Counting rainbow triangles in edge-colored graphs

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Abstract

Let G be an edge-colored graph on n vertices. The minimum color degree of G, denoted by $\delta^c(G)$, is defined as the minimum number of colors assigned to the edges incident to a vertex in G. In 2013, H. Li proved that an edge-colored graph G on n vertices contains a rainbow triangle if $\delta^c(G) \geq \frac{n+1}{2}$. In this paper, we obtain several estimates on the number of rainbow triangles through one given vertex in G. As consequences, we prove counting results for rainbow triangles in edge-colored graphs. One main theorem states that the number of rainbow triangles in G is at least $\frac{1}{6}\delta^c(G)(2\delta^c(G) - n)n$, which is best possible by considering the rainbow k-partite Turán graph, where its order is divisible by k. This means that there are $\Omega(n^2)$ rainbow triangles in G if $\delta^c(G) \geq \frac{n+1}{2}$, and $\Omega(n^3)$ rainbow triangles in G if $\delta^c(G) \geq cn$ when $c > \frac{1}{2}$. Both results are tight in sense of the order of the magnitude. We also prove a counting version of a previous theorem on rainbow triangles under a color neighborhood union condition due to Broersma et al., and an asymptotically tight color degree condition forcing a colored friendship subgraph F_k (i.e., k rainbow triangles sharing a common vertex).

1 Introduction

Throughout this paper, we only consider finite undirected simple graphs. Let G be a graph. By an edge-coloring of G, we mean a function $C : E \to \mathbb{N}$, where $\mathbb{N}$ is the set of non-negative integers. If G has such an edge-coloring, we call G an edge-colored graph and denote it by (G, C). For a vertex $v \in V(G)$, the color neighborhood $CN_G(v)$

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is defined as the set $\{C(e) : e \textit{ is incident with } v\}$, and the *color degree* of v is denoted by $d^c_G(v) := |CN_G(v)|$. We denote by $\delta^c(G) := \min\{d^c_G(v) : v \in V(G)\}$, and by $c(G)$ the number of colors appearing on $E(G)$. Let $\overline{\sigma}^c_2(G) = \min\{d^c(x) + d^c(y) : xy \in E(G)\}$. For a vertex $v \in V(G)$, the *monochromatic degree* of v (in G), denoted by $d^{mon}_G(v)$, is defined as the maximum number of edges incident to v colored with a same color. A subgraph H of G is called *properly-colored* if every two incident edges are assigned with different colors, and is called *rainbow* if all of its edges have distinct colors. When there is no possibility of confusion, we will drop the subscript G. For example, we use δ^c instead of $\delta^c(G)$. For notation and terminology not defined here, we refer to Bondy and Murty [3].

Rainbow and properly-colored subgraph problems have received much attention from graph theorists, see [1, 4, 5, 7, 15]. For surveys, see [6, 18]. In 2013, H. Li [21] proved a minimum color degree condition for rainbow triangles, which was conjectured in [22].

Theorem 1 ([21])**.** *Let (G, C) be an edge-colored graph on $n \geq 3$ vertices. If $\delta^c(G) \geq \frac{n+1}{2}$ then G contains a rainbow triangle.*

A slightly stronger Dirac-type result was proved by B. Li, Ning, Xu, and Zhang in [20].

Theorem 2 ([20])**.** *Let (G, C) be an edge-colored graph on $n \geq 5$ vertices. If $\delta^c(G) \geq \frac{n}{2}$ then G contains a rainbow triangle unless G is a properly colored $K_{\frac{n}{2},\frac{n}{2}}$.*

Theorem 1 motivated much attention on rainbow subgraphs. Czygrinow, Molla, Nagle, and Oursler [7] recently proved that the same condition in Theorem 1 ensures a rainbow ℓ-cycle C_ℓ whenever $n > 432\ell$, which is sharp for a fixed odd integer $\ell \geq 3$ when n is sufficiently large. The authors in [20] proposed a new type condition, i.e., every edge-colored graph (G, C) on n vertices satisfying $e(G) + c(G) \geq \frac{n(n+1)}{2}$ contains a rainbow triangle, where $e(G)$ is the number of edges in G and $c(G)$ is the number of all colors appearing on $E(G)$. This motivated further studies on rainbow cliques [24] and properly-colored C_4's [25].

The original purpose of this article is to study the supersaturation problem of rainbow triangles in edge-colored graphs. This problem is obviously motivated by the study of supersaturation problem of triangles in graphs. It studies the following function: for triangle C_3 and for integers $n, t \geq 1$,

$$h_{C_3}(n, t) = min\{t(G) : |V(G)| = n, |E(G)| = ex(n, C_3) + t\},$$

where $t(G)$ is the number of C_3 in G and $ex(n, C_3)$ is the Turán function of C_3. Improving Mantel's theorem, Rademacher (unpublished, see [9]) proved that $h_{C_3}(n, 1) \geq \lfloor \frac{n}{2} \rfloor$. Erdős [10, 11] proved that $h_{C_3}(n, k) \geq k\lfloor \frac{n}{2} \rfloor$ where $k \leq cn$ for some constant c. In fact, Erdős conjectured that $h_{C_3}(n, k) \geq k\lfloor \frac{n}{2} \rfloor$ for all $k < \lfloor \frac{n}{2} \rfloor$, which was finally resolved by Lovász and Simonovits [23].

One can ask for a rainbow analog of the above Erdős' conjecture. In this direction, answering an open problem in [16], Ehard and Mohr [8] proved there are at least k rainbow triangles in an edge-colored graph (G, C) such that $e(G) + c(G) \geq \binom{n+1}{2} + k - 1$. If we consider $e(G) + c(G)$ as a variant of Turán function in edge-colored graphs, then the theorem above tells us that the supersaturation phenomenon of rainbow triangles under this type of condition is quite different from the original one. On the other hand, the problem of finding a counting version of Theorem 1 is still open.

We denote by $\mathcal{G}_n^*$ the family of edge-colored graphs on n vertices with the minimum color degree at least $\frac{n+1}{2}$, by $rt(G)$ the number of rainbow triangles in an edge-colored graph G, and by $rt(G; v)$ be the number of rainbow triangles through a vertex v in G. Denote by

$$f(n) := \min\{rt(G) : G \in \mathcal{G}_n^*\}.$$

Proving a special case of a conjecture which states that every edge-colored graph on $n \geq 20$ vertices contains two disjoint rainbow triangles if the minimum color degree is at least $\frac{n+2}{2}$, Hu, Li, and Yang developed a key lemma [17, Lemma 1], from which one can easily obtain $f(n) = \Omega(n)$. One may dare to guess that $f(n) = \Omega(n^2)$. Our first humble contribution confirms this.

Remark 1. Throughout this paper, we sometimes assume that an edge-colored graph (G, C) satisfies $\delta^c(G) \geq \frac{n+1}{2}$ and subject to this, $e(G)$ is minimal. Here the word "minimal" means that deleting any edge e in G will result in the inequality $\delta^c(G - e) < \frac{n+1}{2}$. It follows that G contains no monochromatic C_3 or P_4 (a path of order 4). Furthermore, we can see that a spanning subgraph of G with a same color should be a star forest.

One of our main results is as follows.

Theorem 3. *Let (G, C) be an edge-colored graph with vertex set $V(G)$. Let $n = |V(G)|$. If $\delta^c \geq \frac{n+1}{2}$, and furthermore, $e(G)$ is minimal, then we have,*

$$rt(G) \geq \frac{1}{6} \sum_{v \in V(G)} \left((n - d(v) - 1)(d(v) - d^c(v)) + \sum_{a \in N_G(v)} (d^c(v) + d^c(a) - n) \right).$$

As a consequence of Theorem 3, we obtain a counting version of Theorem 1.

Theorem 4. *Let (G, C) be an edge-colored graph on n vertices. Then*

$$rt(G) \geq \frac{1}{6}\delta^c(G)(2\delta^c(G) - n)n.$$

In particular, if $\delta^c(G) > cn$ for $c > \frac{1}{2}$, then

$$rt(G) \geq \frac{c(2c - 1)}{6} n^3.$$

One may wonder the tightness of Theorem 4. The following example shows that Theorem 4 is best possible.

Example 1. Let G be a rainbow k-partite Turán graph on n vertices where $k|n$ and $k \geq 3$. Then there are exactly $\binom{k}{3}(\frac{n}{k})^3 = \frac{(k-1)(k-2)}{6k^2}n^3$ rainbow triangles. By Theorem 4, there are at least $\binom{k}{3}(\frac{n}{k})^3 = \frac{(k-1)(k-2)}{6k^2}n^3$ rainbow triangles.

Setting $\delta^c(G) = \frac{n+1}{2}$ in Theorem 4, we obtain the right hand of the following.

Proposition 5. *For even $n \geq 4$, we have $\frac{n^2}{4} \geq f(n) \geq \frac{n^2+2n}{6}$; for odd $n \geq 3$, we have $\frac{n^2-1}{8} \geq f(n) \geq \frac{n^2+n}{12}$.*

For Proposition 5, the leftmost of each inequality (for $f(n)$) of Proposition 5 was shown by the following two examples. From Proposition 5, we infer $f(n) = \Theta(n^2)$.

Example 2. Let (G, C) be a rainbow graph of order n where n is divisible by 4. Let $V(G) = X_1 \cup X_2$, $|X_1| = |X_2| = \frac{n}{2}$, and each of $G[X_1]$ and $G[X_2]$ consists of a perfect matching of size $\frac{n}{4}$. In addition, $G - E(X_1) - E(X_2)$ is balanced and complete bipartite. For each edge $e \in E(X_1)$, it is contained in exactly $\frac{n}{2}$ rainbow triangles. So does each edge in $G[X_2]$. Therefore, there are exactly $\frac{n^2}{4}$ rainbow triangles in G.

Example 3. Let (G, C) be a rainbow graph of order n where $n \equiv 1 \pmod 4$. Let $V(G) = X_1 \cup X_2$, $|X_1| = \frac{n+1}{2}$ and $|X_2| = \frac{n-1}{2}$, and $G[X_1]$ consists of a perfect matching of size $\frac{n+1}{4}$. In addition, $G - E(X_1)$ is complete bipartite. For each edge $e \in E(X_1)$, it is contained in exactly $\frac{n-1}{2}$ rainbow triangles. Therefore, there are exactly $\frac{n^2-1}{8}$ rainbow triangles in G.

In 2005, Broersma, X. Li, Woeginger, and Zhang [4] proved that an edge-colored graph (G, C) on $n \geq 4$ vertices contains a rainbow C_3 or a rainbow C_4 if $|CN(u) \cup CN(v)| \geq n-1$ for every pair of vertices $u, v \in V(G)$. Define G to be a rainbow $K_{\frac{n}{2},\frac{n}{2}}$ where n is even. Then $|CN(u) \cup CN(v)| = n-1$ for each pair of vertices u and v, and G contains no rainbow triangles. Thus, one need slightly increase the color degree condition when finding rainbow triangles. Broersma et al.'s theorem was generalized by Fujita, Ning, Xu and Zhang [16] to the one forcing rainbow triangles under the same condition.

In this paper, we extend both theorems mentioned above to a counting version as follows.

Theorem 6. *Let (G, C) be an edge-colored graph of order $n \geq 4$ such that $|CN(u) \cup CN(v)| \geq n$ for every pair of vertices $u, v \in V(G)$. Then G contains $\frac{n^2-2n}{24}$ rainbow triangles.*

We also prove some better estimate on the number of rainbow triangles through vertices with high monochromatic degree.

Theorem 7. *Let (G, C) be an edge-colored graph on n vertices with $\delta^c(G)$ and furthermore, $e(G)$ is minimal. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $d_G^{mon}(v_1) \geq d_G^{mon}(v_2) \geq \cdots \geq d_G^{mon}(v_n)$. Then for each $1 \leq k \leq \delta^c(G) - 1$,*

$$\sum_{i=1}^{k} rt(G; v_i) \geq \frac{1}{2}\left(\sum_{i=1}^{k} d_G^{mon}(v_i) + k(\delta^c(G) - 1)\right)(\overline{\sigma}_2^c(G) - n) + \frac{\Delta_k(G)}{2}.$$

where

$$\Delta_k(G) = \left(\delta^c(G)\sum_{i=1}^{k} d_G^{mon}(v_i) - k\sum_{i=1}^{\delta^c(G)} d_G^{mon}(v_i)\right).$$

The above theorem has the following simple but useful corollary.

Theorem 8. *Let (G, C) be an edge-colored graph on n vertices with $\delta^c(G) \geq \frac{n+1}{2}$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $d_G^{mon}(v_1) \geq d_G^{mon}(v_2) \geq \cdots \geq d_G^{mon}(v_n)$. Then for each $1 \leq k \leq \delta^c(G) - 1$,*

$$\sum_{i=1}^{k} rt(G; v_i) \geq \frac{k\delta^c(G)}{2}.$$

The *friendship graph F_k* is a graph consisting of k triangles sharing a common vertex. Finally, we obtain some color degree condition for the existence of some rainbow triangles sharing one common vertex, i.e., the underlying graph is a friendship subgraph. This extends Theorem 1 in another way.

Theorem 9. *Let $k \geq 2$ and $n \geq 50k^2$. Let (G, C) be an edge-colored graph on n vertices. If $\delta^c(G) \geq \frac{n}{2} + k - 1$ then G contains k rainbow triangles sharing one common vertex.*

This paper is organised as follows. In Section 2, we prove one technical lemmas which gives an estimate on the number of rainbow triangles through one given vertex. We also prove another estimate on the number of rainbow triangles through vertices with high monochromatic degree. In Section 3, we prove Theorems 3, 4, 6 and 7. In Section 4, we prove a theorem slightly stronger than Theorem 9. We conclude this paper with some open problems in the last section.

2 Rainbow triangles through a specified vertex

In this section, we first prove one key lemma, whose proof is partly inspired by [17, Lemma 1].

Let G be an edge-colored graph. Without loss of generality, assume that $CN_G(v) = \{1, 2, \ldots, s\}$, where $s = d^c(v)$. Let $N_j(v) := \{u : C(uv) = j, u \in N_G(v)\}$ and $d_j(v) := |N_j(v)|$, where $1 \leq j \leq s$. Furthermore, assume that $d_1(v) \geq d_2(v) \geq \cdots \geq d_s(v)$. So $d^{mon}(v) = d_1(v)$.

Lemma 1. *Let (G, C) be an edge-colored graph on n vertices with $\delta^c(G) \geq \frac{n+1}{2}$ and furthermore, $e(G)$ is minimal. Then for each $v \in V(G)$,*

$$rt(G; v) \geq \frac{1}{2}((n - d(v) - 1)(d(v) - d^c(v)) + \sum_{1 \leq j \leq d^c(v)} \sum_{a \in N_j(v)} (d_j(v) - d_j(a))$$
$$+ \sum_{a \in N(v)} (d^c(v) + d^c(a) - n)).$$

Proof. Since G is edge-minimal, there is no monochromatic path of length 3 and no monochromatic triangle in G.

For the vertex $v \in V(G)$, define a digraph D_v on $N(v)$ as follows: $\overrightarrow{ab} \in A(D_v)$ if and only if $ab \in E(G)$ and $C(ab) \neq C(va)$, i.e., vab is a rainbow path of length 2. Therefore, for any two vertices $x, y \in N_j(v)$ (if $|N_j(v)| \geq 2$), there is either a 2-cycle xyx or no arc between x and y; since otherwise, there is a monochromatic C_3, a contradiction. (Recall Remark 1!)

For $a \in N(v)$, let $S_a \subseteq N(v) \backslash \{a\}$ be maximal such that $C(au), C(au')$ and $C(av)$ are distinct for any two vertices $u, u' \in S_a$. According to the definition of D_v, every edge au, $u \in S_a$, corresponds to an out-arc from a to u in D_v. Notice that

$$d^+_{D_v}(a) \geq |S_a| \geq |CN_{G[N(v) \cup \{v\}]}(a)| - 1 \geq d^c(a) - 1 - |V(G) \setminus (N(v) \cup \{v\})|.$$

Thus, we have

$$d^+_{D_v}(a) \geq d^c(a) + d_G(v) - n.$$

Therefore,

$$\begin{aligned} \sum_{a \in N(v)} d^+_{D_v}(a) &\geq d_G(v)(d_G(v) - n) + \sum_{a \in N(v)} d^c(a) \\ &= d_G(v) \left(\sum_{j=1}^{d^c(v)} (d_j(v) - 1) \right) + \sum_{a \in N(v)} (d^c(a) + d^c(v) - n). \end{aligned} \tag{1}$$

Next, consider $\sum_{a \in N_v} d^-_{D_v}(a)$. For $1 \leq j \leq s$, let n_j be the number of 2-cycles in $D_v[N_j(v)]$. Let n_0 be the number of all 2-cycles xyx in D_v such that $C(xv) \neq C(yv)$. That is, $n_0 = rt(G; v)$.

Thus, we have

$$\sum_{a \in N(v)} d^-_{D_v}(a) \leq \sum_{1 \leq j \leq d^c(v), d_j(v)=1} \sum_{a \in N_j(v)} (d_j(a) - 1) + 2n_0 + 2 \sum_{j=1}^{s} n_j(v). \tag{2}$$

In fact, for any neighbor of v, say u, when we want to compute the value $d^{-}_{D_v}(u)$, there are several different cases. First, for any other neighbor of v, say w, if $uw \notin E(G)$, then there is no in-arc wv for $A(D)$. Secondly, we know $uvwu$ (if $uw \in E(G)$) cannot give us a monochromatic C_3 (as G is edge-minimal).

We divide the corresponding terms into the following different types: (i) $C(uw) = C(uv)$. For this case, we know $d_{C(uv)}(v) = 1$, since otherwise there is a monochromatic P_4 or monochromtic C_3. If $wuvw$ is a triangle in G, then we know $C(uw) = C(uv)$ and $C(vw) \neq C(uw)$, and so there is an in-arc wu in D_v for the vertex u. By the definition of the maximum monochromatic degree of G, we have at most $d_{C(vu)}(u) - 1$ such in-arcs for u. (ii) $C(uw) \neq C(uv)$ but $C(uv) = C(wv)$. For this case, there are two arcs uw and wu (which is a 2-cycle) in D_v. Each such 2-cycle contributes 2 to the sum of all in-degrees (one for u, one for w). In the inequality (2), this can explain where the term $2\sum_{j=1}^{s} n_j(v)$ comes from. (iii) $C(uw) \neq C(uv)$ and $C(uv) \neq C(wv)$. For this case, $uwvu$ gives us a rainbow triangle. We also have a 2-cycle uwu in D_v, and each such 2-cycle contributes 2 to the sum of all in-degrees. In the inequality (2), this can explain why the term $2n_0$ comes from.

Now we have proved (2).

Since

$$2n_j \leq d_j(v)(d_j(v) - 1),$$

from (2), we can obtain that

$$\sum_{a \in N_v} d^{-}_{D_v}(a) \leq \sum_{1 \leq j \leq d^c(v), d_j(v)=1} \sum_{a \in N_j(v)} (d_j(a) - 1) + 2n_0 + \sum_{j=1}^{s} d_j(v)(d_j(v) - 1). \tag{3}$$

As

$$\sum_{a \in N_v} d^{+}_{D_v}(a) = \sum_{a \in N_v} d^{-}_{D_v}(a),$$

combining (1) and (3), we have

$$\begin{aligned} 2n_0 \geq & \sum_{a \in N_v} (d^c(v) + d^c(a) - n) + d(v) \left(\sum_{j=1}^{s} (d_j(v) - 1) \right) - 2 \sum_{j=1}^{s} d_j(v)(d_j(v) - 1) \\ & - \sum_{1 \leq j \leq s, d_j(v)=1} \sum_{a \in N_j(v)} (d_j(a) - 1) + \sum_{j=1}^{s} d_j(v)(d_j(v) - 1). \end{aligned} \tag{4}$$

Set

$$A = d(v) \left(\sum_{j=1}^{s} (d_j(v) - 1) \right) - 2 \sum_{j=1}^{s} d_j(v)(d_j(v) - 1),$$

and

$$B = -\sum_{1\le j\le s, d_j(v)=1}\ \sum_{a\in N_j(v)} (d_j(a)-1) + \sum_{j=1}^{s} d_j(v)(d_j(v)-1).$$

Then (4) is equivalent to the following

$$2n_0 \ge \sum_{a\in N_v} (d^c(v)+d^c(a)-n) + A + B. \tag{5}$$

By simple algebra,

$$A = \sum_{j=1}^{s}(d(v)-2d_j(v))(d_j(v)-1) = \sum_{j=1, d_j(v)\ge 2}^{s} (d(v)-2d_j(v))(d_j(v)-1).$$

As

$$d_1(v) \le d(v) - d^c(v) + 1$$

and

$$d^c(v) \ge \frac{n+1}{2},$$

we have

$$d(v)-2d_j(v) \ge d(v)-2d_1(v) \ge d(v)-2(d(v)-d^c(v)+1) = 2d^c(v)-d(v)-2 \ge n-d(v)-1,$$

and so

$$A \ge \sum_{j=1}^{s}(n-d(v)-1)(d_j(v)-1) = (n-d(v)-1)(d(v)-d^c(v)). \tag{6}$$

Furthermore, we obtain

$$\begin{aligned} B &= -\sum_{1\le j\le s, d_j(v)=1}\ \sum_{a\in N_j(v)} (d_j(a)-d_j(v)) + \sum_{1\le j\le s, d_j(v)\ge 2}\ \sum_{a\in N_j(v)} (d_j(v)-d_j(a)) \\ &= \sum_{1\le j\le d^c(v)}\ \sum_{a\in N_j(v)} (d_j(v)-d_j(a)), \end{aligned} \tag{7}$$

where $d_j(a) = 1$ when $d_j(v) \ge 2$, since G contains no monochromatic path of length three.

Now, together with (5), (6), and (7), we infer

$$\begin{aligned} 2n_0 \ge & \left(\sum_{a\in N_v} (d^c(v)+d^c(a)-n)\right) + (n-d(v)-1)(d(v)-d^c(v)) \\ & + \sum_{1\le j\le d^c(v)}\ \sum_{a\in N_j(v)} (d_j(v)-d_j(a)). \end{aligned}$$

This proves Lemma 1. □

We then obtain a better estimate of the number of rainbow triangles through a specified vertex when the monochromatic degree of this vertex is large. Before the proof, we need to introduce some additional notation.

For a vertex $v \in V(G)$, let X_v be the maximal subset of $N_G(v)$ such that $c(va) = c(vb)$ for any two distinct vertices $a, b \in X_v$. Then $|X_v| = d^{mon}(v)$. In the following, X_v is called a maximum monochromatic neighborhood of v. Let $Y_v \subseteq N_G(v)\backslash X_v$ such that $c(va) \neq c(vb)$ for any two vertices $a, b \in Y_v$. Thus, we have $|Y_v| \leq d^c(v) - 1$. In the following, set

$$f(v) := \min\{d^c(u) + |Y_v| + 1 : u \in X_v \cup Y_v\}.$$

For a vertex $v \in V(G)$, define a digraph D_v on $X_v \cup Y_v$ as follows: $\overrightarrow{ab} \in A(D_v)$ if and only if $ab \in E(G)$ and $c(ab) \neq c(va)$. Let n_1^* be the number of 2-cycles in $D_v[X]$. Let n_0 be the number of other 2-cycles in D_v. Then, $rt(G; v) \geq n_0$.

We now prove the following lemma, whose proof is a variant of Lemma 1.

Lemma 2. *Let (G, C) be an edge-colored graph on n vertices with $\delta^c(G)$, and subject to this, $e(G)$ is minimal. For each $v \in V(G)$, fix a Y_v defined above. Then, we have*

$$rt(G; v) \geq n_0 \geq \frac{1}{2}\left((d_G^{mon}(v) + |Y_v|)(f(v) - n) + (|Y_v| d_G^{mon}(v) - \sum_{a \in Y_v} d_G^{mon}(a))\right). \quad (8)$$

Proof. For $a \in X_v \cup Y_v$, let $S \subset (X_v \cup Y_v)\backslash\{a\}$ be maximal such that $c(au), c(au'), c(av)$ are pairwise different for two distinct vertices $u, u' \in X_v \cup Y_v$. According to the definition of D_v, every edge au, $u \in S$ gives an out-arc of a in D_v. Hence, we have

$$\begin{aligned} d_{D_v}^+(a) &\geq d^c(a) - 1 - |V(G) \setminus (X_v \cup Y_v \cup \{v\})| \\ &\geq f(v) + d^{mon}(v) - n - 1. \end{aligned}$$

Therefore,

$$\sum_{a \in X_v \cup Y_v} d_{D_v}^+(a) \geq (d^{mon}(v) + |Y_v|)(f(v) + d^{mon}(v) - n - 1). \quad (9)$$

Next, consider $\sum_{a \in X_v \cup Y_v} d_{D_v}^-(a)$. By reasoning the proof of Lemma 1 and a similar analysis, we obtain

$$\sum_{a \in X_v \cup Y_v} d_{D_v}^-(a) \leq \sum_{a \in Y_v} d^{mon}(a) - |Y_v| + 2(n_1^* + n_2^*). \quad (10)$$

Since

$$\sum_{a \in X_v \cup Y_v} d_{D_v}^+(a) = \sum_{a \in X_v \cup Y_v} d_{D_v}^-(a)$$

and

$$2n_1^* \leq d^{mon}(v)(d^{mon}(v) - 1),$$

by combining (9) and (10), we have

$$
\begin{aligned}
2n_0 &\geq d^{mon}(v)(f(v)-n) + |Y_v|(f(v)+d^{mon}(v)-n) - \sum_{a\in Y_v} d^{mon}(a)\\
&= (d^{mon}(v)+|Y_v|)(f(v)-n) + (|Y_v|d^{mon}(v) - \sum_{a\in Y_v} d^{mon}(a)).
\end{aligned}
$$

Hence,

$$
rt(G;v) \geq n_0 \geq \frac{1}{2}\left((d^{mon}(v)+|Y_v|)(f(v)-n) + (|Y_v|d^{mon}(v) - \sum_{a\in Y_v} d^{mon}(a))\right).
$$

The proof is complete. □

If Y_v is maximal, then $|Y_v| = d^c(v)-1$. Then

$$
f(v) := \min\{d^c(v)+d^c(u) : u \in X_v \cup Y_v\} \geq \overline{\sigma}_2^c(G),
$$

and Lemma 2 has the following form:

Lemma 3. *Let (G,C) be an edge-colored graph on n vertices with $\delta^c(G)$ and furthermore, $e(G)$ is minimal. Let $V(G)=\{v_1,v_2,\ldots,v_n\}$ such that $d^{mon}(v_1)\geq d^{mon}(v_2)\geq\cdots\geq d^{mon}(v_n)$. Then for each $1\leq i\leq \delta^c(G)$,*

$$
rt(G;v_i) \geq \frac{1}{2}\left((d^{mon}(v_i)+d^c(v_i)-1)(\overline{\sigma}_2^c(G)-n) + (|Y_{v_i}|d^{mon}(v_i) - \sum_{a\in Y_{v_i}} d^{mon}(a))\right).
$$

Next, we prove a technical proposition, which is helpful to the proof of Theorem 7.

Proposition 10. *Let (G,C) be an edge-colored graph on n vertices with $\delta^c(G)$ and furthermore, $e(G)$ is minimal. Let $V(G)=\{v_1,v_2,\ldots,v_n\}$ such that $d^{mon}(v_1)\geq d^{mon}(v_2)\geq\cdots\geq d^{mon}(v_n)$. Let Y_{v_i} be defined as in Lemma 2 with $|Y_{v_i}|=\delta^c(G)-1$. Then for each $1\leq k\leq \delta^c(G)-1$,*

$$
\sum_{i=1}^{k}(|Y_{v_i}|d^{mon}(v_i) - \sum_{a\in Y_{v_i}} d^{mon}(a)) \geq \left(\delta^c\sum_{i=1}^{k} d^{mon}(v_i) - k\sum_{i=1}^{\delta^c} d^{mon}(v_i)\right) \geq 0.
$$

Proof. Note that for $v_i\in V(G)$, $v_i\notin Y_i := Y_{v_i}$. Hence, for $i<\delta^c(G)$,

$$
\sum_{a\in Y_i} d^{mon}(a) \leq \sum_{j=1}^{i-1} d^{mon}(v_j) + \sum_{j=i+1}^{\delta^c} d^{mon}(v_j).
$$

Thus,

$$0 \leq \sum_{i=1}^{k}\sum_{a\in Y_i} d^{mon}(a) \leq \sum_{i=1}^{k}\left(\sum_{j=1}^{i-1} d^{mon}(v_j) + \sum_{j=i+1}^{\delta^c} d^{mon}(v_j)\right) = k\sum_{i=1}^{\delta^c} d^{mon}(v_i) - \sum_{i=1}^{k} d^{mon}(v_i).$$

It follows that

$$\sum_{i=1}^{k}\left(d^{mon}(v_i)(\delta^c - 1) - \sum_{a\in Y_i} d^{mon}(a)\right) \geq \left(\delta^c \sum_{i=1}^{k} d^{mon}(v_i) - k\sum_{i=1}^{\delta^c} d^{mon}(v_i)\right).$$

The proof of Proposition 10 is complete. □

3 Proofs

In this section, we prove Theorems 3, 4, 6 and 7.

By simple technique of counting in two ways, we have the following.

Proposition 11. [1] *Let (G, C) be an edge-colored graph with vertex set $V(G)$ and $\delta^c(G) \geq \frac{n+1}{2}$, and furthermore, $e(G)$ is minimal. For $k \in CN_G(v)$, $N_k(v) := \{u \in N_v : C(uv) = k\}$ and $d_k(v) := |N_k(v)|$. Then*

$$\sum_{v\in V(G)}\sum_{k\in CN_G(v)}\sum_{a\in N_k(v)} (d_k(v) - d_k(a)) = 0. \tag{11}$$

Proof. By definition of $N_k(v)$, we can see

$$\sum_{k\in CN_G(v)}\sum_{a\in N_k(v)} (d_k(v) - d_k(a)) = \sum_{a\in N_G(v)} d_{C(va)}(v) - d_{C(va)}(a).$$

By counting in two ways, we have

$$\begin{aligned}&\sum_{v\in V(G)}\sum_{a\in N_G(v)} (d_{C(va)}(v) - d_{C(va)}(a))\\ &= \sum_{xy\in E(G)} (d_{C(xy)}(x) - d_{C(xy)}(y)) + (d_{C(xy)}(y) - d_{C(xy)}(x)) = 0.\end{aligned}$$

This proves Proposition 11. □

[1] Note that in the proof of Lemma 1, without loss of generality, we assume that $CN_G(v) = \{1, 2, \ldots, d^c(v)\}$ for simplicity. In fact, for distinct vertices $u, v \in V(G)$, $CN_G(u)$ may be not equal to $CN_G(v)$, and may be not a subset of $[1, C(G)]$.

Now we can prove one main result in this paper.

Proof of Theorem 3. The theorem follows from Lemma 1, Proposition 11, and the fact that

$$3rt(G) = \sum_{v \in V(G)} rt(G; v).$$

■

We deduce Theorem 4 from Theorem 3.

Proof of Theorem 4. For any vertex $v \in V$ and $a \in N_G(v)$, we have

$$(n - d(v) - 1)(d(v) - d^c(v)) \geq 0,$$

$e(G) \geq \frac{\delta(G)n}{2} \geq \frac{\delta^c(G)n}{2}$, and $d^c(v) + d^c(a) - n \geq 2\delta^c(G) - n$. Thus, we derive that

$$rt(G) \geq \frac{1}{6}\delta^c(G)(2\delta^c(G) - n)n.$$

If $\delta^c(G) > cn$ for $c > \frac{1}{2}$, then by the inequality above,

$$rt(G) \geq \frac{c(2c-1)}{6}n^3.$$

This proves Theorem 4. ■

Finally, we give proofs of Theorem 7 and Theorem 6.

Proof of Theorem 7. This theorem directly follows from Lemma 3 and Proposition 10. ■

Proof of Theorem 6. If $\delta^c \geq \frac{n+1}{2}$, then by Proposition 5, G contains $\frac{n^2+n}{12}$ rainbow triangles. Thus, $\delta^c \leq \frac{n}{2}$. Choose $v \in V(G)$ such that $d^c_G(v) = \delta^c \leq \frac{n}{2}$. Set $G' = G - v$.

First we furthermore suppose that $d^c_G(v) \leq \frac{n-1}{2}$. For a vertex u adjacent to v, $|CN(u) \cup CN(v)| \geq n$. It follows that

$$d^c_G(u) + d^c_G(v) = |CN(u) \cup CN(v)| + |CN(u) \cap CN(v)| \geq n + 1.$$

It follows that $d^c_G(u) \geq \frac{n+3}{2}$. For a vertex u non-adjacent to v, we also have

$$d^c_G(u) + d^c_G(v) = |CN(u) \cup CN(v)| + |CN(u) \cap CN(v)| \geq n.$$

Thus, $d^c_G(u) \geq \frac{n+1}{2}$. It follows that $d_{G'}(u) \geq \frac{n+1}{2} > \frac{|G'|+1}{2}$. Then by Theorem 4, we have

$$rt(G') \geq \frac{1}{6} \cdot \frac{n+1}{2}\left(2 \cdot \frac{n+1}{2} - (n-1)\right)n \geq \frac{n^2+n}{6}.$$

So $d^c(v) = \frac{n}{2}$, i.e., $\delta^c = \frac{n}{2}$. In this case, for an edge $uv \in E(G)$,

$$d^c(u) + d^c(v) = |CN(u) \cup CN(v)| + |CN(u) \cap CN(v)| \geq n + 1.$$

By setting $k = \delta^c - 1$ in Theorem 7, recall $\delta^c = \frac{n}{2}$ for this case, we have

$$\begin{aligned} rt(G) &\geq \frac{1}{3}\sum_{i=1}^{\delta^c-1} rt(G; v_i) \\ &\geq \frac{1}{6}(\delta^c - 1)\delta^c(n+1-n) + \frac{\Delta_k(G)}{6} \\ &= \frac{n^2-2n}{24} + \frac{\Delta_k(G)}{6}, \end{aligned}$$

where $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $d^{mon}(v_1) \geq d^{mon}(v_2) \geq \cdots \geq d^{mon}(v_n)$. Furthermore, we have

$$\begin{aligned} \Delta_k(G) &= \delta^c(G)\sum_{i=1}^{\delta^c(G)-1} d_G^{mon}(v_i) - (\delta^c(G)-1)\sum_{i=1}^{\delta^c(G)} d_G^{mon}(v_i) \\ &= \sum_{i=1}^{\delta^c(G)-1} d_G^{mon}(v_i) - (\delta^c(G)-1)d_G^{mon}(v_{\delta^c}) \\ &\geq 0 \end{aligned}$$

as $d_G^{mon}(v_i) \geq d_G^{mon}(v_{\delta^c})$ for $i \in [1, \delta^c(G) - 1]$. Thus, $rt(G) \geq \frac{n^2-2n}{24}$. This proves the theorem. ■

4 Edge-colored friendship subgraphs

In this section, we shall prove a result slightly stronger than Theorem 9. For a graph G, we denote by $\Delta^{mon}(G) := \max\{d_G^{mon}(v) : v \in V(G)\}$.

Theorem 12. *Let k, n be positive integers, and G be an edge-colored graph on n vertices with $n \geq 50k^2$ where $k \geq 2$, and $\delta^c(G) \geq \frac{n}{2} + k - 1$. Let $v \in V(G)$ such that $d_G^{mon}(v) = \Delta^{mon}(G)$. Then G contains k rainbow triangles sharing only the vertex v as the center (i.e., the underly graph is F_k with v as its center).*

The following result on Turán number of friendship graphs is well known.

Theorem 13 ([12])**.** *For every $k \geq 1$ and every $n \geq 50k^2$, if a graph G of order n satisfies $e(G) > ex(n, F_k)$, then G contains a copy of a k-friendship graph, where $ex(n, F_k) = \lfloor \frac{n^2}{4} \rfloor + k^2 - k$ if k is odd; and $ex(n, F_k) = \lfloor \frac{n^2}{4} \rfloor + k^2 - \frac{3k}{2}$ if k is even.*

The *matching number* of a graph G, denoted by $\alpha'(G)$, is defined to be the maximum number of pairwise disjoint edges in G. Our proof of Theorem 12 uses a famous result on Turán number of a matching with given number of edges due to Erdős and Gallai [13].

Theorem 14 ([13]). *Let G be a graph on n vertices. If $\alpha'(G) \leq k$ then $e(G) \leq \max\{\binom{2k+1}{2}, \binom{n}{2} - \binom{n-k}{2}\}$.*

We also need a special case of the next lemma.

Lemma 4. *Let (G, C) be an edge-colored graph on n vertices with $\delta^c(G)$ such that $e(G)$ is minimal. Then for a vertex $v \in V(G)$ with $d_G^{mon}(v) = \triangle^{mon}(G)$, we have*

$$rt(G; v) \geq n_0 \geq \frac{1}{2} \left(\Delta^{mon}(G) + d_G^c(v) - 1)(\delta^c(G) + d_G^c(v) - n)\right).$$

Proof. Let v be such that $d^{mon}(v) = \Delta^{mon}(G)$, X_v be the maximum monochromatic neighborhood of v (in G), and $Y_v \subset N(v) \backslash X_v$ (such that for each $u, u' \in Y_v$, we have $C(uv) \neq C(u'v)$) and $|Y_v| = d^c(v) - 1$ in Lemma 2. From the fact

$$|Y_v| d^{mon}(v) - \sum_{a \in Y_v} d^{mon}(a) \geq 0,$$

we obtain the lemma. □

Proof of Theorem 12. Without loss of generality, assume that G is edge-minimal subject to the condition $\delta^c \geq \frac{n}{2} + k - 1$. We prove the theorem by contradiction. Choose $v \in V(G)$ such that $d^{mon}(v) = \Delta^{mon}(G)$.

If $\Delta^{mon}(G) = 1$, then G is properly-colored. Note that $e(G) \geq \frac{\delta^c n}{2} \geq \frac{n^2}{4} + \frac{kn}{2} - \frac{n}{2}$, and $ex(n, F_k) \leq \left\lfloor \frac{n^2}{4} \right\rfloor + k^2 - \frac{3k}{2}$ when $n \geq 50k^2$ by Theorem 13. When $n \geq 50k^2$, we have

$$\frac{n^2}{4} + \frac{kn}{2} - \frac{n}{2} > \left\lfloor \frac{n^2}{4} \right\rfloor + k^2 - \frac{3k}{2}$$

(recall $k \geq 2$), and so G contains a properly-colored F_k, and hence k rainbow triangles sharing one common vertex. Next we assume that $\Delta^{mon}(G) \geq 2$.

By Lemma 4,

$$n_0 \geq \frac{1}{2} \left((d^{mon}(v) + d^c(v) - 1)(\delta^c + d^c(v) - n)\right) \geq (k-1)(d^{mon}(v) + d^c(v) - 1). \quad (12)$$

Recall D_v is the digraph defined on $X_v \cup Y_v$, where X_v is the maximum monochromatic neighborhood of v and $Y_v \subset N(v) \backslash X_v$ such that for any $u_1, u_2 \in Y_v$, we have $C(u_1 v) \neq C(u_2 v)$ and $|Y_v|$ is maximal. Furthermore, n_1^* is the number of 2-cycles in $D_v[X_v]$, and n_0 is the number of other 2-cycles in D_v. Observe that the 2-cycle in $D_v[X_v]$ do not correspond to a rainbow triangles through v, but a 2-cycle contributing to n_0 can correspond to such one.

Consider the subgraph of G on vertex set $X_v \cup Y_v$, denoted by G', with edge set consisting of ones which correspond to the 2-cycles in D_v (of the number n_0). Then

$|G'| = d^{mon}(v) + d^c(v) - 1 \geq \frac{n}{2} + k$. Notice that each edge in G' corresponds to a rainbow triangle through the vertex v. From (12), we have that

$$e(G') \geq (k-1)(d^{mon}(v) + d^c(v) - 1). \tag{13}$$

Since G contains no k rainbow triangles sharing one common vertex, G' contains no matching of size k. That is, $\alpha'(G') \leq k-1$. So by Theorem 14,

$$e(G') \leq \max\left\{\binom{2k-1}{2}, \binom{k-1}{2} + (k-1)(|G'| - k + 1)\right\}. \tag{14}$$

By simple algebra, we have $\binom{2k-1}{2} < \frac{(k-1)(n+2k)}{2}$ when $n \geq 2k-3$. Furthermore,

$$\begin{aligned}&(k-1)(d^{mon}(v) + d^c(v) - 1) - \binom{k-1}{2} - (k-1)(|G'| - k + 1)\\ &= -\binom{k-1}{2} + (k-1)^2 > 0\end{aligned}$$

Thus, (13) contradicts (14) since $n \geq 2k-3$. The proof is complete. ■

5 Concluding remarks

In this paper, we give a tight color degree condition (up to a constant) for k rainbow triangles sharing one common vertex (when k is a fixed integer), and highly suspect the tight one is $\frac{n+1}{2}$ for $n = \Omega(k^2)$ (by considering Theorem 13).

Erdős et al. [12] conjectured Theorem 13 holds for $n \geq 4k$. If the answer to this conjecture is positive, then Theorem 9 can be improved to all graphs with order $n \geq 4k$. On the other hand, maybe an answer to the following is positive.

Problem 1. Let n, k be two positive integers. Let (G, C) be an edge-colored graph on n vertices with $\delta^c(G) \geq \frac{n+1}{2}$. Does there exist a constant c, such that if $n \geq ck$ then G contains a properly-colored F_k?

Recall that $f(n) := \min\{rt(G) : G \in \mathcal{G}_n^*\}$ (see Section 1). We conclude this paper with the following more feasible problem.

Problem 2. Determine the value of $\lim_{n\to\infty} \frac{f(n)}{n^2}$.

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References

[1] R. Aharoni, M. DeVos, and R. Holzman, Rainbow triangles and the Caccetta-Häggkvist conjecture, *J. Graph Theory* **92** (2019), no. 4, 347–360.

[2] J. Balogh, P. Hu, B. Lidický, F. Pfender, J. Volec, and M. Young, Rainbow triangles in three-colored graphs, *J. Combin. Theory, Ser. B* **126** (2017), 83–113.

[3] J.A. Bondy, U.S.R. Murty, Graph theory, Graduate Texts in Mathematics, 244. Springer, New York, 2008. xii+651 pp. ISBN: 978-1-84628-969-9.

[4] H.J. Broersma, X. Li, G. Woeginger, and S. Zhang, Paths and cycles in colored graphs, *Australas. J. Combin.* **31** (2005), 299–311.

[5] R. Čada, A. Kaneko, Z. Ryjáček, and K. Yoshimoto, Rainbow cycles in edge-colored graphs, *Discrete Math.* **339** (2016), no. 4, 1387–1392.

[6] H. Chen, Long rainbow paths and rainbow cycles in edge colored graphs–A survey, *App. Math. Comput.* **317** (2018), no. 15, 187–192.

[7] A. Czygrinow, T. Molla, B. Nagle, and R. Oursler, On odd rainbow cycles in edge-colored graphs, *European J. Combin.* 94 (2021), Paper No. 103316, 10 pp.

[8] S. Ehard, E. Mohr, Rainbow triangles and cliques in edge-colored graphs, *European J. Combin.* **84** (2020), 103037, 12 pp.

[9] P. Erdős, Some theorems on graphs, *Riv. Lemat.* **9** (1955), 13–17. (in Hebrew with English summary)

[10] P. Erdős, On a theorem of Rademacher-Turán, *Illinois J. Math.* **6** (1962), 122–127.

[11] P.Erdős, On the number of complete subgraphs contained in certain graphs, *Magy. Tud. Acad. Mat. Kut. Int. Kőzl.* **7** (1962), 459–474.

[12] P. Erdős, Z. Füredi, R. J. Gould, and D. S. Gunderson, Extremal graphs for intersecting triangles, *J. Combin. Theory. Ser. B* **64** (1995), no.1, 89–100.

[13] P. Erdős, T. Gallai, On maximal paths and circuits of graphs, *Acta Math. Acad. Sci. Hungar* **10** (3–4) (1959) 337–356.

[14] P. Erdős, M. Simonovits, and V. T. Sós, Anti-Ramsey theorems, in Coll. Math. Soc. J. Bolyai 10, 1973, pp. 633-643.

[15] S. Fujita, R. Li, and S. Zhang, Color degree and monochromatic degree conditions for short properly colored cycles in edge-colored graphs, *J. Graph Theory* **87** (2018), no. 3, 362–373.

[16] S. Fujita, B. Ning, C. Xu, and S. Zhang, On sufficient conditions for rainbow cycles in edge-colored graphs, *Discrete Math.* **342** (2019), no. 7, 1956–1965.

[17] J. Hu, H. Li, and D. Yang, Vertex-disjoint rainbow triangles in edge-colored graphs, *Discrete Math.* **343** (2020), no. 12, 112117, 5 pp.

[18] M. Kano, X. Li, Monochromatic and heterochromatic subgraphs in edge-colored graphsa survey, *Graphs Combin.* **24** (2008), no. 4, 237–263.

[19] R. Li, H.J. Broersma, and S. Zhang, Vertex-disjoint properly edge-colored cycles in edge-colored complete graphs, *J. Graph Theory* **94** (2020), no. 3, 476–493.

[20] B. Li, B. Ning, C. Xu, and S. Zhang, Rainbow triangles in edge-colored graphs, *European J. Combin.* **36** (2014), 453–459.

[21] H. Li, Rainbow C_3's and C_4's in edge-colored graphs, *Discrete Math.* **313** (2013), no. 19, 1893–1896.

[22] H. Li, G. Wang, Color degree and heterochromatic cycles in edge-colored graphs, *European J. Combin.* **33** (2012), no. 8, 1958–1964.

[23] L. Lovász, M. Simonovits, On the Number of Complete Subgraphs of a Graph, II, in: Studies in Pure Math, Birkhäuser, 459–495, 1983.

[24] C. Xu, X. Hu, W. Wang, and S. Zhang, Rainbow cliques in edge-colored graphs, *European J. Combin.* **54** (2016), 193–200.

[25] C. Xu, C. Magnant, and S. Zhang, Properly colored C_4's in edge-colored graphs, *Discrete Math.* **343** (2020), no. 12, 112116, 13 pp.