SOME CONGRUENCES FOR 12-COLORED GENERALIZED FROBENIUS PARTITIONS

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Abstract. In his 1984 AMS Memoir, Andrews introduced the family of functions $c\phi_k(n)$, the number of k-colored generalized Frobenius partitions of n. In 2019, Chan, Wang and Yang systematically studied the arithmetic properties of $\mathrm{C}\Phi_k(q)$ for $2 \leq k \leq 17$ by utilizing the theory of modular forms, where $\mathrm{C}\Phi_k(q)$ denotes the generating function of $c\phi_k(n)$. In this paper, we first establish another expression of $\mathrm{C}\Phi_{12}(q)$ with integer coefficients, then prove some congruences modulo small powers of 3 for $c\phi_{12}(n)$ by using some parameterized identities of theta functions due to A. Alaca, S. Alaca and Williams. Finally, we conjecture three families of congruences modulo powers of 3 satisfied by $c\phi_{12}(n)$.

1. Introduction

Throughout this paper, we always assume that q is a complex number such that $|q| < 1$ and adopt the following standard notation:

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k).$$

In his 1984 AMS Memoir, Andrews [2] defined the notion of a generalized Frobenius partition of n, which is a two-rowed array of nonnegative integers of the form:

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

wherein each row, which is of the same length, is arranged in weakly decreasing order with $n = r + \sum_{i=1}^{r}(a_i + b_i)$. Furthermore, Andrews studied a variant of generalized Frobenius partitions whose parts are taken from k copies of the nonnegative integers, which is called k-colored generalized Frobenius partitions. For any $k \geq 1$, let $c\phi_k(n)$ denote the number of k-colored generalized Frobenius partitions of n. Among many other things, Andrews [2, Corollary 10.1] proved that for any $n \geq 0$,

$$c\phi_2(5n+3) \equiv 0 \pmod{5}.$$

From then on, many scholars extensively investigated a number of congruence properties for $c\phi_k(n)$ with different moduli. Baruah and Sarmah [3, 4] derived some congruences

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modulo small powers of 2 for $c\phi_4(n)$ and some congruences modulo small powers of 3 for $c\phi_6(n)$. Congruence properties modulo powers of 5 for $c\phi_3(n)$ and $c\phi_4(n)$ were subsequently considered by Ono [28], Lovejoy [26], Xiong [39], Sellers [31], Xia [38], Hirschhorn and Sellers [21], Chan, Wang and Yang [7], and Wang and Zhang [34]. Congruence properties modulo 7 for $c\phi_4(n)$ were investigated by Lin [25], and Zhang and Wang [41]. Congruence properties of $c\phi_6(n)$ modulo powers of 3 were successively investigated by Xia [37], Hirschhorn [16], Gu, Wang and Xia [14], and the third author [32]. The third author [33] also established congruence properties modulo 5 for $c\phi_8(n)$ and $c\phi_9(n)$. There are other studies on congruence properties for $c\phi_k(n)$; see, for example, [9–13, 22–24, 27, 29, 30, 36].

In 2019, Chan, Wang and Yang [8] systematically investigated the arithmetic properties of $\mathrm{C}\Phi_k(q)$ for $2 \leq k \leq 17$, where $\mathrm{C}\Phi_k(q)$ denotes the generating function of $c\phi_k(n)$. In particular, they [8, Eq. (6.26)] proved that (some typos have been corrected)

$$\begin{aligned}\mathrm{C}\Phi_{12}(q) = \frac{1}{\Theta_3(q)(q;q)_\infty^{12}}\Big(&-\frac{36207}{160}B_{12,1} + \frac{923091}{4000}B_{12,4} + \frac{35829}{100}B_{12,5} + \frac{891}{4}B_{12,6} \\ &- \frac{1485}{8}B_{12,7} - \frac{143247}{250}B_{12,8} - \frac{891}{4}B_{12,9} - \frac{8109}{160}B_{12,10} - \frac{582717}{16000}B_{12,11} \\ &+ \frac{227691}{200}B_{12,12} + \frac{714249}{8000}B_{12,13} + \frac{8109}{80}B_{12,14} + \frac{33}{8}B_{12,15} \\ &+ \frac{294109}{500}B_{12,16} - \frac{16503}{400}B_{12,17} - \frac{99}{8}B_{12,18} + \frac{10559}{200}B_{12,19} \\ &- \frac{128807}{100}B_{12,20} + \frac{25647}{160}B_{12,21} + \frac{727}{160}B_{12,22}\Big),\end{aligned} \tag{1.1}$$

where the $B_{12,i}$ for $i \in \{1, 4, 5, \ldots, 22\}$ are some functions involving the following two theta functions, given by

$$\Theta_2(q) = \sum_{j=-\infty}^{\infty} q^{(j+1/2)^2} = 2q^{1/4}\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty},$$

$$\Theta_3(q) = \sum_{j=-\infty}^{\infty} q^{n^2} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}.$$

It is easy to see that the coefficients of many terms in (1.1) are not integers. Therefore, a natural question is whether there is another expression with integral coefficients for $\mathrm{C}\Phi_{12}(q)$. The first purpose of this paper is to establish the following expression for $\mathrm{C}\Phi_{12}(q)$. For the sake of convenience, denote

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} \qquad \text{and} \qquad E(q^k) := (q^k;q^k)_\infty. \tag{1.2}$$

Theorem 1.1.

$$\begin{aligned}\mathrm{C\Phi}_{12}(q) = \frac{1}{E(q)^{12}}\Bigg\{ & a(q)^4\left(\frac{E(q^6)^8E(q^{12})}{E(q^3)^4E(q^{24})^2}+8q^3\frac{E(q^{12})^3E(q^{24})^2}{E(q^6)^2}\right)\\ & +108qa(q)^2\frac{E(q^3)^6}{E(q)^2}\left(\frac{E(q^4)E(q^6)^4E(q^8)}{E(q^2)^2E(q^{24})}+q\frac{E(q^2)^3E(q^3)^2E(q^{12})E(q^{24})}{E(q)^2E(q^6)E(q^8)}\right)\\ & +216q^2a(q)\frac{E(q^3)^9}{E(q)^3}\left(\frac{E(q^2)E(q^4)E(q^6)^5E(q^{24})}{E(q)E(q^3)E(q^8)E(q^{12})^2}+2\frac{E(q^6)E(q^8)E(q^{12})^3}{E(q^2)E(q^{24})}\right)\\ & +486q^2\frac{E(q^3)^{12}}{E(q)^4}\left(\frac{E(q^4)^2E(q^6)^{11}E(q^{24})}{E(q^2)E(q^3)^4E(q^8)E(q^{12})^5}+4q\frac{E(q^8)E(q^{12})^6}{E(q^4)E(q^6)^2E(q^{24})}\right)\Bigg\}.\end{aligned} \tag{1.3}$$

By utilizing a general congruence relation [8, Theorem 5.3], Chan et al. [8, Eqs. (6.28) and (6.29)] derived that for any $n \geq 0$,

$$\begin{aligned} c\phi_{12}(3n+1) &\equiv 0 \pmod 9,\\ c\phi_{12}(3n+2) &\equiv 0 \pmod 9. \end{aligned} \tag{1.4}$$

The other purpose of this paper is to prove the following congruences modulo 27 and 81 enjoyed by $c\phi_{12}(n)$.

Theorem 1.2. *For any $n \geq 0$,*

$$c\phi_{12}(3n+2) \equiv 0 \pmod{27}, \tag{1.5}$$

$$c\phi_{12}(9n+5) \equiv 0 \pmod{81}, \tag{1.6}$$

$$c\phi_{12}(9n+8) \equiv 0 \pmod{81}. \tag{1.7}$$

Remark 1.3. Obviously, (1.5) is a stronger form of (1.4). By computation, one sees that $c\phi_{12}(2) = 4644 \not\equiv 0 \pmod{81}$. From this perspective, the modulus in (1.5) is best possible. So does (1.6) and (1.7).

Actually, (1.5)–(1.7) appear to be just the tip of the iceberg. With the help of a computer, we pose the following three families of conjectural congruences modulo powers of 3 satisfied by $c\phi_{12}(n)$.

Conjecture 1.4. For any $n \geq 0$ and $\alpha \geq 0$,

$$\begin{aligned} c\phi_{12}\left(3^{2\alpha+1}n+\frac{3^{2\alpha+1}+1}{2}\right) &\equiv 0 \pmod{3^{3\alpha+3}},\\ c\phi_{12}\left(3^{2\alpha+2}n+\frac{3^{2\alpha+2}+1}{2}\right) &\equiv 0 \pmod{3^{3\alpha+4}},\\ c\phi_{12}\left(3^{2\alpha+2}n+\frac{5\times 3^{2\alpha+1}+1}{2}\right) &\equiv 0 \pmod{3^{3\alpha+4}}. \end{aligned}$$

The rest of this paper is organized as follows. In Section 2, we collect some necessary lemmas which will be utilized to prove the main results later. The proofs of Theorems 1.1 and 1.2 are presented in Section 3.

2. Some preliminary results

To prove (1.5)–(1.7), we first collect some necessary identities.

Lemma 2.1.

$$E(q)^4 = \frac{E(q^4)^{10}}{E(q^2)^2E(q^8)^4} - 4q\frac{E(q^2)^2E(q^8)^4}{E(q^4)^2}, \tag{2.1}$$

$$\frac{1}{E(q)^4} = \frac{E(q^4)^{14}}{E(q^2)^{14}E(q^8)^4} + 4q\frac{E(q^4)^2E(q^8)^4}{E(q^2)^{10}}. \tag{2.2}$$

Proof. The identities (2.1) and (2.2) are (2.9) and (2.10) in [40], respectively. □

Lemma 2.2.

$$\frac{E(q^2)^5}{E(q)^2E(q^4)^2} = \frac{E(q^{18})^5}{E(q^9)^2E(q^{36})^2} + 2q\frac{E(q^6)^2E(q^9)E(q^{36})}{E(q^3)E(q^{12})E(q^{18})}, \tag{2.3}$$

$$\frac{E(q^2)^2}{E(q)} = \frac{E(q^6)E(q^9)^2}{E(q^3)E(q^{18})} + q\frac{E(q^{18})^2}{E(q^9)}. \tag{2.4}$$

Proof. The identities (2.3) and (2.4) follow from Corollary (i) and (ii) on page 49 of Berndt's book [5], respectively. □

Lemma 2.3.

$$\frac{E(q^2)}{E(q)^2} = \frac{E(q^6)^4E(q^9)^6}{E(q^3)^8E(q^{18})^3} + 2q\frac{E(q^6)^3E(q^9)^3}{E(q^3)^7} + 4q^2\frac{E(q^6)^2E(q^{18})^3}{E(q^3)^6}, \tag{2.5}$$

$$\frac{E(q)}{E(q^2)^2} = \frac{E(q^3)^2E(q^9)^3}{E(q^6)^6} - q\frac{E(q^3)^3E(q^{18})^3}{E(q^6)^7} + q^2\frac{E(q^3)^4E(q^{18})^6}{E(q^6)^8E(q^9)^3}, \tag{2.6}$$

$$\begin{aligned}\frac{E(q^2)}{E(q)E(q^4)} &= \frac{E(q^{18})^9}{E(q^3)^2E(q^9)^3E(q^{12})^2E(q^{36})^3} + q\frac{E(q^6)^2E(q^{18})^3}{E(q^3)^3E(q^{12})^3} \\ &\quad + q^2\frac{E(q^6)^4E(q^9)^3E(q^{36})^3}{E(q^3)^4E(q^{12})^4E(q^{18})^3}.\end{aligned} \tag{2.7}$$

Proof. The identity (2.5) was derived by Hirschhorn and Sellers [19, Theorem 1.1]. The identity (2.6) is equivalent to Lemma 2.2 due to Hirschhorn and Sellers [20]. Moreover, replacing q by $-q$ in (2.6) and utilizing the fact

$$E(-q) = \frac{E(q^2)^3}{E(q)E(q^4)},$$

upon simplification, we obtain (2.7). □

Lemma 2.4. *If $a(q)$ is defined by (1.2), then*

$$a(q) = a(q^3) + 6q\frac{E(q^9)^3}{E(q^3)} \tag{2.8}$$

and

$$\frac{1}{E(q)^3} = \frac{E(q^9)^3}{E(q^3)^{12}}\big(a(q^3)^2E(q^3)^2 + 3qa(q^3)E(q^3)E(q^9)^3 + 9q^2E(q^9)^6\big). \tag{2.9}$$

Proof. The identity (2.8) was established by Hirschhorn, Garvan and Borwein [18, Eq. (1.3)]. The identity (2.9) was proved by Wang [35, Eq. (2.28)]. □

Hirschhorn et al. [18, Eq. (1.5)] also proved that

$$a(q) = 1 + 6\sum_{n=1}^{\infty}\left(\frac{q^{3n-2}}{1-q^{3n-2}} - \sum_{n=1}^{\infty}\frac{q^{3n-1}}{1-q^{3n-1}}\right),$$

from which we find that

$$a(q) \equiv 1 \pmod 3 \qquad \text{and} \qquad a(q)^3 \equiv 1 \pmod 9. \tag{2.10}$$

According to the binomial theorem, one can easily establish the following congruence, which will be used frequently in the sequel.

Lemma 2.5. *For any* $k \geq 1$,

$$E(q^k)^3 \equiv E(q^{3k}) \pmod 3. \tag{2.11}$$

3. Proofs of the main results

To prove Theorem 1.1, we require the following two related lemmas.

Lemma 3.1.

$$\sum_{r_1,r_2,r_3=-\infty}^{\infty} q^{3\sum_{i=1}^{3}r_i^2+3\sum_{1\leq i<j\leq 3}r_ir_j} = \frac{E(q^6)^8E(q^{12})}{E(q^3)^4E(q^{24})^2} + 8q^3\frac{E(q^{12})^3E(q^{24})^2}{E(q^6)^2}, \tag{3.1}$$

$$\sum_{r_1,r_2,r_3=-\infty}^{\infty} q^{3\sum_{i=1}^{3}r_i^2+3\sum_{1\leq i<j\leq 3}r_ir_j+r_1+r_2+2r_3}$$
$$= \frac{E(q^4)E(q^6)^4E(q^8)}{E(q^2)^2E(q^{24})} + q\frac{E(q^2)^3E(q^3)^2E(q^{12})E(q^{24})}{E(q)^2E(q^6)E(q^8)}, \tag{3.2}$$

$$\sum_{r_1,r_2,r_3=-\infty}^{\infty} q^{3\sum_{i=1}^{3}r_i^2+3\sum_{1\leq i<j\leq 3}r_ir_j+3r_1+3r_2+2r_3}$$
$$= \frac{E(q^2)E(q^4)E(q^6)^5E(q^{24})}{E(q)E(q^3)E(q^8)E(q^{12})^2} + 2\frac{E(q^6)E(q^8)E(q^{12})^3}{E(q^2)E(q^{24})}, \tag{3.3}$$

$$\sum_{r_1,r_2,r_3=-\infty}^{\infty} q^{3\sum_{i=1}^{3}r_i^2+3r_1r_2-3r_1r_3-3r_2r_3+2r_1+2r_2}$$
$$= \frac{E(q^4)^2E(q^6)^{11}E(q^{24})}{E(q^2)E(q^3)^4E(q^8)E(q^{12})^5} + 4q\frac{E(q^8)E(q^{12})^6}{E(q^4)E(q^6)^2E(q^{24})}. \tag{3.4}$$

Proof. The main ingredient in proofs of (3.1)–(3.4) is the integer matrix exact covering systems, developed by Cao [6]. Similar treatments were used for deriving the generating functions of 4- and 6-colored generalized Frobenius partitions; see [3, 4] for a detailed account of such applications.

We only present the proof of (3.1), and the remaining cases can be demonstrated in a similar manner.

First, we adopt the matrix

$$B_1 = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

Then the congruences $B_1\mathbf{r} \equiv 0 \pmod 2$ satisfy that

$$\begin{cases} -r_1 + r_2 \equiv 0 & \pmod 2, \\ r_1 + r_3 \equiv 0 & \pmod 2, \\ r_1 - r_3 \equiv 0 & \pmod 2. \end{cases}$$

Then the above congruences contain two solutions. Namely, $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ modulo 2.

Therefore, we get the following integer matrix exact covering systems

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix},$$

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Using the above integer matrix exact covering systems, we obtain that

$$\begin{aligned} &\sum_{r_1,r_2,r_3=-\infty}^{\infty} q^{3\sum_{i=1}^{3} r_i^2 + 3\sum_{1\le i<j\le 3} r_i r_j} \\ &= \sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{6n_1^2+3n_2^2+3n_3^2} + \sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{6n_1^2+6n_1+3n_2^2+3n_2+3n_3^2+3n_3+3} \\ &= \frac{E(q^6)^8 E(q^{12})}{E(q^3)^4 E(q^{24})^2} + 8q^3 \frac{E(q^{12})^3 E(q^{24})^2}{E(q^6)^2}, \end{aligned}$$

which is nothing but (3.1). For (3.2) and (3.3), we also adopt the matrix B_1 and utilizing a similar strategy. However, for (3.4), we need the following matrix

$$B_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}.$$

This completes the proof of Lemma 3.1. □

Lemma 3.2. *The constant term of* $\left(\sum_{r=-\infty}^{\infty} a^r q^{r^2}\right)^{12}$ *is*

$$\begin{aligned}
&\mathrm{CT}_a\left(\left(\sum_{r=-\infty}^{\infty} a^r q^{r^2}\right)^{12}\right)\\
&= a(q^2)^4 \sum_{r_1,r_2,r_3=-\infty}^{\infty} q^{6\sum_{i=1}^{3} r_i^2+6\sum_{1\le i<j\le 3} r_i r_j}\\
&\quad + 108qa(q^2)^2\frac{E(q^6)^6}{E(q^2)^2} \sum_{r_1,r_2,r_3=-\infty}^{\infty} q^{6\sum_{i=1}^{3} r_i^2+6\sum_{1\le i<j\le 3} r_i r_j+2r_1+2r_2+4r_3}\\
&\quad + 216q^4a(q^2)\frac{E(q^6)^9}{E(q^3)^3} \sum_{r_1,r_2,r_3=-\infty}^{\infty} q^{6\sum_{i=1}^{3} r_i^2+6\sum_{1\le i<j\le 3} r_i r_j+6r_1+6r_2+4r_3}\\
&\quad + 486q^4\frac{E(q^6)^{12}}{E(q^2)^4} \sum_{r_1,r_2,r_3=-\infty}^{\infty} q^{6\sum_{i=1}^{3} r_i^2+6r_1r_2-6r_1r_3-6r_2r_3+4r_1+4r_2}. \qquad (3.5)
\end{aligned}$$

Proof. Hirschhorn [15] proved the following identity

$$\begin{aligned}
\left(\sum_{r=-\infty}^{\infty} a^r q^{r^2}\right)^{3} &= a(q^2)\sum_{r=-\infty}^{\infty} a^{3r}q^{3r^2}\\
&\quad + 3q\frac{E(q^6)^3}{E(q^2)}\left(a\sum_{r=-\infty}^{\infty} a^{3r}q^{3r^2+2r} + a^{-1}\sum_{r=-\infty}^{\infty} a^{-3r}q^{3r^2+2r}\right), \qquad (3.6)
\end{aligned}$$

where $a(q)$ is defined as in (1.2). With the help of (3.6), we obtain that

$$\begin{aligned}
&\mathrm{CT}_a\left(\left(\sum_{r=-\infty}^{\infty} a^r q^{r^2}\right)^{12}\right)\\
&= a(q^2)^4 \sum_{r_1,r_2,r_3=-\infty}^{\infty} q^{6\sum_{i=1}^{3} r_i^2+6\sum_{1\le i<j\le 3} r_i r_j}\\
&\quad + 108q^2a(q^2)^2\frac{E(q^6)^6}{E(q^2)^2} \sum_{r_1,r_2,r_3=-\infty}^{\infty} q^{6\sum_{i=1}^{3} r_i^2+6\sum_{1\le i<j\le 3} r_i r_j+2r_1+2r_2+4r_3}\\
&\quad + 216q^4a(q^2)\frac{E(q^6)^9}{E(q^3)^3} \sum_{r_1,r_2,r_3=-\infty}^{\infty} q^{6\sum_{i=1}^{3} r_i^2+6\sum_{1\le i<j\le 3} r_i r_j+6r_1+6r_2+4r_3}\\
&\quad + 486q^4\frac{E(q^6)^{12}}{E(q^2)^4} \sum_{r_1,r_2,r_3=-\infty}^{\infty} q^{6\sum_{i=1}^{3} r_i^2+6r_1r_2-6r_1r_3-6r_2r_3+4r_1+4r_2},
\end{aligned}$$

which is nothing but (3.5).

We therefore complete the proof of Lemma 3.2. □

Now it is time to prove Theorem 1.1.

Proof of Theorem 1.1. In view of (3.5), we deduce that

$$\begin{aligned}
&\mathrm{CT}_a\left(\left(\sum_{r=-\infty}^{\infty} a^r q^{r^2}\right)^{12}\right)\\
&= \sum_{\substack{m_1+m_2+\cdots+m_{12}=0\\ m_1,m_2,\ldots,m_{12}=-\infty}}^{\infty} q^{\sum_{i=1}^{12} m_i^2}\\
&= \sum_{m_1,\ldots,m_{11}=-\infty}^{\infty} q^{2\sum_{i=1}^{11} m_i^2+2\sum_{1\le i<j\le 11} m_i m_j}\\
&= a(q^2)^4 \sum_{r_1,r_2,r_3=-\infty}^{\infty} q^{6\sum_{i=1}^{3} r_i^2+6\sum_{1\le i<j\le 3} r_i r_j}\\
&\quad + 108q^2 a(q^2)^2 \frac{E(q^6)^6}{E(q^2)^2} \sum_{r_1,r_2,r_3=-\infty}^{\infty} q^{6\sum_{i=1}^{3} r_i^2+6\sum_{1\le i<j\le 3} r_i r_j+2r_1+2r_2+4r_3}\\
&\quad + 216q^4 a(q^2) \frac{E(q^6)^9}{E(q^2)^3} \sum_{r_1,r_2,r_3=-\infty}^{\infty} q^{6\sum_{i=1}^{3} r_i^2+6\sum_{1\le i<j\le 3} r_i r_j+6r_1+6r_2+4r_3}\\
&\quad + 486q^4 \frac{E(q^6)^{12}}{E(q^2)^4} \sum_{r_1,r_2,r_3=-\infty}^{\infty} q^{6\sum_{i=1}^{3} r_i^2+6r_1r_2-6r_1r_3-6r_2r_3+4r_1+4r_2}. \qquad (3.7)
\end{aligned}$$

Moreover, Andrews [2, Theorem 5.2] established the following expression for $\mathrm{C}\Phi_k(q)$, namely,

$$\mathrm{C}\Phi_k(q) = \frac{1}{E(q)^k} \sum_{m_1,m_2,\ldots,m_{k-1}=-\infty}^{\infty} q^{\sum_{i=1}^{k-1} m_i^2+\sum_{1\le i<j\le k-1} m_i m_j}. \qquad (3.8)$$

The identity (1.3) follows from (3.1)–(3.4), (3.7) and (3.8).

This finishes the proof of Theorem 1.1. □

Next, we are in a position to prove Theorem 1.2.

In what follows, all congruences are modulo 81 unless otherwise specified.

Proof of Theorem 1.2. According to (2.10) and (2.11), we find that

$$\begin{aligned}
\mathrm{C}\Phi_{12}(q) \equiv{}& \frac{a(q)^4}{E(q)^{12}}\left(\frac{E(q^6)^8E(q^{12})}{E(q^3)^4E(q^{24})^2} + 8q^3\frac{E(q^{12})^3E(q^{24})^2}{E(q^6)^2}\right)\\
&+ 27q\frac{E(q^3)^2}{E(q)^2}\left(\frac{E(q^4)E(q^6)^4}{E(q^2)^2E(q^8)^2} + q\frac{E(q^3)^2E(q^{12})E(q^{24})}{E(q)^2E(q^8)}\right)
\end{aligned}$$

$$+\,54q^2E(q^3)^4\left(\frac{E(q^2)E(q^4)E(q^6)^5E(q^{24})}{E(q)E(q^3)E(q^8)E(q^{12})^2}+2\frac{E(q^6)E(q^8)E(q^{12})^3}{E(q^2)E(q^{24})}\right). \tag{3.9}$$

Next, we consider the following three auxiliary functions, defined by

$$\sum_{n=0}^{\infty} g_1(n)q^n := \frac{a(q)^4}{E(q)^{12}}\left(\frac{E(q^6)^8E(q^{12})}{E(q^3)^4E(q^{24})^2}+8q^3\frac{E(q^{12})^3E(q^{24})^2}{E(q^6)^2}\right), \tag{3.10}$$

$$\sum_{n=0}^{\infty} g_2(n)q^n := 27q\frac{E(q^3)^2}{E(q)^2}\left(\frac{E(q^4)E(q^6)^4}{E(q^2)^2E(q^8)^2}+q\frac{E(q^3)^2E(q^{12})E(q^{24})}{E(q)^2E(q^8)}\right), \tag{3.11}$$

$$\sum_{n=0}^{\infty} g_3(n)q^n := 54q^2E(q^3)^4\left(\frac{E(q^2)E(q^4)E(q^6)^5E(q^{24})}{E(q)E(q^3)E(q^8)E(q^{12})^2}+2\frac{E(q^6)E(q^8)E(q^{12})^3}{E(q^2)E(q^{24})}\right). \tag{3.12}$$

Substituting (2.8) and (2.9) into (3.10), extracting all the terms of the form q^{3n+2}, after simplification, we deduce that

$$\sum_{n=0}^{\infty} g_1(3n+2)q^n \equiv 27a(q)^{10}\frac{E(q^2)^8E(q^3)^{18}E(q^4)}{E(q)^{46}E(q^8)^2}+54qa(q)^{10}\frac{E(q^3)^{18}E(q^4)^3E(q^8)^2}{E(q)^{42}E(q^2)^2}.$$

Thanks to (2.10) and (2.11),

$$\sum_{n=0}^{\infty} g_1(3n+2)q^n \equiv 27E(q^3)^4\left(\frac{E(q^2)^8E(q^4)}{E(q)^4E(q^8)^2}+2q\frac{E(q^4)^3E(q^8)^2}{E(q^2)^2}\right). \tag{3.13}$$

The congruence (1.5) follows from (3.9) and (3.13) immediately.

Moreover, it follows from (3.13) that

$$\sum_{n=0}^{\infty} g_1(3n+2)q^n$$
$$\equiv 27E(q^3)^4\left(\frac{E(q^2)^{10}}{E(q)^4E(q^4)^4}\cdot\frac{E(q^4)^5}{E(q^2)^2E(q^8)^2}+2q\frac{E(q^4)^4}{E(q^2)^2}\cdot\frac{E(q^8)^2}{E(q^4)}\right). \tag{3.14}$$

Substituting (2.3) and (2.4) into (3.14), after some tedious but straightforward calculations, we deduce that

$$\sum_{n=0}^{\infty} g_1(9n+5)q^n \equiv 27\left\{\frac{E(q^2)^8E(q^4)^{11}}{E(q^8)^6}-\frac{E(q^2)^{10}E(q^4)}{E(q^8)^2}\cdot E(q)^4\right.$$
$$\left.-\,qE(q^4)^3E(q^8)^2\cdot\left(E(q)^4\right)^2+q^2E(q^2)^2E(q^4)E(q^8)^6\cdot E(q)^4\right\} \tag{3.15}$$

and

$$\sum_{n=0}^{\infty} g_1(9n+8)q^n \equiv 27\left\{\frac{E(q^4)^{19}}{E(q^2)^8E(q^8)^6}\cdot\left(E(q)^4\right)^2-\frac{E(q^2)^{32}E(q^8)^2}{E(q^4)^{13}}\cdot\left(\frac{1}{E(q)^4}\right)^2\right.$$

$$- q\frac{E(q^4)^{17}}{E(q^2)^6E(q^8)^2}\cdot E(q)^4 - q\frac{E(q^2)^{10}E(q^8)^6}{E(q^4)^7}\cdot E(q)^4\Bigg\}. \tag{3.16}$$

Substituting (2.1) and (2.2) into (3.15) and (3.16), upon simplification, we further obtain that

$$\sum_{n=0}^{\infty} g_1(9n+5)q^n \equiv 27\left(-q\frac{E(q^4)^{23}}{E(q^2)^4E(q^8)^6} + q\frac{E(q^2)^{12}E(q^8)^2}{E(q^4)} + q^3\frac{E(q^2)^4E(q^8)^{10}}{E(q^4)}\right), \tag{3.17}$$

$$\sum_{n=0}^{\infty} g_1(9n+8)q^n \equiv 27\left(\frac{E(q^4)^{39}}{E(q^2)^{12}E(q^8)^{14}} - \frac{E(q^2)^4E(q^4)^{15}}{E(q^8)^6} - q^2\frac{E(q^4)^{15}E(q^8)^2}{E(q^2)^4}\right). \tag{3.18}$$

Now we recall Horschhorn's version of parameterized identities (see [17, Chap. 35, Eqs. (35.1.1)–(35.1.6)]), whose idea comes from [1].

$$E(q) = s^{1/2}t^{1/24}(1-2qt)^{1/2}(1+qt)^{1/8}(1+2qt)^{1/6}(1+4qt)^{1/8}, \tag{3.19}$$

$$E(q^2) = s^{1/2}t^{1/12}(1-2qt)^{1/4}(1+qt)^{1/4}(1+2qt)^{1/12}(1+4qt)^{1/4}, \tag{3.20}$$

$$E(q^3) = s^{1/2}t^{1/8}(1-2qt)^{1/6}(1+qt)^{1/24}(1+2qt)^{1/2}(1+4qt)^{1/24}, \tag{3.21}$$

$$E(q^4) = s^{1/2}t^{1/6}(1-2qt)^{1/8}(1+qt)^{1/2}(1+2qt)^{1/24}(1+4qt)^{1/8}, \tag{3.22}$$

$$E(q^6) = s^{1/2}t^{1/4}(1-2qt)^{1/12}(1+qt)^{1/12}(1+2qt)^{1/4}(1+4qt)^{1/12}, \tag{3.23}$$

$$E(q^{12}) = s^{1/2}t^{1/2}(1-2qt)^{1/24}(1+qt)^{1/6}(1+2qt)^{1/8}(1+4qt)^{1/24}, \tag{3.24}$$

where

$$s := s(q) = \frac{E(q)^2E(q^4)^2E(q^6)^{15}}{E(q^2)^5E(q^3)^6E(q^{12})^6} \quad \text{and} \quad t := t(q) = \frac{E(q^2)^3E(q^3)^3E(q^{12})^6}{E(q)E(q^4)^2E(q^6)^9}.$$

It follows immediately from the parameterized identities (3.19)–(3.24) that

$$\left(-\frac{E(q^2)^{23}}{E(q)^4E(q^4)^6} + \frac{E(q)^{12}E(q^4)^2}{E(q^2)} + q\frac{E(q)^4E(q^4)^{10}}{E(q^2)}\right)\cdot\frac{E(q)^3E(q^4)^2E(q^6)^{12}}{E(q^2)^7E(q^3)^5E(q^{12})^4}$$
$$= -15qs^7t^2(1-2qt)^3(1+qt)^5(1+2qt)(1+4qt) \equiv 0 \pmod 3$$

and

$$\left(\frac{E(q^2)^{39}}{E(q)^{12}E(q^4)^{14}} - \frac{E(q)^4E(q^2)^{15}}{E(q^4)^6} - q\frac{E(q^2)^{15}E(q^4)^2}{E(q)^4}\right)\cdot\frac{E(q)^3E(q^4)^3E(q^6)^{15}}{E(q^2)^8E(q^3)^5E(q^{12})^7}$$
$$= 15qs^7t(1-2qt)^2(1+qt)^4(1+2qt)(1+4qt)^3 \equiv 0 \pmod 3.$$

Since

$$\frac{E(q)^3E(q^4)^2E(q^6)^{12}}{E(q^2)^7E(q^3)^5E(q^{12})^4} \quad \text{and} \quad \frac{E(q)^3E(q^4)^3E(q^6)^{15}}{E(q^2)^8E(q^3)^5E(q^{12})^7}$$

are invertible in the ring $\mathbb{Z}/3\mathbb{Z}[[q]]$, we deduce that

$$-\frac{E(q^2)^{23}}{E(q)^4E(q^4)^6}+\frac{E(q)^{12}E(q^4)^2}{E(q^2)}+q\frac{E(q)^4E(q^4)^{10}}{E(q^2)}\equiv 0 \pmod 3, \tag{3.25}$$

$$\frac{E(q^2)^{39}}{E(q)^{12}E(q^4)^{14}}-\frac{E(q)^4E(q^2)^{15}}{E(q^4)^6}-q\frac{E(q^2)^{15}E(q^4)^2}{E(q)^4}\equiv 0 \pmod 3. \tag{3.26}$$

The congruences (3.17) and (3.18), together with (3.25) and (3.26), imply that for any $n\geq 0$,

$$g_1(9n+5)\equiv g_1(9n+8)\equiv 0. \tag{3.27}$$

Similarly, from (3.11) we find that

$$\sum_{n=0}^{\infty}g_2(n)q^n\equiv 27\Bigg(qE(q^3)E(q^6)^4\cdot\frac{E(q)}{E(q^2)^2}\cdot\frac{E(q^4)}{E(q^8)^2} + q^2E(q^3)^3E(q^{12})E(q^{24})\cdot\frac{E(q^2)}{E(q)E(q^4)}\cdot\frac{E(q^4)}{E(q^2)E(q^8)}\Bigg).$$

With the help of (2.6) and (2.7), we further obtain that

$$\sum_{n=0}^{\infty}g_2(3n+2)q^n\equiv 27\Bigg(-\frac{E(q)^4E(q^2)^6E(q^4)^{11}}{E(q^8)^6}+\frac{E(q^2)^{16}E(q^4)^{17}}{E(q)^8E(q^8)^{10}} + q\frac{E(q^2)^8E(q^4)^9}{E(q^8)^2}-q\frac{E(q)^{12}E(q^4)^3E(q^8)^2}{E(q^2)^2} + q^2E(q)^8E(q^4)E(q^8)^6+q^3\frac{E(q^2)^{14}E(q^8)^{10}}{E(q)^4E(q^4)^5}\Bigg). \tag{3.28}$$

Plugging (2.1) and (2.2) into (3.28), after simplification, we obtain that

$$\sum_{n=0}^{\infty}g_2(3n+2)q^n\equiv 27\Bigg(\frac{E(q^4)^{45}}{E(q^2)^{12}E(q^8)^{18}}-\frac{E(q^2)^4E(q^4)^{21}}{E(q^8)^{10}}+q\frac{E(q^4)^{33}}{E(q^2)^8E(q^8)^{10}} - q\frac{E(q^2)^8E(q^4)^9}{E(q^8)^2}-q^2\frac{E(q^4)^{21}}{E(q^2)^4E(q^8)^2}-q^3E(q^4)^9E(q^8)^6\Bigg). \tag{3.29}$$

According to the parameterized identities (3.19)–(3.24), we find that

$$\Bigg(\frac{E(q^2)^{45}}{E(q)^{12}E(q^4)^{18}}-\frac{E(q)^4E(q^2)^{21}}{E(q^4)^{10}}-q\frac{E(q^2)^{21}}{E(q)^4E(q^4)^2}\Bigg)\cdot\frac{E(q)^3E(q^4)^4E(q^6)^{13}}{E(q^2)^8E(q^3)^5E(q^{12})^6} = 15qs^8t(1-2qt)^3(1+qt)^4(1+2qt)(1+4qt)^4\equiv 0 \pmod 3$$

and

$$\Bigg(\frac{E(q^2)^{33}}{E(q)^8E(q^4)^{10}}-\frac{E(q)^8E(q^2)^9}{E(q^4)^2}-qE(q^2)^9E(q^4)^6\Bigg)\cdot\frac{E(q)E(q^6)^7}{E(q^2)^2E(q^3)^3E(q^{12})^2} = 15qs^8t^2(1-2qt)^3(1+qt)^5(1+2qt)(1+4qt)^3\equiv 0 \pmod 3.$$

Since

$$\frac{E(q)^3E(q^4)^2E(q^6)^{13}}{E(q^2)^8E(q^3)^5E(q^{12})^6} \quad \text{and} \quad \frac{E(q)E(q^6)^7}{E(q^2)^2E(q^3)^3E(q^{12})^2}$$

are invertible in the ring $\mathbb{Z}/3\mathbb{Z}[[q]]$, we obtain that

$$\frac{E(q^2)^{45}}{E(q)^{12}E(q^4)^{18}} - \frac{E(q)^4E(q^2)^{21}}{E(q^4)^{10}} - q\frac{E(q^2)^{21}}{E(q)^4E(q^4)^2} \equiv 0 \pmod 3, \tag{3.30}$$

$$\frac{E(q^2)^{33}}{E(q)^8E(q^4)^{10}} - \frac{E(q)^8E(q^2)^9}{E(q^4)^2} - qE(q^2)^9E(q^4)^6 \equiv 0 \pmod 3. \tag{3.31}$$

According to (3.29)–(3.31), we find that for any $n \geq 0$

$$g_2(3n+2) \equiv 0. \tag{3.32}$$

Finally, from (3.12) we find that

$$\sum_{n=0}^{\infty} g_3(n)q^n = -27\Bigg(q^2\frac{E(q^3)^3E(q^6)^5E(q^{24})}{E(q^{12})^2}\cdot\frac{E(q^2)^2}{E(q)}\cdot\frac{E(q^4)}{E(q^2)E(q^8)} - q^2\frac{E(q^3)^4E(q^6)E(q^{12})^3}{E(q^{24})}\cdot\frac{E(q^4)^2}{E(q^2)}\cdot\frac{E(q^8)}{E(q^4)^2}\Bigg).$$

Thanks to (2.4), (2.5), (2.7) and (2.11), we further arrive at

$$\sum_{n=0}^{\infty} g_3(3n+2)q^n \equiv 27\Bigg(\frac{E(q)^4E(q^2)^6E(q^4)^{11}}{E(q^8)^6} - \frac{E(q)^8E(q^4)^{25}}{E(q^2)^8E(q^8)^{10}} - q\frac{E(q^2)^8E(q^4)^9}{E(q^8)^2} - q^2\frac{E(q)^4E(q^4)^{11}E(q^8)^2}{E(q^2)^2}\Bigg). \tag{3.33}$$

Substituting (2.1) into (3.33), upon simplification, we obtain that

$$\sum_{n=0}^{\infty} g_3(3n+2)q^n \equiv 27\Bigg(-\frac{E(q^4)^{45}}{E(q^2)^{12}E(q^8)^{18}} + \frac{E(q^2)^4E(q^4)^{21}}{E(q^8)^{10}} + q^2\frac{E(q^4)^{21}}{E(q^2)^4E(q^8)^2}\Bigg) + 27q\Bigg(-\frac{E(q^4)^{33}}{E(q^2)^8E(q^8)^{10}} + \frac{E(q^2)^8E(q^4)^9}{E(q^8)^2} + q^2E(q^4)^9E(q^8)^6\Bigg).$$

According to (3.30) and (3.31), we conclude that for any $n \geq 0$,

$$g_3(3n+2) \equiv 0. \tag{3.34}$$

The congruences (1.6) and (1.7) follow from (3.9)–(3.12), (3.27), (3.32) and (3.34).
We therefore complete the proof of Theorem 1.2. □

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