Some bounds on the largest eigenvalue of degree-based weighted adjacency matrix of a graph*

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Dedicated to Professor Xueliang Li on the occasion of his 65th birthday

Abstract

Let $f(x,y) > 0$ be a real symmetric function. For a connected graph G, the weight of edge v_iv_j is equal to the value $f(d_i, d_j)$, where d_i is the degree of vertex v_i. The degree-based weighted adjacency matrix is defined as $A_f(G)$, in which the (i,j)-entry is equal to $f(d_i, d_j)$ if v_iv_j is an edge of G and 0 otherwise. In this paper, we first give some bounds of the weighted adjacency eigenvalue $\lambda_1(A_f(G))$ in terms of $\lambda_1(A_f(H))$, where H is obtained from G by some kinds of graph operations, including deleting vertices, deleting an edge and subdividing an edge, and examples are given to show that bounds are tight. Second, we obtain some bounds for the largest weighted adjacency eigenvalue $\lambda_1(A_f(G))$ of irregular weighted graphs.

Keywords: degree-based edge-weight; weighted adjacency matrix(eigenvalue); irregular weighted graph; topological function-index; graph operation

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1 Introduction

In this paper, we only concern with simple undirected connected graphs. Let G be a graph of order n. The vertex set and the edge set of G are denoted by $V(G) =$

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$\{v_1, v_2, \ldots, v_n\}$ and $E(G)$, respectively. If two distinct vertices v_i and v_j are adjacent, then we have $v_iv_j \in E(G)$. For $v_i \in V(G)$, let d_i and $N_G(v_i)$ be the degree and the set of neighbors of v_i, respectively. The maximum degree of graph G is denoted by Δ. The distance between two distinct vertices $v_i, v_j \in V(G)$, denoted by $d_G(v_i, v_j)$, is the length of a shortest path from v_i to v_j in G. The diameter D of G is the maximum value of distances between any two vertices of $V(G)$. For a subset S of $V(G)$, let $G - S$ be the graph obtained from G by removing the vertex subset S together with all edges incident with S. For an edge e, we use $G - e$ to denote the graph obtained from G by removing the edge e. Subdividing an edge $e = v_iv_j$ means that a new vertex $v_{(n+1)}$ is added to $V(G)$ and the edge v_iv_j is replaced in $E(G)$ by two edges $v_iv_{(n+1)}$ and $v_jv_{(n+1)}$. We use G_e to denote the graph obtained from G by subdividing the edge e. For notation and terminology not defined here, one may refer to [6].

In chemistry, topological index is an important graph invariant. A great variety of topological indices based on degree of vertex have been extensive researched(see [1, 3, 7, 10, 16, 23]). The general form of degree-based topological indices is as follows:

$$TI(G) = \sum_{v_iv_j \in E(G)} f(d_i, d_j),$$

where the edge-weight function $f(x, y)$ is a real symmetric function, and the value $f(d_i, d_j)$ is the weight of the edge v_iv_j of G. If the first partial derivative $f_x^{'}(x, y) \geq (\leq) 0$, then we say that $f(x, y)$ is increasing(decreasing) in variable x. It is clearly that each index maps a graph to a single number. In 2015, Li [12] put forward an idea: using a matrix to represent the structure of an edge-weighted graph, we may keep more structural information than a topological index. For example, the first(second) Zagreb matrix [18], Randić matrix [19], atom-bound connectivity matrix [4], arithmetic-geometric matrix [25], extended adjacency matrix [9] and p-Somber matrix [17] were considered separately.

In 2018, Das et al.[8] proposed the weighted adjacency matrix of a graph G of order n, where

$$(A_f(G))_{ij} = \begin{cases} f(d_i, d_j), & if\ v_iv_j \in E(G), \\ 0, & otherwise. \end{cases}$$

The eigenvalues of the weighted adjacency matrix $A_f(G)$ are denoted by $\lambda_1(A_f(G)) \geq \lambda_2(A_f(G)) \geq \cdots \geq \lambda_n(A_f(G))$, where $\lambda_1(A_f(G))$ is the largest weighted adjacency eigenvalue. Let $\mathbf{x} = (x_1, x_2, ..., x_n)^T$ be the eigenvector corresponding to $\lambda_1(A_f(G))$. We have $A_f(G)\mathbf{x} = \lambda_1(A_f(G))\mathbf{x}$. Moreover, the vector $\mathbf{x}$ can be regarded as a mapping on $V(G)$. For any vertex v_i, the entry of $\mathbf{x}$ corresponding to v_i is denoted by x_i. If $\sum\limits_{v_j \in N_G(v_1)} f(d_1, d_j) = \sum\limits_{v_j \in N_G(v_2)} f(d_2, d_j) = \cdots = \sum\limits_{v_j \in N_G(v_n)} f(d_n, d_j)$, then we say that G is regular. Otherwise, G is an irregular weighted graph. In [8], Das et al. gave some lower and upper bounds on the energy of the weighted adjacency matrix. Based on their results, many earlier established results were obtained as special cases. So, it is interesting to study the weighted adjacency matrix $A_f(G)$. This will improve work efficiency. In 2021, Li and Wang [13] tried to find unified methods to study the trees with extremal spectral radius of weighted adjacency matrices. In 2022, Zheng et al.[24] further investigated trees and unicyclic graphs with the largest and smallest spectral radii of weighted adjacency matrices, respectively. In 2022, Li and Yang [14, 15] obtained uniform interlacing inequalities for the weighted adjacency eigenvalues under some kinds of graph operations, such as deleting a vertex, deleting an edge and subdividing an edge. Next, we consider the change of the largest weighted adjacency eigenvalue $\lambda_1(A_f(G))$ under graph operations.

A classical result in the theory of nonnegative matrices states that for any nonnegative irreducible $n \times n$ matrix A with the largest eigenvalue $\lambda_1(A)$ and the largest row sum S,

$$\lambda_1(A) \leq S \tag{1}$$

with equality holds if and only if the row sums of A are all equal. If $f(d_i, d_j) = 1$ for every edge $v_iv_j \in E(G)$, then we get the adjacency matrix $A(G)$. So $\lambda_1(A(G)) \leq \Delta$, with equality if and only if G is regular. It is natural to ask how small $\Delta - \lambda_1(A(G))$ can be when G is irregular. In [5], Cioabă proved that

$$\Delta - \lambda_1(A(G)) > \frac{1}{nD}. \tag{2}$$

In this paper, we consider a similar result about the weighted adjacency matrix. We also improve the bound in (1) when G is an irregular weighted graph.

At the last of this section, we state the structure of this paper. In section 2, we give some results that will be used in our proof. In section 3, we obtain some bounds of the largest weighted adjacency eigenvalue $\lambda_1(A_f(G))$ in terms of $\lambda_1(A_f(H))$, where H is obtained from G by graph operations. Moreover, some examples are given to prove that bounds are tight. In section 4, we extend inequality (2) to the largest weighted adjacency eigenvalue and show two other bounds of $\lambda_1(A_f(G))$ when G is an irregular weighted graph.

2 Preliminaries

In this section, we list several known results.

Lemma 2.1 *[21] Let A, B be two real symmetric matrices of order n, and let the respective eigenvalues of A, B and $A+B$ be $\lambda_i(A)$, $\lambda_i(B)$ and $\lambda_i(A+B)$, where $1 \leq i \leq n$, each algebraically ordered in nonincreasing order. Then*

$$\lambda_i(A+B) \leq \lambda_j(A) + \lambda_{i-j+1}(B), \quad (1 \leq j \leq i \leq n).$$

Also,

$$\lambda_i(A+B) \geq \lambda_j(A) + \lambda_{i-j+n}(B), \quad (1 \leq i \leq j \leq n).$$

The following result is the Rayleigh quotient theorem.

Lemma 2.2 *[2] Let A be a real symmetric matrix of order n. Then*

$$\lambda_1(A) \geq \frac{\mathbf{y}^T A \mathbf{y}}{\mathbf{y}^T \mathbf{y}},$$

for any nonzero vector $\mathbf{y} \in R^n$.

Let $A = (a_{ij})_{n\times m}$ and $B = (b_{ij})_{n\times m}$ be two matrices. If $a_{ij} \leq b_{ij}$ for all i and j, then we say that $A \leq B$. If $A \leq B$ and $A \neq B$, then we say that $A < B$.

Lemma 2.3 *[11] Let A, B be two $n \times n$ nonnegative symmetric matrices. If $A \leq B$, then*

$$\lambda_1(A) \leq \lambda_1(B).$$

Futhermore, if B is irreducible and $A < B$, then $\lambda_1(A) < \lambda_1(B)$.

Now, we state the famous Perron-Frobenius theorem.

Lemma 2.4 *[6] Let A be an irreducible symmetric matrix with nonnegative entries. Then the largest eigenvalue $\lambda_1(A)$ is simple, with a corresponding eigenvector whose entries are all positive.*

Lemma 2.5 *[20] If $a, b > 0$, then*

$$a(x-y)^2 + by^2 \geq \frac{abx^2}{a+b},$$

with equality if and only if $y = \frac{ax}{a+b}$.

3 Some bounds for the largest weighted adjacency eigenvalue under graph operations

In this section, if $f(x, y)$ is a real symmetric function and decreasing in variable x, then we first give a relation between $\lambda_1(A_f(G))$ and $\lambda_1(A_f(G - v_i))$, where v_i is a vertex of G.

Theorem 3.1 *Assume that $f(x,y) > 0$ is a real symmetric function and decreasing in variable x. Let G be a connected graph of order n. Then*

$$\lambda_1(A_f(G)) \le \frac{\lambda_1(A_f(G-v_i)) + \sqrt{\lambda_1^2(A_f(G-v_i)) + 4\sum\limits_{v_iv_j\in E(G)} f^2(d_i,d_j)}}{2}.$$

Proof. For the convenience of discussion, suppose that $v_i = v_1$ and $N_G(v_1) = \{v_2, v_3, \ldots, v_t\}$. By Lemma 2.4, since G is connected and $f(x,y) > 0$, we can get a positive unit eigenvector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ corresponding to $\lambda_1(A_f(G))$. The matrix $A'_f(G)$ is obtained from $A_f(G)$ by deleting the row and column associated with vertex v_1. Because $f(x,y) > 0$ is a real symmetric function and decreasing in variable x, it follows that $A_f(G-v_1) \ge A'_f(G)$. From Lemma 2.3, we know that $\lambda_1(A_f(G-v_1)) \ge \lambda_1(A'_f(G))$. Set $\mathbf{x}' = (x_2, x_3, \ldots, x_n)^T \neq \mathbf{0}$, where $\mathbf{0}$ is the zero vector, using Lemma 2.2, we have

$$\begin{aligned}
\lambda_1(A'_f(G)) &\ge \frac{(\mathbf{x}')^T A'_f(G)\mathbf{x}'}{(\mathbf{x}')^T\mathbf{x}'} \\
&= \frac{2\sum\limits_{v_iv_j\in E(G)} f(d_i,d_j)x_ix_j - 2\sum\limits_{j=2}^{t} f(d_1,d_j)x_1x_j}{x_2^2 + x_3^2 + \cdots + x_n^2} \\
&= \frac{\lambda_1(A_f(G)) - 2\sum\limits_{j=2}^{t} f(d_1,d_j)x_1x_j}{1 - x_1^2}.
\end{aligned}$$

Since $\lambda_1(A_f(G))x_1 = \sum\limits_{j=2}^{t} f(d_1,d_j)x_j$, by the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\lambda_1^2(A_f(G))x_1^2 &= \left(\sum_{j=2}^{t} f(d_1,d_j)x_j\right)^2 \\
&\le \sum_{j=2}^{t} f^2(d_1,d_j) \sum_{j=2}^{t} x_j^2 \\
&\le \sum_{j=2}^{t} f^2(d_1,d_j)\left(1 - x_1^2\right).
\end{aligned}$$

Thus

$$x_1^2 \le \frac{\sum\limits_{j=2}^{t} f^2(d_1,d_j)}{\sum\limits_{j=2}^{t} f^2(d_1,d_j) + \lambda_1^2(A_f(G))}.$$

Combining the results above, we get

$$\begin{aligned}
\lambda_1(A_f(G-v_1)) \geq \lambda_1(A'_f(G)) &\geq \frac{\lambda_1(A_f(G)) - 2\left(\sum\limits_{j=2}^{t} f(d_1,d_j)x_j\right)x_1}{1-x_1^2} \\
&= \frac{\lambda_1(A_f(G)) - 2\lambda_1(A_f(G))x_1^2}{1-x_1^2} \\
&= 2\lambda_1(A_f(G)) - \frac{\lambda_1(A_f(G))}{1-x_1^2} \\
&\geq 2\lambda_1(A_f(G)) - \frac{\sum\limits_{j=2}^{t} f^2(d_1,d_j) + \lambda_1^2(A_f(G))}{\lambda_1(A_f(G))} \\
&= \lambda_1(A_f(G)) - \frac{\sum\limits_{j=2}^{t} f^2(d_1,d_j)}{\lambda_1(A_f(G))}.
\end{aligned}$$

Hence $\lambda_1^2(A_f(G)) - \lambda_1(A_f(G-v_1))\lambda_1(A_f(G)) - \sum\limits_{j=2}^{t} f^2(d_1,d_j) \leq 0$. Considering this quadratic equation, by a simple computation, it is not difficult for us to get

$$\lambda_1(A_f(G)) \leq \frac{\lambda_1(A_f(G-v_1)) + \sqrt{\lambda_1^2(A_f(G-v_1)) + 4\sum\limits_{j=2}^{t} f^2(d_1,d_j)}}{2}.$$

Thus we complete our proof. □

Remark 1. From the proof of Theorem 3.1, we know that the weighted function $f(x,y)$ is not strictly monotonically decreasing. Let $f(x,y) = c$, where c is a constant, and $G = K_1 + H$ be a graph of order n obtained by joining an isolated vertex to each vertex of a regular graph H. If $G - v_i = H$, then we can deduce that the bound in Theorem 3.1 is tight.

When $f(x,y) > 0$ is a real symmetric function and increasing in variable x, we consider the change of the largest weighted adjacency eigenvalue of a connected graph G after removing a subset S of $V(G)$.

Theorem 3.2 *Assume that $f(x,y) > 0$ is a real symmetric function and increasing in variable x. Let G be a connected graph of order n and $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be a positive unit eigenvector corresponding to $\lambda_1(A_f(G))$. If S is the subset of $V(G)$ and δ' is the minimum degree of $G - S$, then*

$$\lambda_1(A_f(G-S)) \geq \frac{f(\delta',\delta')}{f(\Delta,\Delta)} \cdot \frac{\left(1 - 2\sum\limits_{v_i\in S} x_i^2\right)\lambda_1(A_f(G)) + \sum\limits_{v_i\in S}\sum\limits_{v_j\in S} f(d_i,d_j)x_ix_j}{1 - \sum\limits_{v_i\in S} x_i^2}.$$

Proof. Let d_i' be the degree of v_i in $G-S$. The number of vertices in S is s. Since $\mathbf{x}=(x_1,x_2,\ldots,x_n)^T$ is a positive unit eigenvector corresponding to eigenvalue $\lambda_1(A_f(G))$, we have

$$\sum_{v_iv_j\in E(G)} f(d_i,d_j)x_j=\lambda_1(A_f(G))x_i.$$

Since $f(x,y)$ is increasing in variable x, it is not difficult for us to get

$$\begin{aligned}
\lambda_1(A_f(G)) &= 2\sum_{v_i\in S} x_i \sum_{v_iv_j\in E(G)} f(d_i,d_j)x_j-\sum_{v_i\in S}\sum_{v_j\in S} f(d_i,d_j)x_ix_j+2\sum_{v_iv_j\in E(G-S)} f(d_i,d_j)x_ix_j\\
&= 2\sum_{v_i\in S}\lambda_1(A_f(G))x_i^2-\sum_{v_i\in S}\sum_{v_j\in S} f(d_i,d_j)x_ix_j+2\sum_{v_iv_j\in E(G-S)}\frac{f(d_i,d_j)}{f(d_i',d_j')}f(d_i',d_j')x_ix_j\\
&\le 2\sum_{v_i\in S}\lambda_1(A_f(G))x_i^2-\sum_{v_i\in S}\sum_{v_j\in S} f(d_i,d_j)x_ix_j+\frac{f(\Delta,\Delta)}{f(\delta',\delta')}\cdot 2\sum_{v_iv_j\in E(G-S)} f(d_i',d_j')x_ix_j.
\end{aligned}$$

Set $\mathbf{x}'=(x_{p_1},x_{p_2},\ldots,x_{p_{(n-s)}})^T\neq\mathbf{0}$, where $v_{p_i}\in V(G-S)$ for $1\le i\le n-s$, using Lemma 2.2, we obtain

$$\lambda_1(A_f(G-S))\ge\frac{(\mathbf{x}')^TA_f(G-S)\mathbf{x}'}{(\mathbf{x}')^T\mathbf{x}'}=\frac{2\sum\limits_{v_iv_j\in E(G-S)} f(d_i',d_j')x_ix_j}{1-\sum\limits_{v_i\in S}x_i^2}.$$

This means that

$$2\sum_{v_iv_j\in E(G-S)} f(d_i',d_j')x_ix_j\le\left(1-\sum_{v_i\in S}x_i^2\right)\lambda_1(A_f(G-S)).$$

Combining the inequalities above, we get

$$\lambda_1(A_f(G))\le 2\sum_{v_i\in S}\lambda_1(A_f(G))x_i^2-\sum_{v_i\in S}\sum_{v_j\in S} f(d_i,d_j)x_ix_j+\frac{f(\Delta,\Delta)}{f(\delta',\delta')}\cdot\left(1-\sum_{v_i\in S}x_i^2\right)\lambda_1(A_f(G-S)).$$

This completes the proof. □

Remark 2. For a real symmetric function $f(x,y)>0$, which is increasing in variable x, if G is a complete graph of order n, deleting a subset S of $V(G)$, then we can prove the equality holds in Theorem 3.2.

A subset S of $V(G)$ is independent if no two of its vertices are adjacent. When we delete an independent set from G, using Theorem 3.2, the following conclusion can be obtained.

Corollary 3.3 *Assume that $f(x,y)>0$ is a real symmetric function and increasing in variable x. Let G be a connected graph of order n and $\mathbf{x}=(x_1,x_2,\ldots,x_n)^T$ be a positive* *unit eigenvector corresponding to $\lambda_1(A_f(G))$. If S is an independent set of $V(G)$ and δ' is the minimum degree of $G-S$, then*

$$\lambda_1(A_f(G-S)) \geq \frac{f(\delta',\delta')}{f(\Delta,\Delta)} \cdot \left(2 - \frac{1}{1-\sum\limits_{v_i\in S} x_i^2}\right) \lambda_1\left(A_f(G)\right).$$

Next, we study the relation between $\lambda_1(A_f(G))$ and $\lambda_1(A_f(G-e))$.

Theorem 3.4 *Assume that $f(x,y)>0$ is a real symmetric function. Let G be a connected graph of order n and $e=v_iv_j$ be an edge of G. Then*

$$\begin{aligned}\lambda_1(A_f(G-e)) \geq{}& \lambda_1(A_f(G)) - 2\sum_{\substack{v_iv_k\in E(G)\\ k\neq j}} (f(d_i,d_k)-f(d_i-1,d_k))x_ix_k \\ &- \sqrt{\sum_{\substack{v_jv_k\in E(G)\\ k\neq i}} (f(d_j,d_k)-f(d_j-1,d_k))^2} - 2f(d_i,d_j)x_ix_j.\end{aligned}$$

Proof. For convenience, let the edge e be v_1v_2. There are two $n\times n$ matrices B and C, written as follows:

$$B = \begin{bmatrix} 0 & 0 & f(d_1-1,d_3) & \cdots & f(d_1-1,d_n) \\ 0 & 0 & f(d_2,d_3) & \cdots & f(d_2,d_n) \\ f(d_1-1,d_3) & f(d_2,d_3) & 0 & \cdots & f(d_3,d_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f(d_1-1,d_n) & f(d_2,d_n) & f(d_3,d_n) & \cdots & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & f(d_2,d_3)-f(d_2-1,d_3) & \cdots & f(d_2,d_n)-f(d_2-1,d_n) \\ 0 & f(d_2,d_3)-f(d_2-1,d_3) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & f(d_2,d_n)-f(d_2-1,d_n) & 0 & \cdots & 0 \end{bmatrix}.$$

By calculating, we obtain $det(\lambda I - C) = \lambda^{n-2}\left(\lambda^2 - \sum\limits_{\substack{v_2v_k\in E(G)\\ k\neq 1}} (f(d_2,d_k)-f(d_2-1,d_k))^2\right)$, and so the largest eigenvalue of C is $\sqrt{\sum\limits_{\substack{v_2v_k\in E(G)\\ k\neq 1}} (f(d_2,d_k)-f(d_2-1,d_k))^2}$. Since $B = A_f(G-e)+C$, from Lemma 2.1, we obtain

$$\lambda_1(B) \leq \lambda_1(A_f(G-e)) + \lambda_1(C)$$

$$= \lambda_1(A_f(G-e)) + \sqrt{\sum_{\substack{v_2v_k\in E(G)\\ k\neq 1}} (f(d_2,d_k)-f(d_2-1,d_k))^2}.$$

By Lemma 2.4, since G is connected and $f(x,y)>0$, we get a positive unit eigenvector $\mathbf{x}=(x_1,x_2,\ldots,x_n)^T$ of $A_f(G)$ corresponding to $\lambda_1(A_f(G))$. From Lemma 2.2, we have

$$\begin{aligned}
\lambda_1(B) &\geq \frac{\mathbf{x}^TB\mathbf{x}}{\mathbf{x}^T\mathbf{x}}\\
&= 2\sum_{\substack{v_1v_k\in E(G)\\ k\neq 2}} f(d_1-1,d_k)x_1x_k + 2\sum_{\substack{v_2v_k\in E(G)\\ k\neq 1}} f(d_2,d_k)x_2x_k + 2\sum_{v_kv_l\in E(G-\{v_1,v_2\})} f(d_k,d_l)x_kx_l\\
&= \lambda_1(A_f(G)) - 2f(d_1,d_2)x_1x_2 - 2\sum_{\substack{v_1v_k\in E(G)\\ k\neq 2}} (f(d_1,d_k)-f(d_1-1,d_k))x_1x_k.
\end{aligned}$$

Combining the inequalities above, we get

$$\begin{aligned}
\lambda_1(A_f(G)) - 2f(d_1,d_2)x_1x_2 - 2\sum_{\substack{v_1v_k\in E(G)\\ k\neq 2}} (f(d_1,d_k)-f(d_1-1,d_k))x_1x_k&\\
\leq \lambda_1(A_f(G-e)) + \sqrt{\sum_{\substack{v_2v_k\in E(G)\\ k\neq 1}} (f(d_2,d_k)-f(d_2-1,d_k))^2}&.
\end{aligned}$$

This completes the proof. $\square$

Remark 3. Considering $f(x,y)=(\frac{x+y-\alpha}{xy})^\beta$, where $\beta>0$ and $2\leq\alpha\leq 2n-2$, if we delete an edge v_iv_j satisfying $d_i+d_j=\alpha$ and $d_k=\alpha$ for every vertex $v_k\in(N_G(v_i)\cup N_G(v_j))\setminus\{v_i,v_j\}$, then the bound in Theorem 3.4 is tight.

At last, we give the bound of the largest weighted adjacency eigenvalue $\lambda_1(A_f(G_e))$.

Theorem 3.5 *Assume that $f(x,y)>0$ is a real symmetric function. Let G be a connected graph of order n and $e=v_iv_j$ be an edge of G. Then*

$$\lambda_1(A_f(G)) - f(d_i,d_j) < \lambda_1(A_f(G_e)) < \frac{\lambda_1(A_f(G)) + \sqrt{\lambda_1^2(A_f(G)) + 4(f^2(d_i,2)+f^2(d_j,2))}}{2}.$$

Proof. For convenience, let the edge e be v_1v_2 and $v_{(n+1)}$ be a new vertex to subdivide the edge e. The weighted adjacency matrix $A_f(G_e)$ can be obtained from $A_f(G)$ by adding the row and column associated with the vertex $v_{(n+1)}$ and changing the $(A_f(G))_{12}$ and $(A_f(G))_{21}$ into 0s. Removing the row and column associated with vertex $v_{(n+1)}$ from

$A_f(G_e)$, we get an $n \times n$ matrix $A'_f(G_e)$. Let $B = A_f(G) - A'_f(G_e)$, written as follows:

$$B = \begin{bmatrix} 0 & f(d_1,d_2) & 0 & 0 & \cdots & 0 \\ f(d_1,d_2) & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

A short computation reveals that $det(\lambda I - B) = \lambda^{n-2}(\lambda^2 - f^2(d_1, d_2))$. Thus the largest eigenvalue of B is $f(d_1, d_2)$.

On the one hand, according to Lemma 2.1, since $A_f(G) = A'_f(G_e) + B$, we get $\lambda_1(A_f(G)) \leq \lambda_1(A'_f(G_e)) + \lambda_1(B)$. That is, $\lambda_1(A'_f(G_e)) \geq \lambda_1(A_f(G)) - \lambda_1(B) = \lambda_1(A_f(G)) - f(d_1, d_2)$. We know that $A'_f(G_e)$ is a principal submatrix of $A_f(G_e)$. From Lemma 2.3, because G is connected, G_e is connected, and $f(x, y) > 0$, we obtain $\lambda_1(A_f(G_e)) > \lambda_1(A'_f(G_e))$. Therefore, the lower bound of $\lambda_1(A_f(G_e))$ holds.

On the other hand, by Lemma 2.4, since G_e is connected and $f(x, y) > 0$, we get a positive unit eigenvector $\mathbf{x} = (x_1, x_2, \ldots, x_{(n+1)})^T$ of G_e corresponding to $\lambda_1(A_f(G_e))$, that is, $A_f(G_e)\mathbf{x} = \lambda_1(A_f(G_e))\mathbf{x}$. Set $\mathbf{x}' = (x_1, x_2, \ldots, x_n)^T \neq \mathbf{0}$, from Lemma 2.2, we have

$$\begin{aligned} \lambda_1(A'_f(G_e)) &\geq \frac{(\mathbf{x}')^T A'_f(G_e)\mathbf{x}'}{(\mathbf{x}')^T\mathbf{x}'} \\ &= \frac{2\sum\limits_{v_iv_j\in E(G_e)} f(d_i,d_j)x_ix_j - 2f(d_1,2)x_1x_{(n+1)} - 2f(d_2,2)x_2x_{(n+1)}}{x_1^2 + x_2^2 + \cdots + x_n^2} \\ &= \frac{\lambda_1(A_f(G_e)) - 2\lambda_1(A_f(G_e))x_{(n+1)}^2}{1 - x_{(n+1)}^2} \\ &= 2\lambda_1(A_f(G_e)) - \frac{\lambda_1(A_f(G_e))}{1 - x_{(n+1)}^2} \\ &\geq 2\lambda_1\left(A_f(G_e)\right) - \frac{\lambda_1^2\left(A_f(G_e)\right) + f^2(d_1,2) + f^2(d_2,2)}{\lambda_1(A_f(G_e))}. \end{aligned} \tag{3}$$

Since $\lambda_1(A_f(G_e))x_{(n+1)} = f(d_1, 2)x_1 + f(d_2, 2)x_2$, by the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \lambda_1^2(A_f(G_e))x_{(n+1)}^2 &= (f(d_1,2)x_1 + f(d_2,2)x_2)^2 \\ &\leq \left(f^2(d_1,2) + f^2(d_2,2)\right)\left(x_1^2 + x_2^2\right) \\ &\leq \left(f^2(d_1,2) + f^2(d_2,2)\right)\left(1 - x_{(n+1)}^2\right). \end{aligned}$$

This means that

$$x_{(n+1)}^2 \leq \frac{f^2(d_1,2) + f^2(d_2,2)}{\lambda_1^2(A_f(G_e)) + f^2(d_1,2) + f^2(d_2,2)},$$

it follows that the inequality (3) above holds.

Note that $B = A_f(G) - A'_f(G_e)$ is a nonnegative nonzero matrix. According to Lemma 2.3, since G is connected, we get $\lambda_1(A_f(G)) > \lambda_1(A'_f(G_e))$. Until now, we have

$$\lambda_1^2(A_f(G_e)) - \lambda_1(A_f(G))\lambda_1\left(A_f(G_e)\right) - \left(f^2(d_1, 2) + f^2(d_2, 2)\right) < 0.$$

By computing, we conclude that $\lambda_1(A_f(G_e)) < \frac{\lambda_1(A_f(G)) + \sqrt{\lambda_1^2(A_f(G)) + 4(f^2(d_1,2) + f^2(d_2,2))}}{2}$. This proof is complete. $\square$

4 Some bounds for the largest weighted adjacency eigenvalue of irregular weighted graphs

First, using (2), we give a result of the largest weighted adjacency eigenvalue about diameter.

Theorem 4.1 *Assume that $f(x, y) > 0$ is a real symmetric function. Let G be a connected irregular weighted graph with order n, maximum degree Δ and diameter D. If $\Omega = \max\{f(d_i, d_j) : v_iv_j \in E(G)\}$, then*

$$\Delta\Omega - \lambda_1(A_f(G)) > \frac{\Omega}{nD}.$$

Proof. Since $\Omega = \max\{f(d_i, d_j) : v_iv_j \in E(G)\}$, for the adjacency matrix $A(G)$ and the weighted adjacency matrix $A_f(G)$, we have $A_f(G) \leq \Omega A(G)$. By Lemma 2.3, we get $\lambda_1(A_f(G)) \leq \Omega\lambda_1(A(G))$. Because G is irregular, from (2), it follows that $\Delta\Omega - \lambda_1(A_f(G)) > \frac{\Omega}{nD}$. $\square$

Next, we give a bound of the largest weighted adjacency eigenvalue by taking minimum degree and diameter into account.

Theorem 4.2 *Assume that $f(x, y) > 0$ is a real symmetric function. Let G be a connected irregular weighted graph with order n, minimum degree δ and diameter D. If $\omega = \min\{f(d_i, d_j) : v_iv_j \in E(G)\}$ and $\Theta = \max\left\{\sum_{v_j \in N_G(v_i)} f(d_i, d_j) : 1 \leq i \leq n\right\}$, then*

$$\Theta - \lambda_1(A_f(G)) > \frac{(n\Theta - 2TI(G))\omega}{((n - \delta - 1)D + 1)(n\Theta - 2TI(G)) + n\omega}.$$

Proof. Because G is connected and $f(x, y) > 0$, from Lemma 2.4, we get a positive unit eigenvector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ corresponding to the eigenvalue $\lambda_1(A_f(G))$. Let v_s be a vertex of $V(G)$ such that $x_s = \max\{x_i : v_i \in V(G)\}$ and v_t be a vertex such that $x_t = \min\{x_i : v_i \in V(G)\}$. Hence,

$$\begin{aligned}
\Theta - \lambda_1(A_f(G)) &= \Theta \sum_{i=1}^{n} x_i^2 - 2 \sum_{v_iv_j \in E(G)} f(d_i, d_j) x_i x_j \\
&= \Theta \sum_{i=1}^{n} x_i^2 - \sum_{v_iv_j \in E(G)} f(d_i, d_j)(x_i^2 + x_j^2) + \sum_{v_iv_j \in E(G)} f(d_i, d_j)(x_i - x_j)^2 \\
&= \Theta \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} \left(\sum_{v_j \in N_G(v_i)} f(d_i, d_j) \right) x_i^2 + \sum_{v_iv_j \in E(G)} f(d_i, d_j)(x_i - x_j)^2 \\
&= \sum_{i=1}^{n} \left(\Theta - \sum_{v_j \in N_G(v_i)} f(d_i, d_j) \right) x_i^2 + \sum_{v_iv_j \in E(G)} f(d_i, d_j)(x_i - x_j)^2 \\
&\geq (n\Theta - 2TI(G)) x_t^2 + \sum_{v_iv_j \in E(G)} f(d_i, d_j)(x_i - x_j)^2. \qquad (4)
\end{aligned}$$

Let $P : v_s = v_{w_0} v_{w_1} \cdots v_{w_p} = v_t$ be a shortest path from v_s to v_t. Then $p \leq D$. Therefore, by the Cauchy-Schwarz inequality and Lemma 2.5, we have

$$\begin{aligned}
\Theta - \lambda_1(A_f(G)) &\geq (n\Theta - 2TI(G)) x_t^2 + \sum_{v_iv_j \in E(P)} f(d_i, d_j)(x_i - x_j)^2 \\
&\geq (n\Theta - 2TI(G)) x_t^2 + \frac{\omega}{p}(x_s - x_t)^2 \\
&\geq \frac{(n\Theta - 2TI(G))\omega}{p(n\Theta - 2TI(G)) + \omega} x_s^2 \\
&\geq \frac{(n\Theta - 2TI(G))\omega}{D(n\Theta - 2TI(G)) + \omega} x_s^2. \qquad (5)
\end{aligned}$$

Note that the minimum degree of G is δ. Let $v_{y_1}, v_{y_2}, \ldots, v_{y_\delta}$ be the neighbors of v_t. Using the Cauchy-Schwarz inequality and Lemma 2.5 again, we have

$$\begin{aligned}
\Theta - \lambda_1(A_f(G)) &\geq (n\Theta - 2TI(G)) x_t^2 + \sum_{i=1}^{\delta} f(d_{y_i}, d_t)(x_{y_i} - x_t)^2 \\
&\geq \sum_{i=1}^{\delta} \left(\frac{n\Theta - 2TI(G)}{\delta} x_t^2 + \omega (x_{y_i} - x_t)^2 \right) \\
&\geq \sum_{i=1}^{\delta} \frac{(n\Theta - 2TI(G))\omega}{n\Theta - 2TI(G) + \delta\omega} x_{y_i}^2.
\end{aligned}$$

That is,

$$\sum_{i=1}^{\delta} x_{y_i}^2 \leq \frac{n\Theta - 2TI(G) + \delta\omega}{(n\Theta - 2TI(G))\omega} (\Theta - \lambda_1(A_f(G))).$$

Since G is irregular and $f(x,y) > 0$, it follows that $\sum\limits_{v_iv_j \in E(G)} f(d_i,d_j)(x_i - x_j)^2 > 0$. Thus, from (4), we get

$$x_t^2 < \frac{\Theta - \lambda_1(A_f(G))}{n\Theta - 2TI(G)}.$$

In addition, we know that $\delta + 1 \leq \Delta \leq n-1$, so $n \geq \delta + 2$. Recall that $\mathbf{x}$ is a positive unit eigenvector, then we have $\sum\limits_{i=1}^{\delta} x_{y_i}^2 + x_t^2 + (n-\delta-1)x_s^2 \geq 1$. Thus

$$\begin{aligned} x_s^2 &\geq \frac{1 - x_t^2 - \sum\limits_{i=1}^{\delta} x_{y_i}^2}{n-\delta-1} \\ &> \frac{1}{n-\delta-1} - \frac{n\Theta - 2TI(G) + (\delta+1)\omega}{(n-\delta-1)(n\Theta - 2TI(G))\omega}(\Theta - \lambda_1(A_f(G))). \end{aligned}$$

From (5), we get

$$(1 + \frac{n\Theta - 2TI(G)) + (\delta+1)\omega}{(D(n\Theta - 2TI(G)) + \omega)(n-\delta-1)})(\Theta - \lambda_1(A_f(G))) > \frac{(n\Theta - 2TI(G))\omega}{(D(n\Theta - 2TI(G)) + \omega)(n-\delta-1)}$$

By calculation, it is not difficult for us to have

$$\Theta - \lambda_1(A_f(G)) > \frac{(n\Theta - 2TI(G))\omega}{((n-\delta-1)D + 1)(n\Theta - 2TI(G)) + n\omega},$$

as desired. □

If G is k-connected irregular weighted graph, we obtain the following bound.

Theorem 4.3 *Assume that $f(x,y) > 0$ is a real symmetric function. Let G be a k-connected irregular weighted graph with order n and minimum degree δ. If $\omega = \min\{f(d_i,d_j) : v_iv_j \in E(G)\}$ and $\Theta = \max\left\{\sum\limits_{v_j \in N_G(v_i)} f(d_i,d_j) : 1 \leq i \leq n\right\}$, then*

$$\Theta - \lambda_1(A_f(G)) > \frac{(n\Theta - 2TI(G))k^2\omega}{((n-\delta-1)(n+k-2) + k^2)(n\Theta - 2TI(G)) + nk^2\omega}.$$

Proof. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be a positive unit eigenvector of $A_f(G)$ corresponding to $\lambda_1(A_f(G))$. We assume that v_s and v_t are two vertices, such that $x_s = \max\{x_i : v_i \in V(G)\}$ and $x_t = \min\{x_i : v_i \in V(G)\}$, respectively. Then by the proof of Theorem 4.2, we have

$$\Theta - \lambda_1(A_f(G)) = \Theta \sum_{i=1}^{n} x_i^2 - 2 \sum_{v_iv_j \in E(G)} f(d_i,d_j)x_ix_j$$

$$\geq (n\Theta - 2TI(G))x_t^2 + \sum_{v_iv_j \in E(G)} f(d_i, d_j)(x_i - x_j)^2. \tag{6}$$

Note that G is k-connected. Suppose that $P_1, P_2, \ldots, P_k$ are internally vertex disjoint v_sv_t-paths, then $\sum\limits_{l=1}^{k} |V(P_l)| \leq n + 2k - 2$. Thus according to the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}
\sum_{v_iv_j \in E(G)} f(d_i, d_j)(x_i - x_j)^2 &\geq \sum_{l=1}^{k} \sum_{v_iv_j \in E(P_l)} f(d_i, d_j)(x_i - x_j)^2 \\
&\geq \frac{\omega}{\sum\limits_{l=1}^{k}(|V(P_l)| - 1)} \left(\sum_{l=1}^{k} \sum_{v_iv_j \in E(P_l)} (x_i - x_j) \right)^2 \\
&\geq \frac{k^2\omega}{\sum\limits_{l=1}^{k}(|V(P_l)| - 1)} (x_s - x_t)^2 \\
&\geq \frac{k^2\omega}{n + k - 2}(x_s - x_t)^2.
\end{aligned}$$

Now, by Lemma 2.5 and from (6), we get

$$\begin{aligned}
\Theta - \lambda_1(A_f(G)) &\geq (n\Theta - 2TI(G))x_t^2 + \frac{k^2\omega}{n + k - 2}(x_s - x_t)^2 \\
&\geq \frac{(n\Theta - 2TI(G))k^2\omega}{(n + k - 2)(n\Theta - 2TI(G)) + k^2\omega} x_s^2. \qquad (7)
\end{aligned}$$

About vertex v_t, we assume that $v_{y_1}, v_{y_2}, \ldots, v_{y_\delta}$ are the neighbors of it. Using Lemma 2.5 and from (6) again, we have

$$\begin{aligned}
\Theta - \lambda_1(A_f(G)) &\geq (n\Theta - 2TI(G))x_t^2 + \sum_{i=1}^{\delta} f(d_{y_i}, d_t)(x_{y_i} - x_t)^2 \\
&\geq \sum_{i=1}^{\delta} \left(\frac{n\Theta - 2TI(G)}{\delta} x_t^2 + \omega(x_{y_i} - x_t)^2 \right) \\
&\geq \sum_{i=1}^{\delta} \frac{(n\Theta - 2TI(G))\omega}{n\Theta - 2TI(G) + \delta\omega} x_{y_i}^2 \\
&= \frac{(n\Theta - 2TI(G))\omega}{n\Theta - 2TI(G) + \delta\omega} \sum_{i=1}^{\delta} x_{y_i}^2.
\end{aligned}$$

That is,

$$\sum_{i=1}^{\delta} x_{y_i}^2 \leq \frac{n\Theta - 2TI(G) + \delta\omega}{(n\Theta - 2TI(G))\omega}(\Theta - \lambda_1(A_f(G))).$$

Since G is irregular, it follows that $\sum\limits_{v_iv_j\in E(G)} f(d_i,d_j)(x_i-x_j)^2>0$. Hence, (6) means

$$x_t^2<\frac{\Theta-\lambda_1(A_f(G))}{n\Theta-2TI(G)}.$$

Furthermore, it is not difficult for us to have

$$\begin{aligned} x_s^2 &\geq \frac{1-x_t^2-\sum\limits_{i=1}^{\delta}x_{y_i}^2}{n-\delta-1} \\ &> \frac{1}{n-\delta-1}-\frac{n\Theta-2TI(G)+(\delta+1)\omega}{(n-\delta-1)(n\Theta-2TI(G))\omega}(\Theta-\lambda_1(A_f(G))). \end{aligned}$$

Thus, (7) yields

$$\Theta-\lambda_1(A_f(G))>\frac{(n\Theta-2TI(G))k^2\omega}{((n-\delta-1)(n+k-2)+k^2)(n\Theta-2TI(G))+nk^2\omega}.$$

This completes the proof. □

In [22], Xie et al. considered a upper bound on the spectral radius of k-connected irregular weighted graphs in which the edge weights are positive numbers.

Theorem 4.4 *Let G be a k-connected irregular weighted graph with order n and minimum degree δ. If $\tilde{A}(G)$ is the weighted matrix of G and S is the sum of all the entries in $\tilde{A}(G)$, then*

$$\lambda_1(\tilde{A}(G))<\Theta-\frac{(n\Theta-S)k^2\omega}{(n^2-2(n-k))(n\Theta-S)+nk^2\omega}.$$

Let us compare this bound and our bound obtained in Theorem 4.3. Since $k\leq\delta<n-1$, we have

$$(n-\delta-1)(n+k-2)+k^2=n^2-2n+(k-\delta)(n-1+k)+\delta-(n-2)<n^2-2n+2k.$$

which means that our bound is better.

5 Concluding

In this paper, our focus is on several bounds of the largest weighted adjacency eigenvalue. On the one hand, we get the relations between the largest weighted adjacency eigenvalue $\lambda_1(A_f(G))$ and $\lambda_1(A_f(H))$, where H is obtained from G by graph operations, including deleting vertices, deleting an edge and subdividing an edge. On the other hand, similar to the idea "How small $\Delta(G) - \lambda_1(A(G))$ can be when G is irregular?", we have some bounds for the largest weighted adjacency eigenvalue of irregular weighted graphs.

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