Vertex-bipancyclicity in a bipartite graph collection

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Abstract

Let $G = \{G_1, \ldots, G_{2n}\}$ be a bipartite graph collection on the common vertex bipartition $(X, Y)$ with $|X| = |Y| = n$. We say that $G$ is bipancyclic if there exists a partial $G$-transversal isomorphic to an $\ell$-cycle for each even integer $\ell \in [4, 2n]$, while $G$ is vertex-bipancyclic if any vertex $v \in X \cup Y$ is contained in a partial $G$-transversal isomorphic to an $\ell$-cycle for each even integer $\ell \in [4, 2n]$. Bradshaw in [Transversals and bipancyclicity in bipartite graph families, Electron. J. Comb., 2021] showed that for each $i \in [2n]$, if $d_{G_i}(x) > \frac{n}{2}$ for each $x \in X$ and $d_{G_i}(y) \geq \frac{n}{2}$ for each $y \in Y$, then $G$ is bipancyclic, which generalizes a classical result of Schmeichel and Mitchem in [Bipartite graphs with cycles of all even lengths, J. Graph Theory, 1982] on the bipancyclicity of bipartite graphs to the setting of graph transversals. Motivated by their work, we study vertex-bipancyclicity in bipartite graph collections and prove that if $\delta(G_i) \geq \frac{n+1}{2}$ for any $i \in [2n]$, then $G$ is vertex-bipancyclic unless $n = 3$ and $G$ consists of 6 identical copies of a 6-cycle. Moreover, we also show the Hamiltonian connectivity of $G$.

Keywords: bipartite graph collection; transversal; vertex-bipancyclicity; Hamiltonian connectivity; minimum degree.

AMS subject classification 2020: 05C38, 05C45, 05C15, 05C07.

1 Introduction

Over the last decades, there has been much research on Hamiltonicity of graphs, which is one of the most fundamental topics in graph theory. The classical Dirac’s theorem [12] in 1952 states that every $n$-vertex graph with minimum degree at least $\frac{n}{2}$ is Hamiltonian. In 1971, Bondy [4] proved that every $n$-vertex graph is pancyclic under the same degree condition except for $K_{\frac{n}{2}, \frac{n}{2}}$. Later, Bondy in [5] posed the following meta-conjecture: Almost any nontrivial condition on a graph which implies that the graph is Hamiltonian
also implies that it is pancyclic (except for possibly a simple family of exceptional graphs). This has been verified for many sufficient conditions for Hamiltonicity. Actually, some sufficient conditions forcing Hamiltonicity can even guarantee vertex-pancyclicity with minor adjustments. For example, in 1990, Hendry in [14] proved that every $n$-vertex graph with minimum degree at least $\frac{n+1}{2}$ is vertex-pancyclic. Note that when the graph is a balanced bipartite graph, it is natural to study whether conditions forcing Hamiltonicity can guarantee bipancyclicity, or even vertex-bipancyclicity. Indeed, in 1963, Moon and Moser in [17] established a minimum degree condition for the existence of Hamiltonian cycle in a balanced bipartite graph, which can be seen as a bipartite analogue of Dirac’s theorem. Later in 1982, Schmeichel and Mitchem in [21] generalized the above result to bipancyclicity under the same degree condition.

Very recently, the study of transversals over graph collections has received much attention, and some classical results in extremal graph theory have been extended to the setting of graph transversals. The concept of a graph transversal was first raised by Joos and Kim [15] in 2020. Let $G = \{G_1, \ldots, G_s\}$ be a graph collection with common vertex set $V$ and $H$ be a graph with $V(H) \subseteq V$. We say that $(H, \phi)$ is a partial $G$-transversal if there exists an injection $\phi : E(H) \rightarrow [s]$ such that $e \in E(G_{\phi(e)})$ for each $e \in E(H)$. In particular, if $|E(H)| = s$, then we call $(H, \phi)$ a $G$-transversal. Aharoni, DeVos, González Hermosillo de la Maza, Montejano and Šámal [2] considered Mantel’s theorem in the setting of graph transversals and proposed the following conjecture motivated by Dirac’s theorem.

**Conjecture 1** ([2]). Let $G = \{G_1, \ldots, G_n\}$ be a graph collection on the common vertex set $V$ of size $n$. If the minimum degree of $G_i$ is at least $\frac{n}{2}$ for each $i \in [n]$, then there exists a $G$-transversal isomorphic to a Hamiltonian cycle on $V$.

This conjecture was verified asymptotically by Cheng, Wang and Zhao in [11], and completely by Joos and Kim in [15]. Besides Hamiltonian cycles, results on other structures in extremal graph theory have also been generalized to the setting of graph transversals, including cycles [7, 9, 20], matchings [1, 3, 13], trees [8, 16] and factors [10, 18]. Bradshaw in [7] initiated the study of (partial) transversals in bipartite graph collections and obtained the following theorem analogous to Moon and Moser’s result in [17].

**Theorem 1.1** ([7]). Let $G = \{G_1, \ldots, G_{2n}\}$ be a bipartite graph collection on the common vertex bipartition $(X, Y)$ with $|X| = |Y| = n$. If for each $i \in [2n]$, $d_{G_i}(x) \geq \frac{n}{2}$ and $d_{G_i}(y) > \frac{n}{2}$ for any $x \in X$ and $y \in Y$, then there exists a $G$-transversal isomorphic to a Hamiltonian cycle on $X \cup Y$.

For convenience, in the following context we always use $G$ to denote a bipartite graph collection $\{G_1, \ldots, G_{2n}\}$ on the common vertex bipartition $(X, Y)$ with $|X| = |Y| = n$, unless otherwise stated. We define $\delta(G) = \min\{\delta(G_i) : i \in [2n]\}$, where $\delta(G)$ denotes the minimum degree of a graph $G$. We say that $G$ is bipancyclic if there exists a partial
Figure 1: $F = (\{x\} \cup X_1 \cup X_2, \{y\} \cup Y_1 \cup Y_2)$

A graph $G$-transversal isomorphic to an $\ell$-cycle for each even integer $\ell \in [4, 2n]$, while $G$ is vertex-bipancyclic if each vertex $v \in X \cup Y$ is contained in a partial $G$-transversal isomorphic to an $\ell$-cycle for each even integer $\ell \in [4, 2n]$. Bradshaw in [7] proved a stronger result which states that $G$ is bipancyclic under the same degree condition as Theorem 1.1, generalizing the result of Schmeichel and Mitchem in [21]. Hence, the degree condition in Theorem 1.1 fits Bondy’s meta-conjecture in the setting of graph transversals. This motivates us to verify if this degree condition (with a minor adjustment) can guarantee the vertex-bipancyclicity of $G$, since vertex-bipancyclicity implies bipancyclicity. It is noteworthy that $K_{\frac{n}{2}, \frac{n}{2}} \cup K_{\frac{n}{2}, \frac{n}{2}}$ contains no Hamiltonian cycles. Thus, the condition $\delta(G) = \frac{n+1}{2}$ can not guarantee the vertex-bipancyclicity of $G$. As a result, we get a positive answer as follows.

**Theorem 1.2.** If $\delta(G) \geq \frac{n+1}{2}$, then $G$ is vertex-bipancyclic, unless $n = 3$ and $G$ consists of six identical copies of a 6-cycle.

Hamiltonian connectivity is closely related with Hamiltonicity and it is a significant property in graph theory (see e.g. [19]). Inspired by this, we study the Hamiltonian connectivity of bipartite graph collections. We say that $G$ is Hamiltonian connected if for any two vertices $x \in X$ and $y \in Y$, there is a partial $G$-transversal isomorphic to a Hamiltonian path from $x$ to $y$. To state our result, we define the following graph collection: Let $n$ be an odd integer and $F = (\{x\} \cup X_1 \cup X_2, \{y\} \cup Y_1 \cup Y_2)$ be a balanced bipartite graph with $|X_i| = |Y_i| = \frac{n-1}{2}$ for $i = 1, 2$ such that $E(F)$ consists of edges of 4 complete bipartite graphs $F[\{x\}, Y_1 \cup Y_2]$, $F[\{y\}, X_1 \cup X_2]$ and $F[X_i, Y_i]$ for $i = 1, 2$. Let $F = \{F_1, \cdots, F_{2n}\}$ be a bipartite graph collection on the common vertex bipartition such that $F_i = F$ or $F_i = F \cup \{xy\}$ for each $i \in [2n]$. Note that $\delta(F) = \frac{n+1}{2}$ and there is no partial $F$-transversal isomorphic to a Hamiltonian path from $x$ to $y$, see Figure 1. Hence, we give the following result on Hamiltonian connectivity of bipartite graph collections.

**Theorem 1.3.** If $\delta(G) \geq \frac{n+1}{2}$, then $G$ is Hamiltonian connected or $G = F$.

In Section 3, we will give the proofs of Theorems 1.2 and 1.3.
2 Preliminaries

We first give some necessary notation and lemmas in this section, which will be used in next sections.

2.1 Notation

In this paper, we only consider finite, undirected, connected and simple graphs. For terminology and notation used but not defined here, we refer the reader to [6]. Let $G$ be a graph. We use $V(G)$ and $E(G)$ to denote the set of vertices and the set of edges of $G$, respectively. For a vertex subset $U \subseteq V(G)$, let $G[U]$ denote the subgraph of $G$ induced by $U$. We use $G_1 \cup G_2$ to denote the union of two vertex-disjoint graphs $G_1$ and $G_2$. A path or cycle of order $k$ is called a $k$-path or $k$-cycle, respectively. For two distinct vertices $v_i$ and $v_j$ in a cycle $C = v_1v_2 \cdots v_ℓv_1$, the segment $v_iv_{i+1} \cdots v_jv_j$ and $v_iv_{i-1} \cdots v_jv_j$ are denoted by $v_iCv_j$ and $v_iCv_j$, respectively, where the subscripts are taken modulo $|V(C)|$. Meanwhile, we use $u_iPu_j$ to denote the subpath of $P = u_1u_2 \cdots u_ℓ$ between $u_i$ and $u_j$. For two positive integers $a < b$, let $[a] = \{1, 2, \ldots, a\}$ and $[a, b] = \{a, a+1, \ldots, b\}$.

Recall that when $(H, φ)$ is a partial $G$-transversal, $φ$ is an injection from $E(H)$ to $[2n]$. Let $im(φ)$ be the image of $φ$. For an integer $i \in [2n]$, if $i \notin im(φ)$, then we say that $i$ is missed by $(H, φ)$. If there exists a partial $G$-transversal $(H, φ)$, then we also say that $G$ contains a partial transversal $(H, φ)$. When there is no possible confusion, we replace $(H, φ)$ by $H$ in the following context.

2.2 Lemmas

Now we will give some lemmas which will be used in the sequel.

Lemma 2.1. Let $G = \{G_i : i \in [2n]\}$ be a bipartite graph collection on the same bipartition $V = (X, Y)$ with $|X| = |Y| = n$. If $δ(G) \geq \frac{n}{2}$, then one of the following statements holds:

(1) $G$ contains a partial transversal isomorphic to a Hamiltonian path;

(2) $n$ is even and $G_1 = \cdots = G_{2n} = K\frac{n}{2}, \frac{n}{2} \cup K\frac{n}{2}, \frac{n}{2}$.

Proof. Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Suppose that neither statements (1) nor (2) holds. Let $P$ be a partial $G$-transversal isomorphic to a path with $|V(P)|$ maximum. We divide into two cases to discuss depending on the parity of $|V(P)|$.

Case 1. $P$ is a partial $G$-transversal isomorphic to a $2ℓ$-path.

Evidently, $ℓ ≤ n − 1$. We first prove an easy but crucial claim.

Claim 2.2. There is no partial $G$-transversal isomorphic to a $2ℓ$-cycle.
Proof. Suppose $G$ has a partial transversal isomorphic to a cycle $C = x_1y_1x_2y_2 \cdots x_ly_lx_1$. Without loss of generality, assume that $C$ has an associated injection $\phi : E(C) \to [2n]$ and $[2n] \setminus \text{im}(\phi) = [2\ell + 1, 2n]$.

We assert that there is no edge between $V(C)$ and $V \setminus V(C)$ in $G_i$ for each $i \in [2\ell + 1, 2n]$. Otherwise, if there exists some $i \in [2\ell + 1, 2n]$, $j \in [\ell]$ and a vertex $x \in V \setminus V(C)$ (let $x \in X$ by symmetry) such that $xy_j \in E(G_i)$, then $P_i = xy_jcx_j$ is a partial $G$-transversal isomorphic to a $(2\ell + 1)$-path, contradicting with the maximality of $P$. Since $\delta(G_i) \geq \frac{n}{2}$ for each $i \in [2\ell + 1, 2n]$, it follows that $|V(C)| \geq n$ and $|V \setminus V(C)| \geq n$. Then $n$ is even and $G_{2\ell + 1} = \cdots = G_{2n} = K_{\frac{2\ell}{2}, \frac{n}{2}} \cup K_{\frac{n}{2}, \frac{2\ell - 1}{2}}$.

We also assert that there is no edge between $V(C)$ and $V \setminus V(C)$ in $G_i$ for each $i \in [2\ell]$. Otherwise, there exists some $i \in [2\ell]$, $j \in [\ell]$ and a vertex $x \in V \setminus V(C)$ (let $x \in X$ by symmetry) such that $xy_j \in E(G_i)$. Let $uv$ be the edge of $E(C)$ with $\phi(uv) = i$. As $G_{2n} = K_{\frac{2\ell}{2}, \frac{n}{2}} \cup K_{\frac{n}{2}, \frac{2\ell - 1}{2}}$, we have $uv \in E(G_{2n})$. Then $(P_2, \phi_1)$ with $P_2 = xy_jcx_j$ is a partial $G$-transversal isomorphic to a $(2\ell + 1)$-path where $\phi_1$ arises from $\phi$ by setting $\phi_1(uv) := 2n$ (if $uv$ exists in $P_2$) and $\phi_1(xy_j) := i$, a contradiction. Similarly, we can deduce that $G_1 = \cdots = G_{2\ell} = K_{\frac{2\ell}{2}, \frac{n}{2}} \cup K_{\frac{n}{2}, \frac{2\ell - 1}{2}}$.

Hence, Lemma 2.1(2) holds and the assumption is wrong. Thus Claim 2.2 follows. $\square$

Without loss of generality, let $P = x_1y_1x_2y_2 \cdots x_ly_l$ have an associated injection $\phi : E(P) \to [2n]$ with $[2n] \setminus \text{im}(\phi) = [2\ell, 2n]$. Choose two arbitrary integers $c_1, c_2 \in [2\ell, 2n]$. Let $I_{c_1}$ and $I_{c_2}$ be the following sets:

$$I_{c_1} := \{i \in [\ell - 1] : x_iy_i \in E(G_{c_1})\},$$
$$I_{c_2} := \{i \in [2, \ell] : x_iy_i \in E(G_{c_2})\}.$$ 

By the maximality of $P$ and Claim 2.2, we have $N_{G_{c_1}}(x_1) \subseteq V(P) \setminus \{y_1\}$ and $N_{G_{c_2}}(y_1) \subseteq V(P) \setminus \{x_1\}$. We also have $I_{c_1} \cap I_{c_2} = \emptyset$, otherwise there exists a partial $G$-transversal isomorphic to a $2\ell$-cycle. Therefore, $n \leq |I_{c_1}| + |I_{c_2}| = |I_{c_1} \cup I_{c_2}| \leq \ell \leq n - 1$, a contradiction.

Case 2. $P$ is a partial $G$-transversal isomorphic to a $(2\ell - 1)$-path.

Let $P = x_1y_1x_2y_2 \cdots x_{\ell - 1}y_{\ell - 1}x_\ell$ with its associated injection $\phi^* : E(P) \to [2n]$. Without loss of generality, set $[2n] \setminus \text{im}(\phi^*) = [2\ell - 1, 2n]$. Obviously, $\ell \leq n$. We first prove the following result.

Claim 2.3. There is a partial $G$-transversal isomorphic to a $(2\ell - 2)$-cycle.

Proof. Suppose $G$ has no partial transversal isomorphic to a $(2\ell - 2)$-cycle. Then $x_1y_{\ell - 1} \notin E(G_{2n})$ and $x_\ell y_1 \notin E(G_{2n - 1})$. We consider the following two sets:

$$I_{2n} := \{i \in [\ell - 2] : x_iy_i \in E(G_{2n})\},$$
$$I_{2n - 1} := \{i \in [3, \ell] : x_\ell y_{i - 1} \in E(G_{2n - 1})\}.$$
By the maximality of \( P \), \( N_{G_{2n}}(x_1) \subseteq V(C) \setminus \{y_{\ell-1}\} \) and \( N_{G_{2n-1}}(x_\ell) \subseteq V(C) \setminus \{y_1\} \). No partial \( G \)-transversal isomorphic to a \((2\ell-2)\)-cycle guarantees that \( I_{2n} \cap I_{2n-1} = \emptyset \), which implies that \( n \leq |I_{2n}| + |I_{2n-1}| = |I_{2n} \cup I_{2n-1}| \leq \ell \leq n \). Hence \( \ell = n \). Now we define the following sets:

\[
I'_{2n} := \{ i \in [2, n - 1] : y_nx_i \in E(G_{2n}) \},
I'_{2n-1} := \{ i \in [n - 2] : x_1y_i \in E(G_{2n-1}) \}.
\]

We can deduce \( I'_{2n} \cap I'_{2n-1} = \emptyset \), otherwise, there is a partial \( G \)-transversal isomorphic to a Hamiltonian path. It follows that \( n \leq |I'_{2n}| + |I'_{2n-1}| = |I'_{2n} \cup I'_{2n-1}| \leq n - 1 \), a contradiction. Thus the claim follows.

Without loss of generality, let \((C, \phi)\) be a partial \( G \)-transversal isomorphic to a \((2\ell - 2)\)-cycle with \( C = x_1y_1x_2y_2 \cdots x_{\ell-1}y_{\ell-1}x_1 \) and \( \phi \) missing \([2\ell - 1, 2n]\). We assert that \( G_i[V \setminus V(C)] \) contains at least one edge for some integer \( i \in [2\ell - 1, 2n]\). Otherwise, \( G_i[V \setminus V(C)] \) is an empty graph for each \( i \in [2\ell - 1, 2n] \). We define

\[
I_{2n} := \{ i \in [\ell - 1] : x_iy \in E(G_{2n}) \},
I_{2n-1} := \{ i \in [\ell - 1] : xy_i \in E(G_{2n-1}) \},
\]

where \( x \in X \setminus V(C) \), \( y \in Y \setminus V(C) \). We have \( I_{2n} \cap I_{2n-1} = \emptyset \), otherwise there exists some \( i \in I_{2n} \cap I_{2n-1} \) such that \( P_i = yx_iC'yx \) is a partial \( G \)-transversal isomorphic to a \( 2\ell \)-path, contradicting with the maximality of \( P \). Hence, \( n \leq |I_{2n}| + |I_{2n-1}| = |I_{2n} \cup I_{2n-1}| \leq \ell - 1 \leq n - 1 \), a contradiction.

Without loss of generality, assume that \( G_{2n}[V \setminus V(C)] \) contains at least one edge. Choose an arbitrary edge \( xy \) in \( G_{2n}[V \setminus V(C)] \). We assert that \( N_{G_j}(x) \cap V(C) = \emptyset \) and \( N_{G_j}(y) \cap V(C) = \emptyset \) for each \( j \in [2\ell - 1, 2n - 1] \). Otherwise, there exists some \( j \in [2\ell - 1, 2n - 1] \) and \( k \in [\ell - 1] \) such that \( xy_k \in E(G_j) \). Then \( P_{2n} = yxy_kCx_k \) is a partial \( G \)-transversal isomorphic to a \( 2\ell \)-path, a contradiction.

Fix an integer \( j \in [2\ell - 1, 2n - 1] \), since \( d_{G_j}(x) \geq \frac{n}{2} \) and \( d_{G_j}(y) \geq \frac{n}{2} \), we have that \( |V \setminus V(C)| \geq n \) and \( G_j[V \setminus V(C)] \) is not empty. Choosing an edge in \( G_j[V \setminus V(C)] \) and repeating the above discussion, we have \( N_{G_{2n}}(x) \cap V(C) = \emptyset \) and \( N_{G_{2n}}(y) \cap V(C) = \emptyset \). Thus, for each \( v \in V \setminus V(C) \) and \( i \in [2\ell - 1, 2n] \), we have \( N_{G_i}(v) \cap V(C) = \emptyset \). It follows that there is no edge between \( V(C) \) and \( V \setminus V(C) \) in \( G_i \) for each \( i \in [2\ell - 1, 2n] \). Hence, \( n \) is even and \( G_{2n-1} = \cdots = G_{2n} = K_{\frac{n}{2}, \frac{n}{2}} \cup K_{\frac{n}{2}, \frac{n}{2}} \).

Next we consider \( G_i \) for each \( i \in [2\ell - 2] \). Suppose that there exists a vertex \( x \in X \setminus V(C) \) such that \( xy_k \in E(G_i) \) for some \( k \in [\ell - 1] \). Let \( uv \) be the edge of \( C \) with \( uv \in E(G_i) \). As \( G_j = K_{\frac{n}{2}, \frac{n}{2}} \cup K_{\frac{n}{2}, \frac{n}{2}} \) for each \( j \in [2\ell - 1, 2n] \), we have \( uv \in E(G_{2n-1}) \) and \( xy \in E(G_{2n}) \) for each \( y \in Y \setminus V(C) \). Then \( (P_4, \phi'' \circ \phi) \) is a partial \( G \)-transversal isomorphic to a \( 2\ell \)-path such that \( P_4 = yxy_kCx_k \) and \( \phi'' \) arises from \( \phi \) by letting \( \phi''(uv) := 2n - 1 \) (if \( uv \) exists in \( P_4 \)), \( \phi''(xy) := 2n \) and \( \phi''(xy_k) := i \), a contradiction. Hence for each \( v \in V \setminus V(C) \) and \( i \in [2\ell - 2] \), we have \( N_{G_i}(v) \cap V(C) = \emptyset \), which implies that \( G_1 = \cdots = G_{2\ell - 2} = K_{\frac{n}{2}, \frac{n}{2}} \cup K_{\frac{n}{2}, \frac{n}{2}} \).
Thus Lemma 2.1 (2) holds. This completes the proof of Lemma 2.1.

Lemma 2.4. Let $G = \{G_i : i \in [2n]\}$ be a bipartite graph collection on the same bipartition $(X, Y)$ with $|X| = |Y| = n - 1$ such that $\delta(G) \geq \frac{n-1}{2}$. Let $P = x_1 y_1 \cdots x_{n-1} y_{n-1}$ be a partial $G$-transversal isomorphic to a $(2n - 2)$-path. Then one of the following statements holds:

(1) there is a partial $G$-transversal isomorphic to a $(2n - 2)$-cycle;

(2) there is a partial $G$-transversal isomorphic to the disjoint union of a $(2n - 4)$-cycle and a copy of $K_2$;

(3) $n - 1$ is even and for each $i$ missed by $P$, we have $N_{G_i}(x_i) = \{y_1, y_2, \ldots, y_{n-1}\}$ and $N_{G_i}(y_{n-1}) = \{x_{\frac{n+1}{2}}, x_{\frac{n+3}{2}}, \ldots, x_{n-1}\}$.

Proof. Without loss of generality, let $\phi : E(P) \to [2n]$ be the associated injection of $P$ with $\phi(x_i y_i) = 2i - 1$ for $i \in [n - 1]$ and $\phi(y_i x_{i+1}) = 2i$ for $i \in [n - 2]$. Then $[2n] \setminus \text{im}(\phi) = [2n - 2, 2n]$. Define the following sets:

$$I_{2n} := \{i \in [n - 1] : x_i y_i \in E(G_{2n})\}$$

$$I_{2n-1} := \{i \in [n - 1] : x_i y_{n-1} \in E(G_{2n-1})\}$$

If $I_{2n} \cap I_{2n-1} \neq \emptyset$, then there exists some $i \in I_{2n} \cap I_{2n-1}$. Thus $C_1 = x_1 P x_i y_{n-1} P y_i x_1$ is a partial $G$-transversal isomorphic to a $(2n - 2)$-cycle and so statement (1) holds. If $I_{2n} \cap I_{2n-1} = \emptyset$, then $n - 1 \leq |I_{2n}| + |I_{2n-1}| = |I_{2n} \cup I_{2n-1}| \leq n - 1$. It follows that $n - 1$ is even, $I_{2n} \cup I_{2n-1} = [n - 1]$ and $|I_{2n}| = |I_{2n-1}| = \frac{n-1}{2}$. If there exists some $i \in I_{2n-1}$ such that $i + 1 \in I_{2n}$, then $C_2 = x_i P x_i y_{n-1} P y_{i+1} x_{i+1}$ and $y_{i+1} x_{i+1}$ is a partial $G$-transversal isomorphic to the disjoint of a $(2n - 4)$-cycle and a copy of $K_2$ and statement (2) thus holds. Otherwise, $I_{2n} = [\frac{n-1}{2}]$ and $I_{2n-1} = [\frac{n+1}{2}, n - 1]$, which implies that $N_{G_{2n}}(x_1) = \{y_1, y_2, \ldots, y_{\frac{n-1}{2}}\}$ and $N_{G_{2n-1}}(y_{n-1}) = \{x_{\frac{n+1}{2}}, x_{\frac{n+3}{2}}, \ldots, x_{n-1}\}$. By symmetry, we have $N_{G_i}(x_1) = \{y_1, y_2, \ldots, y_{\frac{n-1}{2}}\}$ and $N_{G_i}(y_{n-1}) = \{x_{\frac{n+1}{2}}, x_{\frac{n+3}{2}}, \ldots, x_{n-1}\}$ for each $i \in [2n - 2, 2n]$. Hence, statement (3) holds.

3 Proofs of Theorems 1.2 and 1.3

After the preparations of the above section, we are ready to give the proofs of our main results, Theorems 1.2 and 1.3.

3.1 Proof of Theorem 1.2

Given a positive integer $d$ and a set $A \subseteq \mathbb{Z}$, let $A + d = \{i + d : i \in A\}$ and $A - d = \{i - d : i \in A\}$. We first introduce a useful tool obtained in \[7\].
Lemma 3.1 ([7]). Let $n$ be an integer and $(\mathbb{Z}_2n, +)$ be the cyclic group of $2n$ elements. Let $A \subseteq \mathbb{Z}_2n$ and $B = (A + d) \cup (A - d)$ with $d \in [2n - 1]$. If $|A| = |B|$, then $A = A + 2d$.

Now we are ready to prove Theorem 1.2.

**Proof.** By Theorem 1.1 each vertex of $X \cup Y$ is contained in a $G$-transversal isomorphic to a Hamiltonian cycle. It remains to show that each vertex of $X \cup Y$ is contained in a partial $G$-transversal isomorphic to an $\ell$-cycle for every even integer $\ell \in [4, 2n - 2]$.

First, we consider the case $n = 3$. Let $(C, \phi)$ be a $G$-transversal isomorphic to a 6-cycle $C = v_1v_2 \cdots v_6v_1$ and its associated injection $\phi$ satisfying $\phi(v_{i+1}) := i$ for each $i \in [6]$ (identify $v_7$ with $v_1$). Without loss of generality, assume that $v_1$ is not contained in any partial $G$-transversal isomorphic to a 4-cycle. Now we show that $v_1v_2, v_2v_3, v_3v_4 \notin E(G_i)$ for all $i \in [6]$. If $v_1v_2 \notin E(G_i)$ for some $i \in [6]$, then $v_1v_2v_3v_4v_5$ is a partial $G$-transversal isomorphic to a 4-cycle, a contradiction. Since $d_{G_i}(v_1) \geq 2$ and $v_1, v_3, v_5$ belong to the same part, we have $v_1v_2, v_1v_6 \in E(G_i)$ for each $i \in [6]$. If $v_2v_5 \notin E(G_i)$ for some $i \in [6]\{5\}$, then $v_1v_2v_5v_6v_1$ is a partial $G$-transversal isomorphic to a 4-cycle, a contradiction. So, $v_2v_5 \notin E(G_i)$ for each $i \in [6]\{5\}$. It follows from $d_{G_i}(v_5) \geq 2$ that $v_5v_6 \in E(G_i)$ for each $i \in [6]\{5\}$. Therefore, $v_2v_5 \notin E(G_5)$, otherwise, $v_1v_2v_5v_6v_1$ is a partial $G$-transversal isomorphic to a 4-cycle. By symmetry, we have $v_3v_6 \notin E(G_i)$ for each $i \in [6]$. Thus, $G_i = C$ for all $i \in [6]$.

Next we consider the case $n \geq 4$. Fix an arbitrary vertex $x \in X$ and suppose that $x$ is not contained in a partial $G$-transversal isomorphic to an $\ell'$-cycle for some even integer $\ell' \in [4, 2n - 2]$. Since $\delta(G) \geq \frac{n+1}{2}$, we have $\sum_{i=1}^{2n} d_{G_i}(x) \geq n(n+1)$. By an averaging argument, we can find a vertex $y \in Y$ such that the edge $xy$ appears on at least $n + 1$ graphs of $G$. Set $M_i = G_i - \{x, y\}$ for each $i \in [2n]$ and $M' = \{M_1, \cdots, M_{2n-2}\}$. Then $|M_i| = 2n - 2$ and $\delta(M_i) \geq \frac{n-1}{2}$ for each $i \in [2n]$. By Lemma 2.1 $M'$ contains a partial transversal isomorphic to a $(2n - 2)$-path or $n - 1$ is even and $M_1 = \cdots = M_{2n-2} = K_{n-1, \frac{n-2}{2}} \cup K_{\frac{n-2}{2}, \frac{n-3}{2}}$.

We first assume that $n - 1$ is even and $M_1 = \cdots = M_{2n-2} = K_{n-1, \frac{n-2}{2}} \cup K_{\frac{n-2}{2}, \frac{n-3}{2}}$. Set $X - \{x\} = X_1 \cup X_2$ and $Y - \{y\} = Y_1 \cup Y_2$ with $X_1 = \{x_1, x_2, \cdots, x_{\frac{n-2}{2}}\}, X_2 = \{x_{\frac{n-2}{2}+1}, x_{\frac{n-2}{2}+2}, \cdots, x_{n-1}\}, Y_1 = \{y_1, y_2, \cdots, y_{\frac{n-2}{2}}\}$ and $Y_2 = \{y_{\frac{n-2}{2}+1}, y_{\frac{n-2}{2}+2}, \cdots, y_{n-1}\}$. Furthermore, $M_i[X_j \cup Y_j] = K_{\frac{n-2}{2}, \frac{n-1}{2}}$ for every $i \in [2n - 2]$ and $j \in \{1, 2\}$. As $\delta(G) \geq \frac{n+1}{2}$, we have $x_iy_j, y_ix_j \in E(G_i)$ for each $i \in [2n - 2]$ and $j \in [n - 1]$. It is not difficult to find that $x$ is contained in a partial $G$-transversal isomorphic to an $\ell$-cycle for every even integer $\ell \in [4, 2n - 2]$, a contradiction.

Now we assume that $M'$ contains a partial transversal, denoted by $P$, isomorphic to a $(2n - 2)$-path. It is clear that $P$ is also a partial transversal isomorphic to a $(2n - 2)$-path in $M = \{M_1, \cdots, M_{2n}\}$. Thus applying Lemma 2.4 on $M$, we know that one of the three statements in Lemma 2.3 holds. Next, we proceed with our proof by distinguishing three cases according to the three statements.

**Case 1.** There is a partial $M$-transversal isomorphic to a $(2n - 2)$-cycle.
Assume that $M$ has a partial transversal isomorphic to a cycle $C = v_1v_2 \cdots v_{2n-3}v_{2n-2}v_1$. Without loss of generality, let $\phi$ be its associated injection with $\phi(v_i,v_{i+1}) = i$ for each $i \in [2n-2]$ (identify $v_{2n-1}$ with $v_1$). Set $X = \{v_1,v_3, \cdots , v_{2n-3}, x\}$ and $Y = \{v_2,v_4, \cdots , v_{2n-2},y\}$. Let $\ell$ be an even integer with $\ell \in [4,2n-2]$. We define
\[
I_{2n} := \{i \in [2n-2] \cap 2\mathbb{Z} : xv_i \in E(G_{2n})\}
\]
and
\[
I_{2n-1} := \{i \in [2n-2] \cap 2\mathbb{Z} : xv_{i+\ell-2} \in E(G_{2n-1})\}.
\]
If $I_{2n} \cap I_{2n-1} \neq \emptyset$, then choose an integer $i \in I_{2n} \cap I_{2n-1}$, and so $v_iCv_{i+\ell-2}xv_i$ is a partial $G$-transversal isomorphic to an $\ell$-cycle, a contradiction. Hence, $I_{2n} \cap I_{2n-1} = \emptyset$. Then, 
\[
\frac{n-1}{2} + \frac{n-1}{2} \leq |I_{2n}| + |I_{2n-1}| \leq n - 1,
\]
which implies that $I_{2n} \cup I_{2n-1} = [2n-2] \cap 2\mathbb{Z}$ and $|I_{2n}| = |I_{2n-1}| = \frac{n-1}{2}$. Since $\delta(G) \geq \frac{n+1}{2}$, we have $xy \in E(G_{2n}) \cap E(G_{2n-1})$. In fact, for any partial $M$-transversal $(C',\phi')$ isomorphic to a $(2n-2)$-cycle with $V(C') = V(C)$, we have $xy \in E(G_i)$ for each $i \in [2n]$ missed by $(C',\phi')$.

**Claim 3.2.** For any two consecutive edges $e$ and $f$ on $C$, we have $xy \in E(G_{\phi(e)}) \cup E(G_{\phi(f)})$.

**Proof.** Without loss of generality, we only need to prove $xy \in E(G_1) \cup E(G_2)$. Suppose that $xy \notin E(G_1)$ and $xy \notin E(G_2)$. If $v_1v_2 \in E(G_{2n})$, then $(C,\phi_1)$ is a partial $M$-transversal isomorphic to a $(2n-2)$-cycle where $\phi_1$ arises from $\phi$ by setting $\phi_1(v_1v_2) := 2n$. Observe that $(C,\phi_1)$ misses 1, and then $xy \in E(G_1)$, a contradiction. Thus $v_1v_2 \notin E(G_{2n})$.

By symmetry, $v_1v_2 \notin E(G_{2n-1})$.

For each even integer $j \in [2n-2]$, we pair $\{v_1,v_j\}$ with $\{v_2,v_{j+1}\}$. Note that if $v_1v_j \notin E(G_{2n})$, then $v_2v_{j+1} \notin E(G_{2n-1})$. Otherwise, $(C',\phi_2)$ with $C' = C - v_1v_2 - v_jv_{j+1} + v_1v_j + v_2v_{j+1}$ is a partial $M$-transversal isomorphic to a $(2n-2)$-cycle where $\phi_2$ arises from $\phi$ by setting $\phi_2(v_1v_j) := 2n$ and $\phi_2(v_2v_{j+1}) := 2n - 1$. Observe that $(C',\phi_2)$ misses 1, which implies $xy \in E(G_1)$, a contradiction. Therefore, $|N_{G_{2n}}(v_1) \cap V(C)| + |N_{G_{2n-1}}(v_2) \cap V(C)| \leq n - 1$. On the other hand, since $\delta(G_{2n}) \geq \frac{n+1}{2}$ and $\delta(G_{2n-1}) \geq \frac{n+1}{2}$, we have $|N_{G_{2n}}(v_1) \cap V(C)| + |N_{G_{2n-1}}(v_2) \cap V(C)| \leq n - 1$. Then, $|N_{G_{2n}}(v_1) \cap V(C)| = |N_{G_{2n-1}}(v_2) \cap V(C)| = \frac{n-1}{2}$. Hence for each even integer $j \in [2n-2]$, either $v_1v_j \in E(G_{2n})$ and $v_2v_{j+1} \notin E(G_{2n-1})$ or $v_1v_j \notin E(G_{2n})$ and $v_2v_{j+1} \in E(G_{2n-1})$. When $j = 2$, we have $v_2v_3 \in E(G_{2n-1})$ since $v_1v_2 \notin E(G_{2n})$. Therefore, $(C,\phi_3)$ is a partial $M$-transversal isomorphic to a $(2n-2)$-cycle where $\phi_3$ arises from $\phi$ by setting $\phi_3(v_2v_3) := 2n - 1$. So, $(C,\phi_3)$ misses 2 which implies $xy \in E(G_2)$, a contradiction. The claim thus follows. \( \square \)

Recall that $I_{2n} := \{i \in [2n-2] \cap 2\mathbb{Z} : xv_i \in E(G_{2n})\}$ and $|I_{2n}| = \frac{n-1}{2}$. In fact, we will see that $I_{2n}$ can be seen as a subgroup of $\mathbb{Z}_{2n-2}$. We consider the following sets
\[
B := (I_{2n} + (\ell - 2)) \cup (I_{2n} - (\ell - 2)),
\]
\[
B' := (I_{2n} + (\ell - 3)) \cup (I_{2n} - (\ell - 3)),
\]

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where \( \ell \in [4, 2n - 2] \cap 2\mathbb{Z} \). So, \( |B| \geq |I_{2n}| \) and \( |B'| \geq |I_{2n}| \).

Set \( A := \{ j \in [2n - 2] \cap 2\mathbb{Z} : xv_j \in E(G_{2n-1}) \} \setminus B \). If \( |B| \geq |I_{2n}| + 1 \), then \(|A| \leq n - 1 - (|I_{2n}| + 1) \leq \frac{n - 3}{2} < \frac{n - 1}{2} \). So, there exists some \( i \in I_{2n} \) satisfying \( j = i + (\ell - 2) \) or \( j = i - (\ell - 2) \) such that \( xv_j \in E(G_{2n-1}) \). Then for every even integer \( \ell \in [4, 2n - 2] \), \((C_1, \phi_1)\) with \( C_1 = xv_iCv_{i+\ell-2}x \) (or \( C_1 = xv_iC^{-1}v_{i-(\ell-2)}x \)) is a partial \( G \)-transversal isomorphic to an \( \ell \)-cycle, where \( \phi_1 \) arises from \( \phi \) by setting \( \phi_1(xv_i) := 2n \) and \( \phi_1(xv_{i+\ell-2}) := 2n - 1 \) (or \( \phi_1(xv_{i-(\ell-2)}) := 2n - 1 \)), a contradiction. Then \( |I_{2n}| = |B| \), and by Lemma 3.1 \( I_{2n} = I_{2n} + (2\ell - 4) \).

Set \( A' := \{ k \in [2n - 2] \setminus (2\mathbb{Z} \cup B') : yv_k \in G_{2n-1} \} \). If \( |B'| \geq |I_{2n}| + 1 \), then \(|A'| \leq \frac{n - 3}{2} < \frac{n - 1}{2} \). Thus, there exists some \( i \in I_{2n} \) such that \( k = i + (\ell - 3) \) or \( k = i - (\ell - 3) \) such that \( yv_k \in E(G_{2n-1}) \). By Claim 3.2 we know that \( xy \in E(G_{i-1}) \) or \( xy \in E(G_{i+1}) \), and \( xy \in E(G_i) \) or \( xy \in E(G_{i+1}) \). Without loss of generality, assume that \( xy \in E(G_{i-1}) \) and \( xy \in E(G_i) \). Then we obtain that \((C_2, \phi_2)\) with \( C_2 = xv_iCv_{i+\ell-3}yx \) (or \( C_2 = xv_iC^{-1}v_{i-(\ell-3)}yx \)) is a partial \( G \)-transversal isomorphic to an \( \ell \)-cycle, where \( \phi_2 \) arises from \( \phi \) by setting \( \phi_2(xv_i) := 2n \), \( \phi_2(yv_{i+\ell-3}) := 2n - 1 \) (or \( \phi_2(yv_{i-(\ell-3)}) := 2n - 1 \)) and \( \phi_2(xy) := i - 1 \) (or \( \phi_2(xy) := i \)), a contradiction. Then \( |I_{2n}| = |B| \), and by Lemma 3.1 \( I_{2n} = I_{2n} + (2\ell - 6) \).

Since \( I_{2n} = I_{2n} + (2\ell - 4) \) and \( I_{2n} = I_{2n} + (2\ell - 6) \), it follows that \( I_{2n} = I_{2n} + 2 \). Then \( |I_{2n}| = n - 1 \), which contradicts with \( |I_{2n}| = \frac{n - 1}{2} \). Therefore for each even integer \( \ell \in [4, 2n - 2] \), there exists a partial \( G \)-transversal isomorphic to an \( \ell \)-cycle containing \( x \), a contradiction.

**Case 2.** There is a partial \( M \)-transversal isomorphic to the disjoint union of a \((2n - 4)\)-cycle and a copy of \( K_2 \).

Let \( C = v_1v_2 \cdots v_{2n-4}v_1 \) and \( C \cup \{ wz \} \) be the partial \( M \)-transversal isomorphic to the disjoint union of a \((2n - 4)\)-cycle and a copy of \( K_2 \). Let \( \phi \) be its associated injection with \( \phi(wz) = 2n - 3 \) and \( \phi(v_1v_{i+1}) = i \) for each \( i \in [2n - 4] \) (identify \( v_{2n-3} \) with \( v_1 \)). Set \( X = \{ v_2, v_4, \cdots, v_{2n-4}, x, z \} \) and \( Y = \{ v_1, v_3, \cdots, v_{2n-5}, y, w \} \). We define
\[
I_{2n} = \{ i \in [2n - 4] \cap 2\mathbb{Z} : xv_{i+1} \in E(G_{2n}) \}, \\
I_{2n-1} = \{ i \in [2n - 4] \cap 2\mathbb{Z} : wv_i \in E(G_{2n-1}) \}.
\]
Since \( M \) contains no partial transversal isomorphic to a \((2n - 2)\)-cycle, we have \( I_{2n} \cap I_{2n-1} = \emptyset \). Note that \( d_{G_{2n}}[v(C)](z) \geq \frac{n - 3}{2} \) and \( d_{G_{2n-1}}[v(C)](w) \geq \frac{n - 3}{2} \). So, \( n - 3 \leq |I_{2n}| + |I_{2n-1}| = |I_{2n} \cup I_{2n-1}| \leq n - 2 \).

If \( |I_{2n} \cup I_{2n-1}| = n - 2 \), then \( I_{2n} \cup I_{2n-1} = [2n - 4] \cap 2\mathbb{Z} \). So, there exists some \( i \in [2n - 4] \cap 2\mathbb{Z} \) satisfying \( i \in I_{2n} \) and \( i + 2 \in I_{2n-1} \). Therefore, \((C', \phi')\) is partial \( M \)-transversal isomorphic to a \((2n - 2)\)-cycle with \( C' = zvw_i+2Cv_{i+1}z \) and \( \phi' \) originating from \( \phi \) by adding \( \phi'(v_{i+1}z) := 2n \) and \( \phi'(v_{i+2}w) := 2n - 1 \), a contradiction. Hence \( |I_{2n} \cup I_{2n-1}| = n - 3 \). It follows that \( n \) is odd and \( |I_{2n}| = |I_{2n-1}| = \frac{n - 3}{2} \). There exists some \( i \in [2n - 4] \cap 2\mathbb{Z} \) such that \( i \notin I_{2n} \cup I_{2n-1} \). Without loss of generality, we assume
2n − 4 \notin I_{2n} \cup I_{2n−1}, implying that I_{2n} \cup I_{2n−1} = [2n − 6] \cap 2\mathbb{Z}. As \delta(G) \geq \frac{n + 1}{2}, we have zw, zy \in E(G_{2n}) and wx, wz \in E(G_{2n−1}).

If there exists an i \in [2n − 6] \cap 2\mathbb{Z} such that i \in I_{2n} and i + 2 \in I_{2n−1}, then (C_1, \phi_1) is a partial M-transversal isomorphic to a (2n − 2)-cycle with C_1 = zwv_{i+2}Cv_{i+1}z and \phi_1 obtained from \phi by setting \phi_1(zv_{i+1}) := 2n and \phi_1(wv_{i+2}) := 2n − 1, a contradiction. Therefore, I_{2n} = \{n − 1, n + 1, \cdots, 2n − 6\} and I_{2n−1} = \{2, 4, \cdots, n − 3\}, which means that N_{G_{2n}}(z) = \{vn, v_{n+2}, \cdots, v_{2n−5}, y, w\} and N_{G_{2n−1}}(w) = \{v_2, v_4, \cdots, v_{n−3}, x, z\}. By symmetry, for each i \in [2n − 3, 2n], we have N_{G_i}(z) = \{vn, v_{n+2}, \cdots, v_{2n−5}, y, w\} and N_{G_i}(w) = \{v_2, v_4, \cdots, v_{n−3}, x, z\}.

Note that G − \{x, w\} contains no partial transversal isomorphic to a (2n − 2)-cycle, since otherwise, the proof of Case 1 ensures that x is contained in a partial G-transversal isomorphic to an ℓ-cycle for each even integer ℓ \in [4, 2n − 2], a contradiction. Since yz \in E(G_{2n−3}), we have that C \cup \{yz\} is also a partial transversal isomorphic to the disjoint union of a (2n − 4)-cycle and a copy of K_2 in G. Moreover, so far we have not used the property that xy appears on at least n + 1 graphs of G. Hence, we can exchange the roles of y and w in the above proof of Case 2 and obtain that N_{G_i}(y) = \{v_2, v_4, \cdots, v_{n−3}, x, z\} for each i \in [2n − 3, 2n].

Actually, for any partial M-transversal (C', \phi') with V(C') = V(C), we have xy \in E(G_i) for each i \in [2n] missed by (C', \phi'). Next, we prove one more property for the edge xy.

Claim 3.3. For any three consecutive edges e, f, g on C, we have xy \in E(G_{\phi(e)}) \cup E(G_{\phi(f)}) \cup E(G_{\phi(g)}).

Proof. Without loss of generality, we only need to consider the case that e = v_1v_2, f = v_2v_3 and g = v_3v_4, and prove that xy \in E(G_1) \cup E(G_2) \cup E(G_3). Suppose to the contrary that xy \notin E(G_1) \cup E(G_2) \cup E(G_3).

If v_2v_3 \in E(G_{2n}), then (C, \phi') is a partial M-transversal isomorphic to a (2n − 4)-cycle where \phi' arises from \phi by setting \phi'(v_2v_3) := 2n. So, (C, \phi') misses 2. Hence xy \notin E(G_2), a contradiction. Thus v_2v_3 \notin E(G_{2n}). By symmetry, we have v_2v_3 \notin E(G_i) for each i \in [2n − 3, 2n]. Similarly, we have v_iy_{i+1} \notin E(G_j) for each i \in \{1, 2, 3\} and each j \in [2n − 3, 2n].

For each odd integer a \in [2n − 4], we pair \{v_2, v_a\} with \{v_3, v_{a+1}\}. The total number of such pairs is n − 2. For any \{j_1, j_2\} \subseteq [2n − 3, 2n], if v_2v_{a} \in E(G_{j_1}), then v_3v_{a+1} \notin E(G_{j_2}). Otherwise, (C', \phi'') is a partial transversal isomorphic to a (2n − 4)-cycle with C' = v_2v_{a}C'v_{a+1}Cv_{a+2}, where \phi'' arises from \phi by setting \phi''(v_2v_{a}) = j_1 and \phi''(v_{a+1}v_{a+2}) = j_2. Observing that V(C') = V(C) and (C', \phi'') misses 2, we have xy \in G_2, a contradiction.

Notice that \{v_1, v_2\} and \{v_2, v_3\} are a pair, while \{v_2, v_3\} and \{v_3, v_4\} are a pair as well. Recall that v_iy_{i+1} \notin E(G_j) for each i \in \{1, 2, 3\} and j \in [2n − 3, 2n]. Then v_1v_2 \notin E(G_{j_1}), v_2v_3 \notin E(G_{j_2}), v_3v_4 \notin E(G_{j_1}) and v_3v_4 \notin E(G_{j_2}), which indicates that |N_{G_{j_1}}(v_2) \cap V(C)| + |N_{G_{j_2}}(v_3) \cap V(C)| \leq n − 4. Since \delta(G) \geq \frac{n + 1}{2}, we have |N_{G_{j_1}}(v_2) \cap
$V(C) + |N_{G_{2n}}(v_3) \cap V(C)| \geq n - 3$, a contradiction. 

\[ \text{Claim 3.4.} \text{ For each } i \in [2n - 3, 2n], \text{ we have } N_{G_i}(x) = \{v_n, v_{n+2}, \ldots, v_{2n-5}, y, w\}. \]

Proof. Suppose to the contrary that there exists some $i \in [2n - 3, 2n]$ and an odd integer $a \in [n - 1]$ such that $xv_a \in E(G_i)$. Without loss of generality, we assume that $xv_a \in E(G_{2n})$. Recalling that $n$ is odd, we have $a \neq n - 1$.

If $a = n - 2$, then we can find a partial $G$-transversal isomorphic to an $\ell$-cycle containing $x$ for each even integer $\ell \in [4, 2n - 2]$. In fact, for each even integer $\ell \in [4, n - 1]$, $(C_2, \phi_2)$ is a partial $G$-transversal isomorphic to an $\ell$-cycle containing $x$, where $C_2 = yv_{n-\ell+1}Cv_{n-2}xy$ and $\phi_2$ arises from $\phi$ by setting $\phi_2(xv_{n-2}) := 2n, \phi(yv_{n-\ell+1}) := 2n - 1, \phi(xy) := 2n - 2$. For each even integer $\ell \in [n + 3, 2n - 2]$, $(C_2, \phi_2)$ with $C_2 = yv_{n-\ell+1}C^*-v_{n-2}xy$ is a partial $G$-transversal isomorphic to an $\ell$-cycle containing $x$. For $\ell = n + 1$, $(C_2, \phi_2)$ is a partial $G$-transversal isomorphic to an $\ell$-cycle containing $x$, where $C_2 = xyzwv_2Cv_{n-2}x$ and $\phi_2$ arises from $\phi$ by setting $\phi_2(xv_{n-2}) := 2n, \phi(yz) := 2n - 1, \phi(zw) := 2n - 2, \phi(wv_2) := 2n - 3$, $\phi(xy) \in \{n - 2, n - 1, n\}$ (by Claim 3.3). In conclusion, for each even integer $\ell \in [4, 2n - 2]$, $x$ is contained in a partial $G$-transversal isomorphic to an $\ell$-cycle, a contradiction.

Now we assume $a \neq n - 2$. Then the odd integer $a$ is in $[n - 4]$. For each even integer $\ell \in [4, n - a]$, $(C_3, \phi_3)$ is a partial $G$-transversal isomorphic to an $\ell$-cycle containing $x$, where $C_3 = xv_aCv_{a+\ell-3}xy$ and $\phi_3$ arises from $\phi$ by setting $\phi_3(xv_a) := 2n, \phi_3(yv_{a+\ell-3}) := 2n - 1, \phi_3(xy) := 2n - 2$. By symmetry, for each even integer $\ell \in [n + a + 2, 2n - 2]$, set $\ell^* + \ell = 2n + 2$, then $\ell^* \in [4, n - a]$. Thus, $(C_3, \phi_3)$ with $C_3 = xv_aC^-v_{a+\ell-3}xy$ is a partial $G$-transversal isomorphic to an $\ell$-cycle containing $x$.

For $\ell = n - a + 2$, $(C_3, \phi_3)$ is a partial $G$-transversal isomorphic to an $(n + 2 - a)$-cycle with $C_3 = v_1Cv_{n-3}wzyvxv_a$ and $\phi_3$ obtained from $\phi$ by setting $\phi_3(wv_{n-3}) := 2n - 1, \phi_3(yz) := 2n - 2, \phi_3(zw) := 2n - 3, \phi_3(xy) \in \{a - 3, a - 2, a - 1\}$ (by Claim 3.3).

For each even integer $\ell \in [n - a + 4, 2n - a - 1]$, $(C_3, \phi_3)$ with $C_3 = xv_aCv_{a+\ell-4}zyxy$ is a partial $G$-transversal isomorphic to an $\ell$-cycle, where $\phi_3$ arises from $\phi$ by setting $\phi_3(xv_a) := 2n, \phi_3(xy) := 2n - 1, \phi_3(yz) := 2n - 2, \phi_3(zv_{a+\ell-4}) := 2n - 3$. Likewise, for each even integer $\ell \in [a + 5, n + a]$, set $\ell^* + \ell = 2n + 4$, then $\ell^* \in [n - a + 4, 2n - a - 1]$. Thus, $(C_3, \phi_3)$ with $C_3 = xv_aC^-v_{a+\ell-4}zyxy$ is a partial $G$-transversal isomorphic to an $\ell$-cycle, where $\phi_3$ arises from $\phi$ by setting $\phi_3(xv_a) := 2n, \phi_3(xy) := 2n - 1, \phi_3(yz) := 2n - 2, \phi_3(zv_{a+\ell-4}) := 2n - 3$.

Since $a \leq n - 4$, we have $2n - a - 1 > a + 5$. Consequently, $x$ is contained in a partial $G$-transversal isomorphic to an $\ell$-cycle for every even integer $\ell \in [2n - 2]$, a contradiction. Therefore, we get that $N_{G_{2n}}(x) = \{v_n, v_{n+2}, \ldots, v_{2n-5}, y, w\}$. By symmetry, $N_{G_i}(x) = \{v_n, v_{n+2}, \ldots, v_{2n-5}, y, w\}$ for each $i \in [2n - 3, 2n]$. The claim thus follows. 

Therefore, for each even integer $\ell \in [6, n + 1]$, $(C', \phi')$ is a partial $G$-transversal isomorphic to an $\ell$-cycle with $C' = yv_{n-3}Cv_{a+\ell-6}xy$ and $\phi'$ arising from $\phi$ by setting
$\phi'(xy) := 2n, \phi'(xv_{\ell-6}) := 2n-1, \phi'(yv_2) := 2n-2$. For each even integer $\ell \in [n+1, 2n-4]$, $(C''', \phi'')$ is a partial $G$-transversal isomorphic to an $\ell$-cycle with $C'' = yv_3C_{\ell-1}xy$ and $\phi''$ arising from $\phi$ by setting $\phi''(xy) := 2n, \phi''(xv_{\ell-1}) := 2n-1, \phi''(yv_2) := 2n-2$. Apparently, $xyvuwz$ and $xyv_3C_{\ell-1}$ are partial $G$-transversals isomorphic to a 4-cycle and a $(2n-2)$-cycle, respectively. Hence, $x$ is contained in a partial $G$-transversal isomorphic to an $\ell$-cycle with any even integer $\ell \in [2n-2]$, a contradiction.

**Case 3.** Lemma 2.4 (3) holds in $M$.

Without loss of generality, we assume that $P = x_1y_1 \cdots x_{n-1}y_{n-1}$ with $\phi(x_iy_i) = 2i-1$ for each $i \in [n-1]$ and $\phi(y_jx_{j-1}) = 2j$ for each $j \in [n-2]$.

**Claim 3.5.** For each $i \in \{\phi(x_1y_1), \phi(x_{n-1}y_{n-1})\}$ or $i$ missed by $P$, we have $N_{G_1}(x_1) = \{y_1, y_2, \ldots, y_{n-1}, y\}$ and $N_{G_1}(y_{n-1}) = \{x_{n-1}, x_{n-1}/2, \ldots, x, y\}$.

**Proof.** Note that $P$ misses $2n-2, 2n-1$ and $2n$. By Lemma 2.4 (3), we know that $N_{M_1}(x_1) = \{y_1, y_2, \ldots, y_{n-1}\}$ and $N_{M_1}(y_{n-1}) = \{x_{n-1}, x_{n-1}/2, \ldots, x, y\}$ for each $i \in [2n-2, 2n]$. Since $\delta(G) \geq \frac{n+1}{2}$, we have $N_{G_1}(x_1) = \{y_1, y_2, \ldots, y_{n-1}, y\}$ and $N_{G_1}(y_{n-1}) = \{x_{n-1}, x_{n-1}/2, \ldots, x, y\}$ for each $i \in [2n-2, 2n]$.

Let $(P, \phi_1)$ be a partial $M$-transversal isomorphic to a $(2n-2)$-path with $\phi_1$ arising from $\phi$ by setting $\phi_1(x_1y_1) := 2n$. Note that $(P, \phi_1)$ misses 1. Then by Lemma 2.4 (3) and $\delta(G_1) \geq \frac{n+1}{2}$, we get that $N_{G_1}(x_1) = N_{G_{2n}}(x_1)$ and $N_{G_1}(y_{n-1}) = N_{G_{2n}}(y_{n-1})$. Since $x_{n-1}y_{n-1} \in E(G_{2n-1})$, by the similar analysis, we can deduce that $N_{G_{2n-3}}(y_{n-1}) = N_{G_{2n-1}}(y_{n-1})$ and $N_{G_{2n-1}}(x_1) = N_{G_{2n-1}}(x_1)$. The claim thus follows.

Denote $C(uv) := \{i \in [2n] : uv \in E(G_i)\}$. Let $(P, \phi_1)$ be a partial $M$-transversal isomorphic to a $(2n-2)$-path with $P_i = x_iP_{x_i}y_{y_{i-1}}$ and $\phi_1$ arising from $\phi$ by setting $\phi_1(x_1y_i) := 2n$ for each $i \in [2, \frac{n+1}{2}]$. Observe that $(P, \phi_1)$ misses 1 and $\phi_1(x_1y_{n-1}) = 2i-2$. By Claim 3.5, we know that $[1, n-2] \cup [2n-3, 2n] \subseteq C(x_{y_{n-1}})$. Let $(P_1', \phi_1')$ be a partial $M$-transversal isomorphic to a $(2n-2)$-path with $P_1' = y_1P_{y_{1-1}}x_1Px_1$ and $\phi_1'$ arising from $\phi$ by setting $\phi_1'(x_iy_{i-1}) := 2n$ for each $i \in [\frac{n+1}{2}, n-2]$. By the similar analysis, we have $\{1\} \cup [n/2, 2n] \subseteq C(x_{y_{i-1}})$.

Since $|C(xy)| \geq n+1 > 1$, there exists some $a \in C(xy)$ satisfying $a \neq n-1$. If $a \in [2n-2, 2n]$, say $a = 2n$, then $yx_1P_{y_{n-1}}xy$ is a partial $G$-transversal isomorphic to a Hamiltonian cycle and $xy_{n-1}x_{n-1}y_{n-2}$ is a partial $G$-transversal isomorphic to a 4-cycle. Moreover, for each $k \in [3, n-1]$, $(C_{2k}, \phi_{2k})$ is a partial transversal isomorphic to a $2k$-cycle with $C_{2k} = y_1P_{y_{1-1}}x_1P_{x_{1-1}}y_{n-1}x_{n-2}xy$ and $\phi_{2k}$ arising from $\phi$ by adding $\phi_{2k}(x_1y_{n-1} - \frac{x_{n-1}}{2}) := 1, \phi_{2k}(x_{n-1}y_{n-1} - \frac{x_{n-1}}{2}) := 2n-3, \phi_{2k}(y_{2k}xy) := 2n, \phi_{2k}(y_{2k}(x_{2k}y_{1})) := 2n-1, \phi_{2k}(x_{2k}y_{n-1}) := 2n-2$, a contradiction. Furthermore, for any Hamiltonian $G$-transversal $C$ satisfying $xy \in E(C)$, there exists a partial $G$-transversal isomorphic to a $2k$-cycle containing $x$ for each $k \in [2, n]$.

Next, we consider the case $a < n-1$. If $\phi(x_jy_j) = a$, then $(P', \phi')$ is a partial $M$-transversal isomorphic to a $(2n-2)$-path with $P' = P_j$ and $\phi' = \phi_j$. Observe that $(P', \phi')$
misses \( a \). By Claim 3.5, \((C, \phi^*)\) is a \( G \)-transversal isomorphic to a Hamiltonian cycle with
\[ C = yx_jP'y_{n-1}xy, \]
where \( \phi^* \) arises from \( \phi' \) by setting \( \phi^*(xy) := a, \phi^*(x_jy) := 2n - 1 \) and \( \phi^*(xy_{n-1}) := 2n - 2, \) a contradiction.

If \( \phi(y_jx_{j+1}) = a \), then \((P_{j+1}, \phi_{j+1})\) is a partial \( M \)-transversal isomorphic to a \((2n - 2)\)-path. Applying Claim 3.5 yields that \( y_jx_{j+1} \in E(G_{2n-1}) \). Therefore \((P', \phi')\) is a partial transversal isomorphic to a \((2n - 2)\)-path as well, with \( P' = P_{j+1} \) and \( \phi' \) arising from \( \phi_{j+1} \) by setting \( \phi'(y_jx_{j+1}) := 2n - 1 \). So, \((P', \phi')\) misses \( a \) and \( 2j + 1 \). Hence, \((C, \phi^*)\) is a Hamiltonian \( G \)-transversal with \( C = yx_jP'y_{n-1}xy \), where \( \phi^* \) arises from \( \phi' \) by setting \( \phi^*(xy) := a, \phi^*(x_{j+1}y) := 2n - 2 \) and \( \phi^*(xy_{n-1}) := 2j + 1 \), a contradiction. By symmetry, we deduce that if \( a \in [n, 2n - 3] \), then \( x \) is contained in a partial \( G \)-transversal isomorphic to a \( 2k \)-cycle for each \( k \in [2, n] \), a contradiction.

This complete the proof for the case \( n \geq 4 \), and Theorem 2.1 thus follows. \( \square \)

### 3.2 Proof of Theorem 1.3

**Proof.** Take arbitrary vertices \( x \in X \) and \( y \in Y \). Set \( M_i = G_i - \{x, y\} \) for each \( i \in [2n] \) with bipartition \( X_M = X \setminus \{x\} \) and \( Y_M = Y \setminus \{y\} \). Let \( M = \{M_1, M_2, \ldots, M_{2n}\} \). Then \( |M_i| = 2n - 2 \) and \( \delta(M_i) \geq \frac{n - 1}{2} \) for each \( i \in [2n] \). By Lemma 2.1, \( M \) contains a partial transversal isomorphic to a \((2n - 2)\)-path or \( M_1 = \cdots = M_{2n} = K_{\frac{n-1}{2}, \frac{n-1}{2}} \cup K_{\frac{n+1}{2}, \frac{n+1}{2}} \) (\( n \) is odd).

If \( M_1 = \cdots = M_{2n} = K_{\frac{n-1}{2}, \frac{n-1}{2}} \cup K_{\frac{n+1}{2}, \frac{n+1}{2}} \), then without loss of generality, let \( X_M = X_1 \cup X_2, Y_M = Y_1 \cup Y_2 \) with \( X_1 = \{x_1, x_2, \ldots, x_{\frac{n-1}{2}}\}, X_2 = \{x_{\frac{n+1}{2}}, x_{\frac{n+1}{2} + 1}, \ldots, x_{n-1}\}, Y_1 = \{y_1, y_2, \ldots, y_{\frac{n-1}{2}}\}, Y_2 = \{y_{\frac{n+1}{2}}, y_{\frac{n+1}{2} + 1}, \ldots, y_{n-1}\} \). Moreover, \( M_i[X_j \cup Y_j] = K_{\frac{n-1}{2}, \frac{n-1}{2}} \) for every \( i \in [2n] \) and \( j \in \{1, 2\} \). As \( \delta(G) \geq \frac{n + 1}{2} \), we have \( xy_i, yx_i \in E(G_j) \) for each \( i \in [n - 1] \) and \( j \in [2n] \). Therefore \( G = F \).

Note that each partial \( M \)-transversal is also a partial \( G \)-transversal. If \( M \) contains a partial transversal isomorphic to a \((2n - 2)\)-path, then \( M \) satisfies one of the three statements by Lemma 2.4. Next, we distinguish the following three cases to proceed the proof.

**Case 1.** There is a partial \( M \)-transversal isomorphic to a \((2n - 2)\)-cycle.

Assume that \( C = x_1y_1 \cdots x_{n-1}y_{n-1}x_1 \) is a partial \( M \)-transversal isomorphic to a \((2n - 2)\)-cycle. Let \( \phi \) be an associated injection of \( C \) (identify \( x_n \) with \( x_1 \) with \( \phi(x_iy_i) = 2i - 1 \) and \( \phi(y_ix_{i+1}) = 2i \) for each \( i \in [n - 1] \). Then \([2n] \setminus \text{im}(\phi) = \{2n - 1, 2n\} \). We define
\[ T_{2n} := \{i \in [n - 1]: xy_i \in E(G_{2n})\}, \]
\[ T_{2n-1} := \{i \in [n - 1]: x_iy \in E(G_{2n-1})\}. \]
If \( T_{2n} \cap T_{2n-1} \neq \emptyset \), then \( xy, Cx_iy \) is a partial transversal isomorphic to a Hamiltonian path from \( x \) to \( y \). If \( T_{2n} \cap T_{2n-1} = \emptyset \), then \( \frac{n-1}{2} + \frac{n-1}{2} \leq |T_{2n}| + |T_{2n-1}| \leq n - 1 \), which implies that \( T_{2n} \cup T_{2n-1} = [n - 1] \) and \( |T_{2n}| = |T_{2n-1}| = \frac{n-1}{2} \). Therefore, there exists some
\( i \in [n - 1] \) such that \( i \in T_{2n} \) and \( i + 1 \in T_{2n-1} \). Then \( yx_{i+1}Cy_i x \) is a partial transversal isomorphic to a Hamiltonian path from \( y \) and \( x \).

**Case 2.** There is a partial \( \mathbf{M} \)-transversal isomorphic to the disjoint union of a \((2n - 4)\)-cycle and a copy of \( K_2 \).

Let \( C = x_1y_1 \cdots x_{n-2}y_{n-2}x_1 \) (identify \( x_{n-1} \) with \( x_1 \)) and \( C \cup \{ wz \} \) \((z \in X_M \) and \( w \in Y_M \) be the partial \( \mathbf{M} \)-transversal isomorphic to the disjoint union of a \((2n - 4)\)-cycle and a copy of \( K_2 \) with associated injection \( \phi \), where \( \phi(wz) = 2n - 3 \) and \( \phi(x_iy_i) = 2i - 1 \) and \( \phi(y_i x_{i+1}) = 2i \) for each \( i \in [n - 2] \). We define the following sets

\[
T_{2n} := \{ i \in [n - 2] : zy_i \in E(G_{2n}) \}, \\
T_{2n-1} := \{ i \in [n - 2] : wx_i \in E(G_{2n-1}) \}.
\]

Note that if \( \mathbf{M} \) contains a partial transversal isomorphic to a \((2n - 2)\)-cycle, then we are done by Case 1. So, we have \( T_{2n} \cap T_{2n-1} = \emptyset \). Then, \( n - 3 \leq |T_{2n}| + |T_{2n-1}| = |T_{2n} \cup T_{2n-1}| \leq n - 2 \).

If \( |T_{2n} \cup T_{2n-1}| = n - 2 \), then we have \( T_{2n} \cup T_{2n-1} = [n - 2] \). Observe that there exists some \( i \in [n - 2] \) satisfying \( i \in T_{2n} \), \( i + 1 \in T_{2n-1} \). Thus, \((C, \phi')\) is a partial \( \mathbf{M} \)-transversal isomorphic to a \((2n - 2)\)-cycle such that \( C' = y_1zwx_{i+1}Cy_i \) and \( \phi' \) arises from \( \phi \) by setting \( \phi'(wy) := 2n, \phi'(wx_{i+1}) := 2n - 1 \), a contradiction.

If \( |T_{2n} \cup T_{2n-1}| = n - 3 \), then we have \( |T_{2n}| = |T_{2n-1}| = \frac{n-3}{2} \), which indicates that \( n \) is odd. Hence, there exists some \( i \in [n - 2] \) such that \( i \notin T_{2n} \cup T_{2n-1} \). Without loss of generality, say \( i = 1 \), implying that \( T_{2n} \cup T_{2n-1} = [2, n - 2] \). As \( \delta(G) \geq \frac{n+1}{2} \), we have \( zw, zy \in E(G_{2n}), wx, wz \in E(G_{2n-1}) \).

If there exists some \( i \in [n - 3] \) such that \( i \in T_{2n} \) and \( i + 1 \in T_{2n-1} \), then \( y_i zwx_{i+1}Cy_i \) is a partial transversal isomorphic to a \((2n - 2)\)-cycle, a contradiction. Hence, \( T_{2n-1} = [2, \frac{n-1}{2}] \) and \( T_{2n} = [\frac{n+1}{2}, n - 2] \). It follows that \( N_{G_{2n-1}}(w) = \{ x_2, x_3, \ldots, x_{\frac{n-1}{2}}, x, z \} \) and \( N_{G_{2n}}(z) = \{ y_{\frac{n+1}{2}}, y_{\frac{n+3}{2}}, \ldots, y_{n-2}, y, w \} \). By symmetry, we deduce that for any \( i \in [2n - 3, 2n] \), \( N_G(w) = \{ x_2, x_3, \ldots, x_{\frac{n+1}{2}}, x, z \} \) and \( N_G(z) = \{ y_{\frac{n-1}{2}}, y_{\frac{n-3}{2}}, \ldots, y_{n-2}, y, w \} \).

Observe that \((C' \cup \{ x_1y_1 \}), \phi') \) is a partial \( \mathbf{M} \)-transversal with \( C' = wx_2Cy_{n-2}zw \) (identify \( x_{n-1} \) with \( z \)), where \( \phi' \) arises from \( \phi \) by setting \( \phi'(wxz) := 2n, \phi'(zyy_{n-2}) := 2n - 1, \phi'(zw) := 2n - 2 \). Let

\[
T_{2n-3} := \{ i \in [2, n - 2] : x_1y_i \in E(G_{2n-3}) \}, \\
T_{2n-4} := \{ i \in [n - 2] : x_{i+1}y_i \in E(G_{2n-4}) \}.
\]

Since \( x_1 \notin N_{G_{2n-3}}(w) \), by the similar analysis we obtain that \( |T_{2n-3}| = |T_{2n-4}| = \frac{n-3}{2} \), \( T_{2n-3} = [2, \frac{n-1}{2}] \) and \( T_{2n-4} = [\frac{n+1}{2}, n - 2] \). Recall that \( x_{n-1} = z \). Hence, \( N_{G_{2n-3}}(x_1) = \{ y, y_1, y_2, \ldots, y_{\frac{n-1}{2}} \} \) and \( N_{G_{2n-4}}(y_1) = \{ x, x_1, x_{\frac{n+3}{2}}, \ldots, x_{n-2}, z \} \). By symmetry, \( N_{G_{2n-3}}(y_i) = N_{G_{2n-4}}(y_i) \), contradicting with \( y_1 \notin N_{G_{2n-3}}(z) \).

**Case 3.** Lemma 2.4 (3) holds in \( \mathbf{M} \).
Let \( P = x_1y_1 \cdots x_{n-1}y_{n-1} \) be a partial \( M \)-transversal isomorphic to a \((2n - 2)\)-path with \( \phi(x_iy_i) = 2i - 1 \) for each \( i \in [n - 1] \) and \( \phi(y_ix_{i+1}) = 2i \) for each \( i \in [n - 2] \). Then \([2n]\backslash \text{im}(\phi) = [2n - 2, 2n]\). By Lemma 2.4 (3) and \( \delta(G_j) \geq \frac{n+1}{2} \) for each \( j \in \{2n - 1, 2n\} \), we have \( N_{G_{2n}}(x_1) = \{y_1, y_2, \cdots, y_{\frac{n-1}{2}}, y\} \) and \( N_{G_{2n-1}}(y_{n-1}) = \{x_{\frac{n+1}{2}}, x_{\frac{n+3}{2}}, \cdots, x_{n-1}, x\} \). Hence, \((P', \phi')\) is a partial \( G \)-transversal isomorphic to a Hamiltonian path from \( x \) to \( y \) such that \( P' = xy_{n-1}P'x_1y \) and \( \phi' \) arises from \( \phi \) by setting \( \phi'(xy_{n-1}) := 2n - 1 \) and \( \phi'(yx_1) := 2n \).

This completes the proof of Theorem 1.3.

Note that Theorem 1.3 is about the rainbow Hamiltonian connectivity of a collection of \( 2n \) bipartite graphs rather than \( 2n - 1 \) bipartite graphs. The main reason is that in the above proof, we need to use two colors not appearing on the rainbow cycle \( C \) of length \( 2n - 2 \) to find a rainbow Hamiltonian path for for any two vertices \( x \in X \) and \( y \in Y \). It would be interesting to study the rainbow Hamiltonian connectivity of a collection \( 2n - 1 \) bipartite graphs under the same degree condition.

### 4 Acknowledgment

The authors would like to thank the Associate Editor and two anonymous referees for their constructive comments and helpful suggestions. The work was supported by the National Natural Science Foundation of China (Nos. 12131013, 12161141006 and 12301443), the China Postdoctoral Science Foundation (No.2023M741826) and the Tianjin Research Innovation Project for Postgraduate Students (No.2022BKY039).

### References


