Journal of Combinatorial Theory, Series A 207 (2024) 105909

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

journal homepage: www.elsevier.com/locate/jcta

Spectra of power hypergraphs and signed graphs via parity-closed walks

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ARTICLE INFO

Article history:
Received 20 August 2023
Received in revised form 8 May 2024
Accepted 9 May 2024
Available online 22 May 2024

Keywords:
Power hypergraphs
Adjacency tensor
Signed graphs
Spectral moments
Characteristic polynomial
Spectral radius
Closed walks

ABSTRACT

The k-power hypergraph $G^{(k)}$ is the k-uniform hypergraph that is obtained by adding $k-2$ new vertices to each edge of a graph G, for $k \geq 3$. A parity-closed walk in G is a closed walk that uses each edge an even number of times. In an earlier paper, we determined the eigenvalues of the adjacency tensor of $G^{(k)}$ using the eigenvalues of signed subgraphs of G. Here, we express the entire spectrum (that is, we determine all multiplicities and the characteristic polynomial) of $G^{(k)}$ in terms of parity-closed walks of G. Moreover, we give an explicit expression for the multiplicity of the spectral radius of $G^{(k)}$. As a side result, we show that the number of parity-closed walks of given length is the corresponding spectral moment averaged over all signed graphs with underlying graph G. By extrapolating the characteristic polynomial of $G^{(k)}$ to $k = 2$, we introduce a pseudo-characteristic function which is shown to be the geometric mean of the characteristic polynomials of all signed graphs on G. This supplements a result by Godsil and Gutman that the arithmetic mean of the characteristic polynomials of all signed graphs on G equals the matching polynomial of G.

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https://doi.org/10.1016/j.jcta.2024.105909

1. Introduction

For a given graph $G = (V, E)$ and a sign function $\pi : E \to \{\pm 1\}$, we denote by $\phi_\pi(\lambda)$ the characteristic polynomial of the signed adjacency matrix of a signed graph G_π on G. Let Π denote the set of all sign functions on E. Godsil and Gutman [18] proved the remarkable result that the arithmetic mean of the characteristic polynomials of all signed graphs with underlying graph G equals the matching polynomial $\alpha(\lambda)$ of G, i.e.,

$$\alpha(\lambda) = 2^{-|E|} \sum_{\pi \in \Pi} \phi_\pi(\lambda).$$

The result of Godsil and Gutman was used by Marcus, Spielman, and Srivastava [25] to show the existence of an infinite family of d-regular bipartite Ramanujan graphs, for any $d \geq 3$.

The k-power hypergraph $G^{(k)}$ is the k-uniform hypergraph that is obtained by adding $k-2$ new vertices to each edge of a graph G, for $k \geq 3$ [22]. A parity-closed walk in G is a closed walk that uses each edge an even number of times. In an earlier paper [5], we determined all eigenvalues of the adjacency tensor of $G^{(k)}$ using the eigenvalues of signed subgraphs of G.

Here, we will extend combinatorial methods that are common for ordinary graphs to power hypergraphs in the sense that we will find a relation between the spectral moments of $G^{(k)}$ and counts of parity-closed walks in G. In order to determine these spectral moments, we will mainly exploit the so-called trace formula for tensors by Shao, Qi, and Hu [27].

In particular, we will determine the entire spectrum (and the characteristic polynomial) of $G^{(k)}$ by determining expressions for the multiplicities of the eigenvalues in terms of parity-closed walks of G. Moreover, we give an explicit expression for the multiplicity of the spectral radius $\rho(G^{(k)})$.

Note that from the Perron-Frobenius Theorem (for matrices), it is known that the spectral radius of a connected graph is an eigenvalue of the graph with multiplicity 1, and the number of eigenvalues whose modulus is equal to the spectral radius is 2 (or 1) if and only if the spectrum is symmetric (or asymmetric). Part of the Perron-Frobenius Theorem has been generalized to tensors and it is known that the spectral radius of a uniform hypergraph is an eigenvalue [2]. Additionally, the number of distinct eigenvalues whose modulus is equal to the spectral radius (called the cyclic index) is l if and only if the spectrum of the hypergraph is l-symmetric [3,15,16]. However, the multiplicity of the spectral radius of a hypergraph is unknown. The cyclic index of a power hypergraph $G^{(k)}$ has been proven to be k [21], which implies that the spectrum of the power hypergraph is k-symmetric. We consider the total multiplicity of eigenvalues of $G^{(k)}$ whose modulus is equal to the spectral radius $\rho(G^{(k)})$ and thus obtain the multiplicity of $\rho(G^{(k)})$.

As a side result (that has nothing to do with hypergraphs), we show that the number of parity-closed walks of given length in a graph G is the corresponding spectral moment averaged over all signed graphs with underlying graph G. We also extrapolate the

characteristic polynomial of $G^{(k)}$ to $k = 2$, thereby introducing a pseudo-characteristic function. Among other results, we show that this function is the geometric mean of the characteristic polynomials of all signed graphs on G and characterize when it is a polynomial. This supplements the result by Godsil and Gutman [18] that the arithmetic mean of the characteristic polynomials of all signed graphs on G equals the matching polynomial of G.

1.1. Organization of the paper

The rest of this paper is organized as follows: In Section 2, we introduce some notation and give some lemmas about the trace of tensors (Section 2.1), the spectral moments of hypergraphs (Section 2.2), and the spectral radius of signed graphs (Section 2.3).

In Section 3, we show that the number of parity-closed walks of given length in a graph G is the arithmetic mean of spectral moments of all signed graphs on G. This is not a result about hypergraphs, nor eigenvalues, but it will be used later in the paper.

Section 4 is the main part that relates concepts in the power hypergraph $G^{(k)}$ to concepts in G. The section is divided into a part on Eulerian walks in related digraphs (Section 4.1), a part on the number of spanning trees (Section 4.2), and a final part (Section 4.3), where we ultimately express the spectral moments of the power hypergraph in terms of counts of parity-closed walks in G in Proposition 4.8.

Our main results are obtained in Sections 5-7. Specifically, these results are presented in Theorems 5.1, 6.2, and 7.2.

In Section 5, eigenvalues and their multiplicities come into play. In Theorem 5.1, we determine expressions for the multiplicities and the characteristic polynomial of a power hypergraph $G^{(k)}$, all in terms of k, counts of parity-closed walks in G, and eigenvalues of signed subgraphs of G.

In Section 6, we extend the expressions for the characteristic polynomial of $G^{(k)}$ to the (hypothetical) case $k = 2$ and study the obtained pseudo-characteristic function. In particular, in Theorem 6.2 we show that it is the geometric mean of the characteristic polynomials of all signed graphs on G.

In Section 7, we finish the paper by determining the multiplicity of the spectral radius of a power hypergraph.

2. Preliminaries

In this section, we introduce some basic notation and give auxiliary lemmas on the trace of tensors, the spectral moments of hypergraphs, and the spectral radius of signed graphs.

2.1. The trace of tensors

For a positive integer n, let $[n] = \{1, \dots, n\}$. A k-order n-dimensional complex tensor $A = (t_{i_1 \cdots i_k})$ is a multidimensional array with n^k entries in $\mathbb{C}$, where $i_j \in [n]$, $j = 1, \dots, k$.

For $x = (x_1, \ldots, x_n)^\top \in \mathbb{C}^n$, we let $x^{[k-1]} = \left(x_1^{k-1}, \ldots, x_n^{k-1}\right)^\top$. The i-th component of the vector $Ax^{k-1} \in \mathbb{C}^n$ is defined as

$$\left(Ax^{k-1}\right)_i = \sum_{i_2,\ldots,i_k=1}^{n} a_{ii_2\cdots i_k} x_{i_2} \cdots x_{i_k}.$$

If there exists a nonzero vector $x = (x_1, x_2, \ldots, x_n)^\top \in \mathbb{C}^n$ such that $Ax^{k-1} = \lambda x^{[k-1]}$, then λ is called an *eigenvalue* of A and x is an *eigenvector* of A corresponding to λ [24,26]. The *characteristic polynomial* of A is defined as the resultant of the polynomial system $(\lambda x^{[k-1]} - Ax^{k-1})$ [26]. The (algebraic) multiplicity of an eigenvalue is the multiplicity as a root of the characteristic polynomial.

Also the trace of a tensor can be defined such that it generalizes the trace of a matrix. Shao, Qi, and Hu [27] obtained a useful expression for the trace of general tensors. Using this, we will give an expression for the spectral moments of a hypergraph H in terms of the number of connected subhypergraphs of H in Section 2.2.

In order to describe the tensor trace formula of Shao et al. [27], we introduce some related notation. Let $\mathcal{F}_d = \{(i_1\alpha_1, \ldots, i_d\alpha_d) : 1 \le i_1 \le \cdots \le i_d \le n; \alpha_1, \ldots, \alpha_d \in [n]^{k-1}\}$. Let $f = (i_1\alpha_1, \ldots, i_d\alpha_d) \in \mathcal{F}_d$, where $i_j\alpha_j \in [n]^k$, $j = 1, \ldots, d$. For a k-order n-dimensional tensor $A = (a_{i_1\cdots i_k})$, let $\pi(f) = \prod_{j=1}^{d} a_{i_j\alpha_j}$. We will now construct a multi-digraph D_f, in which we let (v_1, v_2) denote an arc from vertex v_1 to vertex v_2. For $i_j\alpha_j = i_j v_1 \ldots v_{k-1}$, we let the set of arcs from i_j to $v_1, v_2, \ldots, v_{k-1}$ be $\theta(i_j\alpha_j) = \{(i_j, v_1), \ldots, (i_j, v_{k-1})\}$. Let $\Theta(f)$ be the multi-set $\bigcup_{j=1}^{d} \theta(i_j\alpha_j)$. Let $b(f)$ be the product of the factorials of the multiplicities of the arcs in $\Theta(f)$. Let $c(f)$ be the product of the factorials of the outdegrees of all vertices in $\Theta(f)$. Denote the set of all closed walks using all arcs in $\Theta(f)$ by $\mathcal{W}(f)$.

Lemma 2.1. *[27, (3.5)] Let A be a k-order n-dimensional tensor. Then*

$$\mathrm{Tr}_d(A) = (k-1)^{n-1} \sum_{f \in \mathcal{F}_d} \frac{b(f)}{c(f)} \pi_f(A) |\mathcal{W}(f)|.$$

2.2. The spectral moments of hypergraphs

A hypergraph $H = (V_H, E_H)$ is called *k-uniform* if each edge of H contains exactly k vertices. Similar to the relation between graphs and matrices, there is a natural correspondence between uniform hypergraphs and tensors. For a k-uniform hypergraph H with n vertices, its *adjacency tensor* $A_H = (a_{i_1 i_2 \ldots i_k})$ is a k-order n-dimensional tensor, where

$$a_{i_1 i_2 \ldots i_k} = \begin{cases} \frac{1}{(k-1)!}, & \text{if } \{i_1, i_2, \ldots, i_k\} \in E_H, \\ 0, & \text{otherwise.} \end{cases}$$

When $k = 2$, A_H is the usual adjacency matrix of the graph H. The characteristic polynomial of the adjacency tensor A_H is called the characteristic polynomial of hypergraph H. We note that expressions for the coefficients of the characteristic polynomial of uniform hypergraphs have been obtained, by Clark and Cooper [6], but we will not use these.

It is known that the d-th order spectral moment of a graph G is equal to the d-th order trace of the adjacency matrix A_G [9]. Similar to the case of graphs, the d-th order spectral moment $\mathrm{S}_d(H)$ of a hypergraph H (i.e., the sum of d-th powers of all eigenvalues of H) is equal to the d-th order trace $\mathrm{Tr}_d(A_H)$ of the adjacency tensor A_H [20].

From Lemma 2.1, we have

$$\mathrm{S}_d(H) = \mathrm{Tr}_d(A_H) = (k-1)^{|V(H)|-1} \sum_{f \in \mathcal{F}_d} \frac{b(f)}{c(f)} \pi_f(A_H) |\mathcal{W}(f)|. \tag{2.1}$$

Without loss of generality, we only consider $f = (i_1\alpha_1, \ldots, i_d\alpha_d) \in \mathcal{F}_d$ for which $\pi_f(A_H)|W(f)| \neq 0$. Let $i_j\alpha_j = i_j v_1^{(j)} v_2^{(j)} \ldots v_{k-1}^{(j)}$ for all $j \in [d]$. Since $\pi_f(A_H) = \prod_{j=1}^{d} (A_H)_{i_j\alpha_j}$, we know that $e_j = \{i_j, v_1^{(j)}, v_2^{(j)}, \ldots, v_{k-1}^{(j)}\}$ is a hyperedge of H. We interpret $i_j\alpha_j$ as the hyperedge e_j with i_j as the root. It implies that f consists of d rooted hyperedges. We construct a k-uniform hypergraph H_f such that $V(H_f) = \bigcup_{j=1}^{d} e_j$ and $E(H_f) = \bigcup_{j=1}^{d} \{e_j\}$. Obviously, H_f is a subhypergraph of H. Consider next a set of representatives of isomorphic such subhypergraphs (so-called motifs)

$$\mathcal{H}_d = \{\widehat{H} : H_f \cong \widehat{H} \text{ and } \pi_f(A_H)|W(f)| \neq 0 \text{ for some } f \in \mathcal{F}_d\}.$$

For $\widehat{H} \in \mathcal{H}_d$, let $\mathcal{F}_d(\widehat{H}) = \{f\colon f \text{ consists of } d \text{ rooted hyperedges of } \widehat{H} \text{ and } H_f \cong \widehat{H}\}$. We use $N_H(\widehat{H})$ to denote the number of subhypergraphs of H which are isomorphic to $\widehat{H}$. By (2.1), we have

$$\mathrm{S}_d(H) = (k-1)^{|V(H)|-1}((k-1)!)^{-d} \sum_{\widehat{H} \in \mathcal{H}_d} N_H(\widehat{H}) \sum_{f \in \mathcal{F}_d(\widehat{H})} \frac{b(f)}{c(f)} |\mathcal{W}(f)|.$$

We call the coefficient

$$c_d(\widehat{H}) = ((k-1)!)^{-d} \sum_{f \in \mathcal{F}_d(\widehat{H})} \frac{b(f)}{c(f)} |\mathcal{W}(f)| \tag{2.2}$$

the d-th order *spectral moment coefficient* of $\widehat{H}$. We note that for a graph G, the coefficient $c_d(G)$ equals the number of covering closed walks of length d, that is, closed walks that use each edge at least once. We thus give the following subgraph structure interpretation of the spectral moments of hypergraphs:

$$\mathrm{S}_d(H) = (k-1)^{|V(H)|-1} \sum_{\widehat{H} \in \mathcal{H}_d} c_d(\widehat{H}) N_H(\widehat{H}). \tag{2.3}$$

How to reduce and calculate the d-th order spectral moment coefficient $c_d(\widehat{H})$ is the key question in Section 4. Here we make a first few steps in this direction. For $f \in \mathcal{F}_d(\widehat{H})$, we constructed in Section 2.1 a multi-digraph $D_f = (V(D_f), E(D_f))$ with $V(D_f) = V(\widehat{H})$ and $E(D_f) = \Theta(f)$. Since $\mathcal{W}(f)$ is the set of all closed walks with all arcs in $\Theta(f)$, we know that $|\mathcal{W}(f)| \neq 0$ if and only if D_f is Eulerian.

Because isomorphic graphs D_f contribute the same to the spectral moment coefficient in (2.2), we consider the set of representatives of isomorphic Eulerian multi-digraphs

$$\mathfrak{D}_d(\widehat{H}) = \{D : \ D_f \cong D \text{ is Eulerian for some } f \in \mathcal{F}_d(\widehat{H})\}.$$

Then

$$c_d(\widehat{H}) = ((k-1)!)^{-d} \sum_{D \in \mathfrak{D}_d(\widehat{H})} |\{f : f \in \mathcal{F}_d(\widehat{H}) \text{ and } D_f \cong D\}| \frac{b(f)}{c(f)} |\mathcal{W}(f)|. \quad (2.4)$$

In (2.4), $|\mathcal{W}(f)|$ is the number of the Eulerian walks in D. An expression for the number of the Eulerian walks in a multi-digraph was given by the so-called "BEST Theorem" [17].

Lemma 2.2. *[17, Theorem 6] Let $D = (V(D), E(D))$ be a Eulerian multi-digraph. Let $b(D)$ denote the product of the factorials of the multiplicities of the arcs in $E(D)$. Then the number of Eulerian walks in D is*

$$\frac{|E(D)|}{b(D)} t(D) \prod_{v \in V(D)} (\deg_D^+(v) - 1)!,$$

where $\deg_D^+(v)$ is the outdegree of vertex v and $t(D)$ is the number of spanning trees of D.

In (2.4), we have $b(f) = b(D)$ and $c(f) = \prod_{v \in V(D)} \deg_D^+(v)!$. From Lemma 2.2, we get the following expression for the d-th order spectral moment coefficient of a fixed hypergraph $\widehat{H}$.

Lemma 2.3. *The d-th order spectral moment coefficient of a k-uniform hypergraph $\widehat{H}$ is*

$$c_d(\widehat{H}) = d(k-1)((k-1)!)^{-d} \sum_{D \in \mathfrak{D}_d(\widehat{H})} \frac{\left|\{f : f \in \mathcal{F}_d(\widehat{H}) \text{ and } D_f \cong D\}\right| t(D)}{\prod_{v \in V(D)} \deg_D^+(v)}.$$

We will use this result to further reduce the spectral moment coefficients of power hypergraphs in Section 4.

The spectrum of a hypergraph is said to be k-symmetric if it is invariant under a rotation of an angle $2\pi/k$ in the complex plane [7]. The spectrum of a k-power hypergraph

is indeed k-symmetric [21], and hence $\mathrm{S}_d(G^{(k)}) = 0$ for $k \nmid d$. Note that Lemma 2.4 below shows k-symmetry of the eigenvalues, but it does not say anything about the corresponding multiplicities.

From (2.3), we then have

$$\mathrm{S}_d(G^{(k)}) = \begin{cases} (k-1)^{|V(G^{(k)})|-1} \sum_{\widehat{G}\in\mathcal{G}_d} c_d(\widehat{G}^{(k)}) N_G(\widehat{G}), & k \mid d, \\ 0, & k \nmid d, \end{cases} \tag{2.5}$$

where $\mathcal{G}_d = \{\widehat{G}\colon \widehat{G}$ is a non-empty subgraph of G and $\mathfrak{D}_d(\widehat{G}^{(k)}) \neq \emptyset\}$.

2.3. Signed graphs and the spectral radius

All the distinct eigenvalues of the power hypergraph $G^{(k)}$ are given by eigenvalues of signed subgraphs as follows [5].

Lemma 2.4. *[5] The complex number λ is an eigenvalue of $G^{(k)}$ if and only if*

(a) some signed induced subgraph of G has an eigenvalue σ such that $\sigma^2 = \lambda^k$, when $k = 3$;

(b) some signed subgraph of G has an eigenvalue σ such that $\sigma^2 = \lambda^k$, when $k \geq 4$.

We call signed graphs G_π and $G_{\pi'}$ switching equivalent if there is a diagonal matrix D with diagonal entries ± 1 such that $A(G_{\pi'}) = D^{-1}A(G_\pi)D$. Clearly, switching equivalent signed graphs have the same spectrum. The signed graphs G_+ and G_- are the ones with all signs $+1$ and all signs -1, respectively. If G_π is switching equivalent to G_+, then G_π is called a balanced signed graph. Note that G_+ and G_- are switching equivalent if and only if G is bipartite.

Lemma 2.5. *[28, Theorem 3.1] A signed graph G_π contains a balanced spanning subgraph, say $H_{\tilde{\pi}}$, which satisfies $\lambda_{\max}(G_\pi) \leq \lambda_{\max}(H_{\tilde{\pi}}) = \rho(H)$.*

We obtain the following result from Lemma 2.5.

Lemma 2.6. *Let G be a connected graph and $\pi \in \Pi$. Then $\rho(G_\pi) \leq \rho(G)$, with equality if and only if G_π is switching equivalent to G_+ or G_-. Moreover, G_π has an eigenvalue $\rho(G)$ if and only if it is switching equivalent to G_+, and it has an eigenvalue $-\rho(G)$ if and only if it is switching equivalent to G_-.*

Proof. Note that $\rho(G_\pi) = \max\{\lambda_{\max}(G_\pi), \lambda_{\max}(G_{-\pi})\}$. It is clear that if G_π is switching equivalent to G_+, then it has an eigenvalue $\rho(G)$, and if it is switching equivalent to G_-, then it has an eigenvalue $-\rho(G)$, hence in both cases $\rho(G_\pi) = \rho(G)$.

From Lemma 2.5, we have that $\rho(G_\pi) = \lambda_{\max}(G_{\pm\pi}) \leq \rho(H) \leq \rho(G)$. If equality holds, then $H = G$, and hence $G_{\pm\pi} = H_{\bar{\pi}}$, which is balanced, i.e., G_π is switching equivalent to G_+ or G_-.

If G_π has an eigenvalue $\rho(G)$, and it would not be switching equivalent to G_+, then by the above it must be switching equivalent to G_-, in which case it also has an eigenvalue $-\rho(G)$. But then G is bipartite, in which case G_+ is switching equivalent to G_-, and hence to G_π after all. Similarly it follows that if G_π has an eigenvalue $-\rho(G)$, then it is switching equivalent to G_-. □

3. The number of parity-closed walks via spectral moments of signed graphs

Our first goal is to show that the number of parity-closed walks P_d of length d in a graph G is the arithmetic mean of spectral moments of all signed graphs with underlying graph G. Although this result does not concern hypergraphs, it will become relevant when we apply the results on hypergraphs in Section 6.

Recall that Π denotes the set of all sign functions on the edge set E of G.

Theorem 3.1. *Let G be a graph. Then*

$$P_d = 2^{-|E|} \sum_{\pi \in \Pi} \mathrm{S}_d(G_\pi).$$

Proof. We denote the edges of G by $e_1, e_2, \ldots, e_m$, where $m = |E|$. Let $\mathcal{A} = \mathcal{A}(a_1, a_2, \ldots, a_m)$ denote the variable adjacency matrix of G with entries

$$\mathcal{A}_{uv} = \begin{cases} a_i, & \text{if } \{u, v\} = e_i, \\ 0, & \text{otherwise,} \end{cases}$$

where a_i is a variable for each $i \in [m]$. Let $\mathcal{W}_d$ be the set of all closed walks of length d in G. For $w \in \mathcal{W}_d$, let $t_i(w)$ be the number of times edge e_i is used in the closed walk w. The d-th order trace of the variable adjacency matrix $\mathcal{A}$ is a homogeneous polynomial of degree d with respect to $a_1, a_2, \ldots, a_m$, i.e.,

$$\mathrm{trace}(\mathcal{A}^d) = \sum_{w \in \mathcal{W}_d} \prod_{e_i \in E(w)} a_i^{t_i(w)}.$$

If $t_i(w)$ is even for every $a_i \in E(w)$, then the closed walk w is a parity-closed walk in G. In order to account for parity-closed walks only, we will remove all monomials containing variables of odd degree from $\mathrm{trace}(\mathcal{A}^d)$. We then obtain the number of such walks by substituting $a_1 = a_2 = \cdots = a_m = 1$.

Let f_{a_i} be an operation on a polynomial p such that

$$f_{a_i} \circ p(a_1, a_2, \ldots, a_m) = \frac{1}{2} p(a_1, \ldots, a_i, \ldots, a_m) + \frac{1}{2} p(a_1, \ldots, -a_i, \ldots, a_m).$$

It follows that $f_{a_i} \circ p$ is a polynomial obtained by removing all terms for which the degree of a_i is odd in the polynomial p. We therefore have to use m such operations, and observe that

$$f_{a_m} \circ f_{a_{m-1}} \circ \cdots \circ f_{a_1} \circ p(a_1, a_2, \ldots, a_m) = 2^{-m} \sum_{\pi \in \Pi} p(\pi(e_1)a_1, \pi(e_2)a_2, \ldots, \pi(e_m)a_m)$$

Thus, the number of parity-closed walks of length d equals

$$\begin{aligned} P_d &= \left(f_{a_m} \circ f_{a_{m-1}} \circ \cdots \circ f_{a_1} \circ \mathrm{trace}(\mathcal{A}^d)\right)|_{a_1=a_2=\cdots=a_m=1} \\ &= 2^{-|E|} \sum_{\pi \in \Pi} \mathrm{trace}(A(G_\pi)^d) = 2^{-|E|} \sum_{\pi \in \Pi} \mathrm{S}_d(G_\pi). \quad \square \end{aligned}$$

4. The spectral moments of power hypergraphs via the number of parity-closed walks

Let $\mathcal{G}(m)$ be the set of all connected non-empty unlabeled subgraphs (so-called motifs) of G with at most m edges. For $\widehat{G} \in \mathcal{G}(m)$, let $N_G(\widehat{G})$ denote the number of subgraphs of G isomorphic to $\widehat{G}$. A closed walk in $\widehat{G}$ is called *covering* if it uses each edge at least once. We use $p_d(\widehat{G})$ to denote the number of covering parity-closed walks of length d in $\widehat{G}$. Define a function

$$\mathcal{S}_d(k) = \begin{cases} (k-1)^{|V|+|E|(k-2)-1} \sum\limits_{\widehat{G} \in \mathcal{G}(\frac{d}{k})} \frac{2^{|E(\widehat{G})|-|V(\widehat{G})|} k^{|E(\widehat{G})|(k-3)+|V(\widehat{G})|}}{(k-1)^{|V(\widehat{G})|+|E(\widehat{G})|(k-2)-1}} p_{\frac{2d}{k}}(\widehat{G}) N_G(\widehat{G}), & k \mid d, \\ 0, & k \nmid d, \end{cases} \tag{4.1}$$

that depends on G, d, k and involves parity-closed walks. By substituting $k = 2$ in (4.1), it follows that

$$\mathcal{S}_d(2) = \sum_{\widehat{G} \in \mathcal{G}(\frac{d}{2})} p_d(\widehat{G}) N_G(\widehat{G}) = P_d, \tag{4.2}$$

so the definition $S_d(k)$ extends a natural decomposition of counting parity-closed walks in G to larger k.

The goal of this section is to show that $\mathcal{S}_d(k)$ is an expression for the spectral moments of the power hypergraph $G^{(k)}$ for $k \geq 3$. In order to accomplish this, we first need to look closer at the Eulerian digraphs D_f and their numbers of spanning trees.

4.1. Eulerian digraphs

A vertex $v \in V(G^{(k)}) \setminus V(G)$ is called a *core vertex* of $G^{(k)}$. For $\{i, j\} \in E(G)$, we use $\{i, j\}^{(k)}$ to denote the hyperedge of $G^{(k)}$ formed by adding $k-2$ core vertices to $\{i, j\}$. We denote the set of core vertices in $\{i, j\}^{(k)}$ by $\mathcal{N}_{ij}$.

We provide the following lemma to give the multiplicity of every arc of the multi-digraph D in $\mathfrak{D}_d(\widehat{G}^{(k)})$. This will allow us to find all multi-digraphs $D \in \mathfrak{D}_d(\widehat{G}^{(k)})$ to compute the spectral moment coefficient $c_d(\widehat{G}^{(k)})$ from Lemma 2.3.

Lemma 4.1. *Let $k \geq 3$. For $D \in \mathfrak{D}_d(\widehat{G}^{(k)})$, let $m_D(v,u)$ denote the multiplicity of the arc (v,u) in $E(D)$. Let $\{i,j\} \in E(\widehat{G})$. Then $m_D(v,u) = m_D(v,u')$ for any three distinct vertices v,u,u' in $\{i,j\}^{(k)}$ and $m_D(i,j) + m_D(j,i) = 2m_D(v,i) = 2m_D(v,j)$ for every core vertex v of $\{i,j\}^{(k)}$.*

Proof. From the construction of $D_f \cong D$, it follows that $m_D(v,u) = m_D(v,u')$ for any three distinct vertices v,u,u' in $\{i,j\}^{(k)}$.

Because a core vertex $v \in \mathcal{N}_{ij}$ occurs in only one hyperedge, it follows that $\deg_D^+(v) = \sum_{u \in \{i,j\}^{(k)} \setminus \{v\}} m_D(v,u) = (k-1)m_D(v,i)$ and

$$\begin{aligned}\deg_D^-(v) &= \sum_{u \in \{i,j\}^{(k)} \setminus \{v\}} m_D(u,v) \\ &= m_D(i,v) + m_D(j,v) + \sum_{u \in \mathcal{N}_{ij} \setminus \{v\}} m_D(u,v) \\ &= m_D(i,j) + m_D(j,i) + \sum_{u \in \mathcal{N}_{ij} \setminus \{v\}} m_D(u,i).\end{aligned}$$

Since $D \in \mathfrak{D}_d(\widehat{G}^{(k)})$ is Eulerian, we have $\deg_D^-(v) = \deg_D^+(v)$. It yields that

$$km_D(v,i) = m_D(i,j) + m_D(j,i) + \sum_{u \in \mathcal{N}_{ij}} m_D(u,i) \tag{4.3}$$

for all $v \in \mathcal{N}_{ij}$. It follows that $m_D(v,i) = m_D(u,i)$ for any $v,u \in \mathcal{N}_{ij}$. By (4.3), we have $m_D(i,j) + m_D(j,i) = 2m_D(v,i) = 2m_D(v,j)$ for all $v \in \mathcal{N}_{ij}$. □

For $D \in \mathfrak{D}_d(\widehat{G}^{(k)})$, let D^* be the multi-digraph obtained by removing all core vertices from D and let $\mathfrak{D}^* = \mathfrak{D}_d^*(\widehat{G}^{(k)}) = \{D^* : D \in \mathfrak{D}_d(\widehat{G}^{(k)})\}$. It follows from Lemma 4.1 that there is a one-one correspondence between $\mathfrak{D}_d(\widehat{G}^{(k)})$ and $\mathfrak{D}_d^*(\widehat{G}^{(k)})$, i.e., D can be reconstructed from D^* using the derived equations in the statement of Lemma 4.1.

Lemma 4.2. *Let $k \geq 3$ and $D \in \mathfrak{D}_d(\widehat{G}^{(k)})$. Then D^* is Eulerian.*

Proof. From Lemma 4.1, we have that

$$\deg_D^+(i) = \sum_{j:\{i,j\} \in E(\widehat{G})} (k-1)m_D(i,j)$$

and

$$\deg_D^-(i) = \sum_{j:\{i,j\} \in E(\widehat{G})} \left(m_D(j,i) + \sum_{v \in \mathcal{N}_{ij}} m_D(v,i) \right)$$

$$= \sum_{j:\{i,j\}\in E(\widehat{G})} \left(m_D(j,i) + (k-2)\frac{m_D(i,j)+m_D(j,i)}{2} \right)$$

for any $i \in V(\widehat{G})$. Since D is Eulerian, we have $\deg_D^+(i) = \deg_D^-(i)$, which implies that

$$\sum_{j:\{i,j\}\in E(\widehat{G})} m_D(i,j) = \sum_{j:\{i,j\}\in E(\widehat{G})} m_D(j,i),$$

i.e., $\deg_{D^*}^+(i) = \deg_{D^*}^-(i)$ for $i \in V(\widehat{G}) = V(D^*)$. Thus, D^* is Eulerian. □

Using Lemma 4.1, we can now give an intuitive description of the set $\mathcal{G}_d$ from (2.5). Recall that $\mathcal{G}(m)$ is the set of all connected non-empty unlabeled subgraphs (motifs) of $G = (V, E)$ with at most m edges.

Lemma 4.3. *Let $k \geq 3$. Then $\mathcal{G}_d = \mathcal{G}(\frac{d}{k})$ and $|E(D^*)| = \frac{2d}{k}$ for $D \in \mathfrak{D}_d(\widehat{G}^{(k)})$.*

Proof. We will first prove that $\mathcal{G}_d \subseteq \mathcal{G}(\frac{d}{k})$.

For any $\widehat{G} \in \mathcal{G}_d$ and $D \in \mathfrak{D}_d(\widehat{G}^{(k)})$, first note that $|E(D)| = d(k-1)$. On the other hand, using Lemma 4.1, we have that

$$\begin{aligned}
|E(D)| &= \sum_{\{i,j\}\in E(\widehat{G})} \sum_{\substack{u,v\in\{i,j\}^{(k)} \\ u\neq v}} m_D(u,v) \\
&= \sum_{\{i,j\}\in E(\widehat{G})} \left(\sum_{v\in\{i,j\}^{(k)}\setminus\{i\}} m_D(i,v) + \sum_{v\in\{i,j\}^{(k)}\setminus\{j\}} m_D(j,v) + \sum_{u\in\mathcal{N}_{ij}} \sum_{v\in\{i,j\}^{(k)}\setminus\{u\}} m_D(u,v) \right) \\
&= \sum_{\{i,j\}\in E(\widehat{G})} \left((k-1)m_D(i,j) + (k-1)m_D(j,i) + \sum_{u\in\mathcal{N}_{ij}} \sum_{v\in\{i,j\}^{(k)}\setminus\{u\}} m_D(u,v) \right).
\end{aligned}$$

Because $m_D(u,v) = \frac{m_D(i,j)+m_D(j,i)}{2}$ for any $u \in \mathcal{N}_{ij}$ and any $v \in \{i,j\}^{(k)} \setminus \{u\}$, it now readily follows that

$$|E(D)| = k(k-1) \sum_{\{i,j\}\in E(\widehat{G})} \frac{m_D(i,j)+m_D(j,i)}{2}.$$

Since $|E(D)| = d(k-1)$, we thus have that

$$\sum_{\{i,j\}\in E(\widehat{G})} \frac{m_D(i,j)+m_D(j,i)}{2} = \frac{d}{k}. \tag{4.4}$$

From Lemma 4.1, we know that $\frac{m_D(i,j)+m_D(j,i)}{2}$ is a positive integer for any $\{i,j\} \in E(\widehat{G})$. Hence we obtain that $|E(\widehat{G})| \leq \frac{d}{k}$, i.e., $\widehat{G} \in \mathcal{G}(\frac{d}{k})$.

From (4.4), it also follows that

$$|E(D^*)| = \sum_{i\in V(\widehat{G})} \deg_{D^*}(i) = \sum_{i\in V(\widehat{G})} \sum_{j:\{i,j\}\in E(\widehat{G})} m_D(i,j)$$
$$= \sum_{\{i,j\}\in E(\widehat{G})} m_D(i,j) + m_D(j,i) = \frac{2d}{k}.$$

Finally, we will show that $\mathcal{G}_d \supseteq \mathcal{G}(\frac{d}{k})$. For any $\widehat{G} \in \mathcal{G}(\frac{d}{k})$, there exists a closed walk w in $\widehat{G}$ with length $\frac{2d}{k}$ such that all edges of $\widehat{G}$ are used an even number of times. We first construct the multi-digraph $D_w = (V(\widehat{G}), E(w))$ without core vertices from the closed walk w. Then D_w is clearly Eulerian. Recall that $m_{D_w}(i,j)$ denotes the multiplicity of the arc (i,j) in $E(w)$, and note that $\frac{m_{D_w}(i,j)+m_{D_w}(j,i)}{2}$ is a positive integer. Next, we will add core vertices and their incident arcs to D_w in order to obtain a multi-digraph D. For any $\{i,j\} \in E(\widehat{G})$ and any core vertex $v \in \mathcal{N}_{ij}$, we construct D such that $m_D(v,u) = \frac{m_{D_w}(i,j)+m_{D_w}(j,i)}{2}$ for any $u \in \{i,j\}^{(k)} \setminus \{v\}$ and $m_D(i,v) = m_D(i,j) = m_{D_w}(i,j)$. It is easy to see that D is Eulerian and $D \in \mathfrak{D}_d(\widehat{G}^{(k)})$, i.e., $\mathfrak{D}_d(\widehat{G}^{(k)})$ is not empty, since D_w is Eulerian. Thus, $\widehat{G} \in \mathcal{G}_d$. □

An additional consequence of the above proof is that every covering parity-closed walk w gives rise to a Eulerian walk in some $D^* = D_w \in \mathfrak{D}^*_d(\widehat{G}^{(k)}) = \mathfrak{D}^*$. This leads to the following result.

Lemma 4.4. *Let $k \geq 3$ and $\widehat{G} \in \mathcal{G}(\frac{d}{k})$. Then*

$$p_{2d/k}(\widehat{G}) = \sum_{D^*\in\mathfrak{D}^*} \frac{|E(D^*)|}{b(D^*)} \prod_{v\in V(D^*)} (\deg^+_{D^*}(v)-1)!\, t(D^*).$$

Proof. As remarked, every covering parity-closed walk w gives rise to a Eulerian walk in some D^*. On the other hand, consider a Eulerian walk in some $D^* \in \mathfrak{D}^*$. First, note that the walk gives rise to a covering closed walk in the underlying graph $\widehat{G}$. Next, let D be the corresponding multi-digraph in $\mathfrak{D}_d(\widehat{G}^{(k)})$. Then $m_D(i,j) = m_{D^*}(i,j)$ for any two non-core vertices i,j. By Lemma 4.1, we have $m_D(i,j) + m_D(j,i) = 2m_D(v,i)$ for every core vertex v of $\{i,j\}^{(k)}$, and hence $m_{D^*}(i,j)+m_{D^*}(j,i)$ is even. This implies that every edge in the corresponding walk is covered an even number of times, and hence the (covering) walk is parity-closed. The result now follows by Lemma 2.2 (the "BEST theorem"). □

4.2. The number of spanning trees

In order to obtain the spectral moments of $G^{(k)}$ from (2.5), we need to further reduce the spectral moment coefficients $c_d(\widehat{G}^{(k)})$, and hence we have to reduce the number of spanning trees $t(D)$ from Lemma 2.3. By the Matrix-Tree theorem [11] and an expression

for the determinant involving the Schur complement [1], the number of spanning trees of a larger graph can be reduced to the number of spanning trees of a smaller weighted graph [8,13,19,29]. Using a similar trick, we reduce the number of spanning trees $t(D)$ of the digraph D to $t(D^*)$ as follows.

Lemma 4.5. *Let $k \geq 3$ and $D \in \mathfrak{D}_d(\widehat{G}^{(k)})$. Then*

$$t(D) = t(D^*)k^{|E(\widehat{G})|(k-3)+|V(\widehat{G})|-1}2^{|E(\widehat{G})|-V(\widehat{G})+1} \prod_{\{i,j\}\in E(\widehat{G})} \left(\frac{m_{D^*}(i,j)+m_{D^*}(j,i)}{2}\right)^{k-2}. \tag{4.5}$$

Proof. We write the Laplacian matrix of D as a block matrix

$$L_D = \begin{bmatrix} M & N \\ P & Q \end{bmatrix}.$$

The matrix M is a $|V(\widehat{G})| \times |V(\widehat{G})|$ matrix, with $M_{ii} = \deg_D^+(i)$ and $M_{ij} = -m_D(i,j)$ for $i \in V(\widehat{G})$.

Denote the edges of G by $e_1, e_2, \ldots, e_{|E(\widehat{G})|}$. The matrix Q is a block diagonal matrix with diagonal blocks $Q_{mm} = \frac{m_D(i,j)+m_D(j,i)}{2}(kI - J)$, where $\{i,j\} = e_m$, I is an identity matrix and J is an all-ones matrix of size $(k-2)\times(k-2)$.

The matrix N is a block matrix with $|V(\widehat{G})|\times|E(\widehat{G})|$ blocks as follows. Let $\mathbf{1}$ be the all-ones column vector of size $(k-2)$. Then $N_{im} = -m_D(i,j)\mathbf{1}^\top$ if $e_m = \{i,j\}$ and $N_{im} = 0$ if $i \notin e_m$. Similarly, the matrix P is a block matrix with blocks $P_{mi} = -\frac{m_D(i,j)+m_D(j,i)}{2}\mathbf{1}$ if $e_m = \{i,j\}$ and $P_{mi} = 0$ if $i \notin e_m$.

Let $\widehat{L_D}$ denote the submatrix of L_D obtained by deleting the first row and column. Then we can write it as block matrix

$$\widehat{L_D} = \begin{bmatrix} \widehat{M} & \widehat{N} \\ \widehat{P} & Q \end{bmatrix}.$$

From the Matrix-tree Theorem [11], we have $t(D) = \det(\widehat{L_D})$. Note that the matrix Q is invertible and $(Q^{-1})_{mm} = (Q_{mm})^{-1} = k^{-1}(m_D(i,j)+m_D(j,i))^{-1}(2I+J)$, where $\{i,j\} = e_m$. From the determinant formula involving the Schur complement [1], we have

$$t(D) = \det(\widehat{L_D}) = \det(Q)\det(\widehat{M} - \widehat{N}Q^{-1}\widehat{P}). \tag{4.6}$$

Furthermore,

$$\left(\widehat{M} - \widehat{N}Q^{-1}\widehat{P}\right)_{ii} = \widehat{M}_{ii} - \sum_{m=1}^{|E(\widehat{G})|} \widehat{N}_{im}\left(Q^{-1}\right)_{mm}\widehat{P}_{mi}$$

$$
\begin{aligned}
&= \deg_D^+(i) - \sum_{j:\{i,j\}\in E(\widehat{G})} (2k)^{-1} m_D(i,j)\mathbf{1}^\top(2I+J)\mathbf{1} \\
&= \deg_D^+(i) - \frac{k-2}{2} \sum_{j:\{i,j\}\in E(\widehat{G})} m_D(i,j) \\
&= (k-1)\deg_{D^*}^+(i) - \frac{k-2}{2}\deg_{D^*}^+(i) = \frac{k}{2}\deg_{D^*}^+(i)
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\widehat{M} - \widehat{N}Q^{-1}\widehat{P}\right)_{ij} &= \widehat{M}_{ij} - \sum_{m=1}^{\left|E(\widehat{G})\right|} \widehat{N}_{im}\left(Q^{-1}\right)_{mm}\widehat{P}_{mj} \\
&= -m_D(i,j) - \frac{k-2}{2}m_D(i,j) = -\frac{k}{2}m_{D^*}(i,j).
\end{aligned}
$$

Thus, for the Schur complement, we have $\widehat{M} - \widehat{N}Q^{-1}\widehat{P} = \frac{k}{2}\widehat{L_{D^*}}$, where $\widehat{L_{D^*}}$ is obtained from the Laplacian matrix of D^* by removing the same row and column as in M. Hence $\det(\widehat{M} - \widehat{N}Q^{-1}\widehat{P}) = \det(\frac{k}{2}\widehat{L_{D^*}}) = (\frac{k}{2})^{|V(\widehat{G})|-1}t(D^*)$.

Note also that if $e_m = \{i,j\}$, then

$$
\begin{aligned}
\det(Q_{mm}) &= \left(\frac{m_{D^*}(i,j) + m_{D^*}(j,i)}{2}\right)^{k-2} \det(kI - J) \\
&= 2k^{k-3}\left(\frac{m_{D^*}(i,j) + m_{D^*}(j,i)}{2}\right)^{k-2}.
\end{aligned}
$$

By (4.6), we thus have that

$$
\begin{aligned}
t(D) &= (\frac{k}{2})^{|V(\widehat{G})|-1}t(D^*)\det(Q) = (\frac{k}{2})^{|V(\widehat{G})|-1}t(D^*)\prod_{m=1}^{|E(\widehat{G})|}\det(Q_{mm}) \\
&= t(D^*)k^{|E(\widehat{G})|(k-3)+|V(\widehat{G})|-1}2^{|E(\widehat{G})|-V(\widehat{G})+1} \\
&\quad \times \prod_{\{i,j\}\in E(\widehat{G})}\left(\frac{m_{D^*}(i,j) + m_{D^*}(j,i)}{2}\right)^{k-2}. \quad \square
\end{aligned}
$$

4.3. The number of covering closed walks

In a related paper [4], the spectral moment coefficients of power hypertrees were reduced to the spectral moment coefficients of trees as follows.

Lemma 4.6. *[4] Let $k \geq 3$ and let $\widehat{T}$ be a tree. Then*

$$c_{\ell k}(\widehat{T}^{(k)}) = \frac{k^{|E(\widehat{T})|(k-2)+1}}{2(k-1)^{|V(\widehat{T})|+|E(\widehat{T})|(k-2)-1}} c_{2\ell}(\widehat{T}).$$

Note that $c_{2\ell}(\widehat{T})$ is the number of covering closed walks of length 2ℓ in $\widehat{T}$. We recall that a closed walk in which every edge is used an even number of times is called a parity-closed walk and that we denote the number of covering parity-closed walks of length 2ℓ in $\widehat{G}$ by $p_{2\ell}(\widehat{G})$. It is clear that for trees, every closed walk is parity-closed, so $c_{2\ell}(\widehat{T}) = p_{2\ell}(\widehat{T})$.

Lemma 4.6 does not hold in the case of general graphs, but using the earlier results in this section, it can be generalized as follows.

Lemma 4.7. *Let $k \geq 3$ and let $\widehat{G} \in \mathcal{G}(\ell)$. Then*

$$c_{\ell k}(\widehat{G}^{(k)}) = \frac{2^{|E(\widehat{G})|-|V(\widehat{G})|} k^{|E(\widehat{G})|(k-3)+|V(\widehat{G})|}}{(k-1)^{|V(\widehat{G})|+|E(\widehat{G})|(k-2)-1}} p_{2\ell}(\widehat{G}). \tag{4.7}$$

Proof. From the spectral moment coefficient in Lemma 2.3, we have

$$c_{\ell k}(\widehat{G}^{(k)}) = \ell k(k-1)((k-1)!)^{-\ell k} \sum_{D \in \mathfrak{D}_{\ell k}(\widehat{G}^{(k)})} \frac{\left|\{f : f \in \mathcal{F}_{\ell k}(\widehat{G}^{(k)}) \text{ and } D_f \cong D\}\right| t(D)}{\prod_{v \in V(D)} \deg^+_D(v)}. \tag{4.8}$$

First recall how $D_f \cong D$ is constructed from ℓk rooted hyperedges of $\widehat{G}^{(k)}$. In order to count the number of f that give rise to a given D, note that we can permute the $k-1$ non-roots of each hyperedge (without changing D_f). We can also permute the hyperedges with the same root. If this root is a core vertex, then this however leads to the same f because the permuted hyperedges are the same. If the root, i say, is not a core vertex, then it occurs $\deg^+_{D^*}(i)$ times, but again, permuting the same hyperedges (with hyperedge $\{i,j\}^{(k)}$ with root i occurring $m_{D^*}(i,j)$ times) leads to the same f. This implies that

$$\left|\{f : f \in \mathcal{F}_{\ell k}(\widehat{G}^{(k)}) \text{ and } D_f \cong D\}\right| = ((k-1)!)^{\ell k} \frac{\prod_{i \in V(\widehat{G})} (\deg^+_{D^*}(i))!}{\prod_{(i,j) \in E(D^*)} m_{D^*}(i,j)!}. \tag{4.9}$$

From Lemma 4.1, it follows that if $v \in \mathcal{N}_{ij}$, then $\deg^+_D(v) = (k-1)\frac{m_D(i,j)+m_D(j,i)}{2}$, which implies that

$$\prod_{v \in V(D)} \deg^+_D(v) = \prod_{v \in V(D^*)} \deg^+_D(v) \prod_{v \in V(D) \setminus V(D^*)} \deg^+_D(v)$$

$$= \prod_{v\in V(D^*)} (k-1)\deg^+_{D^*}(v) \prod_{\{i,j\}\in E(\widehat{G})} \left((k-1)\frac{m_{D^*}(i,j)+m_{D^*}(j,i)}{2}\right)^{k-2}$$

$$= (k-1)^{|V(\widehat{G})|+(k-2)|E(\widehat{G})|} \prod_{v\in V(\widehat{G})} \deg^+_{D^*}(v) \prod_{\{i,j\}\in E(\widehat{G})} \left(\frac{m_{D^*}(i,j)+m_{D^*}(j,i)}{2}\right)^{k-2}. \tag{4.10}$$

Next, recall that there is a one-one correspondence between $\mathfrak{D}^* = \mathfrak{D}^*_d(\widehat{G}^{(k)})$ and $\mathfrak{D}_d(\widehat{G}^{(k)})$. After substituting (4.5), (4.9) and (4.10) into (4.8), eliminating and collecting terms, we then obtain that

$$c_{\ell k}(\widehat{G}^{(k)}) = \ell \frac{k^{|E(\widehat{G})|(k-3)+|V(\widehat{G})|} 2^{|E(\widehat{G})|-|V(\widehat{G})|+1}}{(k-1)^{|V(\widehat{G})|+|E(\widehat{G})|(k-2)-1}} \sum_{D^*\in\mathfrak{D}^*} \frac{\prod_{v\in V(D^*)}(\deg^+_{D^*}(v)-1)!t(D^*)}{\prod_{(i,j)\in E(D^*)} m_{D^*}(i,j)!}$$

$$= \frac{2^{|E(\widehat{G})|-|V(\widehat{G})|} k^{|E(\widehat{G})|(k-3)+|V(\widehat{G})|}}{(k-1)^{|V(\widehat{G})|+|E(\widehat{G})|(k-2)-1}} \sum_{D^*\in\mathfrak{D}^*} \frac{|E(D^*)|}{b(D^*)} \prod_{v\in V(D^*)} (\deg^+_{D^*}(v)-1)!t(D^*),$$

where in the second equality, we used that $|E(D^*)| = 2\ell$ by Lemma 4.3 and $b(D^*)$ equals the product of the factorials of the multiplicities of the edges. By Lemma 4.4, (4.7) now follows. □

From (2.5), (4.1), Lemmas 4.3 and 4.7, we finally obtain our claimed expression for the spectral moment of k-power hypergraphs.

Proposition 4.8. *Let $k \geq 3$. Then*

$$\mathrm{S}_d(G^{(k)}) = \mathcal{S}_d(k).$$

5. The characteristic polynomial of power hypergraphs

So far, we have used the "trace formula" for tensors to derive the spectral moments of the power hypergraph $G^{(k)}$. From Lemma 2.4, we can find the eigenvalues of $G^{(k)}$ from the eigenvalues of the signed graphs on G. In this section, we will discuss how to derive the multiplicities of the eigenvalues using these eigenvalues, numbers of parity-closed walks in G, and spectral moments. The derived expressions will depend explicitly on k.

Let Σ denote the set of squares of the nonzero eigenvalues of all signed subgraphs of G, let $\varsigma = |\Sigma|$, and denote the elements of Σ by $\sigma_1^2, \sigma_2^2, \ldots, \sigma_\varsigma^2$.

By Lemma 2.4 and the fact that the spectrum of $G^{(k)}$ is k-symmetric [21] (see also [27, Theorem 3.1]), we can write the characteristic polynomial $\phi_k(\lambda)$ of $G^{(k)}$ as

$$\phi_k(\lambda) = \lambda^{\mu_0(k)} \prod_{i=1}^{\varsigma} (\lambda^k - \sigma_i^2)^{\mu_i(k)}, \tag{5.1}$$

where $\mu_i(k)$ is the multiplicity of the corresponding eigenvalues λ. Note that $\sigma_i^2 \in \Sigma$ does not necessarily give rise to an eigenvalue of $G^{(3)}$ if $\pm|\sigma_i|$ is not an eigenvalue of a signed *induced* subgraph. In this case, we have $\mu_i(3) = 0$. Let

$$\mathrm{M} = \begin{bmatrix} \sigma_1^2 & \sigma_2^2 & \cdots & \sigma_\varsigma^2 \\ \sigma_1^4 & \sigma_2^4 & \cdots & \sigma_\varsigma^4 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1^{2\varsigma} & \sigma_2^{2\varsigma} & \cdots & \sigma_\varsigma^{2\varsigma} \end{bmatrix}$$

be a matrix of coefficients. Then M is invertible (cf. Vandermonde). Let the multiplicity vector be $\mu(k) = (\mu_1(k), \mu_2(k), \ldots, \mu_\varsigma(k))^\top$ and the spectral moments vector be $S(k) = \big(\mathrm{S}_k(G^{(k)}), \mathrm{S}_{2k}(G^{(k)}), \ldots, \mathrm{S}_{\varsigma k}(G^{(k)})\big)^\top$.

From (5.1), the spectral moments can be written as $\mathrm{S}_{\ell k}(G^{(k)}) = \sum_{i=1}^{\varsigma} k\mu_i(k)\sigma_i^{2\ell}$ for all positive integers ℓ, which leads to the (nonsingular) system of equations

$$k\mathrm{M}\mu(k) = S(k)$$

for the multiplicities.

Next, we will rewrite $S(k)$ using the subgraph structural interpretation (4.1) of the spectral moments (Proposition 4.8). Let $\chi = |\mathcal{G}(\varsigma)|$, denote the (non-isomorphic) subgraphs of G with at most ς edges by $\widehat{G}_1, \widehat{G}_2, \ldots, \widehat{G}_\chi$ and let the parity-closed walk matrix be

$$\mathrm{P} = \begin{bmatrix} p_2(\widehat{G}_1) & p_2(\widehat{G}_2) & \cdots & p_2(\widehat{G}_\chi) \\ p_4(\widehat{G}_1) & p_4(\widehat{G}_2) & \cdots & p_4(\widehat{G}_\chi) \\ \vdots & \vdots & \ddots & \vdots \\ p_{2\varsigma}(\widehat{G}_1) & p_{2\varsigma}(\widehat{G}_2) & \cdots & p_{2\varsigma}(\widehat{G}_\chi) \end{bmatrix}.$$

Moreover, consider the subgraph number vector $N = \Big(N_G(\widehat{G}_1), N_G(\widehat{G}_2), \ldots, N_G(\widehat{G}_\chi)\Big)^\top$ and the diagonal $\chi \times \chi$ matrix $\mathrm{D}(k)$ with

$$(\mathrm{D}(k))_{ii} = \frac{2^{|E(\widehat{G}_i)|-|V(\widehat{G}_i)|}k^{|E(\widehat{G}_i)|(k-3)+|V(\widehat{G}_i)|}}{(k-1)^{|V(\widehat{G}_i)|+|E(\widehat{G}_i)|(k-2)-1}}$$

for $i \in [\chi]$. From Proposition 4.8 and (4.1), with the additional remark that $p_{2\ell}(\widehat{G_i}) = 0$ when $|E(\widehat{G_i})| > \ell$ (i.e., $\mathrm{P}_{\ell i} = 0$ when $\widehat{G_i} \notin \mathcal{G}(\ell)$), we then have

$$\mathrm{S}_{\ell k}(G^{(k)}) = (k-1)^{|V|+(k-2)|E|-1}\sum_{i=1}^{\chi}\mathrm{D}(k)_{ii}\mathrm{P}_{\ell i}N_i,$$

and hence

$$\mu(k) = k^{-1}\mathrm{M}^{-1}S(k) = \frac{(k-1)^{|V|+(k-2)|E|-1}}{k}\mathrm{M}^{-1}\mathrm{PD}(k)N.$$

Thus, we have expressions for the multiplicities in $\mu(k)$. The remaining multiplicity $\mu_0(k)$ clearly follows from these, and we may conclude the following result.

Theorem 5.1. *Let $k \geq 3$. Then the characteristic polynomial of $G^{(k)}$ is*

$$\phi_k(\lambda) = \lambda^{\mu_0(k)} \prod_{i=1}^{\varsigma} (\lambda^k - \sigma_i^2)^{\mu_i(k)},$$

where $\mu_i(k) = \frac{(k-1)^{|V|+(k-2)|E|-1}}{k}(\mathrm{M}^{-1}\mathrm{PD}(k)N)_i$ for $i \in [\varsigma]$ and

$$\mu_0(k) = (|V| + (k-2)|E|)(k-1)^{|V|+(k-2)|E|-1} - k\sum_{i=1}^{\varsigma} \mu_i(k).$$

For example, for $G = C_3$, it is easy to get $\Sigma = \{\sigma_1^2, \sigma_2^2, \sigma_3^2\} = \{1, 2, 4\}$, which implies that $\mathrm{M} = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 4 & 16 \\ 1 & 8 & 64 \end{pmatrix}$. Note that $\mathcal{G}(3) = \{P_2, P_3, C_3\}$ and $N = (3, 3, 1)^\top$. From Lemma 4.4, we get $\mathrm{P} = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 4 & 0 \\ 2 & 12 & 24 \end{pmatrix}$. From Theorem 5.1, the following result from [10] can be obtained as a consequence.

Corollary 5.2. *[10, Theorem 4.2] For $k \geq 3$, the characteristic polynomial of $C_3^{(k)}$ is*

$$\phi_k(\lambda) = \lambda^a(\lambda^k - 1)^b(\lambda^k - 2)^c(\lambda^k - 4)^d,$$

where

$$\begin{cases} a = 3(k-1)^{3k-3} - 3k^{k-1}(k-1)^{2k-3} + 3k^{2k-3}(k-1)^{k-2} - 3k^{3k-6} \\ b = 3k^{k-2}(k-1)^{2k-3} - 6k^{2k-4}(k-1)^{k-2} + 8k^{3k-7} \\ c = 3k^{2k-4}(k-1)^{k-2} - 6k^{3k-7} \\ d = k^{3k-7}. \end{cases}$$

6. The geometric mean of the characteristic polynomials of all signed graphs with the same underlying graph

Using the expression in Theorem 5.1, we can extend the characteristic polynomial $\phi_k(\lambda)$ of a power hypergraph to $k = 2$. We thus define the pseudo-characteristic function

$$\beta(\lambda) \equiv \phi_2(\lambda) = \lambda^{\mu_0(2)} \prod_{i=1}^{\varsigma} (\lambda^2 - \sigma_i^2)^{\mu_i(2)},$$

where $\mu_i(2) = \frac{1}{2}(\mathrm{M}^{-1}\mathrm{P}N)_i$ for $i \in [\varsigma]$ (note that $\mathrm{D}(2) = I$) and $\mu_0(2) = |V| - 2\sum_{i=1}^{\varsigma} \mu_i(2)$. Note that $\beta(\lambda)$ is not a polynomial, in general, because the $\mu_i(2)$ are not necessarily integer for $i = 0, 1, \ldots, \varsigma$.

It is important to note that also for $k = 2$, the "spectral moments" (of $\beta(\lambda)$) are determined by the equation $2\mathrm{M}\mu(2) = S(2)$. On the other hand, we defined $\mu(2) = \frac{1}{2}\mathrm{M}^{-1}\mathrm{P}N$, which implies by (4.2) that

$$S(2) = \mathrm{P}N = (\mathcal{S}_2(2), \mathcal{S}_4(2), \ldots, \mathcal{S}_{2\varsigma}(2))^\top.$$

Thus, it follows that Proposition 4.8 extends to $k = 2$ in the sense that the spectral moment $\mathrm{S}_d(\beta) = \sum_{i=1}^{\varsigma} 2\mu_i(2)\sigma_i^d$ of the pseudo-characteristic function equals $\mathcal{S}_d(2)$, for d even and $d \leq 2\varsigma$ (and note that both equal 0 for d odd).

Proposition 6.1. *Let G be a graph and d be even with $d \leq 2\varsigma$. Then the number of parity-closed walks in G equals $P_d = \sum_{i=1}^{\varsigma} 2\mu_i(2)\sigma_i^d$.*

Because $P_d = 2^{-|E|} \sum_{\pi \in \Pi} \mathrm{S}_d(G_\pi)$ by Theorem 3.1, we will next show that the multiplicity $\mu_i(2)$ is the average multiplicity of $\pm|\sigma_i|$ over all signed graphs on G, and obtain the following theorem.

Theorem 6.2. *The pseudo-characteristic function $\beta(\lambda)$ is the geometric mean of the characteristic polynomials of all signed graphs with underlying graph G, i.e.,*

$$\beta(\lambda) = \prod_{\pi \in \Pi} \phi_\pi(\lambda)^{2^{-|E|}}.$$

Moreover, $\beta(\lambda)$ is the characteristic polynomial of G if and only if G is a forest.

Proof. Note first that $|\sigma_i|$ and $-|\sigma_i|$ have the same average multiplicity over all signed graphs as sign functions π and $-\pi$ give opposite eigenvalues. Let $\overline{\mu}_i$ then be the average multiplicity of $\pm|\sigma_i|$ over all signed graphs on G and let $\overline{\mu}$ be the corresponding vector. Because $P_d = 2^{-|E|} \sum_{\pi \in \Pi} \mathrm{S}_d(G_\pi)$ by Theorem 3.1, it is clear that $\mathrm{P}N = 2\mathrm{M}\overline{\mu}$. Because $2\mathrm{M}\mu(2) = S(2) = \mathrm{P}N$ and M is invertible, it follows that $\overline{\mu} = \mu(2)$.

Now it follows that $\beta(\lambda)^{2^{|E|}}$ is a monic polynomial with the same spectral moments as $\prod_{\pi \in \Pi} \phi_\pi(\lambda)$. From Newton's identities, also known as the Girard-Newton formulae, it is known that a monic polynomial is determined by its spectral moments [12,23], which finishes the proof of the main statement.

We are left to determine when $\beta(\lambda)$ is the characteristic polynomial $\phi(G)$ of G. Note first that if G is a forest, then $\phi_\pi(\lambda) = \phi(G)$ for all $\pi \in \Pi$, and hence $\beta(\lambda) = \prod_{\pi \in \Pi} \phi_\pi(\lambda)^{2^{-|E|}} = \phi(G)$.

On the other hand, if $\beta(\lambda) = \phi(G)$, then their spectral moments are the same, hence $P_d = S_d(G)$ for every d, which implies that every closed walk is a parity-closed walk. But then G cannot have cycles, so it is a forest.

Thus, $\beta(\lambda) = \phi(G)$ if and only if G is a forest. □

Corollary 6.3. *Let G be a cycle. Then $\beta(\lambda) = \phi(\lambda^2 - 2)^{\frac{1}{2}}$.*

Proof. A signed cycle (C_n, π) is called positive (resp. negative) if the product of signs of all edges of (C_n, π) is positive (resp. negative). Note that all positive (resp. negative) cycles are switching equivalent. We use $\phi_+(\lambda)$ and $\phi_-(\lambda)$ to denote the characteristic polynomial of a positive and negative cycle, respectively. It is known that $\phi_+(\lambda) = \prod_{i=1}^{n}(\lambda - 2\cos\frac{2i\pi}{n})$ [9, p. 72] and $\phi_-(\lambda) = \prod_{i=1}^{n}(\lambda - 2\cos\frac{(2i-1)\pi}{n})$ [14, Lemma 2.3]. From Theorem 6.2, we have

$$\begin{aligned}\beta^2(\lambda) = \phi_+(\lambda)\phi_-(\lambda) &= \prod_{i=1}^{2n}(\lambda - 2\cos\tfrac{i\pi}{n}) = \prod_{i=1}^{n}(\lambda^2 - 4\cos^2\tfrac{i\pi}{n}) \\ &= \prod_{i=1}^{n}(\lambda^2 - 2 - 2\cos\tfrac{2i\pi}{n}) = \phi(\lambda^2 - 2),\end{aligned}$$

where we have used the well-known identities $\cos\theta = -\cos(\theta - \pi)$ and $2\cos^2\theta = 1 + \cos 2\theta$. □

For example, $\beta(\lambda) = (\lambda^2 - 1)(\lambda^2 - 4)^{\frac{1}{2}}$ for $G = C_3$, and note that this is not a polynomial. In fact, when G is connected, but not a tree, then $\beta(\lambda)$ is not a polynomial because the multiplicity of the spectral radius is not an integer. Indeed, note that if G is connected, then its spectral radius has multiplicity 1, and hence G_π has at most one eigenvalue $\rho(G)$ for all sign functions π. If G is not a tree, then there is a sign function π such that G_π is not switching equivalent to G_+ and hence, by Lemma 2.6, G_π does not have eigenvalue $\rho(G)$. Thus, the average multiplicity of eigenvalue $\rho(G)$ over all sign functions is less than 1, and hence $\beta(\lambda)$ is not a polynomial. In the next section, we will make this more precise and determine the multiplicity of $\rho(G)$ as a root of β.

From the arithmetic-geometric mean inequality, we can get an inequality between the matching polynomial $\alpha(\lambda)$ and the pseudo-characteristic function $\beta(\lambda)$ of G.

Corollary 6.4. *Let λ_0 be a real number such that $\phi_\pi(\lambda_0) \geq 0$ for all $\pi \in \Pi$. Then $\alpha(\lambda_0) \geq \beta(\lambda_0)$, with equality if and only if $\phi_\pi(\lambda_0)$ is constant over all $\pi \in \Pi$.*

Godsil and Gutman [18] showed that the matching polynomials α of some standard graphs can be expressed in terms of some classical orthogonal polynomials. For example $\alpha_{C_n}(\lambda) = 2T_n(\lambda/2)$, where T_n is a Chebyshev polynomial of the first kind. For the complete graphs K_n, they obtained that $\alpha_{K_n}(\lambda) = He_n(\lambda)$, where He_n denotes the

probabilist's Hermite polynomial. Thus, Corollary 6.4 implies that $\beta_{K_n}(\lambda_0) \leq He_n(\lambda_0)$, for example.

7. The multiplicity of the spectral radius of a power hypergraph

Let $n_\rho(G^{(k)})$ be the total multiplicity of eigenvalues of the power hypergraph $G^{(k)}$ whose modulus is equal to the spectral radius $\rho(G^{(k)})$. We note that it follows from Theorem 5.1 that this number is k times the multiplicity of the spectral radius.

Lemma 7.1. *Let $k \geq 3$ and let G be a connected graph. Then*

$$n_\rho(G^{(k)}) = 2^{|E|-|V|}k^{|E|(k-3)+|V|}\lim_{\ell\to\infty}\frac{P_{2\ell}}{\rho(G)^{2\ell}}.$$

Proof. It is known that the spectral radius of a power hypergraph $G^{(k)}$ is a positive number and $\rho(G^{(k)})^k = \rho(G)^2$ [30] (this follows also from Lemmas 2.4 and 2.6). Since the ℓk-th order spectral moment $\mathrm{S}_{\ell k}(G^{(k)})$ is the sum of the ℓk-th powers of all eigenvalues of a power hypergraph $G^{(k)}$, it follows that

$$n_\rho(G^{(k)}) = \lim_{\ell\to\infty}\frac{\mathrm{S}_{\ell k}(G^{(k)})}{\rho(G^{(k)})^{\ell k}} = \lim_{\ell\to\infty}\frac{\mathrm{S}_{\ell k}(G^{(k)})}{\rho(G)^{2\ell}}.$$

By Proposition 4.8, we then have

$$n_\rho(G^{(k)}) = \lim_{\ell\to\infty}\frac{\sum_{\widehat{G}\in\mathcal{G}(\ell)} l_G(\widehat{G})p_{2\ell}(\widehat{G})}{\rho(G)^{2\ell}}, \tag{7.1}$$

where $l_G(\widehat{G}) = 2^{|E(\widehat{G})|-|V(\widehat{G})|}(k-1)^{|V(G^{(k)})|-|V(\widehat{G}^{(k)})|}k^{|E(\widehat{G})|(k-3)+|V(\widehat{G})|}N_G(\widehat{G})$. We claim that in the summation over $\mathcal{G}(\ell)$, only G itself contributes to the limit in (7.1). Indeed, for any graph $H \in \mathcal{G}(\ell)$ with $H \neq G$, let $\mathcal{H}(\ell)$ be the set of connected subgraphs (motifs) of the graph H with at most ℓ edges. By (7.1), we have

$$n_\rho(H^{(k)}) = \lim_{\ell\to\infty}\frac{\sum_{\widehat{G}\in\mathcal{H}(\ell)} l_H(\widehat{G})p_{2\ell}(\widehat{G})}{\rho(H)^{2\ell}},$$

which implies that

$$\lim_{\ell\to\infty}\frac{\sum_{\widehat{G}\in\mathcal{H}(\ell)} l_H(\widehat{G})p_{2\ell}(\widehat{G})}{\rho(G)^{2\ell}} = 0,$$

because $\rho(H) < \rho(G)$.

Note that $l_H(\widehat{G})p_{2\ell}(\widehat{G}) \geq 0$ and $l_H(\widehat{G}) > 0$ for any $\widehat{G} \in \mathcal{H}(\ell)$ and any $H \in \mathcal{G}(\ell)$ $(H \neq G)$, hence it follows that

$$\lim_{\ell\to\infty}\frac{p_{2\ell}(\widehat{G})}{\rho(G)^{2\ell}}=0$$

for any $\widehat{G}\in\mathcal{G}(\ell)$ with $\widehat{G}\neq G$. By (7.1), we thus have

$$\begin{aligned}n_\rho(G^{(k)})&=2^{|E|-|V|}k^{|E|(k-3)+|V|}\lim_{\ell\to\infty}\frac{p_{2\ell}(G)}{\rho(G)^{2\ell}}\\&=2^{|E|-|V|}k^{|E|(k-3)+|V|}\lim_{\ell\to\infty}\frac{\sum_{\widehat{G}\in\mathcal{G}(\ell)}p_{2\ell}(\widehat{G})N_G(\widehat{G})}{\rho(G)^{2\ell}}\\&=2^{|E|-|V|}k^{|E|(k-3)+|V|}\lim_{\ell\to\infty}\frac{P_{2\ell}}{\rho(G)^{2\ell}}.\quad\square\end{aligned}$$

From Theorem 3.1, we have that $P_{2\ell}=2^{-|E|}\sum_{\pi\in\Pi}\mathrm{S}_{2\ell}(G_\pi)$. Using Lemma 2.6 about the spectral radius of signed graphs, we obtain $\lim_{\ell\to\infty}\frac{P_{2\ell}}{\rho(G)^{2\ell}}$ and determine the multiplicity of the spectral radius of $G^{(k)}$.

Theorem 7.2. *For a connected graph G and $k\geq 3$, the multiplicity of the spectral radius of $G^{(k)}$ is $k^{|E|(k-3)+|V|-1}$.*

Proof. From Theorem 3.1 and Lemma 7.1, we have that

$$\begin{aligned}n_\rho(G^{(k)})&=2^{-|V|}k^{|E|(k-3)+|V|}\lim_{\ell\to\infty}\frac{\sum_{\pi\in\Pi}\mathrm{S}_d(G_\pi)}{\rho(G)^{2\ell}}\\&=2^{-|V|}k^{|E|(k-3)+|V|}\lim_{\ell\to\infty}\frac{\sum_{\pi\in\Pi}\sum_{\lambda_\pi\in\sigma(G_\pi)}\lambda_\pi^{2\ell}}{\rho(G)^{2\ell}},\end{aligned}\tag{7.2}$$

where $\sigma(G_\pi)$ is the spectrum of G_π.

Let $\mathcal{D}$ be the set of all $|V|\times|V|$ diagonal matrices with diagonal entries ±1. Let Π_+ and Π_- denote the sets of sign functions $\pi\in\Pi$ such that $A(G_\pi)=D^{-1}A(G)D$ and $\Pi_+=D^{-1}\left(-A(G)\right)D$ for some $D\in\mathcal{D}$, respectively. By Lemma 2.6, Π_+ contains the sign functions for which G_π has an eigenvalue $\rho(G)$, and Π_- contains the sign functions for which G_π has an eigenvalue $-\rho(G)$.

We observe that since G is connected, $D^{-1}A(G)D=A(G)$ implies that $D=I$ or $D=-I$. This implies that there is a two-one correspondence between $\mathcal{D}$ and Π_+ and similarly between $\mathcal{D}$ and Π_-. Thus,

$$|\Pi_+|=|\Pi_-|=\frac{|\mathcal{D}|}{2}=2^{|V|-1}.$$

From Lemma 2.6, we now have that

$$\lim_{\ell\to\infty}\frac{\sum_{\pi\in\Pi}\sum_{\lambda_\pi\in\sigma(G_\pi)}\lambda_\pi^{2\ell}}{\rho(G)^{2\ell}}=\lim_{\ell\to\infty}\frac{\sum_{\pi\in\Pi_+}\rho(G_\pi)^{2\ell}+\sum_{\pi\in\Pi_-}(-\rho(G_\pi))^{2\ell}}{\rho(G)^{2\ell}}$$

$$= |\Pi_+| + |\Pi_-| = 2^{|V|}$$

(note that it is not relevant whether G is bipartite, in which case $\Pi_+ = \Pi_-$, or not). By (7.2), we thus have $n_\rho(G^{(k)}) = k^{|E|(k-3)+|V|}$, and hence the multiplicity of the spectral radius of the power hypergraph $G^{(k)}$ is $k^{|E|(k-3)+|V|-1}$. □

As a corollary of the proof, we can also extend the above result to the case $k = 2$. Indeed, as Π_+ contains all sign functions π such that G_π has an eigenvalue $\rho(G)$, each with multiplicity 1, and the multiplicity of the spectral radius $\rho(G)$ as a root of β is the average multiplicity over all sign functions, it follows that this multiplicity equals $2^{-|E|}|\Pi_+|$.

Proposition 7.3. *For a connected graph G, the multiplicity of the spectral radius $\rho(G)$ as a root of β is $2^{-|E|+|V|-1}$.*

Consequently, we confirm our earlier observation that for a connected graph, β is a polynomial only for a tree.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgment

This work is supported by the National Natural Science Foundation of China (No. 12071097, 12371344), the Natural Science Foundation for The Excellent Youth Scholars of the Heilongjiang Province (No. YQ2022A002), the Fundamental Research Funds for the Central Universities and CSC.

References

[1] Richard Brualdi, Hans Schneider, Determinantal identities: Gauss, Schur, Cauchy, Sylvester, Kronecker, Jacobi, Binet, Laplace, Muir, and Cayley, Linear Algebra Appl. 52 (1983) 769–791.
[2] Kungching Chang, Kelly Pearson, Tan Zhang, Perron-Frobenius theorem for nonnegative tensors, Commun. Math. Sci. 6 (2) (2008) 507–520.
[3] Kungching Chang, Kelly Pearson, Tan Zhang, On eigenvalue problems of real symmetric tensors, J. Math. Anal. Appl. 350 (2009) 416–422.
[4] Lixiang Chen, Changjiang Bu, Jiang Zhou, Spectral moments of hypertrees and their applications, Linear Multilinear Algebra 70 (2022) 6297–6311.

[5] Lixiang Chen, Edwin R. van Dam, Changjiang Bu, All eigenvalues of the power hypergraph and signed subgraphs of a graph, Linear Algebra Appl. 676 (2023) 205–210.
[6] Gregory Clark, Joshua Cooper, A Harary-Sachs theorem for hypergraphs, J. Comb. Theory, Ser. B 149 (2021) 1–15.
[7] Joshua Cooper, Aaron Dutle, Spectra of uniform hypergraphs, Linear Algebra Appl. 436 (9) (2012) 3268–3292.
[8] Douglas Crabtree, Applications of M-matrices to non-negative matrices, Duke Math. J. 33 (1) (1966) 197–208.
[9] Dragoš Cvetković, Horst Sachs, Michael Doob, Spectra of Graphs: Theory and Applications, VEB Deutscher Verlag der Wissenschaften, 1980.
[10] Cunxiang Duan, Edwin R. van Dam, Ligong Wang, The characteristic polynomials of uniform double hyperstars and uniform hypertriangles, Linear Algebra Appl. 678 (2023) 16–32.
[11] Art Duval, Caroline Klivans, Jeremy Martin, Simplicial matrix-tree theorems, Trans. Am. Math. Soc. 361 (11) (2009) 6073–6114.
[12] J.A. Eidswick, A proof of Newton's power sum formulas, Am. Math. Mon. 75 (4) (1968) 396–397.
[13] Ky Fan, Note on M-matrices, Chin. Q. J. Math. 11 (1) (1960) 43–49.
[14] Yizheng Fan, Largest eigenvalue of a unicyclic mixed graph, Appl. Math. J. Chin. Univ. Ser. B 19 (2) (2004) 140–148.
[15] Yizheng Fan, Tao Huang, Yanhong Bao, Chenluzhuan Sun, Yaping Li, The spectral symmetry of weakly irreducible nonnegative tensors and connected hypergraphs, Trans. Am. Math. Soc. 372 (3) (2019) 2213–2233.
[16] Yizheng Fan, Mengyu Tian, Min Li, The stabilizing index and cyclic index of the coalescence and Cartesian product of uniform hypergraphs, J. Comb. Theory, Ser. A 185 (2022) 105537.
[17] Matthew Farrell, Lionel Levine, Multi-Eulerian tours of directed graphs, Electron. J. Comb. 23 (2) (2016) #P2.21.
[18] Christopher Godsil, Ivan Gutman, On the matching polynomial of a graph, in: L. Lovász, V.T. Sós (Eds.), Algebraic Methods in Graph Theory, in: Colloquia Mathematica Societatis János Bolyai, vol. I, Papers from the Conference Held in Szged, August 24-31, 1978, North-Holland, New York, August 1978, pp. 241–249.
[19] Alexander Griffing, Benjamin Lynch, Eric Stone, Structural properties of the minimum cut of partially-supplied graphs, Discrete Appl. Math. 177 (2014) 152–157.
[20] Shenglong Hu, Zhenghai Huang, Chen Ling, Liqun Qi, On determinants and eigenvalue theory of tensors, J. Symb. Comput. 50 (2013) 508–531.
[21] Shenglong Hu, Liqun Qi, The eigenvectors associated with the zero eigenvalues of the Laplacian and signless Laplacian tensors of a uniform hypergraph, Discrete Appl. Math. 169 (2014) 140–151.
[22] Shenglong Hu, Liqun Qi, Jiayu Shao, Cored hypergraphs, power hypergraphs and their Laplacian H-eigenvalues, Linear Algebra Appl. 439 (2013) 2980–2998.
[23] Dan Kalman, A matrix proof of Newton's identities, Math. Mag. 73 (4) (2000) 313–315.
[24] Lek-Heng Lim, Singular values and eigenvalues of tensors: a variational approach, in: 1st IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing, IEEE, 2005, pp. 129–132.
[25] Adam Marcus, Daniel Spielman, Nikhil Srivastava, Interlacing families I: bipartite Ramanujan graphs of all degrees, Ann. Math. 182 (1) (2015) 307–325.
[26] Liqun Qi, Eigenvalues of a real supersymmetric tensor, J. Symb. Comput. 40 (6) (2005) 1302–1324.
[27] Jiayu Shao, Liqun Qi, Shenglong Hu, Some new trace formulas of tensors with applications in spectral hypergraph theory, Linear Multilinear Algebra 63 (5) (2015) 971–992.
[28] Zoran Stanić, Bounding the largest eigenvalue of signed graphs, Linear Algebra Appl. 573 (2019) 80–89.
[29] Jiang Zhou, Changjiang Bu, The enumeration of spanning tree of weighted graphs, J. Algebraic Comb. 54 (1) (2021) 75–108.
[30] Jiang Zhou, Lizhu Sun, Wenzhe Wang, Changjiang Bu, Some spectral properties of uniform hypergraph, Electron. J. Comb. 21 (2014) 4–24.