslant 3$ and $d_G(S) \leqslant 6$. With the condition (II), we obtain that $S = \{a_1, b_1\}$ and $d_G(S) = 6$. This implies that, $d_G(u) = 6$, and both a_1 and b_1 have three neighbors in $N_G(u)$. So G is isomorphic to the signed plane graph as shown in Figure 6(b). Clearly, $G - u$ contains an ordinary graph W_4 with a_1 as a center, which contradicts the condition (V). This verifies statement (4) that u is incident with at least one 4^+-face if $|L_G(u)| = 2$.

Now, we prove that

$$w(T) < \frac{11}{6} \text{ if one of } a_1 \text{ and } b_1 \text{ is a center of an ordinary subgraph } W_4 \text{ of } G. \tag{5}$$

Without loss of generality, assume that a_1 is a center of W_4. With the condition (V), we have $a_1 \in N_G(u) \setminus L_G(u)$. Clearly, $w(u) \leqslant \frac{|L_G(u)|}{2} + \frac{d_G(u) - |L_G(u)|}{3} - \frac{d_G(u)-2}{2}$. Moreover, we have $|L_G(u)| \leqslant 2$ since $d_G(u) \leqslant 6$. If $\mu_G(a_1, b_1) = 1$, then we have $w(v) \leqslant \frac{1}{3} \times 5 - \frac{3}{2} = \frac{1}{6}$ for each $v \in (N_G(u) \setminus L_G(u)) \cup (J_1 \setminus (\{u\} \cup L_G(u)))$ and $w(v) \leqslant \frac{1}{3} \times 4 + \frac{1}{2} - \frac{3}{2} = \frac{1}{3}$ for each $v \in L_G(u)$. So

$$w(T) \leqslant w(u) + \frac{d_G(u) - 2|L_G(u)| + 3}{6} + \frac{|L_G(u)|}{3} = \frac{3}{2} + \frac{|L_G(u)|}{6} \leqslant \frac{11}{6}.$$

If the equality holds, then u is not incident with 4^+-faces and $|L_G(u)| = 2$, a contradiction to statement (4). Hence we assume $\mu_G(a_1, b_1) = 2$ and $|N_G(a_1)| = 4$. Now we consider two cases according to the different locations of b_1. First, assume $b_1 \in N_G(u) \setminus L_G(u)$. It is clear that either $|J_1| \leqslant 3$ or $\{c_1, d_1\} \cap N_G[u] \neq \emptyset$. We may assume $d_1 \notin J_1$ or $d_1 \in J_1 \cap N_G[u]$. Then we have $w(c_1) \leqslant \frac{1}{3} \times 5 - \frac{3}{2} = \frac{1}{6}$ if $c_1 \in V(G) \setminus N_G[u]$, $w(v) \leqslant \frac{1}{3} \times 5 - \frac{3}{2} = \frac{1}{6}$ for each $v \in N_G(u) \setminus (L_G(u) \cup \{a_1, b_1\})$ and $w(v) \leqslant \frac{1}{3} \times 4 + \frac{1}{2} - \frac{3}{2} = \frac{1}{3}$ for each $v \in L_G(u) \cup \{a_1, b_1\}$. Hence by $|L_G(u)| \leqslant 2$ we have

$$w(T) \leqslant w(u) + \frac{d_G(u) - 2|L_G(u)| - 2 + 1}{6} + \frac{|L_G(u)| + 2}{3} = \frac{3}{2} + \frac{|L_G(u)|}{6} \leqslant \frac{11}{6}.$$

If the equality holds, then u is not incident with 4^+-faces and $|L_G(u)| = 2$, a contradiction to statement (4). Hence we obtain a strict inequality that $w(T) < \frac{11}{6}$. Next, assume $b_1 \notin N_G[u]$. For each $v \in \{c_1, d_1\}$, we have $v \in N_G(u)$ if v exists, since a_1 is the center of a W_4. Clearly, $w(v) \leqslant \frac{1}{3} \times 5 - \frac{3}{2} = \frac{1}{6}$ for each $v \in N_G(u) \setminus (L_G(u) \cup \{a_1\})$ and $w(v) \leqslant \frac{1}{3} \times 4 + \frac{1}{2} - \frac{3}{2} = \frac{1}{3}$ for each $v \in L_G(u) \cup \{a_1, b_1\}$. Hence, $w(T) \leqslant w(u) +$

$\frac{d_G(u)-2|L_G(u)|-1}{6} + \frac{|L_G(u)|+2}{3} = \frac{3}{2} + \frac{|L_G(u)|}{6} \leqslant \frac{11}{6}$ since $|L_G(u)| \leqslant 2$. If the equality holds, then u is not incident with 4^+-faces and $|L_G(u)| = 2$, a contradiction to statement (4). So we conclude that statement (5) holds, i.e., $w(T) < \frac{11}{6}$ if one of a_1 and b_1 is a center of an ordinary subgraph W_4 of G.

Hence by (5), we assume that the center of any ordinary subgraph W_4 is neither a_1 nor b_1. Let $T_1 = J_1 \setminus (\{u\} \cup L_G(u))$ and $T_2 = T \setminus T_1$. Clearly, $\{a_1, b_1\} \subseteq T_1$ and $w(T) = w(T_1) + w(T_2)$. To complete the proof, we will prove below that

$$w(T_1) \leqslant \frac{2}{3}, \tag{6}$$

and

$$w(T_2) < \frac{4}{3} \text{ if } |L_G(u)| = 3,\ w(T_2) < \frac{7}{6} \text{ if } |L_G(u)| = 2, \text{ and } w(T_2) < \frac{5}{4} \text{ if } |L_G(u)| \leqslant 1. \tag{7}$$

The proof of statement (6) is similar to that of (b). With the condition (IV), each vertex $v \in T_1$ is not incident with positive parallel edges. If $e_1' \notin E(G)$, then $w(v) = \frac{1}{3} \times 5 - \frac{3}{2} = \frac{1}{6}$ for each $v \in T_1$ and $w(T_1) \leqslant \frac{1}{6} \times 4 = \frac{2}{3}$. If $e_1' \in E(G)$, then we consider two cases. First, assume $T_1 = \{a_1, b_1, c_1, d_1\}$. By the assumption and $d_G(a_1) = d_G(b_1) = 5$, each $v \in \{a_1, b_1\}$ is incident with at least one 4^+-face. If a_1 and b_1 are incident with the same 4^+-face, then $d_G(c_1) = 2$ or $d_G(d_1) = 2$ since $d_G(a_1) = 5$, a contradiction. Hence, there are at least two 4^+-faces incident with exactly one of a_1 and b_1, and they are also incident with c_1 or d_1. Then, we have $w(v) \leqslant \frac{1}{2} + \frac{1}{3} \times 3 + \frac{1}{4} - \frac{3}{2} = \frac{1}{4}$ for each $v \in \{a_1, b_1\}$, and $w(c_1) + w(d_1) \leqslant \frac{1}{3} \times 8 + \frac{1}{4} \times 2 - \frac{3}{2} \times 2 = \frac{1}{6}$. Thus, $w(T_1) \leqslant \frac{1}{4} \times 2 + \frac{1}{6} = \frac{2}{3}$. Next, by symmetry, we assume $T_1 \subseteq \{a_1, b_1, c_1\}$. By statement (5), we have $w(v) \leqslant \frac{1}{2} + \frac{1}{3} \times 3 + \frac{1}{4} - \frac{3}{2} = \frac{1}{4}$ for each $v \in \{a_1, b_1\}$ and $w(c_1) \leqslant \frac{1}{3} \times 5 - \frac{3}{2} = \frac{1}{6}$ if it exists. Thus, $w(T_1) \leqslant \frac{1}{4} \times 2 + \frac{1}{6} = \frac{2}{3}$. This proves statement (6).

It remains to prove statement (7): $w(T_2) < \frac{4}{3}$ if $|L_G(u)| = 3$, $w(T_2) < \frac{7}{6}$ if $|L_G(u)| = 2$, and $w(T_2) < \frac{5}{4}$ if $|L_G(u)| \leqslant 1$. Recall that $4 \leqslant d_G(u) \leqslant 6$. Since there exists a 5-vertex in G and $\mu(G) \leqslant 2$, we have $|V(G)| \geqslant 4$. Actually, if $|V(G)| = 4$, then $d_G(u) = 5$ and $|E(G)| = 10$ by Handshaking Theorem. It implies that $|L_G(u)| \geqslant 3$ since $|E_\sigma^-(G)| = 1$ and $G - u$ contains no positive parallel edges, which contradicts $d_G(u) = 5$. So we assume $|V(G)| \geqslant 5$. Since $T_2 = T \setminus T_1$ and $\{a_1, b_1\} \subseteq T_1$, we have that $(\{u\} \cup L_G(u)) \subseteq T_2 \subseteq N_G[u]$ and each vertex $v \in T_2$ is not incident with the negative edge e_1. Then, we consider two cases as follows.

Case A $4 \leqslant d_G(u) \leqslant 5$.

First, suppose $|L_G(u)| \geqslant 1$. Assume $u_1 \in L_G(u)$ and let $S = \{u, u_1\}$. Clearly, $S \neq \{a_1, b_1\}$ and $|S^c| \geqslant 3$ since $u \notin \{a_1, b_1\}$ and $|V(G)| \geqslant 5$. We have $d_G(S) \leqslant 6$ since $d_G(u_1) = 5$, contrary to the condition (II). Next, we suppose $|L_G(u)| = 0$. Clearly, $w(u) \leqslant \frac{d_G(u)}{3} - \frac{d_G(u)-2}{2} = 1 - \frac{d_G(u)}{6}$ and $w(v) \leqslant 5 \times \frac{1}{3} - \frac{3}{2} = \frac{1}{6}$ for each $v \in T_2 \setminus \{u\}$. So $w(T_2) = w(u) + \sum_{v \in T_2 - \{u\}} w(v) \leqslant 1 - \frac{d_G(u)}{6} + \frac{d_G(u)}{6} = 1 < \frac{5}{4}$ as required.

Case B $d_G(u) = 6$.

Since $\mu(G) \leqslant 2$, we have $0 \leqslant |L_G(u)| \leqslant 3$. If $|L_G(u)| = 0$, then $w(u) \leqslant 6 \times \frac{1}{3} - \frac{4}{2} = 0$ and $w(v) \leqslant 5 \times \frac{1}{3} - \frac{3}{2} = \frac{1}{6}$ for each $v \in T_2 \setminus \{u\}$. So $w(T_2) \leqslant 0 + \frac{1}{6} \times 6 = 1 < \frac{5}{4}$.

If $|L_G(u)| = 1$, then assume $L_G(u) = \{u_1\}$. We have $w(u) \leqslant \frac{1}{2} + 5 \times \frac{1}{3} - \frac{4}{2} = \frac{1}{6}$, $w(u_1) \leqslant \frac{1}{2} + 4 \times \frac{1}{3} - \frac{3}{2} = \frac{1}{3}$ and $w(v) \leqslant 5 \times \frac{1}{3} - \frac{3}{2} = \frac{1}{6}$ for each $v \in T_2 \setminus \{u, u_1\}$. So $w(T_2) \leqslant \frac{1}{6} + \frac{1}{3} + \frac{1}{6} \times 4 = \frac{7}{6} < \frac{5}{4}$. If $|L_G(u)| = 2$, then assume that f is a 4^+-face incident with u by statement (4). Hence $w(u) \leqslant 2 \times \frac{1}{2} + 3 \times \frac{1}{3} + \frac{1}{4} - \frac{4}{2} = \frac{1}{4}$. Moreover, since f is incident with at least two vertices in $N_G(u)$, we have

$$\sum_{v \in T_2 - \{u\}} w(v) \leqslant 2 \times (\frac{1}{2} + 4 \times \frac{1}{3} - \frac{3}{2}) + 2 \times (5 \times \frac{1}{3} - \frac{3}{2}) - 2 \times \frac{1}{12} = \frac{5}{6} \quad \text{if } T_2 = N_G[u],$$

and

$$\sum_{v \in T_2 - \{u\}} w(v) \leqslant 2 \times (\frac{1}{2} + 4 \times \frac{1}{3} - \frac{3}{2}) + (5 \times \frac{1}{3} - \frac{3}{2}) = \frac{5}{6} \quad \text{if } T_2 \subset N_G[u].$$

So $w(T_2) \leqslant \frac{1}{4} + \frac{5}{6} = \frac{13}{12} < \frac{7}{6}$.

If $|L_G(u)| = 3$, then we denote $L_G(u) = \{u_1, u_2, u_3\}$. First, suppose that u is incident with a 4^+-face, denoted by f. Without loss of generality, assume that u_1 and u_2 are both incident with f. Then we have $w(u) \leqslant 3 \times \frac{1}{2} + 2 \times \frac{1}{3} + \frac{1}{4} - \frac{4}{2} = \frac{5}{12}$, $w(u_3) \leqslant \frac{1}{2} + 4 \times \frac{1}{3} - \frac{3}{2} = \frac{1}{3}$ and $w(u_i) \leqslant \frac{1}{2} + 3 \times \frac{1}{3} + \frac{1}{4} - \frac{3}{2} = \frac{1}{4}$ for each $i \in \{1, 2\}$. So $w(T_2) \leqslant \frac{5}{12} + 2 \times \frac{1}{4} + \frac{1}{3} = \frac{5}{4} < \frac{4}{3}$ as required. Next, assume instead that u is incident with three 3-faces. Let $S = N_G[u]$. Then we have $|S^c| \geqslant 1$ since $|V(G)| \geqslant 5$. Moreover, we observe that $d_G(S) \leqslant 3$, which contradicts that G is 4-edge-connected. This completes the proof of statement (7), as well as Lemma 23. □

4 Proof of Theorem 9

As an overview of our proof techniques and to keep the paper self-contained, we shall present a short proof of Theorem 9 (without using Theorems 8 and 10) in this section, utilizing tools developed in Sections 2 and 3. We copy Theorem 9 here for convenience.

Theorem 9. *Let G be an ordinary planar graph with $u \in V(G)$.*
(i) If $\kappa'(G) \geqslant 5$ and $d_G(u) \leqslant 7$, then G is $\mathcal{M}_3$-extendable at u.
(ii) If $\kappa'_o(G) \geqslant 5$, $d_G(u) \leqslant 5$ and u is not a cut-vertex, then G is $\mathcal{M}_3$-extendable at u.

Proof of Theorem 9(i) By way of contradiction, suppose that Theorem 9(i) is false and let G be a counterexample. That is, G satisfies $\kappa'(G) \geqslant 5$, but there exists a vertex u with $d_G(u) \leqslant 7$ and a pre-orientation τ_u of $E_G(u)$ in which u is balanced modulo 3 such that G has no $\mathcal{M}_3$-extension of τ_u. Among all possible counterexamples G, choose the one with $|V(G)| + |E(G - u)|$ minimized. It implies that for any graph G' with $\kappa'(G') \geqslant 5$ and a 7^--vertex u',

$$\text{Theorem 9(i) is applicable for } G' \text{ if } |V(G')| + |E(G' - u')| < |V(G)| + |E(G - u)|. \quad (8)$$

Clearly, we can assume that G is loopless. Theorem 9(i) holds naturally if $|V(G)| \leqslant 2$, so assume $|V(G)| \geqslant 3$. Now we embed G in a plane such that it contains no separating 2-cycles. For notational convenience, the new resulting plane graph is still denoted by G.

Claim I. Both of the following hold.

(i) Every $\mathbb{Z}_3$-connected subgraph of $G-u$ has exactly one vertex, i.e., a singleton K_1. Particularly, $|V(G)| \geqslant 4$ and neither $2K_2$ nor W_4 is contained in $G-u$ by Proposition 17.

(ii) G is 2-connected.

Proof. (i) By contradiction, we assume that G' is a $\mathbb{Z}_3$-connected subgraph of $G-u$ with $|V(G')| \geqslant 2$. By Observation 13(i), we know $\kappa'(G/G') \geqslant 5$. Additionally, $d_{G/G'}(u) \leqslant 7$ and $|V(G/G')|+|E(G/G'-u)| < |V(G)|+|E(G-u)|$. Hence, G/G' has an $\mathcal{M}_3$-extension of τ_u by statement (8), and so does G by Lemma 16, a contradiction.

(ii) Suppose that G is not 2-connected and has a cut-vertex v. Let G_1 and G_2 be two connected subgraphs of G such that $V(G_1) \cap V(G_2) = \{v\}$ and $V(G_1) \cup V(G_2) = V(G)$. Clearly, G_1 and G_2 are both planar and 5-edge-connected. By Observation 14, we have $u \neq v$, and we may assume $u \in V(G_1)$. By Claim I(i), we obtain that G_2 is simple and so G_2 contains a 5^{-}-vertex w. Assign a pre-orientation τ'_w at w in G_2 such that $d^+_{\tau'_w}(w) \equiv d^-_{\tau'_w}(w) \pmod 3$. Since $|V(G_1)| + |E(G_1 - u)| < |V(G)| + |E(G - u)|$ and $|V(G_2)|+|E(G_2-w)| < |V(G)|+|E(G-u)|$, by statement (8), G_1 has an $\mathcal{M}_3$-extension τ_1 of τ_u and G_2 has an $\mathcal{M}_3$-extension τ_2 of τ'_w, respectively. Combining the orientations τ_1 and τ_2, we will get an $\mathcal{M}_3$-extension of τ_u on G, contrary to the assumption of G. $\square$

Claim II. $\kappa'_e(G) \geqslant 8$.

Proof. Suppose to the contrary that there is an essential cut of size at most 7, denoted by $[S, S^c]_G$. Clearly, by Observation 13(i), both $G/G[S]$ and $G/G[S^c]$ are planar and 5-edge-connected. We may assume $u \in S$. By statement (8), $G/G[S^c]$ has an $\mathcal{M}_3$-extension τ_1 of τ_u. Denote by u' the contracting vertex of $G/G[S]$. Then, $d_{G/G[S]}(u') \leqslant 7$ since $|[S, S^c]_G| \leqslant 7$. Moreover, we use the restriction of τ_1 on $G/G[S]$ as a pre-orientation $\tau'_{u'}$ at u'. By statement (8), $G/G[S]$ has an $\mathcal{M}_3$-extension τ_2 of $\tau'_{u'}$. The union of orientations τ_1 and τ_2 provides an $\mathcal{M}_3$-extension of τ_u on G, which contradicts the assumption of G. $\square$

Claim III. Both of the following hold.

(i) $\mu(G) \leqslant 2$ and all parallel edges are incident with u.

(ii) Every vertex except u of G is a 5-vertex.

Proof. We shall give a proof of the following claim firstly:

$$\text{every vertex except } u \text{ of } G \text{ is a } 6^{-}\text{-vertex.} \tag{9}$$

Suppose that G contains a vertex $v \neq u$ of degree more than 6. By Claim I(ii), we have $|N_G(v)| \geqslant 2$. So in G, we can find two consecutive but not parallel edges in $E_G(v)$, denoted by vv_1 and vv_2, to take the splitting operation. Denote by G' the new resulting graph. By restricting τ_u on $E_{G'}(u)$ and setting $\tau'(v_jv_i) = \tau(vv_i)$ if vv_i is pre-oriented in G for $\{i, j\} = \{1, 2\}$, we get a pre-orientation τ'_u on $E_{G'}(u)$. Since $\kappa'(G) \geqslant 5$ and by Claim II, we get $\kappa'(G') \geqslant 5$. Following statement (8), the graph G' has an $\mathcal{M}_3$-extension τ' of τ'_u. Then by Lemma 12, the graph G admits an $\mathcal{M}_3$-extension of τ_u, a contradiction. This confirms statement (9).

(i) Claim I(i) implies that all parallel edges are incident with u. Clearly, $\mu(G) \leqslant 2$. Otherwise there exists a vertex $u_1 \in N_G(u)$ such that $\mu_G(u, u_1) \geqslant 3$. Denote $S = \{u, u_1\}$ and we have $d_G(S) \leqslant 7$ by statement (9). Moreover, by Claim I(i) and Observation 14, we get that $[S, S^c]_G$ is an essential cut, contrary to Claim II. This verifies Claim III(i).

(ii) By contradiction, suppose that G contains a vertex $v \neq u$ of degree more than 5. By Claim III(i) and by statement (9), we have $d_G(v) = 6$ and there are at most two parallel edges in $E_G(v)$. Clearly, the edge set $E_G(v)$ consists of three pairs of edges that each of them are consecutive but not parallel in G. With a similar argument above, we split these three pairs of edges successively and delete the vertex v to obtain a new plane graph G' and a new pre-orientation τ'_u on $E_{G'}(u)$. Note that $\delta(G') \geqslant 5$ since $\delta(G) \geqslant 5$. We conclude $\kappa'(G') \geqslant 5$. Otherwise G' has an essential 4^--cut $[S, S^c]_{G'}$ with $\min\{d_G(S), d_G(S \cup \{v\})\} \leqslant 7$ since $\min\{|[v, S]_G|, |[v, S^c]_G|\} \leqslant 3$, which contradicts Claim II. Following statement (8), G' has an $\mathcal{M}_3$-extension τ' of τ'_u. Then by Lemma 12, it provides an $\mathcal{M}_3$-extension of τ_u on G, a contradiction again. This proves Claim III(ii). $\square$

By Claims II and III, we conclude that G satisfies all conditions in Lemma 23. Hence by Lemma 23(a), G contains a VGC $(H; x_2)$. Let $G^* = GR(G; H; x_2)$ be a Grötzsch Reduction of G as constructed in Definition 20. Note that G^* is connected by Lemma 22.

Claim IV. Except the 2-cut $[x_2^1, V(G^*) \setminus \{x_2^1\}]_{G^*}$, the graph G^* contains no other 4^--cut.

Proof. By contradiction, suppose that $[S, S^c]_{G^*}$ is another 4^--cut in G^* other than $[x_2^1, V(G^*) \setminus \{x_2^1\}]_{G^*}$. Since $\kappa'(G) \geqslant 5$, for each i and j satisfying $1 \leqslant j < i \leqslant 5$ and $i \notin \{j+1, j+4\}$, we have $x_j x_i \notin E(G)$ by Lemma 15. By the construction of G^*, we have $d_{G^*}(x_2^1) = 2$, $|N_{G^*}(x_2^1)| = 2$, $d_{G^*}(u) \geqslant 5$ and $d_{G^*}(v) = 5$ for each $v \in V(G^*) \setminus \{x_2^1, u\}$. Then we know that $[S, S^c]_{G^*}$ is essential. By Lemma 22 and by symmetry, only two cases are needed to consider as follows.

First, we suppose $x_2^1 \in S$ and $\{y, z\} \subseteq S^c$. Clearly, $|S| \geqslant 2$. If $|S| = 2$, then $d_{G^*}(S) \geqslant 5$ since $|N_{G^*}(x_2^1)| = 2$ and $d_{G^*}(v) \geqslant 5$ for each $v \in V(G^*) \setminus \{x_2^1\}$. This contradicts $d_{G^*}(S) \leqslant 4$. Hence we assume $|S| \geqslant 3$. Let $S_1 = S \setminus \{x_2^1\}$. We have that $[S_1, S_1^c]_G$ is an essential cut of G with $d_G(S_1) \leqslant d_{G^*}(S) + |\{e_0, e'_0\}| \leqslant 6$. It is a contradiction to Claim II.

Next, we suppose $z \in S$ and $\{x_2^1, y\} \subseteq S^c$. Let $S_1 = (S \setminus \{z\}) \cup \{x_4, x_5\}$. We have that $[S_1, S_1^c]_G$ is an essential cut satisfying $d_G(S_1) = d_{G^*}(S) + |\{x_3x_4, xx_4, xx_5\}| \leqslant 7$, a contradiction to Claim II again. $\square$

The final step. With a similar argument as in the proof of statement (9), by splitting edges e_0, e'_0 and deleting the vertex x_2^1 in G^*, we get a new planar graph G'' and a new pre-orientation τ''_u on $E_{G''}(u)$. Clearly, G'' is loopless and $\delta(G'') \geqslant 5$. Moreover, by Claim IV and by the construction of G'', we have $\kappa'(G'') \geqslant 5$. Following statement (8), G'' has an $\mathcal{M}_3$-extension of τ''_u. Thus, by Lemmas 12 and 21, the graph G^* has an $\mathcal{M}_3$-extension of τ_u and so does G, which contradicts to the assumption of G. This completes the proof. $\square$

Proof of Theorem 9(ii) By way of contradiction, we study a counterexample G with a 5^--vertex u which is not a cut-vertex, and a pre-orientation τ_u of $E_G(u)$ in which u is balanced modulo 3 such that G does not have an $\mathcal{M}_3$-extension of τ_u. Let $\mathcal{G}$ be the set of counterexamples G with $|V(G)| + |E(G-u)|$ minimized. Assume that G is a graph in $\mathcal{G}$ with $d_G(u)$ maximized. Clearly, each component of G contains at least 4 vertices. Let H be a component of G. If H has an essential 5^--cut $[S, S^c]_H$ with $u \notin S$, then by a similar argument as in Claim II, we apply inductions on the graphs $G/G[S]$ and $H/H[S^c]$ to obtain two modulo 3-orientations and combine them together to get an $\mathcal{M}_3$-extension of τ_u in G. Thus, $\kappa'_e(H) \geqslant 6$. Moreover, if $u \notin V(H)$, then H contains no parallel edges. So H contains a 5^--vertex which is not a cut-vertex by the planarity of H and $\kappa'_e(H) \geqslant 6$. This implies that G has only one component. Hence, G is connected, $|V(G)| \geqslant 4$, and $\kappa'_e(G) \geqslant 6$.

By Theorem 9(i), G must contain a trivial 4^--cut, i.e., a 2-vertex or 4-vertex v. First, suppose $v \neq u$. By a similar argument as in the proof of statement (9), we split two edges incident with v which are consecutive but not parallel to get a new planar graph G' and a new pre-orientation τ'_u on $E_{G'}(u)$. Clearly, $\kappa'_o(G') \geqslant 5$ and u is not a cut-vertex of G' since $\kappa'_o(G) \geqslant 5$ and $\kappa'_e(G) \geqslant 6$. The minimality of G implies that G' has an $\mathcal{M}_3$-extension of τ'_u and so does G by Lemma 12, a contradiction. Next, suppose $v = u$. Replace an edge $e_1 \in E_G(u)$ in G by two parallel edges and assign them the opposite direction of e_1 if $d_G(u) = 4$ (by four parallel edges and assign them the same direction of e_1 if $d_G(u) = 2$, respectively) to get a new graph G'. Clearly, $d_{G'}(u) > d_G(u)$ and $G' \in \mathcal{G}$, which contradicts the choice of G. This completes the proof. $\square$

5 Proof of Theorem 11

We need the following lemma of Zhang [30] before proceeding the next proof.

Lemma 24. (Zhang [30]) *Let G be a graph with a vertex $v \in V(G)$ satisfying $d_G(v) \neq \kappa'_o(G)$. Let $b = d_G(v)$ and use $\{e_1, \ldots, e_b\}$ to arbitrarily label the edges in $E_G(v)$. Then there exists an index $i \in \{1, \ldots, b\}$ such that the new graph G' obtained from G by splitting e_i and e_{i+1} (the index is taken modulo b) satisfies $\kappa'_o(G') = \kappa'_o(G)$.*

Let us recall the statement of Theorem 11. Note that the condition of being connected is necessary to guarantee flow-admissible property of the signed graph below.

Theorem 11. *Every connected signed planar graph G with $\kappa'_o(G) \geqslant 5$ and $|E^-(G)| = 2$ admits a modulo 3-orientation.*

Proof of Theorem 11 By way of contradiction, suppose that Theorem 11 is false and assume that the signed graph G is a minimum counterexample with respect to $|V(G)| + |E(G)|$. It implies that for any signed graph G',

$$\text{Theorem 11 is applicable for } G' \text{ if } |V(G')| + |E(G')| < |V(G)| + |E(G)|. \tag{10}$$

Clearly, G contains no positive loops. Moreover, Theorem 11 holds trivially if G contains at most two vertices. Hence, assume $|V(G)| \geqslant 3$. In the notation of a signed graph, the signature is sometimes omitted if there is no chance for confusion. We embed G in a plane such that it contains no separating 2-cycles. For notational convenience, such a planar embedding of G is still denoted by G. Set $E^-(G) = \{e_1, e_2\}$.

5.1 Properties of a Minimum Counterexample to Theorem 11

In this subsection, we establish some basic properties of the minimum counterexample G.

Claim 25. *Both of the following hold.*

(i) Every $\mathbb{Z}_3$-connected subgraph of the ordinary graph $G-e_1-e_2$ has exactly one vertex. Particularly, neither $2K_2$ nor W_4 is contained in $G-e_1-e_2$ as ordinary subgraphs by Proposition 17.

(ii) G is 2-connected.

Proof. (i) By contradiction, assume that G_1 is a $\mathbb{Z}_3$-connected subgraph of $G-e_1-e_2$ with $|V(G_1)| \geqslant 2$. Clearly, G/G_1 contains neither 1-cut nor 3-cut by Observation 13(i). By statement (10) and Lemma 16, G/G_1 admits a modulo 3-orientation and so does G, a contradiction.

(ii) Suppose that G is not 2-connected and has a cut-vertex v. Similar as the proof of Claim I of Theorem 9, G can be decomposed into two connected signed planar subgraphs without 1-cut or 3-cut, which are denoted by G_1 and G_2. First, by symmetry, suppose that G_1 contains two negative edges. Clearly, G_2 is simple by (i) and so G_2 contains a 5^--vertex u. If u is a cut-vertex of G_2, then $d_{G_2}(u) = 4$ since $\kappa'_o(G) \geqslant 5$. In this case, decompose G_2 into two smaller connected planar subgraphs G_2^1 and G_2^2 without 1-cut or 3-cut such that u is not a vertex-cut in either. Then we obtain that G_1 has a modulo 3-orientation by statement (10), and G_2 (or each of G_2^1 and G_2^2) has a modulo 3-orientation by Theorem 9(ii). Their union is a modulo 3-orientation of G, a contradiction. So by symmetry, suppose that there exists exactly one negative edge in G_1 and the other in G_2. For each $i \in \{1, 2\}$, adding a negative loop e_{i+2} at v in G_i to obtain the signed planar graph G'_i. By statement (10), G_1 admits a modulo 3-orientation τ_1 where e_3 is a sink edge, and G_2 admits a modulo 3-orientation τ_2 where e_4 is a source edge. Let τ'_i be the restriction of τ_i on G. The union of τ'_1 and τ'_2 is a modulo 3-orientation of G, a contradiction again. □

Claim 26. *G is loopless and 5-regular.*

Proof. We first claim that G is loopless. Otherwise assume that G contains a negative loop e_1 with endpoint v. Clearly, v is also incident with a positive edge $e_3 = vv_1$ since G is connected and has no 1-cut. Lift two edges e_1 and e_3 in G to get a new signed planar graph G'. Clearly, G' has exactly two negative edges and contains no 1-cut or 3-cut. Thus, by statement (10) and Lemma 12, G' admits a modulo 3-orientation and so does G, a contradiction. Thus, G is loopless.

Now, let us prove that G is 5-regular. Suppose that G is not 5-regular and so contains a vertex v satisfying $d_G(v) \neq 5$. Since G has no 1-cut or 3-cut, we have $d_G(v) \neq 1$ and $d_G(v) \neq 3$. By Lemma 24, $E_G(v)$ contains two consecutive edges in G such that the signed planar graph G', obtained by splitting this pair of edges, has $\kappa'_o(G') \geqslant 5$. Clearly, G' has zero or two negative edges. So by statement (10) or Theorem 9(ii) and by Lemma 12, G' admits a modulo 3-orientation and so dose G, a contradiction again. □

Claim 27. *G contains no essential 7^--cut $[S,S^c]_G$ such that $G[S]$ contains no negative edge.*

Proof. By contradiction, let $[S,S^c]_G$ be a counterexample to Claim 27 such that $|S|$ is minimum. Denote $G_1 = G/G[S]$ and $G_2 = G/G[S^c]$. Set $V(G_1) = S^c \cup \{u\}$ and $V(G_2) = S \cup \{v\}$. By Observation 13(i), G_1 and G_2 are planar signed graphs without 1-cut or 3-cut. Since G_1 contains two negative edges, G_1 has a modulo 3-orientation τ_1 by statement (10).

Let G' be a signed graph obtained from G_2 by replacing each negative edge in G_2 with a positive edge and deleting the resulting positive loops. Now, we give a pre-orientation τ'_v at v in G' as follows: Transfer the direction of every positive edge in $E_{G_1}(u)$ in τ_1 to its corresponding edge incident with v in G'. For any edge $e = u_1v \in E_{G'}(v)$ whose corresponding edge $e' \in E_{G_1}(u)$ is negative, orienting e away from u_1 and towards v if e' is a sink edge in τ_1, and orienting e away from v and towards u_1 if e' is a source edge in τ_1. Clearly, G' contains no loop, $\kappa'_o(G') \geqslant 5$ and $d^+_{\tau'_v}(v) \equiv d^-_{\tau'_v}(v) \pmod 3$ since u is balanced in τ_1. Since $[S,S^c]_G$ is a cut, $G[S]$ is connected and then the vertex v is not a cut-vertex in G'. Since $d_{G'}(w) = 5$ for each $w \in V(G') \setminus \{v\}$, by the minimality of $|S|$, we have $\kappa'(G') \geqslant 5$ if $d_{G'}(v) \geqslant 5$. Thus, G' admits an $\mathcal{M}_3$-extension of τ'_v by Theorem 9. In addition, this can be extended to an orientation τ_2 of G_2 such that τ_1 and τ_2 agree along $[S,S^c]_G$ and $d^+_{\tau_2}(z) \equiv d^-_{\tau_2}(z) \pmod 3$ for each $z \in S$. Thus, the union of τ_1 and τ_2 provides a modulo 3-orientation of G, contrary to the assumption. □

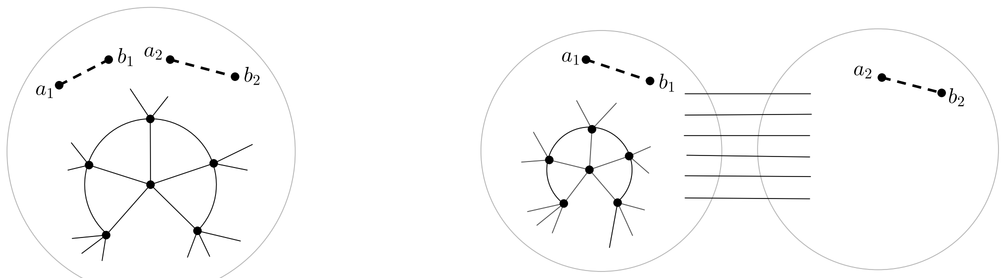
Claim 28. *G is 4-edge-connected.*

Proof. By contradiction, suppose that there is a 2-cut in G which is denoted by $[S,S^c]_G$, since G has no 1-cut or 3-cut. By Claims 26 and 27, we have that $[S,S^c]_G$ is essential and each of $G[S]$ and $G[S^c]$ contains exactly one negative edge. Let $G_1 = G/G[S]$ and $G_2 = G/G[S^c]$. By Observation 13(i), G_1 and G_2 have no 1-cut or 3-cut. Following statement (10), for each $i \in \{1,2\}$, we have that G_i has a modulo 3-orientation τ_i in which e_1 is a sink edge and e_2 is a source edge. Clearly, τ_1 and τ_2 agree along $[S,S^c]_G \cup \{e_1,e_2\}$. Thus, by combining τ_1 and τ_2, G has a modulo 3-orientation, which contradicts the assumption. □

Claim 29. *G contains no adjacent negative edges.*

Proof. To the contrary, suppose $\{e_1,e_2\} \subseteq E_G(v)$. By Claim 26, neither e_1 nor e_2 are loops. So for each $i \in \{1,2\}$, e_i has an endpoint other than v, which is denoted by v_i. Replace the negative edge e_i by a positive edge e'_i in G to get a new signed graph G'. Besides, we obtain a pre-orientation τ'_v at v in G' by orienting e'_1 away from v and towards v_1 and orienting each edge $e = uv \in E_{G'}(v) \setminus \{e'_1\}$ away from u and towards v. Note that G' is 2-connected and $\kappa'_o(G') \geqslant 5$. By Theorem 9(ii), G' has an $\mathcal{M}_3$-extension τ' of τ'_v. With the restriction of τ' on G, we obtain a modulo 3-orientation of G by orienting e_1 as a source edge and orienting e_2 as a sink edge. This leads to a contradiction. □

The idea of the remaining proof is similar to that of Theorem 9. That is, we would like to find a VGC for reductions. In the proof of Theorem 9, we observe that high essential connectivity is necessary for finding a VGC to apply Grötzsch Reduction. So the remaining proof is split into two cases: *Case A. highly essential connected case and Case B. existence of small essential cuts.* For the former Case A, we directly find a VCG in G, which is sufficient for reduction. For the latter Case B, we first choose a minimum subset U such that $[U, U^c]_G$ is a small essential cut, then $G_1 = G/G[U^c]$ will has high essential connectivity, and so we can find a VGC inside $G[U]$ to perform a desired Grötzsch Reduction. See Figure 7. The notations mentioned in Lemma 23 are applied in the remaining proof. Recall that the positive edge parallel to the negative edge e_i (if exists) is denoted by e'_i for each $i \in \{1, 2\}$. By Claim 29, we have $\{a_1, b_1\} \cap \{a_2, b_2\} = \emptyset$.



(a) A VGC in Case A (b) A VGC in Case B

Figure 7: Finding a VGC in Cases A and B.

5.2 Case A. Highly Essential Connected Case

In this subsection, we consider the case that *there is no essential cut* $[W, W^c]_G$ *in* G *with* $d_G(W) \leqslant 6$ *and* $\{W, W^c\} \cap \{\{a_1, b_1\}, \{a_2, b_2\}\} = \emptyset$. Thus, $\kappa'(G) \geqslant 5$ in this case.

By Claims 25-29, we conclude that G satisfies all conditions in Lemma 23. Hence by Lemma 23(b), G contains a VGC $(H; x_2)$. Let $G^* = GR(G; H; x_2)$ be a Grötzsch Reduction of G as constructed in Definition 20. Then we have $|E^-(G^*)| = 2$ by the construction of G^*, and G^* is connected by Lemma 22.

Claim 30. *Except the* 2*-cut* $[x_2^1, V(G^*) \setminus \{x_2^1\}]_{G^*}$*,* G^* *contains no other* 3^-*-cut in Case A.*

Proof. By contradiction, suppose that, except $[x_2^1, V(G^*) \setminus \{x_2^1\}]_{G^*}$, there exists another 3^--cut $[S, S^c]_{G^*}$. Since $\kappa'(G) \geqslant 5$, for each i and j satisfying $1 \leqslant j < i \leqslant 5$ and $i \notin \{j+1, j+4\}$, we have $x_j x_i \notin E(G)$ by Lemma 15. By the construction of G^*, G^* is loopless and we have $d_{G^*}(x_2^1) = 2$, $|N_{G^*}(x_2^1)| = 2$ and $d_{G^*}(v) = 5$ for each $v \in V(G^*) \setminus \{x_2^1\}$. Thus, we know that $[S, S^c]_{G^*}$ is essential. By Lemma 22 and by symmetry, only two cases are needed to consider as follows.

First, we suppose $x_2^1 \in S$ and $\{y, z\} \subseteq S^c$. Clearly, $|S| \geqslant 2$. If $|S| = 2$, then $d_{G^*}(S) \geqslant 5$ since $|N_{G^*}(x_2^1)| = 2$ and each vertex of G^* other than x_2^1 has degree 5. This contradicts $d_{G^*}(S) \leqslant 3$. Hence we assume $|S| \geqslant 3$. Let $S_1 = S \setminus \{x_2^1\}$. Clearly, $[S_1, S_1^c]_G$ is an essential cut of G satisfying $d_G(S_1) \leqslant d_{G^*}(S) + |\{e_0, e'_0\}| \leqslant 5$ by Observation 14, a contradiction to the assumption in Case A.

Next, suppose $z \in S$ and $\{x_2^1, y\} \subseteq S^c$. Let $S_1 = (S \setminus \{z\}) \cup \{x_4, x_5\}$. By Observation 14, we have that $[S_1, S_1^c]_G$ is an essential cut of G with $d_G(S_1) = d_{G^*}(S) + |\{x_3x_4, xx_4, xx_5\}| \leqslant 6$, $|S_1| \geqslant 3$ and $|S_1^c| \geqslant 3$, a contradiction again. This proves Claim 30. □

5.3 Case B. Existence of Small Essential Cuts

In this subsection, we consider the case that *there is an essential cut* $[W, W^c]_G$ *in* G *with* $d_G(W) \leqslant 6$ *and* $\{W, W^c\} \cap \{\{a_1, b_1\}, \{a_2, b_2\}\} = \emptyset$.

Denote by U the minimum subset of $V(G)$ among all possible choices such that $[U, U^c]_G$ is an essential 6^--cut and $\{U, U^c\} \cap \{\{a_1, b_1\}, \{a_2, b_2\}\} = \emptyset$. By the minimality of U, for any subset $U_1 \subset U$ with $|U_1| \geqslant 2$, we have $d_G(U_1) \geqslant 7$ unless $U_1 \in \{\{a_1, b_1\}, \{a_2, b_2\}\}$. Moreover, each of $G[U]$ and $G[U^c]$ has exactly 1 negative edge by Claim 27. Since G is 5-regular and contains no positive parallel edges by Claims 25 and 26, it is easy to check that

$$|U| \geqslant 4 \text{ and } |U^c| \geqslant 4.$$

Recall that e_1 and e_2 are two negative edges of G. Without loss of generality, assume $e_1 \in E(G[U])$ and $e_2 \in E(G[U^c])$.

Let $G_1 = G/G[U^c] - e_2$ and denote $V(G_1) = U \cup \{u\}$, where u is the contracting vertex. Clearly, G_1 is a loopless 2-connected signed plane graph without separating 2-cycles. We have $4 \leqslant d_{G_1}(u) \leqslant 6$, $\delta(G_1) \geqslant 4$ and $e_1 \in E(G_1) \setminus E_{G_1}(u)$. By the choice of U, each essential cut of G_1 has size at least 6, and so $\kappa'(G_1) \geqslant 4$ and $\kappa'_e(G_1) \geqslant 6$.

Claim 31. *$\mu(G_1) \leqslant 2$, and if $\mu_{G_1}(u, v) = 2$ for $v \in N_{G_1}(u)$ then v is not incident with e_1.*

Proof. First, we claim that $\mu(G_1) \leqslant 2$. Otherwise suppose that G_1 has two vertices v and v_1 with $\mu_{G_1}(v, v_1) \geqslant 3$. By the construction of G_1, we get $u \in \{v, v_1\}$ since $\mu(G) \leqslant 2$ by Claims 25 and 29. Without loss of generality, assume $u = v$. It implies that v_1 has at least three neighbors in U^c and at most two incident edges in $G[U]$. Let $U_1 = U \setminus \{v_1\}$. Then we have $|U_1| \geqslant 3$ and $[U_1, U_1^c]_G$ is an essential 5^--cut by Observation 14, contrary to the minimality of U. Hence $\mu(G_1) \leqslant 2$.

Next, we suppose, for a contradiction, that G_1 contains a vertex v such that $\mu_{G_1}(u, v) = 2$ and v is incident with e_1. Let $U_2 = U \setminus \{v\}$. Then we have that $[U_2, U_2^c]_G$ is an essential 7^--cut and $G[U_2]$ contains no negative edge, contrary to Claim 27. □

By the choice of U and the construction of G_1, we conclude that G_1 satisfies $d_{G_1}(u) \leqslant 6$ and all conditions (I)-(V) in Lemma 23 when we set $E^-(G_1) = \{e_1\}$ and $X = \{u\}$. Moreover, Claim 31 implies that $(\{u\} \cup L_G(u)) \cap \{a_1, b_1\} = \emptyset$. Hence, G_1 contains an NGC H_1 if $|L_{G_1}(u)| = 3$ and a VGC H_2 if $|L_{G_1}(u)| \leqslant 2$ by Lemma 23(c). Define $H = H_1$ if $|L_{G_1}(u)| = 3$ and $H = H_2$ if $|L_{G_1}(u)| \leqslant 2$. We claim below that H is indeed a VGC of G.

Claim 32. *H is a VGC of G inside U.*

Proof. Note that, for each $i \in \{1,2\}$, the center of H_i is not in $N_{G_1}[u] \cup \{a_1, b_1\}$. The statement is clear for $|L_{G_1}(u)| \leqslant 2$. It remains to show that, when $|L_{G_1}(u)| = 3$, the NGC $H_1 = H$ of G_1 is a VGC of G inside U.

Assume that the vertices of H are labeled as shown in Figure 3(b). Since $u \notin V(H)$ in G_1, G also contains H as an NGC. Since $\kappa'(G_1) \geqslant 4$ and $\kappa'_e(G_1) \geqslant 6$, G_1 does not contain edges joining x_2 and x_4 by Lemma 15. Neither does G. It implies that x_2 or x_4 is not incident with e_1 in G, and then one of them is not incident with parallel edges in G. So H is a VGC of G. $\square$

Assume that x_2 is a good corner of H. Let $G^* = GR(G; H; x_2)$ be a Grötzsch Reduction of G as constructed in Definition 20 (see Figure 3(d)). Clearly, G^* is connected by Lemma 22 and $|E^-(G^*)| = 2$ by the construction of G^*. Note that $U^c \subseteq V(G^*)$. We further claim the following.

Claim 33. *Except the* 2*-cut* $[x_2^1, V(G^*) \setminus \{x_2^1\}]_{G^*}$*,* G^* *contains no other* 3^-*-cut in Case B.*

Proof. By contradiction, suppose that there exists a 3^--cut $[S, S^c]_{G^*}$ with $[S, S^c]_{G^*} \neq [x_2^1, V(G^*) \setminus \{x_2^1\}]_{G^*}$. Since $\kappa'(G_1) \geqslant 4$ and $\kappa'_e(G_1) \geqslant 6$, by Lemma 15, G_1 contains no edges joining x_i and x_j if $1 \leqslant j < i \leqslant 5$ and $i \notin \{j+1, j+4\}$. Neither does G. So we have $d_{G^*}(x_2^1) = 2$, $|N_{G^*}(x_2^1)| = 2$ and $d_{G^*}(v) = 5$ for each $v \in V(G^*) \setminus \{x_2^1\}$. This implies that G^* is loopless and $[S, S^c]_{G^*}$ is essential.

Subclaim 1. $U^c \cap S \neq \emptyset$ and $U^c \cap S^c \neq \emptyset$.

Proof. By contradiction, suppose $U^c \subseteq S$ or $U^c \subseteq S^c$. By symmetry, we assume $U^c \subseteq S^c$. Clearly, $|S^c| \geqslant |U^c| \geqslant 4$. By Lemma 22, we consider four cases as follows.

First, suppose $x_2^1 \in S$ and $\{y, z\} \subseteq S^c$. Clearly, $|S| \geqslant 2$. If $|S| = 2$, then $d_{G^*}(S) \geqslant 5$ since $|N_{G^*}(x_2^1)| = 2$ and each vertex of G^* other than x_2^1 has degree 5. This contradicts $d_{G^*}(S) \leqslant 3$. Hence, we assume $|S| \geqslant 3$. Let $S_1 = S \setminus \{x_2^1\}$. Clearly, $S_1 \subset U$, and $[S_1, S_1^c]_G$ is an essential cut of G satisfying $d_G(S_1) \leqslant d_{G^*}(S) + |\{e_0, e'_0\}| \leqslant 5$ by Observation 14, contrary to the minimality of U.

Next, suppose $x_2^1 \in S^c$ and $\{y, z\} \subseteq S$. Let $S_1 = (S^c \setminus \{x_2^1\}) \cup \{x_2\}$. We have $S_1^c = V(G) \setminus S_1 \subset U$, and $[S_1, S_1^c]_G$ is an essential cut of G with $d_G(S_1^c) \leqslant d_{G^*}(S) + |\{x_2x_1, x_2x, x_2x_3\}| \leqslant 6$, $|S_1| \geqslant 3$ and $|S_1^c| \geqslant 3$ by Observation 14, leading to a contradiction.

Then, suppose $z \in S$ and $\{x_2^1, y\} \subseteq S^c$. Let $S_1 = (S \setminus \{z\}) \cup \{x_4, x_5\}$. Clearly, $S_1 \subset U$. By Observation 14, we have that $[S_1, S_1^c]_G$ is an essential cut of G with $d_G(S_1) = d_{G^*}(S) + |\{x_3x_4, xx_4, xx_5\}| \leqslant 6$, $|S_1| \geqslant 3$ and $|S_1^c| \geqslant 3$, contradicting the choice of U.

Finally, suppose $z \in S^c$ and $\{x_2^1, y\} \subseteq S$. Let $S_1 = (S^c \setminus \{z\}) \cup \{x_4, x_5\}$. We have $S_1^c = V(G) \setminus S_1 \subset U$ and, by Observation 14, $[S_1, S_1^c]_G$ is an essential cut of G with $d_G(S_1^c) = d_{G^*}(S) + |\{x_3x_4, xx_4, xx_5\}| \leqslant 6$, $|S_1| \geqslant 3$ and $|S_1^c| \geqslant 3$, a contradiction again. $\square$

Note that Subclaim 1 holds for any 3^--cut $[S, S^c]_{G^*}$ of G^* with $[S, S^c]_{G^*} \neq [x_2^1, V(G^*) \setminus \{x_2^1\}]_{G^*}$ (if exists).

Subclaim 2. Such a 3^--cut $[S, S^c]_{G^*}$ does not exist. Hence Claim 33 holds.

Proof. Let $S_1 = U^c \cap S$ and $S_2 = S \setminus S_1$. Denote $S_3 = U^c \cap S^c$ and $S_4 = S^c \setminus S_3$. By Subclaim 1, $S_1 \neq \emptyset$ and $S_3 \neq \emptyset$. Recall that $[S, S^c]_{G^*}$ separates the set $\{x_2^1, y, z\}$. Since $\{x_2^1, y, z\} \subseteq S_2 \cup S_4$, we know $S_2 \neq \emptyset$ and $S_4 \neq \emptyset$ as well. By the construction of G^* from G, only the edges in H are deleted or contracted, and any edge incident with at least one vertex in U^c remains unchanged. Hence $|[S_1, S_3]_G| = |[S_1, S_3]_{G^*}|$, $d_G(S_1) = d_{G^*}(S_1)$ and $d_G(S_3) = d_{G^*}(S_3)$. Thus we have

$$d_G(S_1) + d_G(S_3) = 2|[S_1, S_3]_G| + d_G(U^c) \leqslant 2|[S_1, S_3]_{G^*}| + 6. \tag{11}$$

Recall that e_2 is a negative edge of G and $e_2 \in E(G[U^c])$. Without loss of generality, assume that $G[S_3]$ has no negative edges. By Claim 28, we have $d_G(S_1) \geqslant 4$.

Suppose $|S_3| \geqslant 2$. By Claims 27 and 28, we have $d_G(S_3) \geqslant 8$ and then $|[S_1, S_3]_{G^*}| \geqslant \frac{4+8-6}{2} = 3$ by Equation (11). It implies that $[S_1, S_3]_{G^*} = [S, S^c]_{G^*}$, $|[S_1, S_3]_{G^*}| = 3$ and $|[S_2, S_4]_{G^*}| = 0$ since $d_{G^*}(S) \leqslant 3$. Then we have $|[S_1, S_2]_{G^*}| = 1$ and $|[S_3, S_4]_{G^*}| = 5$ since $d_{G^*}(U^c) \leqslant 6$. Hence $[S_2, S_1 \cup S_3 \cup S_4]_{G^*}$ is a 1-cut of G^*, contrary to Subclaim 1.

Now we assume $|S_3| = 1$ and say $S_3 = \{s_3\}$. Since G is 5-regular, we have $d_{G^*}(S_3) = 5$ and so $|[S_1, S_3]_{G^*}| \geqslant \frac{4+5-6}{2}$ by Equation (11), which implies $|[S_1, S_3]_{G^*}| \geqslant 2$. Hence, it follows from $d_{G^*}(S) \leqslant 3$ that $3 \geqslant |[S_3, S_4]_{G^*}| \geqslant 2$ and $|[S_1, S_2]_{G^*}| \geqslant 1$. Then $|S_4| \geqslant 2$ since $d_{G^*}(x_2^1) = 2$, $|N_{G^*}(x_2^1)| = 2$, $d_{G^*}(v) = 5$ for each $v \in V(G^*) - \{x_2^1\}$ and $d_{G^*}(S) \leqslant 3$. Furthermore, both $G^*[S_1 \cup S_3 \cup S_4]$ and $G^*[S_1 \cup S_3 \cup S_2]$ are connected since $G^*[S]$, $G^*[S^c]$ and $G^*[U^c]$ are all connected. Fix $i \in \{2, 4\}$. If $d_{G^*}(S_i) = 1$ or $d_{G^*}(S_i) = 3$, then there is a 1-cut or 3-cut $[A_i, A_i^c]_{G^*}$ with $A_i \subseteq S_i$ and $U^c \subseteq A_i^c$. This is a contradiction to Subclaim 1. If $d_{G^*}(S_i) = 2$ and $|S_i| \geqslant 2$, then there is a 2^--cut $[A_i, A_i^c]_{G^*}$ other than $[x_2^1, V(G^*) \setminus \{x_2^1\}]_{G^*}$ with $A_i \subseteq S_i$ and $U^c \subseteq A_i^c$, a contradiction to Subclaim 1 again. If $d_{G^*}(S_i) = 2$ and $|S_i| = 1$, then $i = 2$ and $S_2 = \{x_2^1\}$ since G^* contains exactly one 2-vertex x_2^1. We have $|[S_1, S_2]_{G^*}| = 2$ since $d_{G^*}(S_1) \geqslant 4$ and $d_{G^*}(S) \leqslant 3$. It implies that the VGC H of G mentioned in Claim 32 is an NGC but not a VGC of G_1, since a good corner of a VGC of G_1 must be adjacent to at least four vertices in U. Then by Lemma 23(c) and by the choice of H, we have $|L_{G_1}(u)| = 3$ and $d_{G^*}(U^c) = d_G(U^c) = 6$. Moreover, it is clear that $|[S_3, S_4]_{G^*}| = 3$. Since x_2 has two distinct neighbors in $U^c \setminus \{s_3\}$ and $|L_{G_1}(u)| = 3$, there must be two positive edges of $[U, U^c]_G$ which are incident with s_3 and parallel. This is a contradiction to Claim 25. Hence, we conclude that $d_{G^*}(S_2) \geqslant 4$ and $d_{G^*}(S_4) \geqslant 4$. By counting number of edges between those parts and by the fact that $d_{G^*}(U^c) \leqslant 6$, $d_{G^*}(S) \leqslant 3$ and $|[S_1, S_3]_{G^*}| \geqslant 2$, we have $d_{G^*}(S_2) = 4$ and $d_{G^*}(S_4) = 4$. Moreover, $|[S_2, S_4]_{G^*}| = 1$, $|[S_1, S_2]_{G^*}| = 3$ and $|[S_3, S_4]_{G^*}| = 3$. It is clear that $|S_2| \geqslant 2$ and $|S_4| \geqslant 2$ since G^* contains no 4-vertex. Now we set $\{A, B\} = \{S_2, S_4\}$ for convenience. Thus we have that

$$|A| \geqslant 2, \ |B| \geqslant 2, \text{ and } d_{G^*}(A) = d_{G^*}(B) = 4.$$

By symmetry and by Lemma 22, we consider two cases as follows.

First, suppose $x_2^1 \in A$ and $\{y, z\} \subseteq B$. Let $B_1 = (B \setminus \{y, z\}) \cup \{x, x_1, x_2, x_3, x_4, x_5\}$. Then it is clear that $B_1 \subset U$, $|B_1| \geqslant 3$ and $|B_1^c| \geqslant |U^c| \geqslant 4$. Hence, by Observation 14, $[B_1, B_1^c]_G$ is an essential cut of G with $d_G(B_1) \leqslant d_{G^*}(B) + |\{e_0, e_0'\}| = 6$, which contradicts the choice of U.

Then, suppose $z \in A$ and $\{x_2^1, y\} \subseteq B$. Let $A_2 = (A \setminus \{z\}) \cup \{x_4, x_5\}$ and $B_2 = (B \setminus \{x_2^1, y\}) \cup \{x, x_1, x_2, x_3\}$. By Observation 14, we have that $[A_2, A_2^c]_G$ and $[B_2, B_2^c]_G$ are both essential cuts of G with $d_G(A_2) = d_{G^*}(A) + |\{x_3x_4, xx_4, xx_5\}| = 7$ and $d_G(B_2) = d_{G^*}(B) + |\{x_3x_4, xx_4, xx_5\}| = 7$. Clearly, one of $G[A_2]$ and $G[B_2]$ contains no negative edges, which contradicts Claim 27. This proves Claim 33. $\square$

This completes the proof of Claim 33. $\square$

5.4 The final step

By Claims 30 and 33, in any case G^* is a connected signed planar graph with $\kappa'_o(G^*) \geqslant 5$ and $|E^-(G^*)| = 2$. The minimality of G implies that G^* has a modulo 3-orientation, and so does G by Lemma 21, a contradiction. This completes the proof. $\square$

6 Conclusion

In this paper, we generalize Grötzsch's theorem to signed planar graphs by showing that every 4-edge-connected signed planar graph with two negative edges admits a 3-NZF. This is also related to the existence of 3-NZFs in some ordinary graphs which are close to be planar. In [7], it is indicated that Thomassen in 1993 proposed the following problem.

Problem 34. *(Thomassen, see [7] Page 212)* Is it possible to prove the following: if G is a 4-edge-connected graph and there exists an edge $e \in E(G)$ such that $G - e$ is planar, then G admits a 3-NZF ?

Thomassen [7] suggested that an affirmative answer to the above question would imply that every 4-edge-connected graph embedded on the torus admits a 3-NZF. With a simple argument, we can also obtain a corollary of Theorem 11 (in fact, a somehow equivalent form of Theorem 11) below, which is closely related to Problem 34.

Corollary 35. *Let G be an ordinary graph. If there exists an edge $e \in E(G)$ such that $G - e$ is a 4-edge-connected planar graph, then G admits a 3-NZF.*

Proof. Let $e = xy$ be an edge in G such that $G - e$ is a 4-edge-connected planar graph. In $G - e$, we add a negative loop e_1 in x and a negative loop e_2 in y, respectively, and assign positive signs to $E(G) - e$ to obtain a new signed graph (H, σ). Now (H, σ) is a 4-edge-connected signed planar graph with two negative edges, which admits a modulo 3-orientation by Theorem 11. It is worth noting that the difference between the number of sink edges and the number of source edges must be a multiple of 3. Therefore, e_1 and e_2 must receive opposite directions. Without loss of generality, we may assume that e_1 is a source edge and e_2 is a sink edge. By deleting these two negative loops and assigning a direction to the edge e directed from x to y, we can obtain a modulo 3-orientation of G. Thus, G admits a 3-NZF. $\square$

Actually, the methods developed in this paper have been used to solve Problem 34 affirmatively by all authors of this paper together with C.-Q. Zhang [13], and furthermore lead to a proof of Tutte's 3-flow conjecture for all toroidal graphs.

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References

[1] J. A. Bondy and U. S. R. Murty. Graph Theory. Springer, New York, 2008.

[2] A. Bouchet. Nowhere-zero integral flows on a bidirected graph. *J. Combin. Theory Ser. B*, 34: 279–292, 1983.

[3] M. DeVos, J. Li, Y. Lu, R. Luo, C.-Q. Zhang, and Z. Zhang. Flows on flow-admissible signed graphs. *J. Combin. Theory Ser. B*, 149: 198–221, 2021.

[4] H. Grötzsch. Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur., Reihe 8: 109–120, 1959.

[5] F. Jaeger. Flows and generalized coloring theorems in graphs. *J. Combin. Theory Ser. B*, 26: 205–216, 1979.

[6] F. Jaeger, N. Linial, C. Payan, and M. Tarsi. Group connectivity of graphs - A nonhomogeneous analogue of nowhere-zero flow properties. *J. Combin. Theory Ser. B*, 56: 165–182, 1992.

[7] T. R. Jensen and B. Toft. Graph Coloring Problems. John Wiley & Sons, Inc., New York, 1995. A Wiley-Interscience Publication.

[8] M. Kochol. An equivalent version of the 3-flow conjecture. *J. Combin. Theory Ser. B*, 83: 258–261, 2001.

[9] H.-J. Lai. Group connectivity in 3-edge-connected chordal graphs. *Graphs Combin.*, 16: 165–176, 2000.

[10] H.-J. Lai, R. Luo, and C.-Q. Zhang. Integer flow and orientation. in: L. Beineke, R. Wilson (Eds.), Topics in Chromatic Graph Theory, in: Encyclopedia of Mathematics and its Applications, vol. 156: 181–198, 2015.

[11] H. Lebesgue. Quelques conséquences simples de la formule d'Euler. *J. Math. Pures Appl.*, 19: 19–43, 1940.

[12] J. Li, R. Luo, H. Ma, and C.-Q. Zhang. Flow-contractible configurations and group connectivity of signed graphs. *Discrete Math.*, 341: 3227–3236, 2018.

[13] J. Li, Y. Ma, Z. Miao, Y. Shi, W. Wang, and C.-Q. Zhang. Nowhere-zero 3-flows in toroidal graphs. *J. Combin. Theory Ser. B*, 153: 61–80, 2022.

[14] L. M. Lovász, C. Thomassen, Y. Wu, and C.-Q. Zhang. Nowhere-zero 3-flows and modulo k-orientations. *J. Combin. Theory Ser. B*, 103: 587–598, 2013.

[15] E. Máčajová and M. Škoviera. Nowhere-zero flows on signed eulerian graphs. *SIAM J. Discrete Math.*, 31: 1937–1952, 2017.

[16] A. Raspaud and X. Zhu. Circular flow on signed graph. *J. Combin. Theory Ser. B*, 101: 464–479, 2011.

[17] E. Rollová, M. Schubert, and E. Steffen. Flows in signed graphs with two negative edges. *Electron. J. Combin.*, 25: no. 2, #P2.40, 2018.

[18] R. Steinberg and D. H. Younger. Grötzsch's theorem for the projective plane. *Ars Combin.*, 28: 15–31, 1989.

[19] C. Thomassen. Grötzsch's 3-color theorem and its counterparts for the torus and the projective plane. *J. Combin. Theory Ser. B*, 62: 268–279, 1994.

[20] C. Thomassen. A short list color proof of Grötzsch's theorem. *J. Combin. Theory Ser. B*, 88: 189–192, 2003.

[21] C. Thomassen. The weak 3-flow conjecture and the weak circular flow conjecture. *J. Combin. Theory Ser. B*, 102: 521–529, 2012.

[22] W. T. Tutte. On the imbedding of linear graphs in surfaces. *Proc. London Math. Soc.*, Vol. s2-51: 474–483, 1949.

[23] W. T. Tutte. A contribution to the theory of chromatical polynomials. *Canad. J. Math.*, 6: 80–91, 1954.

[24] W. T. Tutte. On the algebraic theory of graph colorings. *J. Combin. Theory*, 1: 15–50, 1966.

[25] X. Wang, Y. Lu, C.-Q. Zhang, and S. Zhang. Six-flows on almost balanced signed graphs. *J. Graph Theory*, 92: 394–404, 2019.

[26] Y. Wu, D. Ye, W. Zang, and C.-Q. Zhang. Nowhere-zero 3-flow in signed graphs. *SIAM J. Discrete Math.*, 28(3): 1628–1637, 2014.

[27] R. Xu and C.-Q. Zhang. On flows in bidirected graphs. *Discrete Math.*, 299: 335–343, 2005.

[28] D. H. Younger. Integer flows. *J. Graph Theory*, 7: 349–357, 1983.

[29] D. A. Youngs. 4-chromatic projective graphs. *J. Graph Theory*, 21: 219–227, 1996.

[30] C.-Q. Zhang. Circular flows of nearly Eulerian graphs and vertex-splitting. *J. Graph Theory*, 40: 147–161, 2002.

[31] C.-Q. Zhang. Integer Flows and Cycle Covers of Graphs. Marcel Dekker, New York, 1997.