THE METHOD OF CONSTANT TERMS AND k-COLORED GENERALIZED FROBENIUS PARTITIONS

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Abstract. In his 1984 AMS memoir, Andrews introduced the family of k-colored generalized Frobenius partition functions. For any positive integer k, let $c\phi_k(n)$ denote the number of k-colored generalized Frobenius partitions of n. Among many other things, Andrews proved that for any $n \geq 0$, $c\phi_2(5n+3) \equiv 0 \pmod 5$. Since then, many scholars subsequently considered congruence properties of various k-colored generalized Frobenius partition functions, typically with a small number of colors.

In 2019, Chan, Wang and Yang systematically studied arithmetic properties of $\mathrm{C}\Phi_k(q)$ with $2 \leq k \leq 17$ by employing the theory of modular forms, where $\mathrm{C}\Phi_k(q)$ denotes the generating function of $c\phi_k(n)$. We notice that many coefficients in the expressions of $\mathrm{C}\Phi_k(q)$ are not integers. In this paper, we first observe that $\mathrm{C}\Phi_k(q)$ is related to the constant term of a family of bivariable functions, then establish a general symmetric and recurrence relation on the coefficients of these bivariable functions. Based on this relation, we next derive many bivariable identities. By extracting and computing the constant terms of these bivariable identities, we establish the expressions of $\mathrm{C}\Phi_k(q)$ with integral coefficients. As an immediate consequence, we prove some infinite families of congruences satisfied by $c\phi_k(n)$, where k is allowed to grow arbitrary large.

1. Introduction

Throughout this paper, we always assume that q is a complex number such that $|q| < 1$ and adopt the following customary notation:

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \tag{1.1}$$

$$(a_1,a_2,\ldots,a_m;q)_\infty := (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty.$$

In his 1984 AMS Memoir, Andrews [2] introduced the notion of a generalized Frobenius partition of n, which is a two-rowed array of nonnegative integers of the form:

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

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wherein each row, which is of the same length, is arranged in weakly decreasing order with $n = r + \sum_{i=1}^{r}(a_i + b_i)$. Furthermore, Andrews studied a variant of generalized Frobenius partitions whose parts are taken from k copies of nonnegative integers, which is called k-colored generalized Frobenius partitions. For any $k \geq 1$, let $c\phi_k(n)$ denote the number of k-colored generalized Frobenius partitions of n. The popular way to study a partition function such as $c\phi_k(n)$ is to investigate its generating function

$$\mathrm{C}\Phi_k(q) = \sum_{n=0}^{\infty} c\phi_k(n) q^n.$$

In [2, Theorem 5.2], Andrews proved that

$$\mathrm{C}\Phi_k(q) = \frac{1}{(q;q)_\infty^k} \sum_{m_1,m_2,\ldots,m_{k-1}=-\infty}^{\infty} q^{Q(m_1,m_2,\cdots,m_{k-1})}, \tag{1.2}$$

where

$$Q(m_1, m_2, \ldots, m_{k-1}) = \sum_{i=1}^{k-1} m_i^2 + \sum_{1\leq i<j\leq k-1} m_i m_j. \tag{1.3}$$

Based on (1.2), Andrews [2, pp. 13, 26] established different expressions of $\mathrm{C}\Phi_k(q)$ with $k \in \{2, 3, 5\}$. To state Andrews' results, we introduce the following two functions given by

$$\Theta_2(q) = \sum_{j=-\infty}^{\infty} q^{(j+1/2)^2} = 2q^{1/4} \frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty},$$

$$\Theta_3(q) = \sum_{j=-\infty}^{\infty} q^{j^2} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2 (q^4;q^4)_\infty^2}.$$

Andrews proved that

$$\mathrm{C}\Phi_2(q) = \frac{\Theta_3(q)}{(q;q)_\infty^2} = \frac{(q^2;q^4)_\infty}{(q;q^2)_\infty^4 (q^4;q^4)_\infty}, \tag{1.4}$$

$$\mathrm{C}\Phi_3(q) = \frac{1}{(q;q)_\infty^3} \left(\Theta_3(q)\Theta_3(q^3) + \Theta_2(q)\Theta_2(q^3)\right) \tag{1.5}$$

$$= \frac{1}{(q;q)_\infty^3} \left(1 + 6 \sum_{j=0}^{\infty} \left(\frac{j}{3}\right) \frac{q^j}{1-q^j}\right) \tag{1.6}$$

and

$$\mathrm{C}\Phi_5(q) = \frac{1}{(q;q)_\infty^5} \left(1 + 25 \sum_{j=1}^{\infty} \left(\frac{j}{5}\right) \frac{q^j}{(1-q^j)^2} - 5 \sum_{j=1}^{\infty} \left(\frac{j}{5}\right) \frac{jq^j}{1-q^j}\right), \tag{1.7}$$

where $\left(\frac{\cdot}{p}\right)$ is the Kronecker symbol. An equivalent form of (1.7) can be found in the work of Kolitsch [26, Lemma 1]. Andrews [2, pp. 13–15] obtained (1.4) and (1.5) by applying

Jacobi's triple product identity (see [2, Eq. (3.1)]) and some properties of theta series. The main ingredient in the proofs of (1.6) and (1.7) is the result of Kloosterman [24, pp. 358, 362]. Andrews [2, p. 26] also remarked that there exists a similar identity for the case $k = 7$, but this identity was not presented in [2]. This missing identity is

$$\mathrm{C}\Phi_7(q) = \frac{1}{(q;q)_\infty^7}\left(1 + \frac{343}{8}\sum_{j=1}^{\infty}\left(\frac{j}{7}\right)\frac{q^j + q^{2j}}{(1-q^j)^3} - \frac{7}{8}\sum_{j=1}^{\infty}\left(\frac{j}{7}\right)\frac{j^2q^j}{1-q^j}\right), \tag{1.8}$$

which was later derived by Kolitsch [26, Lemma 2].

As an immediate consequence of (1.4), Andrews [2, Corollary 10.1] proved that for any $n \geq 0$,

$$c\phi_2(5n+3) \equiv 0 \pmod 5. \tag{1.9}$$

In 1994, Sellers [38] conjectured that (1.9) is the first special case of an infinite family of congruences modulo any powers of 5 enjoyed by $c\phi_2(n)$, namely,

$$c\phi_2\big(5^\alpha n + \delta_\alpha\big) \equiv 0 \pmod{5^\alpha}, \tag{1.10}$$

where δ_α is the least positive integer such that $12\delta_\alpha \equiv 1 \pmod{5^\alpha}$. Later, Eichhorn and Sellers [13] proved that (1.10) is true for $1 \leq \alpha \leq 4$. Paule and Radu [36] confirmed the Sellers conjecture by utilizing the theory of modular forms.

Since then, many scholars successively established other expressions of $\mathrm{C}\Phi_k(q)$ and derived a number of congruences for $c\phi_k(n)$ with different moduli. More specifically, Baruah and Sarmah [3,4] utilized the method described in Cao's work [6] to establish integral expressions of $\mathrm{C}\Phi_k(q)$ for $k \in \{4,5,6\}$, namely,

$$\mathrm{C}\Phi_4(q) = \frac{1}{(q;q)_\infty^4}\big(\Theta_3^3(q^2) + 3\Theta_3(q^2)\Theta_2^2(q^2)\big), \tag{1.11}$$

$$\begin{aligned}\mathrm{C}\Phi_5(q) = \frac{1}{(q;q)_\infty^5}\bigg(&\Theta_3(q^{10})\Theta_3^3(q^2) + 3\Theta_3(q^{10})\Theta_3(q^2)\Theta_2^2(q^2)\\ &+ \frac{1}{2}\Theta_2(q^{5/2})\Theta_2^3(q^{1/2}) + 3\Theta_2(q^{10})\Theta_2(q^2)\Theta_3^2(q) + \Theta_2(q^{10})\Theta_2^3(q^2)\bigg),\end{aligned} \tag{1.12}$$

$$\begin{aligned}\mathrm{C}\Phi_6(q) = \frac{1}{(q;q)_\infty^6}\bigg(&\Theta_3^3(q)\Theta_3(q^2)\Theta_3(q^6)\\ &+ \frac{3}{4}\Theta_2^3(q^{1/2})\Theta_2(q)\Theta_2(q^{3/2}) + \Theta_3^2(q)\Theta_2(q^2)\Theta_2(q^6)\bigg).\end{aligned} \tag{1.13}$$

According to (1.11) and (1.13), Baruah and Sarmah [3,4] established some congruences modulo powers of 2 for $c\phi_4(n)$ and some congruences modulo small powers of 3 for $c\phi_6(n)$. Congruence properties modulo powers of 5 for $c\phi_3(n)$ and $c\phi_4(n)$ were subsequently studied by Ono [35], Lovejoy [33], Xiong [47], Sellers [39], Xia [46], Hirschhorn and Sellers [21], Chan, Wang and Yang [7], and Wang and Zhang [42]. Congruence properties modulo 7 for $c\phi_4(n)$ were considered by Lin [32], and Zhang and Wang [49]. Congruence properties of $c\phi_6(n)$ modulo powers of 3 were successively investigated by Xia [45], Hirschhorn [18], Gu, Wang and Xia [16], and the third author [40]. The third author [41] also proved congruence properties modulo 5 for $c\phi_8(n)$ and $c\phi_9(n)$. There are other studies on congruences and arithmetic properties of $c\phi_k(n)$; see, for example, [9, 10, 12, 14, 15, 22, 25, 27–30, 34, 37].

It is worthwhile to mention that Andrews [2, p. 15] commented that as k increases, the expressions of $\mathrm{C}\Phi_k(q)$ quickly become long and messy. In 2019, Chan, Wang and Yang [8] systematically studied the expressions of $\mathrm{C}\Phi_k(q)$ by utilizing the theory of modular forms, and discovered many surprising properties of $\mathrm{C}\Phi_k(q)$, where k is an integer satisfying $2 \leq k \leq 17$. Based on the results of Chan, Wang and Yang, Wang [44] further proved some congruence families modulo powers of 3 enjoyed by $c\phi_3(n)$ and $c\phi_9(n)$, which improve some previous results of Kolitsch [28, 29]. In particular, Chan, Wang and Yang [8, Theorem 5.3] proved the following general internal congruences for $c\phi_k(n)$, namely,

$$c\phi_{p^\alpha N}(n) \equiv c\phi_{p^{\alpha-1}N}(n/p) \pmod{p^{2\alpha}}, \tag{1.14}$$

where $\alpha \geq 1$, $n \geq 0$, p is a prime number and N is a positive integer which is not divisible by p.

The method developed by Chan, Wang and Yang [8] is a powerful technique to derive different expressions of $\mathrm{C}\Phi_k(q)$. We notice that some coefficients in the expressions of $\mathrm{C}\Phi_k(q)$ are not integers. For example, one result derived by Chan, Wang and Yang [8, Theorem 6.7] is that (some typos have been corrected)

$$\begin{aligned}\mathrm{C}\Phi_{12}(q) = \frac{1}{\Theta_3(q)(q;q)_\infty^{12}}\bigg(&-\frac{36207}{160}B_{12,1} + \frac{923091}{4000}B_{12,4} + \frac{35829}{1000}B_{12,5}\\ &+\frac{891}{4}B_{12,6} - \frac{1485}{8}B_{12,7} - \frac{143247}{250}B_{12,8} - \frac{891}{4}B_{12,9} - \frac{8109}{160}B_{12,10}\\ &-\frac{582717}{16000}B_{12,11} + \frac{227691}{200}B_{12,12} + \frac{714249}{8000}B_{12,13} + \frac{8109}{80}B_{12,14}\\ &+\frac{33}{8}B_{12,15} + \frac{294109}{500}B_{12,16} - \frac{16503}{400}B_{12,17} - \frac{99}{8}B_{12,18}\\ &+\frac{10559}{200}B_{12,19} - \frac{128807}{100}B_{12,20} + \frac{25647}{160}B_{12,21} + \frac{727}{160}B_{12,22}\bigg),\end{aligned} \tag{1.15}$$

where the $B_{12,i}$ for $i \in \{1, 4, 5, \ldots, 22\}$ are some functions involving $\Theta_2(q)$ and $\Theta_3(q)$. Similar phenomena also exist in the expressions of $\mathrm{C}\Phi_k(q)$ for $k \in \{10, 14, 15, 17\}$. Moreover, some expressions of $\mathrm{C}\Phi_k(q)$ derived by Chan, Wang and Yang contain the following Eisenstein series given by

$$E_2(\tau) = 1 - 24\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}, \qquad q = e^{2\pi i\tau}, \ \ \mathrm{Im}(\tau) > 0.$$

These places cause heavy obstacle to further study congruence properties of $c\phi_k(n)$. In a recent paper [11], we established another expression of $\mathrm{C}\Phi_{12}(q)$ with integral coefficients.

Further, we proved some congruences modulo small powers of 3 enjoyed by $c\phi_{12}(n)$ and conjectured several congruence families modulo powers of 3 for $c\phi_{12}(n)$.

The objective of this paper is to present a general strategy which can be applied to establish the expressions of $\mathrm{C}\Phi_k(q)$ with integral coefficients. For this purpose, we first observe that $\mathrm{C}\Phi_k(q)$ is related to the constant term of a family of bivariable functions. For $0<|z|<\infty$, we define

$$f_k(z)=f_k(z,q):=\left(\sum_{n=-\infty}^{\infty} z^n q^{n^2}\right)^k=\sum_{n=-\infty}^{\infty} c_{k,n}(q)z^n. \tag{1.16}$$

Then the **c**onstant **t**erm (on the variable z) of $f_k(z)$ is

$$\begin{aligned} c_{k,0}(q) &= \mathrm{CT}_z\left(\left(\sum_{n=-\infty}^{\infty} z^n q^{n^2}\right)^k\right) \\ &= \sum_{\substack{n_1,n_2,\ldots,n_k=-\infty \\ n_1+n_2+\cdots+n_k=0}}^{\infty} q^{n_1^2+n_2^2+\cdots+n_k^2} = \sum_{n_1,n_2,\ldots,n_{k-1}=-\infty}^{\infty} q^{2Q(n_1,n_2,\ldots,n_{k-1})}, \end{aligned} \tag{1.17}$$

where $Q(n_1,n_2,\ldots,n_{k-1})$ is defined as in (1.3). Combining (1.2) and (1.17), we find that

$$c_{k,0}(q)=(q^2;q^2)_\infty^k \mathrm{C}\Phi_k(q^2). \tag{1.18}$$

By (1.18), to derive the integral expression of $\mathrm{C}\Phi_k(q)$, we only need to compute the constant term in $f_k(z)$. We shall establish a general symmetric and recurrence relation on the coefficients in $f_k(z)$. With the help of this relation, we next derive many bivariable identities for $f_k(z)$. Based on these bivariable identities, we prove three infinite families of congruences satisfied by $c\phi_k(n)$, where k is allowed to grow arbitrary large, namely,

$$c\phi_{9N+3}(3n+2)\equiv 0 \pmod{27}, \tag{1.19}$$

$$c\phi_{9N+6}(3n+2)\equiv 0 \pmod{27}, \tag{1.20}$$

$$c\phi_{4N+4}(4n+3)\equiv 0 \pmod{32}. \tag{1.21}$$

The rest of this paper is organized as follows. In Section 2, we first establish a general symmetric and recurrence relation on the coefficients of a family of bivariable functions. Based on this relation, we next derive many bivariable identities which can be applied to deduce the expressions of $\mathrm{C}\Phi_k(q)$ with integral coefficients. In Section 3, we prove many integral expressions of $\mathrm{C}\Phi_k(q)$. The proofs of (1.19)–(1.21) are presented in Section 4. In the last section, we give several remarks and conjecture that (1.21) also holds for the modulus 256.

2. A general symmetric and recurrence relation and some bivariable identities

In this section, we derive a general symmetric and recurrence relation on the coefficients of a family of bivariable functions and some bivariable identities, which can be utilized to establish the expressions of $\mathrm{C}\Phi_k(q)$ with integral coefficients.

For notational convenience, we denote

$$J_{a,b} := (q^a, q^{b-a}, q^b; q^b)_\infty, \quad \overline{J}_{a,b} := (-q^a, -q^{b-a}, q^b; q^b)_\infty, \quad J_a := J_{a,3a} = (q^a; q^a)_\infty.$$

Ramanujan's theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} = (-a, -b, ab; ab)_\infty, \quad |ab| < 1. \tag{2.1}$$

The last identity in (2.1) is the well-known Jacobi triple product identity [5, p. 35, Entry 19]. Two important special cases of $f(a,b)$ are, respectively, given by

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{J_2^5}{J_1^2 J_4^2}, \tag{2.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=-\infty}^{\infty} q^{2n^2-n} = \frac{1}{2}\sum_{n=-\infty}^{\infty} q^{n(n+1)/2} = \frac{J_2^2}{J_1}. \tag{2.3}$$

Now we introduce the following auxiliary functions involving $\varphi(q)$ and $\psi(q)$, defined by

$$A(q) = \varphi^2(q)\varphi(q^2) + 8q\psi^2(q^2)\psi(q^4) \qquad \text{and} \qquad B(q) = \varphi^2(q) + 2\varphi^2(q^2). \tag{2.4}$$

The following lemma provides another expression of $f_k(z)$ and a general symmetric and recurrence relation on the coefficients of $f_k(z)$. For the sake of convenience, we denote

$$L_{a,b,c,d} = L_{a,b,c,d}(z,q) := \sum_{n=-\infty}^{\infty} z^{an+b} q^{cn^2+dn}.$$

Lemma 2.1. *Let $f_k(z)$ and $c_{k,i}(q)$ be defined as in (1.16). Then for any $k \geq 1$,*

$$f_k(z) = \sum_{i=-\lfloor (k-1)/2 \rfloor}^{\lfloor k/2 \rfloor} c_{k,i}(q) L_{k,i,k,2i}. \tag{2.5}$$

In particular, $c_{k,i}(q) = c_{k,-i}(q)$.

Proof. According to the definition of $f_k(z)$, we find that

$$f_k(1/z) = f_k(z) \qquad \text{and} \qquad f_k(zq^2) = (zq)^{-k} f_k(z),$$

from which we deduce that

$$c_{k,n}(q) = c_{k,-n}(q) \qquad \text{and} \qquad c_{k,n}(q) = q^{2n-k} c_{k,n-k}(q).$$

Therefore, by iteration, we conclude that for $-\lfloor (k-1)/2 \rfloor \le i \le \lfloor k/2 \rfloor$,

$$c_{k,kn+i}(q) = q^{kn^2+2ni} c_{k,i}(q).$$

The identity (2.5) thus follows. □

In the sequel, if we assume that $f_k(z)$ have the following expression:

$$f_k(z) = \sum_{i=-\lfloor (k-1)/2 \rfloor}^{\lfloor k/2 \rfloor} \widehat{c}_{k,i}(q) L_{k,i,k,2i},$$

where $\widehat{c}_{k,i}(q)$ are some functions of q. Then we mean that

$$c_{k,i}(q) = \widehat{c}_{k,i}(q), \qquad \text{for } -\lfloor (k-1)/2 \rfloor \le i \le \lfloor k/2 \rfloor.$$

With the help of Lemma 2.1, we deduce the following necessary lemmas, some of which are main ingredients in the proofs of (1.19)–(1.21).

Lemma 2.2. *We have*

$$f_2(z) = \varphi(q^2) L_{2,0,2,0} + 2q\psi(q^4) L_{2,1,2,2}, \tag{2.6}$$

$$f_3(z) = a(q^2) L_{3,0,3,0} + 3q \frac{J_6^3}{J_2} \big(L_{3,1,3,2} + L_{3,-1,3,-2} \big), \tag{2.7}$$

where $a(q)$ is one of the Borwein cubic functions, given by (see [20])

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}. \tag{2.8}$$

The identity (2.7) was proved earlier by Hirschhorn, Garvan and Borwein [20] and Hirschhorn [17]; see also [19, Chap. 21]. It should be admitted that the proof of (2.7) due to Hirschhorn is very elementary and is simpler than ours to some extent. However, in order to the completeness and self-contained content of the discussion, we present another proof of (2.7) here. More importantly, in the process of proving (2.7), we need some variable substitutions to simplify certain double sums, and we will frequently utilize this technique in what follows.

Proof. By Lemma 2.1, we need to derive $c_{2,0}(q)$, $c_{2,1}(q)$, $c_{3,0}(q)$ and $c_{3,1}(q)$. According to the definition of $f_k(z)$, we arrive at

$$c_{k,i}(q) = \sum_{\substack{n_1,n_2,\dots,n_k=-\infty \\ n_1+n_2+\cdots+n_k=i}}^{\infty} q^{n_1^2+n_2^2+\cdots+n_k^2}. \tag{2.9}$$

It follows from (2.2) and (2.3) that

$$c_{2,0}(q) = \sum_{\substack{n_1,n_2=-\infty \\ n_1+n_2=0}}^{\infty} q^{n_1^2+n_2^2} = \sum_{n=-\infty}^{\infty} q^{2n^2} = \varphi(q^2), \tag{2.10}$$

$$c_{2,1}(q)=\sum_{\substack{n_1,n_2=-\infty\\ n_1+n_2=1}}^{\infty} q^{n_1^2+n_2^2}=\sum_{n=-\infty}^{\infty} q^{2n^2+2n+1}=2q\psi(q^4). \tag{2.11}$$

The identity (2.6) follows by Lemma 2.1, (2.10) and (2.11).

Next, we find that

$$c_{3,0}(q)=\sum_{\substack{n_1,n_2,n_3=-\infty\\ n_1+n_2+n_3=0}}^{\infty} q^{n_1^2+n_2^2+n_3^2}=\sum_{n_1,n_2=-\infty}^{\infty} q^{n_1^2+n_2^2+(n_1+n_2)^2}=a(q^2).$$

We next turn to prove the following identity, namely,

$$c_{3,1}(q)=3q\frac{J_6^3}{J_2}. \tag{2.12}$$

It follows immediately from (2.9) that

$$c_{3,1}(q)=\sum_{\substack{n_1,n_2,n_3=-\infty\\ n_1+n_2+n_3=1}}^{\infty} q^{n_1^2+n_2^2+n_3^2}=\sum_{n_1,n_2=-\infty}^{\infty} q^{2n_1^2+2n_2^2+2n_1n_2-2n_1-2n_2+1}.$$

If $n_1+n_2\equiv 0 \pmod 2$, we take $n_1=r-s$ and $n_2=r+s$; if $n_1+n_2\equiv 1 \pmod 2$, we put $n_1=r-s$ and $n_2=r+s+1$, where r and s are two integers. After simplification, we deduce that

$$\begin{aligned} c_{3,1}(q)&=\sum_{r,s=-\infty}^{\infty} q^{6r^2-4r+2s^2+1}+\sum_{r,s=-\infty}^{\infty} q^{6r^2+2r+2s^2+2s+1}\\ &=q\overline{J}_{2,12}\varphi(q^2)+2q\overline{J}_{4,12}\psi(q^4)=q\frac{J_4^7J_6J_{24}}{J_2^3J_8^3J_{12}}+2q\frac{J_8^3J_{12}^2}{J_4^2J_{24}}. \end{aligned} \tag{2.13}$$

Therefore, in order to prove (2.12), we need to prove that

$$q\frac{J_4^7J_6J_{24}}{J_2^3J_8^3J_{12}}+2q\frac{J_8^3J_{12}^2}{J_4^2J_{24}}=3q\frac{J_6^3}{J_2}, \tag{2.14}$$

or, equivalently,

$$\frac{J_2^7J_3J_{12}}{J_1^3J_4^3J_6}+2\frac{J_4^3J_6^2}{J_2^2J_{12}}-3\frac{J_3^3}{J_1}=0. \tag{2.15}$$

Now we recall Hirschhorn's version of the parameterized identities of theta functions (see [19, Chap. 35, Eqs. (35.1.1)–(35.1.6)]), and the idea comes from [1].

$$J_1=s^{1/2}t^{1/24}(1-2qt)^{1/2}(1+qt)^{1/8}(1+2qt)^{1/6}(1+4qt)^{1/8}, \tag{2.16}$$

$$J_2=s^{1/2}t^{1/12}(1-2qt)^{1/4}(1+qt)^{1/4}(1+2qt)^{1/12}(1+4qt)^{1/4}, \tag{2.17}$$

$$J_3=s^{1/2}t^{1/8}(1-2qt)^{1/6}(1+qt)^{1/24}(1+2qt)^{1/2}(1+4qt)^{1/24}, \tag{2.18}$$

$$J_4=s^{1/2}t^{1/6}(1-2qt)^{1/8}(1+qt)^{1/2}(1+2qt)^{1/24}(1+4qt)^{1/8}, \tag{2.19}$$

$$J_6=s^{1/2}t^{1/4}(1-2qt)^{1/12}(1+qt)^{1/12}(1+2qt)^{1/4}(1+4qt)^{1/12}, \tag{2.20}$$

$$J_{12}=s^{1/2}t^{1/2}(1-2qt)^{1/24}(1+qt)^{1/6}(1+2qt)^{1/8}(1+4qt)^{1/24}, \tag{2.21}$$

where

$$s:=s(q)=\frac{J_1^2J_4^2J_6^{15}}{J_2^5J_3^6J_{12}^6} \qquad \text{and} \qquad t:=t(q)=\frac{J_2^3J_3^3J_{12}^6}{J_1J_4^2J_6^9}.$$

Substituting (2.16)–(2.21) into the left-hand side of (2.15), after simplification, we obtain (2.15). According to Lemma 2.1, we obtain (2.7).

This completes the proof of Lemma 2.2. □

Lemma 2.3. *Let $A(q)$ and $B(q)$ be defined as in* (2.4). *Then*

$$f_4(z)=A(q^2)L_{4,0,4,0}+4q\psi^3(q^2)\big(L_{4,1,4,2}+L_{4,-1,4,-2}\big)+2q^2B(q^2)\psi(q^8)L_{4,2,4,4} \tag{2.22}$$

and

$$\begin{aligned}f_4(z)=&\left(a(q^2)\varphi(q^{12})+6q^2\frac{J_6^3J_8^2J_{12}J_{48}}{J_2J_4J_{16}J_{24}}\right)L_{4,0,4,0}\\&+\left(qa(q^2)\psi(q^6)+3q\frac{J_4J_6^5}{J_2^2J_{12}}\right)\big(L_{4,1,4,2}+L_{4,-1,4,-2}\big)\\&+\left(2q^4a(q^2)\psi(q^{24})+6q^2\frac{J_6^3J_{16}J_{24}^2}{J_2J_8J_{48}}\right)L_{4,2,4,4}.\end{aligned} \tag{2.23}$$

Proof. According to the definition of $f_k(z)$ and using (2.6), we deduce that

$$f_4(z)=\big(f_2(z)\big)^2=\big(\varphi(q^2)L_{2,0,2,0}+2q\psi(q^4)L_{2,1,2,2}\big)^2. \tag{2.24}$$

Therefore, combining (1.16) and (2.24) gives that

$$\begin{aligned}c_{4,0}(q)&=\mathrm{CT}_z\{\varphi^2(q^2)L_{2,0,2,0}^2\}+\mathrm{CT}_z\{4q^2\psi^2(q^4)L_{2,1,2,2}^2\}\\&=\varphi^2(q^2)\sum_{\substack{n_1,n_2=-\infty\\n_1+n_2=0}}^{\infty}q^{2n_1^2+2n_2^2}+4q^2\psi^2(q^4)\sum_{\substack{n_1,n_2=-\infty\\n_1+n_2+1=0}}^{\infty}q^{2n_1^2+2n_1+2n_2^2+2n_2}\\&=\varphi^2(q^2)\varphi(q^4)+8q^2\psi^2(q^4)\psi(q^8)=A(q^2).\end{aligned} \tag{2.25}$$

Next we turn to derive the expression of $c_{4,1}(q)$. It follows from (1.16) and (2.24) that

$$\begin{aligned}c_{4,1}(q)&=4q\varphi(q^2)\psi(q^4)\sum_{\substack{n_1,n_2=-\infty\\n_1+n_2=0}}^{\infty}q^{2n_1^2+2n_2^2+2n_2}\\&=4q\psi^2(q^2)\sum_{n=-\infty}^{\infty}q^{4n^2-2n}=4q\psi^3(q^2).\end{aligned} \tag{2.26}$$

Following a similar strategy of deriving (2.25), we can prove that

$$c_{4,2}(q)=2q^2\psi(q^8)\big(\varphi^2(q^2)+2\varphi^2(q^4)\big)=2q^2B(q^2)\psi(q^8). \tag{2.27}$$

Based on Lemma 2.1, we obtain (2.22) by (2.25)–(2.27).

To prove (2.23), one first readily finds that

$$f_1(z) = L_{3,0,9,0} + q\big(L_{3,1,9,6} + L_{3,-1,9,-6}\big). \tag{2.28}$$

According to the definition of $f_k(z)$, and utilizing (2.7) and (2.28), we find that

$$\begin{aligned} f_4(z) = f_3(z)\cdot f_1(z) = &\left\{a(q^2)(q)L_{3,0,3,0} + 3q\frac{J_6^3}{J_2}\big(L_{3,1,3,2}+L_{3,-1,3,-2}\big)\right\} \\ &\times\left\{L_{3,0,9,0} + q\big(L_{3,1,9,6}+L_{3,-1,9,-6}\big)\right\}. \end{aligned} \tag{2.29}$$

We only derive the expressions of $c_{4,0}(q)$ and $c_{4,1}(q)$ in (2.29). The coefficient $c_{4,2}(q)$ can be demonstrated similarly, and thus, we omit the details.

Combining (1.16) and (2.29) yields that

$$\begin{aligned} c_{4,0}(q) &= a(q^2)\sum_{\substack{n_1,n_2=-\infty\\ n_1+n_2=0}}^{\infty} q^{3n_1^2+9n_2^2} + 3q\frac{J_6^3}{J_2}\sum_{\substack{n_1,n_2=-\infty\\ n_1+n_2=0}}^{\infty} q^{3n_1^2+2n_1+9n_2^2-6n_2+1} \\ &\quad + 3q\frac{J_6^3}{J_2}\sum_{\substack{n_1,n_2=-\infty\\ n_1+n_2=0}}^{\infty} q^{3n_1^2-2n_1+9n_2^2+6n_2+1} \\ &= a(q^2)\varphi(q^{12}) + 6q^2\frac{J_6^3}{J_2}\overline{J}_{4,24} = a(q^2)\varphi(q^{12}) + 6q^2\frac{J_6^3J_8^2J_{12}J_{48}}{J_2J_4J_{16}J_{24}}. \end{aligned} \tag{2.30}$$

Now we prove $c_{4,1}(q)$. Notice that

$$\begin{aligned} c_{4,1}(q) &= a(q^2)\sum_{\substack{n_1,n_2=-\infty\\ n_1+n_2=0}}^{\infty} q^{3n_1^2+9n_2^2+6n_2+1} + 3q\frac{J_6^3}{J_2}\sum_{\substack{n_1,n_2=-\infty\\ n_1+n_2=0}}^{\infty} q^{3n_1^2+2n_1+9n_2^2} \\ &\quad + 3q\frac{J_6^3}{J_2}\sum_{\substack{n_1,n_2=-\infty\\ n_1+n_2=1}}^{\infty} q^{3n_1^2-2n_1+9n_2^2-6n_2+1} \\ &= qa(q^2)\psi(q^6) + 3q\frac{J_6^3}{J_2}\big(\overline{J}_{10,24} + q^2\overline{J}_{2,24}\big). \end{aligned}$$

We also require the following identity, which follows from two identities in Berndt's book [5, Entry 30 (ii) and (iii)]:

$$f(a,b) = f\big(a^3b, ab^3\big) + af\big(b/a, a^5b^3\big). \tag{2.31}$$

Taking $(a,b) = (q^2,q^4)$ in (2.31) yields that

$$\overline{J}_{2,6} = \overline{J}_{10,24} + q^2\overline{J}_{2,24}, \tag{2.32}$$

from which we obtain that

$$c_{4,1}(q) = qa(q^2)\psi(q^6) + 3q\frac{J_6^3}{J_2}\overline{J}_{2,6} = qa(q^2)\psi(q^6) + 3q\frac{J_4J_6^5}{J_2^2J_{12}}.$$

We therefore complete the proof of Lemma 2.3 by utilizing Lemma 2.1. □

Lemma 2.4. *Let $A(q)$ and $B(q)$ be defined as in (2.4). Then*

$$\begin{aligned} f_5(z) &= \big(A(q^2)\varphi(q^{20}) + 8q^2\psi^3(q^2)\psi(q^{10}) + 4q^6B(q^2)\psi(q^8)\psi(q^{40})\big)L_{5,0,5,0} \\ &\quad + \big(qA(q^2)\overline{J}_{12,40} + 4q\psi^3(q^2)\overline{J}_{4,10} + 2q^3B(q^2)\psi(q^8)\overline{J}_{8,40}\big) \\ &\quad\quad \times \big(L_{5,1,5,2} + L_{5,-1,5,-2}\big) \\ &\quad + \big(q^4A(q^2)\overline{J}_{4,40} + 4q^2\psi^3(q^2)\overline{J}_{2,10} + 2q^2B(q^2)\psi(q^8)\overline{J}_{16,40}\big) \\ &\quad\quad \times \big(L_{5,2,5,4} + L_{5,-2,5,-4}\big). \end{aligned} \tag{2.33}$$

Proof. According to (1.16) and (2.22), we find that

$$\begin{aligned} f_5(z) &= \big\{A(q^2)L_{4,0,4,0} + 4q\psi^3(q^2)\big(L_{4,1,4,2} + L_{4,-1,4,-2}\big) + 2q^2B(q^2)\psi(q^8)L_{4,2,4,4}\big\} \\ &\quad \times \big\{L_{4,0,16,0} + q\big(L_{4,1,16,8} + L_{4,-1,16,-8}\big) + q^4L_{4,2,16,16}\big\}. \end{aligned} \tag{2.34}$$

Similar to the proof of (2.23), we only present the proofs of the expressions of $c_{5,0}(q)$ and $c_{5,1}(q)$ in (2.33).

From (1.16) and (2.34) we deduce that

$$\begin{aligned} c_{5,0}(q) &= A(q^2)\sum_{\substack{n_1,n_2=-\infty \\ n_1+n_2=0}}^{\infty} q^{4n_1^2+16n_2^2} + 4q\psi^3(q^2)\sum_{\substack{n_1,n_2=-\infty \\ n_1+n_2=0}}^{\infty} q^{4n_1^2+2n_1+16n_2^2-8n_2+1} \\ &\quad + 4q\psi^3(q^2)\sum_{\substack{n_1,n_2=-\infty \\ n_1+n_2=0}}^{\infty} q^{4n_1^2-2n_1+16n_2^2+8n_2+1} \\ &\quad + 2q^2B(q^2)\psi(q^8)\sum_{\substack{n_1,n_2=-\infty \\ n_1+n_2+1=0}}^{\infty} q^{4n_1^2+4n_1+16n_2^2+16n_2+4} \\ &= A(q^2)\varphi(q^{20}) + 8q^2\psi^3(q^2)\psi(q^{10}) + 4q^6B(q^2)\psi(q^8)\psi(q^{40}). \end{aligned} \tag{2.35}$$

In the same vein, we deduce that

$$\begin{aligned} c_{5,1}(q) &= A(q^2)\sum_{n=-\infty}^{\infty} q^{20n^2-8n+1} + 4q\psi^3(q^2)\sum_{n=-\infty}^{\infty} q^{20n^2+2n} \\ &\quad + 4q\psi^3(q^2)\sum_{n=-\infty}^{\infty} q^{20n^2-18n+4} + 2q^2B(q^2)\psi(q^8)\sum_{n=-\infty}^{\infty} q^{20n^2+12n+1} \\ &= qA(q^2)\overline{J}_{12,40} + 4q\psi^3(q^2)\big(\overline{J}_{18,40} + q^4\overline{J}_{2,40}\big) + 2q^3B(q^2)\psi(q^8)\overline{J}_{8,40} \\ &= qA(q^2)\overline{J}_{12,40} + 4q\psi^3(q^2)\overline{J}_{4,10} + 2q^3B(q^2)\psi(q^8)\overline{J}_{8,40}, \end{aligned} \tag{2.36}$$

where the last step follows from (2.31). Based on Lemma 2.1, we complete the proof of Lemma 2.4. □

Lemma 2.5. *We have*

$$f_6(z) = \big(a(q^4)\varphi^3(q^2) + 24q^2\psi^3(q^2)\psi(q^4)\psi(q^6)\big)L_{6,0,6,0}$$

$$+\left(24q^3\psi^3(q^4)\frac{J_{12}^3}{J_4}+6q\varphi(q^2)\psi^3(q^2)\frac{J_4J_6^2}{J_2J_{12}}\right)\left(L_{6,1,6,2}+L_{6,-1,6,-2}\right)$$
$$+\left(3q^2\varphi^3(q^2)\frac{J_{12}^3}{J_4}+12q^2\psi^3(q^2)\psi(q^4)\frac{J_4J_6^2}{J_2J_{12}}\right)\left(L_{6,2,6,4}+L_{6,-2,6,-4}\right)$$
$$+\left(8q^3a(q^4)\psi^3(q^4)+12q^3\varphi(q^2)\psi^3(q^2)\psi(q^6)\right)L_{6,3,6,6} \quad (2.37)$$

and

$$f_6(z)=\left(a^2(q^2)\varphi(q^6)+18q^2\frac{J_4^2J_6^7J_{24}}{J_2^3J_8J_{12}}\right)L_{6,0,6,0}$$
$$+\left(18q^3\psi(q^{12})\frac{J_6^6}{J_2^2}+6qa(q^2)\frac{J_6^3J_8J_{12}^2}{J_2J_4J_{24}}\right)\left(L_{6,1,6,2}+L_{6,-1,6,-2}\right)$$
$$+\left(9q^2\varphi(q^6)\frac{J_6^6}{J_2^2}+6q^2a(q^2)\frac{J_4^2J_6^4J_{24}}{J_2^2J_8J_{12}}\right)\left(L_{6,2,6,4}+L_{6,-2,6,-4}\right)$$
$$+\left(2q^3a^2(q^2)\psi(q^{12})+18q^3\frac{J_6^6J_8J_{12}^2}{J_2^2J_4J_{24}}\right)L_{6,3,6,6}. \quad (2.38)$$

Proof. We first prove (2.37). From (2.6) we find that

$$\begin{aligned}f_6(z)&=\left(\varphi(q^2)L_{2,0,2,0}+2q\psi(q^4)L_{2,1,2,2}\right)^3\\&=\varphi^3(q^2)L_{2,0,2,0}^3+6q\varphi^2(q^2)\psi(q^4)L_{2,0,2,0}^2L_{2,1,2,2}\\&\quad+12q^2\varphi(q^2)\psi^2(q^4)L_{2,0,2,0}L_{2,1,2,2}^2+8q^3\psi^3(q^4)L_{2,1,2,2}^3.\end{aligned} \quad (2.39)$$

Based on Lemma 2.1, we need to find the expressions of $c_{6,i}(q)$ for $0\le i\le 3$.

First, combining (1.16) and (2.39) gives that

$$\begin{aligned}c_{6,0}(q)&=\varphi^3(q^2)\sum_{\substack{n_1,n_2,n_3=-\infty\\n_1+n_2+n_3=0}}^{\infty}q^{2n_1^2+2n_2^2+2n_3^2}\\&\quad+12q^2\psi^2(q^2)\psi(q^4)\sum_{\substack{n_1,n_2,n_3=-\infty\\n_1+n_2+n_3+1=0}}^{\infty}q^{2n_1^2+2n_2^2+2n_2+2n_3^2+2n_3}\\&=\varphi^3(q^2)\sum_{n_1,n_2=-\infty}^{\infty}q^{4n_1^2+4n_2^2+4n_1n_2}\\&\quad+12q^2\psi^2(q^2)\psi(q^4)\sum_{n_1,n_2=-\infty}^{\infty}q^{4n_1^2+4n_2^2+4n_1n_2+2n_1+4n_2}.\end{aligned} \quad (2.40)$$

Following a similar strategy of proving (2.13), upon simplification, we obtain that

$$\sum_{n_1,n_2=-\infty}^{\infty}q^{4n_1^2+4n_2^2+4n_1n_2+2n_1+4n_2}$$

$$= \sum_{r,s=-\infty}^{\infty} q^{12r^2+6r+4s^2+2s} + \sum_{r,s=-\infty}^{\infty} q^{12r^2+18r+4s^2+6s+8} = 2\psi(q^2)\psi(q^6). \quad (2.41)$$

Substituting (2.41) into (2.40) yields that

$$c_{6,0}(q) = a(q^4)\varphi^3(q^2) + 24q^2\psi^3(q^2)\psi(q^4)\psi(q^6).$$

Next, we prove $c_{6,1}(q)$. We first derive that

$$\begin{aligned}
c_{6,1}(q) &= 8q^3\psi^3(q^4) \sum_{n_1,n_2=-\infty}^{\infty} q^{4n_1^2+4n_2^2+4n_1n_2+4n_1+4n_2} \\
&\quad + 6q\varphi(q^2)\psi^2(q^2) \sum_{n_1,n_2=-\infty}^{\infty} q^{4n_1^2+4n_2^2+4n_1n_2-2n_1-2n_2} \\
&= 8q^3\psi^3(q^4)\left(\sum_{r,s=-\infty}^{\infty} q^{12r^2+8r+4s^2} + \sum_{r,s=-\infty}^{\infty} q^{12r^2-4r+4s^2+4s} \right) \\
&\quad + 6q\varphi(q^2)\psi^2(q^2)\left(\sum_{r,s=-\infty}^{\infty} q^{12r^2+2r+4s^2-2s} + \sum_{r,s=-\infty}^{\infty} q^{12r^2-10r+4s^2+2s+2} \right) \\
&= 8q^3\psi^3(q^4)\left(\varphi(q^4)\overline{J}_{4,24} + 2\psi(q^8)\overline{J}_{8,24}\right) + 6q\varphi(q^2)\psi^3(q^2)\left(\overline{J}_{10,24} + q^2\overline{J}_{2,24}\right) \\
&= 24q^3\psi^3(q^4)\frac{J_{12}^3}{J_4} + 6q\varphi(q^2)\psi^3(q^2)\overline{J}_{2,6} \\
&= 24q^3\psi^3(q^4)\frac{J_{12}^3}{J_4} + 6q\varphi(q^2)\psi^3(q^2)\frac{J_4J_6^2}{J_2J_{12}}, \quad (2.42)
\end{aligned}$$

where we obtain the second step by setting $n_1 = r + s$, $n_2 = -2r$ or $n_1 = r + s + 1$, $n_2 = -2r - 1$, and the penultimate step follows from (2.14) and (2.32). The expressions of $c_{6,2}(q)$ and $c_{6,3}(q)$ can be proved similarly.

Next, we turn to prove (2.38). At this time, it follows from (2.7) that

$$f_6(z) = \left\{ a(q^2)L_{3,0,3,0} + 3q\frac{J_6^3}{J_2}\left(L_{3,1,3,2} + L_{3,-1,3,-2}\right) \right\}^2. \quad (2.43)$$

And we can derive the expressions of $c_{6,i}(q)$ in (2.43) for $0 \le i \le 3$ by following a similar strategy as deriving (2.22), and thus, we omit the details here. We therefore finish the proof of Lemma 2.5 by using Lemma 2.1. □

Lemma 2.6. *Let $A(q)$ and $B(q)$ be defined as in (2.4). Then*

$$\begin{aligned}
f_8(z) &= \left(A^2(q^2)\varphi(q^8) + 32q^2\psi^6(q^2)\psi(q^4) + 8q^4B^2(q^2)\psi^2(q^8)\psi(q^{16})\right)L_{8,0,8,0} \\
&\quad + \left(8qA(q^2)\psi^3(q^2)\overline{J}_{6,16} + 16q^3B(q^2)\psi^3(q^2)\psi(q^8)\overline{J}_{2,16}\right)\left(L_{8,1,8,2} + L_{8,-1,8,-2}\right) \\
&\quad + \left(4q^2A(q^2)B(q^2)\psi(q^4)\psi(q^8) + 16q^2\psi^6(q^2)\varphi(q^8) + 32q^4\psi^6(q^2)\psi(q^{16})\right) \\
&\qquad \times \left(L_{8,2,8,4} + L_{8,-2,8,-4}\right)
\end{aligned}$$

$$+\big(8q^3A(q^2)\psi^3(q^2)\overline{J}_{2,16}+16q^3B(q^2)\psi^3(q^2)\psi(q^8)\overline{J}_{6,16}\big)$$
$$\times\left(L_{8,3,8,6}+L_{8,-3,8,-6}\right)$$
$$+\big(2q^4A^2(q^2)\psi(q^{16})+4q^4B^2(q^2)\varphi(q^8)\psi^2(q^8)+32q^4\psi^6(q^2)\psi(q^4)\big)L_{8,4,8,8}. \quad (2.44)$$

Proof. According to (1.16) and (2.22), we deduce that

$$f_8(z)=\big\{A(q^2)L_{4,0,4,0}+4q\psi^3(q^2)\big(L_{4,1,4,2}+L_{4,-1,4,-2}\big)+2q^2B(q^2)\psi(q^8)L_{4,2,4,4}\big\}^2. \quad (2.45)$$

Similar to the previous lemmas, we only demonstrate the proof of the expression of $c_{8,0}(q)$ in (2.45). From (1.16) and (2.45), we find that

$$\begin{aligned}c_{8,0}(q)&=A^2(q^2)\sum_{\substack{n_1,n_2=-\infty\\ n_1+n_2=0}}^{\infty}q^{4n_1^2+4n_2^2}+32q^2\psi^6(q^2)\sum_{\substack{n_1,n_2=-\infty\\ n_1+n_2=0}}^{\infty}q^{4n_1^2+2n_1+4n_2^2-2n_2}\\&\quad+4q^4B^2(q^2)\psi^2(q^8)\sum_{\substack{n_1,n_2=-\infty\\ n_1+n_2+1=0}}^{\infty}q^{4n_1^2+4n_1+4n_2^2+4n_2}\\&=A^2(q^2)\varphi(q^8)+32q^2\psi^6(q^2)\psi(q^4)+8q^4B^2(q^2)\psi^2(q^8)\psi(q^{16}).\end{aligned}$$

Based on Lemma 2.1, we complete the proof of Lemma 2.6. □

For the sake of convenience, we write

$$C(q)=\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}. \quad (2.46)$$

Lemma 2.7. *We have*

$$\begin{aligned}f_9(z)&=\big(a^3(q^2)a(q^6)+54q^2a(q^2)C^3(q^2)+162q^4C^3(q^2)C(q^6)\big)L_{9,0,9,0}\\&\quad+\big\{27q^3a(q^2)C^2(q^2)\big(\overline{J}_{4,12}\overline{J}_{8,36}+\overline{J}_{2,12}\overline{J}_{10,36}\big)\\&\qquad+9qa^2(q^2)C(q^2)\big(\varphi(q^6)\overline{J}_{14,36}+2q^4\psi(q^{12})\overline{J}_{4,36}\big)\\&\qquad+81q^3C^3(q^2)\big(\varphi(q^6)\overline{J}_{10,36}+2q^2\psi(q^{12})\overline{J}_{8,36}\big)\big\}\big(L_{9,1,9,2}+L_{9,-1,9,-2}\big)\\&\quad+\big\{27q^2a(q^2)C^2(q^2)\big(\overline{J}_{4,12}\overline{J}_{16,36}+q^4\overline{J}_{2,12}\overline{J}_{2,36}\big)\\&\qquad+9q^2a^2(q^2)C(q^2)\big(\varphi(q^6)\overline{J}_{10,36}+2q^2\psi(q^{12})\overline{J}_{8,36}\big)\\&\qquad+81q^4C^3(q^2)\big(\overline{J}_{2,12}\overline{J}_{10,36}+\overline{J}_{4,12}\overline{J}_{8,36}\big)\big\}\big(L_{9,2,9,4}+L_{9,-2,9,-4}\big)\\&\quad+\big\{3q^3a^3(q^2)C(q^6)+27q^3a(q^6)C^3(q^2)\\&\qquad+54q^3a(q^2)C^3(q^2)+81q^5C^3(q^2)C(q^6)\big\}\big(L_{9,3,9,6}+L_{9,-3,9,-6}\big)\\&\quad+\big\{27q^4a(q^2)C^2(q^2)\big(\overline{J}_{2,12}\overline{J}_{14,36}+q^2\overline{J}_{4,12}\overline{J}_{4,36}\big)\\&\qquad+9q^4a^2(q^2)C(q^2)\big(q^2\varphi(q^6)\overline{J}_{2,36}+2\psi(q^{12})\overline{J}_{16,36}\big)\\&\qquad+81q^4C^3(q^2)\big(\varphi(q^6)\overline{J}_{14,36}+2q^4\psi(q^{12})\overline{J}_{4,36}\big)\big\}\big(L_{9,4,9,8}+L_{9,-4,9,-8}\big).\end{aligned} \quad (2.47)$$

Proof. From (2.7), we deduce that

$$f_9(z) = \left\{ a(q^2)L_{3,0,3,0} + 3q\frac{J_6^3}{J_2}\left(L_{3,1,3,2} + L_{3,-1,3,-2}\right) \right\}^3. \tag{2.48}$$

The proof of the expression of $c_{9,0}(q)$ in (2.48) is a little trickier than the previous cases.

According to (1.16) and (2.48), we deduce that

$$\begin{aligned} c_{9,0}(q) = a^3(q^2) &\sum_{\substack{n_1,n_2,n_3=-\infty \\ n_1+n_2+n_3=0}}^{\infty} q^{3n_1^2+3n_2^2+3n_3^2} \\ &+ 27q^3\frac{J_6^9}{J_2^3} \sum_{\substack{n_1,n_2,n_3=-\infty \\ n_1+n_2+n_3+1=0}}^{\infty} q^{3n_1^2+2n_1+3n_2^2+2n_2+3n_3^2+2n_3} \\ &+ 27q^3\frac{J_6^9}{J_2^3} \sum_{\substack{n_1,n_2,n_3=-\infty \\ n_1+n_2+n_3-1=0}}^{\infty} q^{3n_1^2-2n_1+3n_2^2-2n_2+3n_3^2-2n_3} \\ &+ 54q^2a(q^2)\frac{J_6^6}{J_2^2} \sum_{\substack{n_1,n_2,n_3=-\infty \\ n_1+n_2+n_3=0}}^{\infty} q^{3n_1^2+3n_2^2+2n_2+3n_3^2-2n_3} \\ = a^3(q^2) &\sum_{n_1,n_2=-\infty}^{\infty} q^{6n_1^2+6n_2^2+6n_1n_2} \\ &+ 54q^3\frac{J_6^9}{J_2^3} \sum_{n_1,n_2=-\infty}^{\infty} q^{6n_1^2+6n_2^2+6n_1n_2+6n_1+6n_2+1} \\ &+ 54q^2a(q^2)\frac{J_6^6}{J_2^2} \sum_{n_2,n_3=-\infty}^{\infty} q^{6n_2^2+6n_3^2+6n_2n_3+2n_2-2n_3}. \end{aligned}$$

Adopting a similar strategy of deriving (2.13), after simplification, we obtain that

$$\begin{aligned} c_{9,0}(q) = a^3(q^2)a(q^6) &+ 54q^4\frac{J_6^9}{J_2^3}\left(\varphi(q^6)\overline{J}_{6,36} + 2\psi(q^{12})\overline{J}_{12,36}\right) \\ &+ 54q^2a(q^2)\frac{J_6^6}{J_2^2}\left(\varphi(q^{18})\overline{J}_{2,12} + 2q^4\psi(q^{36})\overline{J}_{4,12}\right). \end{aligned} \tag{2.49}$$

Thanks to (2.15),

$$\varphi(q)\overline{J}_{1,6} + 2\psi(q^2)\overline{J}_{2,6} = 3\frac{J_3^3}{J_1},$$

from which we obtain that

$$54q^4\frac{J_6^9}{J_2^3}\left(\varphi(q^6)\overline{J}_{6,36} + 2\psi(q^{12})\overline{J}_{12,36}\right) = 162q^4\frac{J_6^8J_{18}^3}{J_2^3}. \tag{2.50}$$

Therefore, in order to prove

$$c_{9,0}(q) = a^3(q^2)a(q^6) + 54q^2a(q^2)\frac{J_6^9}{J_2^3} + 162q^4\frac{J_6^8J_{18}^3}{J_2^3}, \tag{2.51}$$

we only need to prove that

$$\varphi(q^9)\overline{J}_{1,6} + 2q^2\psi(q^{18})\overline{J}_{2,6} = \frac{J_3^3}{J_1}, \tag{2.52}$$

or, equivalently,

$$3\big(\varphi(q^9)\overline{J}_{1,6} + 2q^2\psi(q^{18})\overline{J}_{2,6}\big) = \varphi(q)\overline{J}_{1,6} + 2\psi(q^2)\overline{J}_{2,6}. \tag{2.53}$$

We also require the following identities which follow from the two identities in Berndt's book [5, p. 345, Entry 1 (ii) and (iii)]:

$$\big(\psi(q^2) - 3q^2\psi(q^{18})\big)^3 = \psi^3(q^2)\left(1 - 9q^2\frac{\psi^4(q^6)}{\psi^4(q^2)}\right), \tag{2.54}$$

$$\big(\varphi(q) - 3\varphi(q^9)\big)^3 = \varphi^3(q)\left(1 - 9\frac{\varphi^4(q^3)}{\varphi^4(q)}\right). \tag{2.55}$$

Combining (2.54) and (2.55) gives that

$$\left(\frac{\psi(q^2) - 3q^2\psi(q^{18})}{\varphi(q) - \varphi(q^9)}\right)^3 = \frac{\psi^3(q^2) - 9q^2\psi^4(q^6)/\psi(q^2)}{\varphi^3(q) - 9\varphi^4(q^3)/\varphi(q)}.$$

Notice that (2.53) is equivalent to

$$\frac{\psi(q^2) - 3q^2\psi(q^{18})}{\varphi(q) - 3\varphi(q^9)} = -\frac{\overline{J}_{1,6}}{2\overline{J}_{2,6}} = -\frac{1}{2}\frac{J_2^2J_3J_{12}}{J_1J_4J_6}\cdot\frac{J_2J_{12}}{J_4J_6^2} = -\frac{J_2^3J_3J_{12}^2}{2J_1J_4^2J_6^3}.$$

Thus, in order to prove (2.53), we need to prove that

$$\frac{\psi^3(q^2) - 9q^2\psi^4(q^6)/\psi(q^2)}{\varphi^3(q) - 9\varphi^4(q^3)/\varphi(q)} = \frac{J_2^9J_3^3J_{12}^6}{-8J_1^3J_4^6J_6^9}. \tag{2.56}$$

Substituting (2.2) and (2.3) into (2.56), we find that (2.56) is equivalent to

$$\frac{J_4^6}{J_2^3} - 9q^2\frac{J_2J_{12}^8}{J_4^2J_6^4} = \left(\frac{J_2^{15}}{J_1^6J_4^6} - 9\frac{J_1^2J_4^2J_6^{20}}{J_2^5J_3^8J_{12}^8}\right)\left(\frac{J_2^9J_3^3J_{12}^6}{-8J_1^3J_4^6J_6^9}\right),$$

or, equivalently,

$$8\frac{J_4^6}{J_2^3} - 72q^2\frac{J_2J_{12}^8}{J_4^2J_6^4} + \frac{J_2^{24}J_3^3J_{12}^6}{J_1^9J_4^{12}J_6^9} - 9\frac{J_2^4J_6^{11}}{J_1J_3^5J_4^4J_{12}^2} = 0. \tag{2.57}$$

Plugging (2.16)–(2.21) into the left-hand side of (2.57), upon simplification, we find that (2.57) is indeed true. The identity (2.51) thus follows.

The proofs of the expressions of $c_{9,i}(q)$ for $1 \le i \le 4$ can be established similarly, and thus, we omit the details. Notice that when we derive the fourth term of $c_{9,3}(q)$, we need to set $n_2 = 1 - n_1 - n_3$ and utilize (2.52).

This completes the proof of Lemma 2.7 based on Lemma 2.1. □

3. Generating functions of k-colored generalized Frobenius partitions

In this section, we derive some expressions of $\mathrm{C}\Phi_k(q)$ with integral coefficients. From (1.18), one sees that for any $k \geq 1$,

$$\mathrm{C}\Phi_k(q) = \frac{c_{k,0}(\sqrt{q})}{J_1^k}. \tag{3.1}$$

Combining (3.1) and these bivariable identities presented in Section 2, we can deduce the integral expressions of $\mathrm{C}\Phi_k(q)$.

3.1. **Case $k = 4$.** The following theorem provides two expressions of $\mathrm{C}\Phi_4(q)$.

Theorem 3.1. *Let $a(q)$ be defined as in (2.8). Then*

$$\mathrm{C}\Phi_4(q) = \frac{1}{J_1^4}\left(\frac{J_2^8 J_4}{J_1^4 J_8^2} + 8q\frac{J_4^3 J_8^2}{J_2^2}\right) \tag{3.2}$$

$$= \frac{1}{J_1^4}\left(a(q)\frac{J_{12}^5}{J_6^2 J_{24}^2} + 6q\frac{J_3^3 J_4^2 J_6 J_{24}}{J_1 J_2 J_8 J_{12}}\right). \tag{3.3}$$

Proof. The identities (3.2) and (3.3) follow immediately from (3.1) and (2.25), and (3.1) and (2.30), respectively. □

Baruah and Sarmah [3, Theorem 2.1] derived an equivalent form of (3.2) (see (1.11) above) by utilizing the method of integer matrix exact covering systems. The identity (1.11) can be derived from (3.2) if one uses the following three identities involving $\varphi(q)$ and $\psi(q)$ (see [5, p. 40, Entry 25 (iv)–(vi)]):

$$\varphi(q)\psi(q^2) = \psi^2(q), \tag{3.4}$$

$$\varphi^2(q) - \varphi^2(-q) = 8q\psi^2(q^4), \tag{3.5}$$

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2). \tag{3.6}$$

3.2. **Case $k = 5$.** The following theorem gives an expression of $\mathrm{C}\Phi_5(q)$.

Theorem 3.2. *Let $A(q)$ and $B(q)$ be defined as in (2.4). Then*

$$\mathrm{C}\Phi_5(q) = \frac{1}{J_1^5}\big(A(q)\varphi(q^{10}) + 8q\psi^3(q)\psi(q^5) + 4q^3 B(q)\psi(q^4)\psi(q^{20})\big). \tag{3.7}$$

Proof. The identity (3.7) follows immediately from (1.18) and (2.35). □

Employing the method of integer matrix exact covering systems, Baruah and Sarmah [3, Theorem 2.2] derived another form of (3.7). However, it seems that there is a misprint in the coefficient of last component in the expression of $\mathrm{C}\Phi_5(q)$ due to Baruah and Sarmah.

3.3. **Case** $k = 6$**.** The identities (2.37) and (2.38) give the following two expressions of $\mathrm{C}\Phi_6(q)$.

Theorem 3.3. *Let $a(q)$ be defined as in (2.8). Then*

$$\mathrm{C}\Phi_6(q) = \frac{1}{J_1^6}\left(a(q^2)\frac{J_2^{15}}{J_1^6 J_4^6} + 24q\frac{J_2^5 J_4^2 J_6^2}{J_1^3 J_3}\right) \tag{3.8}$$

$$= \frac{1}{J_1^6}\left(a^2(q)\frac{J_6^5}{J_3^2 J_{12}^2} + 18q\frac{J_2^2 J_3^7 J_{12}}{J_1^3 J_4 J_6}\right). \tag{3.9}$$

Utilizing the method of integer matrix exact covering systems, Baruah and Sarmah [4, Theorem 2.1] provided an expression of $\mathrm{C}\Phi_6(q)$. Actually, (3.8) is equivalent to the expression of $\mathrm{C}\Phi_6(q)$ derived by Baruah and Sarmah if one observes the following identity [4, p. 368]:

$$a(q) = \varphi(q)\varphi(q^3) + 4q\psi(q^2)\psi(q^6).$$

The identity (3.9) was established by Hirschhorn [18]. Moreover, Gu, Wang and Xia [16], and Tang [40] proved some congruences and internal congruences modulo powers of 3 for $c\phi_6(n)$ based on (3.9).

3.4. **Case** $k = 8$**.** The identity (2.44) implies the following expressions of $\mathrm{C}\Phi_8(q)$.

Theorem 3.4. *Let $A(q)$ and $B(q)$ be defined as in (2.4). Then*

$$\mathrm{C}\Phi_8(q) = \frac{1}{J_1^8}\big(A^2(q)\varphi(q^4) + 8q^2B^2(q)\psi^2(q^4)\psi(q^8) + 32q\psi^6(q)\psi(q^2)\big). \tag{3.10}$$

With the help of modular forms, Chan, Wang and Yang [8, Theorem 6.3] obtained an expression of $\mathrm{C}\Phi_8(q)$ with seven terms. Interestingly, we can further simplify (3.10). For this purpose, we require the following 2-dissection of $\varphi(q)$ (see [5, p. 40, Entry 25 (i) and (ii)]):

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8), \tag{3.11}$$

which is equivalent to

$$\frac{J_2^5}{J_1^2 J_4^2} = \frac{J_8^5}{J_4^2 J_{16}^2} + 2q\frac{J_{16}^2}{J_8}. \tag{3.12}$$

Substituting (3.12) into (3.10), after simplification, we find that

$$\mathrm{C}\Phi_8(q) = \frac{1}{J_1^8}\left(\frac{J_2^{16} J_8}{J_1^8 J_{16}^2} + 64q^2\frac{J_4^4 J_8^9}{J_2^4 J_{16}^2} + 8q^2\frac{J_2^{20} J_8^3 J_{16}^2}{J_1^8 J_4^{10}} + 32q^2\frac{J_4^{18} J_{16}^2}{J_2^8 J_8^5} + 48q\frac{J_2^{11} J_4^2}{J_1^6}\right).$$

3.5. **Case** $k = 9$**.** The identity (2.47) implies the following expression of $\mathrm{C}\Phi_9(q)$.

Theorem 3.5. *Let $a(q)$ be defined as in (2.8). Then*

$$\mathrm{C}\Phi_9(q) = \frac{1}{J_1^9}\left(a^3(q)a(q^3) + 54qa(q)\frac{J_3^9}{J_1^3} + 162q^2\frac{J_3^8J_9^3}{J_1^3}\right). \tag{3.13}$$

In 1996, Kolitsch [30,31] proved that

$$\sum_{n=0}^{\infty}\overline{c\phi}_9(n+1)q^n = 3\sum_{n=0}^{\infty}\overline{c\phi}_3(3n+2)q^n = 81\frac{J_3^8}{J_1^9} + 729q\frac{J_3^8J_9^3}{J_1^{12}}, \tag{3.14}$$

where

$$\overline{c\phi}_k(n) = \sum_{\ell|\gcd(k,n)}\mu(\ell)c\phi_{k/\ell}(n/\ell), \tag{3.15}$$

and $\mu(n)$ is the Möbius function. Based on (1.5), (3.14) and (3.15), one can obtain an integral expression of $\mathrm{C}\Phi_9(q)$.

3.6. **Case** $k = 10$**.** We derive the following expression of $\mathrm{C}\Phi_{10}(q)$.

Theorem 3.6. *Let $A(q)$ and $B(q)$ be defined as in (2.4). Then*

$$\begin{aligned}
\mathrm{C}\Phi_{10}(q) = \frac{1}{J_1^{10}}\big\{&\big(A(q)\varphi(q^{10}) + 16q\psi^3(q)\psi(q^5)\big)A(q)\varphi(q^5)\varphi(q^{10})\\
&+ \big(64q^4\psi^3(q)\psi(q^5) + 16q^6B(q)\psi(q^4)\psi(q^{20})\big)B(q)\varphi(q^5)\psi(q^4)\psi(q^{20})\\
&+ \big(64q^2\psi^6(q)\psi^2(q^5) + 8q^3A(q)B(q)\varphi(q^{10})\psi(q^4)\psi(q^{20})\big)\varphi(q^5)\\
&+ \big(2q\overline{J}_{3,10}\overline{J}_{6,20}^2 + 2q^4\overline{J}_{1,10}\overline{J}_{2,20}^2\big)A^2(q)\\
&+ \big(16q\overline{J}_{2,5}\overline{J}_{3,10}\overline{J}_{6,20} + 16q^3\overline{J}_{1,5}\overline{J}_{1,10}\overline{J}_{2,20}\big)A(q)\psi^3(q)\\
&+ \big(32q\overline{J}_{2,5}^2\overline{J}_{3,10} + 32q^2\overline{J}_{1,5}^2\overline{J}_{1,10}\big)\psi^6(q)\\
&+ \big(8q^2\overline{J}_{3,10}\overline{J}_{4,20}\overline{J}_{6,20} + 8q^3\overline{J}_{1,10}\overline{J}_{2,20}\overline{J}_{8,20}\big)A(q)B(q)\psi(q^4)\\
&+ \big(32q^2\overline{J}_{2,5}\overline{J}_{3,10}\overline{J}_{4,20} + 32q^2\overline{J}_{1,5}\overline{J}_{1,10}\overline{J}_{8,20}\big)B(q)\psi^3(q)\psi(q^4)\\
&+ \big(8q^3\overline{J}_{3,10}\overline{J}_{4,20}^2 + 8q^2\overline{J}_{1,10}\overline{J}_{8,20}^2\big)B^2(q)\psi^2(q^4)\big\}.
\end{aligned} \tag{3.16}$$

Proof. It follows from (1.16) and (2.33) that

$$\begin{aligned}
f_{10}(z) = \big\{&\big(A(q^2)\varphi(q^{20}) + 8q^2\psi^3(q^2)\psi(q^{10}) + 4q^6B(q^2)\psi(q^8)\psi(q^{40})\big)L_{5,0,5,0}\\
&+ \big(qA(q^2)\overline{J}_{12,40} + 4q\psi^3(q^2)\overline{J}_{4,10} + 2q^3B(q^2)\psi(q^8)\overline{J}_{8,40}\big)\\
&\quad\times\big(L_{5,1,5,2} + L_{5,-1,5,-2}\big)\\
&+ \big(q^4A(q^2)\overline{J}_{4,40} + 4q^2\psi^3(q^2)\overline{J}_{2,10} + 2q^2B(q^2)\psi(q^8)\overline{J}_{16,40}\big)\\
&\quad\times\big(L_{5,2,5,4} + L_{5,-2,5,-4}\big)\big\}^2.
\end{aligned}$$

Then we find that

$$\begin{aligned}c_{10,0}(q) &= \big(A(q^2)\varphi(q^{20}) + 8q^2\psi^3(q^2)\psi(q^{10}) + 4q^6B(q^2)\psi(q^8)\psi(q^{40})\big)^2\\ &\qquad\times \sum_{\substack{n_1,n_2=-\infty\\ n_1+n_2=0}}^{\infty} q^{5n_1^2+5n_2^2}\\ &\quad + 2\big(qA(q^2)\overline{J}_{12,40} + 4q\psi^3(q^2)\overline{J}_{4,10} + 2q^3B(q^2)\psi(q^8)\overline{J}_{8,40}\big)^2\\ &\qquad\times \sum_{\substack{n_1,n_2=-\infty\\ n_1+n_2=0}}^{\infty} q^{5n_1^2+2n_1+5n_2^2-2n_2}\\ &\quad + 2\big(q^4A(q^2)\overline{J}_{4,40} + 4q^2\psi^3(q^2)\overline{J}_{2,10} + 2q^2B(q^2)\psi(q^8)\overline{J}_{16,40}\big)^2\\ &\qquad\times \sum_{\substack{n_1,n_2=-\infty\\ n_1+n_2=0}}^{\infty} q^{5n_1^2+4n_1+5n_2^2-4n_2}. \end{aligned} \tag{3.17}$$

Now (3.16) follows by simplifying three double sums on the right-hand side of (3.17) and utilizing (3.1). □

3.7. **Case** $k = 12$**.** We obtain the following expression of $\mathrm{C}\Phi_{12}(q)$.

Theorem 3.7. *Let $A(q)$ and $B(q)$ be defined as in (2.4), and let $a(q)$ be defined as in (2.8). Then*

$$\begin{aligned}\mathrm{C}\Phi_{12}(q) &= \frac{1}{J_1^{12}}\bigg(a(q^4)A^3(q) + 96qA(q)\frac{J_2^{11}J_4^2J_6^5}{J_1^6J_3^2J_{12}^2}\\ &\qquad + 24q^2A(q)B^2(q)\frac{J_8^4J_{12}^2}{J_2J_6} + 192q^2B(q)\frac{J_2^{17}J_8^2J_{12}^2}{J_1^8J_4^3J_6}\bigg).\end{aligned} \tag{3.18}$$

Proof. According to (1.16) and (2.22), we deduce that

$$f_{12}(z) = \big\{A(q^2)L_{4,0,4,0} + 4q\psi^3(q^2)\big(L_{4,1,4,2} + L_{4,-1,4,-2}\big) + 2q^2B(q^2)\psi(q^8)L_{4,2,4,4}\big\}^3,$$

from which we obtain that

$$\begin{aligned}c_{12,0}(q) &= A^3(q^2)\sum_{\substack{n_1,n_2,n_3=-\infty\\ n_1+n_2+n_3=0}}^{\infty} q^{4n_1^2+4n_2^2+4n_3^2}\\ &\quad + 96q^2A(q^2)\psi^6(q^2)\sum_{\substack{n_1,n_2,n_3=-\infty\\ n_1+n_2+n_3=0}}^{\infty} q^{4n_1^2+4n_2^2+2n_2+4n_3^2-2n_3}\\ &\quad + 12q^4A(q^2)B^2(q^2)\psi^2(q^8)\sum_{\substack{n_1,n_2,n_3=-\infty\\ n_1+n_2+n_3+1=0}}^{\infty} q^{4n_1^2+4n_2^2+4n_2+4n_3^2+4n_3}\end{aligned}$$

$$+ 96q^4B(q^2)\psi^6(q^2)\psi(q^8)\sum_{\substack{n_1,n_2,n_3=-\infty\\ n_1+n_2+n_3+1=0}}^{\infty} q^{4n_1^2+2n_1+4n_2^2+2n_2+4n_3^2+4n_3}$$
$$+ 96q^4B(q^2)\psi^6(q^2)\psi(q^8)\sum_{\substack{n_1,n_2,n_3=-\infty\\ n_1+n_2+n_3=0}}^{\infty} q^{4n_1^2-2n_1+4n_2^2-2n_2+4n_3^2+4n_3}. \tag{3.19}$$

We next simplify the five triple sums in (3.19) by applying a similar strategy of deriving (2.13), after tedious but straightforward calculations, we conclude that

$$c_{12,0}(q) = a(q^8)A^3(q^2) + 96q^2A(q^2)\frac{J_4^{11}J_8^2J_{12}^5}{J_2^6J_6^2J_{24}^2}$$
$$+ 24q^4A(q^2)B^2(q^2)\frac{J_{16}^4J_{24}^2}{J_4J_{12}} + 192q^4B(q^2)\frac{J_4^{17}J_{16}^2J_{24}^2}{J_2^8J_8^3J_{12}}. \tag{3.20}$$

Notice that when we simplify the second term on the right-hand side of (3.19), we set $n_1 = -n_2 - n_3$. Moreover, we also need to utilize (3.11) to derive (3.20). The identity (3.18) follows from (3.1) and (3.20). □

Note that

$$f_{13}(z) = \big(f_4(z)\big)^3 \cdot f_1(z), \qquad f_{14}(z) = \big(f_4(z)\big)^3 \cdot \big(f_1(z)\big)^2, \qquad f_{15}(z) = \big(f_5(z)\big)^3.$$

Of course, we can also analyse the corresponding integral expressions of $\mathrm{C}\Phi_{13}(q)$, $\mathrm{C}\Phi_{14}(q)$ and $\mathrm{C}\Phi_{15}(q)$ similarly. However, the process will be lengthier and trickier because we need to use the method of integer matrix exact covering systems to simplify some multiple q-series sums. Therefore, we do not consider these cases here.

3.8. **Case $k = 16$.** We obtain the following expression of $\mathrm{C}\Phi_{16}(q)$.

Theorem 3.8. *Let $A(q)$ and $B(q)$ be defined as in (2.4). Then*

$$\mathrm{C}\Phi_{16}(q) = \frac{1}{J_1^{16}}\bigg(A^4(q)A(q^4) + 192qA^2(q)\frac{J_2^{11}J_8^3J_{16}^3}{J_1^6J_{32}^2} + 512q^2B(q)\frac{J_2^{24}J_8^2}{J_1^{12}J_4}$$
$$+ 768q^2A(q)B(q)\frac{J_2^{16}J_8^4}{J_1^8J_4^2} + 48q^2A^2(q)B^2(q)\frac{J_4J_8^6}{J_2^2}$$
$$+ 1536q^3B^2(q)\frac{J_2^{11}J_8J_{16}^7}{J_1^6J_{32}^2} + 1536q^4B^2(q)\frac{J_2^{11}J_8^9J_{32}^2}{J_1^6J_4^2J_{16}^3}$$
$$+ 64q^4B^4(q)\frac{J_8^4J_{16}^9}{J_4^4J_{32}^2} + 32q^4B^4(q)\frac{J_8^{18}J_{32}^2}{J_4^8J_{16}^5} + 768q^4A^2(q)\frac{J_2^{11}J_4^2J_{16}J_{32}^2}{J_1^6J_8}\bigg). \tag{3.21}$$

Proof. It follows from (1.16) and (2.44) that

$$f_{16}(z) = \big\{\big(A^2(q^2)\varphi(q^8) + 8q^4B^2(q^2)\psi^2(q^8)\psi(q^{16}) + 32q^2\psi^6(q^2)\psi(q^4)\big)L_{8,0,8,0}$$
$$+ \big(8qA(q^2)\psi^3(q^2)\overline{J}_{6,16} + 16q^3B(q^2)\psi^3(q^2)\psi(q^8)\overline{J}_{2,16}\big)\big(L_{8,1,8,2} + L_{8,-1,8,-2}\big)$$
$$+ \big(4q^2A(q^2)B(q^2)\psi(q^4)\psi(q^8) + 16q^2\psi^6(q^2)\varphi(q^8) + 32q^4\psi^6(q^2)\psi(q^{16})\big)$$

$$\times \big(L_{8,2,8,4}+L_{8,-2,8,-4}\big)$$
$$+\big(8q^3A(q^2)\psi^3(q^2)\overline{J}_{2,16}+16q^3B(q^2)\psi^3(q^2)\psi(q^8)\overline{J}_{6,16}\big)$$
$$\times \big(L_{8,3,8,6}+L_{8,-3,8,-6}\big)$$
$$+\big(2q^4A^2(q^2)\psi(q^{16})+4q^4B^2(q^2)\varphi(q^8)\psi^2(q^8)+32q^4\psi^6(q^2)\psi(q^4)\big)L_{8,4,8,8}\big\}^2,$$

from which we obtain that

$$\begin{aligned}
c_{16,0}(q) = {} & \big(A^2(q^2)\varphi(q^8)+8q^4B^2(q^2)\psi^2(q^8)\psi(q^{16})+32q^2\psi^6(q^2)\psi(q^4)\big)^2 \\
& \times \sum_{\substack{n_1,n_2=-\infty \\ n_1+n_2=0}}^{\infty} q^{8n_1^2+8n_2^2} \\
& + 2\big(8qA(q^2)\psi^3(q^2)\overline{J}_{6,16}+16q^3B(q^2)\psi^3(q^2)\psi(q^8)\overline{J}_{2,16}\big)^2 \\
& \times \sum_{\substack{n_1,n_2=-\infty \\ n_1+n_2=0}}^{\infty} q^{8n_1^2+2n_1+8n_2^2-2n_2} \\
& + 2\big(4q^2A(q^2)B(q^2)\psi(q^4)\psi(q^8)+16q^2\varphi(q^8)\psi^6(q^2)+32q^4\psi^6(q^2)\psi(q^{16})\big)^2 \\
& \times \sum_{\substack{n_1,n_2=-\infty \\ n_1+n_2=0}}^{\infty} q^{8n_1^2+4n_1+8n_2^2-4n_2} \\
& + 2\big(8q^3A(q^2)\psi^3(q^2)\overline{J}_{2,16}+16q^3B(q^2)\psi^3(q^2)\psi(q^8)\overline{J}_{6,16}\big)^2 \\
& \times \sum_{\substack{n_1,n_2=-\infty \\ n_1+n_2=0}}^{\infty} q^{8n_1^2+6n_1+8n_2^2-6n_2} \\
& + \big(2q^4A^2(q^2)\psi(q^{16})+4q^4B^2(q^2)\varphi(q^8)\psi^2(q^8)+32q^4\psi^6(q^2)\psi(q^4)\big)^2 \\
& \times \sum_{\substack{n_1,n_2=-\infty \\ n_1+n_2+1=0}}^{\infty} q^{8n_1^2+8n_1+8n_2^2+8n_2}.
\end{aligned} \tag{3.22}$$

After simplification, one finds that the five double sums in (3.22) can be expressed as certain single Ramanujan's theta function (2.1). When we simplify (3.22), we also need the following results.

First, we shall consider the simplification of the following expression, namely,

$$\overline{J}_{6,16}^2\overline{J}_{12,32}+q^4\overline{J}_{2,16}^2\overline{J}_{4,32}. \tag{3.23}$$

We need to utilize the following identity involving $f(a,b)$, which can be found in Berndt's book [5, p. 45, Entry 29].

$$f(a,b)f(c,d)=f(ac,bd)f(ad,bc)+af\big(b/c,ac^2d\big)f\big(b/d,acd^2\big). \tag{3.24}$$

Taking $a=c=q^6$ and $b=d=q^{10}$ in (3.24) first and setting $a=c=q^2$ and $b=d=q^{14}$ in (3.24) second yield that

$$\overline{J}_{6,16}^2=\overline{J}_{12,32}\varphi(q^{16})+2q^6\overline{J}_{4,32}\psi(q^{32}), \tag{3.25}$$

$$\overline{J}_{2,16}^2=\overline{J}_{4,32}\varphi(q^{16})+2q^2\overline{J}_{12,32}\psi(q^{32}). \tag{3.26}$$

Substituting (3.25) and (3.26) into (3.23), we get that

$$\overline{J}_{6,16}^2\overline{J}_{12,32}+q^4\overline{J}_{2,16}^2\overline{J}_{4,32}=\big(\overline{J}_{12,32}^2+q^4\overline{J}_{4,32}^2\big)\varphi(q^{16})+4q^6\overline{J}_{4,32}\overline{J}_{12,32}\psi(q^{32}).$$

We also recall from [5, p. 51, Example (iv)] that

$$2\overline{J}_{3,8}^2=\varphi(q^2)\psi(q)+\varphi(-q^2)\psi(-q),$$

$$2q\overline{J}_{1,8}^2=\varphi(q^2)\psi(q)-\varphi(-q^2)\psi(-q),$$

from which we find that

$$\overline{J}_{3,8}^2+q\overline{J}_{1,8}^2=\varphi(q^2)\psi(q). \tag{3.27}$$

With the help of (3.27), we further have

$$\begin{aligned}\overline{J}_{6,16}^2\overline{J}_{12,32}+q^4\overline{J}_{2,16}^2\overline{J}_{4,32}&=\varphi(q^8)\varphi(q^{16})\psi(q^4)+4q^6\overline{J}_{4,32}\overline{J}_{12,32}\psi(q^{32})\\&=\frac{J_{16}^3J_{32}^3}{J_4J_{64}^2}+4q^6\frac{J_8^2J_{32}J_{64}^2}{J_4J_{16}}.\end{aligned}$$

Next, notice that

$$\overline{J}_{2,16}\overline{J}_{6,16}\big(\overline{J}_{12,32}+q^2\overline{J}_{4,32}\big)=\overline{J}_{2,16}\overline{J}_{6,16}\overline{J}_{2,8}=\frac{J_4^4J_{16}^2}{J_2^2J_8},$$

where the first identity follows from (2.31) with $a=q^2$ and $b=q^6$. Finally, we also need to use (2.2)–(2.4) and (3.11) to simplify (3.22).

Through these simplifications, (3.21) thus follows. We therefore complete the proof of Theorem 3.8. □

4. Congruences for k-colored generalized Frobenius partitions

To investigate congruence properties of $c\phi_k(n)$, we collect some necessary identities.

Lemma 4.1. *We have*

$$\frac{J_1^2}{J_2}=\frac{J_9^2}{J_{18}}-2q\frac{J_3J_{18}^2}{J_6J_9}, \tag{4.1}$$

$$\frac{J_1J_4}{J_2}=\frac{J_3J_{12}J_{18}^5}{J_6^2J_9^2J_{36}^2}-q\frac{J_9J_{36}}{J_{18}}. \tag{4.2}$$

The identities (4.1) and (4.2) follow by replacing q by $-q$ in (i) and (ii) of [5, p. 49, Corollary], respectively.

Lemma 4.2. [20, 43] *Let $a(q)$ be defined as in (2.8). Then*

$$a(q) = a(q^3) + 6q\frac{J_9^3}{J_3}, \tag{4.3}$$

$$J_1^3 = a(q^3)J_3 - 3qJ_9^3 \tag{4.4}$$

and

$$\frac{1}{J_1^3} = \frac{J_9^3}{J_3^{12}}\big(a^2(q^3)J_3^2 + 3qa(q^3)J_3J_9^3 + 9q^2J_9^6\big). \tag{4.5}$$

The identities (4.3) and (4.4) were established by Hirschhorn, Garvan and Borwein [20, Eqs. (1.3) and (1.4)]. The identity (4.5) was proved by Wang [43, Eq. (2.28)].

According to the binomial theorem, one can easily establish the following congruence, which will be used frequently without mention.

Lemma 4.3. *For any $k \geq 1$ prime number p,*

$$J_k^p \equiv J_{pk} \pmod{p}.$$

First, we prove the following infinite family of congruences modulo 27.

Theorem 4.4. *For any $N \geq 0$ and $n \geq 0$,*

$$c\phi_{9N+3}(3n+2) \equiv 0 \pmod{27}. \tag{4.6}$$

Proof. The congruence $c\phi_3(3n+2) \equiv 0 \pmod{27}$ was proved by Kolitsch [28]. Therefore, we only need to consider the case $N \geq 1$ in (4.6).

With the help of (2.7), we obtain that, modulo 27,

$$\begin{aligned} f_{9N+3}(z) &= \big\{a(q^2)L_{3,0,3,0} + 3qC(q^2)\big(L_{3,1,3,2} + L_{3,-1,3,-2}\big)\big\}^{3N+1} \\ &\equiv a^{3N+1}(q^2)L_{3,0,3,0}^{3N+1} + 3(3N+1)qC(q^2)a^{3N}(q^2)L_{3,0,3,0}^{3N}\big(L_{3,1,3,2} + L_{3,-1,3,-2}\big) \\ &\quad + (27N(3N+1)/2)q^2C^2(q^2)a^{3N-1}(q^2)L_{3,0,3,0}^{3N-1}\big(L_{3,1,3,2} + L_{3,-1,3,-2}\big)^2, \end{aligned} \tag{4.7}$$

where $C(q)$ is defined as in (2.46). Extracting the constant term in (4.7), we find that, modulo 27,

$$\begin{aligned} \mathrm{CT}_z\big\{f_{9N+3}(z)\big\} &\equiv \mathrm{CT}_z\big\{a^{3N+1}(q^2)L_{3,0,3,0}^{3N+1}\big\} \\ &= a^{3N+1}(q^2)\sum_{\substack{n_1,n_2,\ldots,n_{3N+1}=-\infty \\ n_1+n_2+\cdots+n_{3N+1}=0}}^{\infty} q^{3n_1^2+3n_2^2+\cdots+3n_{3N+1}^2} \\ &= a^{3N+1}(q^2)\sum_{n_1,n_2,\ldots,n_{3N}=-\infty}^{\infty} q^{6\left(\sum_{i=1}^{3N} n_i^2 + \sum_{1\leq i<j\leq 3N} n_in_j\right)}. \end{aligned} \tag{4.8}$$

Combining (3.1) and (4.8) yields that

$$\mathrm{C}\Phi_{9N+3}(q) \equiv \frac{a^{3N+1}(q)}{J_1^{9N+3}}\sum_{n_1,n_2,\ldots,n_{3N}=-\infty}^{\infty} q^{3\left(\sum_{i=1}^{3N} n_i^2 + \sum_{1\leq i<j\leq 3N} n_in_j\right)} \pmod{27}. \tag{4.9}$$

Thanks to (4.3) and (4.4),

$$a^3(q)J_3^3 = a^3(q^3)J_3^3 + 18qa^2(q^3)J_3^2J_9^3 + 108q^2a(q^3)J_3J_9^6 + 216q^3J_9^9,$$
$$J_1^9 = a^3(q^3)J_3^3 - 9qa^2(q^3)J_3^2J_9^3 + 27q^2a(q^3)J_3J_9^6 - 27q^3J_9^9,$$

from which we obtain that

$$J_1^{9N} \equiv a^{3N}(q)J_3^{3N} \pmod{27}. \tag{4.10}$$

Substituting (4.3), (4.5) and (4.10) into (4.9), we find that

$$\mathrm{C}\Phi_{9N+3}(q) \equiv \frac{J_9^3}{J_3^{3N+12}}\left(a^2(q^3)J_3^2 + 3qa(q^3)J_3J_9^3 + 9q^2J_9^6\right)\left(a(q^3) + 6q\frac{J_9^3}{J_3}\right)$$
$$\times \sum_{n_1,n_2,\ldots,n_{3N}=-\infty}^{\infty} q^{3\left(\sum_{i=1}^{3N} n_i^2 + \sum_{1\le i<j\le 3N} n_in_j\right)} \pmod{27}. \tag{4.11}$$

Taking all the terms of the form q^{3n+2} in (4.11), upon simplification, we conclude that

$$\sum_{n=0}^{\infty} c\phi_{9N+3}(3n+2)q^n \equiv 27a(q)\frac{J_3^9}{J_1^{3N+12}} \sum_{n_1,n_2,\ldots,n_{3N}=-\infty}^{\infty} q^{\left(\sum_{i=1}^{3N} n_i^2 + \sum_{1\le i<j\le 3N} n_in_j\right)}$$
$$\equiv 0 \pmod{27}.$$

Therefore, we complete the proof of (4.6). □

Next, we prove another infinite family of congruences modulo 27.

Theorem 4.5. *For any $N \ge 0$ and $n \ge 0$,*

$$c\phi_{9N+6}(3n+2) \equiv 0 \pmod{27}. \tag{4.12}$$

Proof. The case $N = 0$ of (4.12) was proved by Xia [45]. Therefore, we consider that N is a positive integer in (4.12).

According to (2.38) and (2.47), we find that, modulo 27,

$$\begin{aligned}
c_{9N+6,0}(q) &= \mathrm{CT}_z\{f_{9N+6}(z)\} = \mathrm{CT}_z\big\{f_9^N(z)\cdot f_6(z)\big\} \\
&\equiv c_{9,0}^N(q)c_{6,0}(q)S_{9N+6,1}(q) + c_{9,0}^N(q)c_{6,3}(q)S_{9N+6,2}(q) \\
&\quad + Nc_{9,0}^{N-1}(q)c_{9,3}(q)c_{6,0}(q)\big(S_{9N+6,3}(q) + S_{9N+6,4}(q)\big) \\
&\quad + Nc_{9,0}^{N-1}(q)c_{9,3}(q)c_{6,3}(q)\big(S_{9N+6,5}(q) + S_{9N+6,6}(q)\big) \\
&\quad + (N(N-1)/2)c_{9,0}^{N-2}(q)c_{9,3}^2(q)c_{6,0}(q) \\
&\qquad \times \big(S_{9N+6,7}(q) + S_{9N+6,8}(q) + 2S_{9N+6,9}\big) \\
&\quad + (N(N-1)/2)c_{9,0}^{N-2}(q)c_{9,3}^2(q)c_{6,3}(q) \\
&\qquad \times \big(S_{9N+6,10}(q) + S_{9N+6,11}(q) + 2S_{9N+6,12}\big) =: S_{9N+6}(q),
\end{aligned}$$

where the $S_{9N+6,i}(q)$ for $1 \leq i \leq 12$ are some functions of q^3. For example,

$$S_{9N+6,1}(q) = \sum_{\substack{n_1,n_2,\ldots,n_{N+1}=-\infty \\ 3n_1+3n_2+\cdots+3n_N+2n_{N+1}=0}}^{\infty} q^{9n_1^2+9n_2^2+\cdots+9n_N^2+6n_{N+1}^2},$$

$$S_{9N+6,2}(q) = \sum_{\substack{n_1,n_2,\ldots,n_{N+1}=-\infty \\ 3n_1+3n_2+\cdots+3n_N+2n_{N+1}+1=0}}^{\infty} q^{9n_1^2+9n_2^2+\cdots+9n_N^2+6n_{N+1}^2+6n_{N+1}}.$$

From (3.1) we have

$$\mathrm{C}\Phi_{9N+6}(q^2) = \frac{c_{9N+6,0}(q)}{J_2^{9N+6}} \equiv \frac{S_{9N+6}(q)}{J_2^{9N+6}} \pmod{27}.$$

Therefore, in order to prove (4.12), we need to prove that the coefficients of q^{3n+1} for any $n \geq 0$ in $S_{9N+6}(q)/J_2^{9N+6}$ vanish modulo 27. Next, we shall prove that for any $n \geq 0$, the coefficients of q^{3n+1} in

$$c_{9,0}^N(q)c_{6,0}(q)/J_2^{9N+6}, \qquad c_{9,0}^N(q)c_{6,3}(q)/J_2^{9N+6},$$
$$c_{9,0}^{N-1}(q)c_{9,3}(q)c_{6,0}(q)/J_2^{9N+6}, \qquad c_{9,0}^{N-1}(q)c_{9,3}(q)c_{6,3}(q)/J_2^{9N+6},$$
$$c_{9,0}^{N-2}(q)c_{9,3}^2(q)c_{6,0}(q)/J_2^{9N+6} \quad \text{and} \quad c_{9,0}^{N-2}(q)c_{9,3}^2(q)c_{6,3}(q)/J_2^{9N+6}$$

vanish modulo 27. For this purpose, we first notice that, modulo 27,

$$c_{9,0}(q) \equiv a^3(q^2)a(q^6) \qquad \text{and} \qquad c_{9,3}(q) \equiv 3q^3a^3(q^2)C(q^6).$$

Moreover, we find that, modulo 27,

$$c_{9,0}^N(q)c_{6,0}(q) \equiv a^{3N}(q^2)a^N(q^6)\left(a^2(q^2)\varphi(q^6) + 18q^2\frac{J_4^2J_6^7J_{24}}{J_2^3J_8J_{12}}\right), \tag{4.13}$$

$$c_{9,0}^N(q)c_{6,3}(q) \equiv a^{3N}(q^2)a^N(q^6)\left(2q^3a^2(q^2)\psi(q^{12}) + 18q^3\frac{J_6^6J_8J_{12}^2}{J_2^2J_4J_{24}}\right), \tag{4.14}$$

$$c_{9,0}^{N-1}(q)c_{9,3}(q)c_{6,0}(q) \equiv 3q^3a^{3N+2}(q^2)a^{N-1}(q^6)C(q^6)\varphi(q^6), \tag{4.15}$$

$$c_{9,0}^{N-1}(q)c_{9,3}(q)c_{6,3}(q) \equiv 6q^6a^{3N+2}(q^2)a^{N-1}(q^6)C(q^6)\psi(q^{12}), \tag{4.16}$$

$$c_{9,0}^{N-2}(q)c_{9,3}^2(q)c_{6,0}(q) \equiv 9q^6a^{3N+2}(q^2)a^{N-2}(q^6)C^2(q^6)\varphi(q^6), \tag{4.17}$$

$$c_{9,0}^{N-2}(q)c_{9,3}^2(q)c_{6,3}(q) \equiv 18q^9a^{3N+2}(q^2)a^{N-2}(q^6)C^2(q^6)\psi(q^{12}). \tag{4.18}$$

Based on (4.13)–(4.18), we consider the following three auxiliary functions, given by

$$\sum_{n=0}^{\infty} A_1(n)q^n = \frac{a^{3N+2}(q^2)}{J_2^{9N+6}}, \qquad \sum_{n=0}^{\infty} A_2(n)q^n = 18q^2a^{3N}(q^2)\frac{J_4^2J_6^7J_{24}}{J_2^{9N+9}J_8J_{12}},$$

$$\sum_{n=0}^{\infty} A_3(n)q^n = 18a^{3N}(q^2)\frac{J_6^6J_8J_{12}^2}{J_2^{9N+8}J_4J_{24}}.$$

Therefore, to obtain (4.12), the ultimate task is to prove that for any $n \geq 0$,

$$A_1(3n+1) \equiv A_2(3n+1) \equiv A_3(3n+1) \equiv 0 \pmod{27}.$$

Thanks to (4.3) and (4.5), we deduce that, modulo 27,

$$\begin{aligned} a^{3N+2}(q^2) &\equiv a^{3N+2}(q^6) + 6(3N+2)q^2a^{3N+1}(q^6)C(q^6) \\ &\qquad + 18(3N+2)(3N+1)q^4a^{3N}(q^6)C^2(q^6), \\ \frac{1}{J_2^{9N+6}} &\equiv \frac{J_{18}^{9N+6}}{J_6^{36N+24}}\left(a^{6N+4}(q^6)J_6^{6N+4} + 3(3N+2)q^2a^{6N+3}(q^6)J_6^{6N+3}J_{18}^3\right), \end{aligned}$$

from which we obtain that

$$\sum_{n=0}^{\infty} A_1(3n+1)q^n \equiv 54(3N+2)(2N+1)q\frac{J_6^{9N+12}}{J_2^{30N+22}}a^{9N+4}(q^2) \equiv 0 \pmod{27}.$$

This implies that for any $n \geq 0$,

$$A_1(3n+1) \equiv 0 \pmod{27}.$$

Moreover, it follows from (4.3) that

$$\begin{aligned} \sum_{n=0}^{\infty} A_2(n)q^n &\equiv 18q^2a^{4N}(q^6)\frac{J_{24}}{J_6^{3N-4}J_{12}} \cdot \frac{J_4^2}{J_8} \pmod{27}, \\ \sum_{n=0}^{\infty} A_3(n)q^n &\equiv 18a^{4N}(q^6)\frac{J_{12}^2}{J_6^{3N-3}J_{24}} \cdot \frac{J_2J_8}{J_4} \pmod{27}. \end{aligned}$$

Thanks to (4.1) and (4.2), we conclude that for any $n \geq 0$,

$$A_2(3n+1) \equiv A_3(3n+1) \equiv 0 \pmod{27}.$$

We therefore prove (4.12). □

Remark 4.6. The congruence (1.14) implies that for any $N \geq 0$ and $n \geq 0$,

$$c\phi_{9N+3}(3n+2) \equiv 0 \pmod{9}, \tag{4.19}$$
$$c\phi_{9N+6}(3n+2) \equiv 0 \pmod{9}. \tag{4.20}$$

From this perspective, (4.6) and (4.12) are the corresponding generalizations of (4.19) and (4.20), respectively.

Finally, we derive the following infinite family of congruences modulo 32.

Theorem 4.7. *For any $N \geq 0$ and $n \geq 0$,*

$$c\phi_{4N+4}(4n+3) \equiv 0 \pmod{32}. \tag{4.21}$$

Proof. Baruah and Sarmah [3, Eq. (3.2)] proved that for any $n \geq 0$,

$$c\phi_4(4n+3) \equiv 0 \pmod{256}. \tag{4.22}$$

With the help of (1.14), Chan, Wang and Yang [8, Eq. (6.14)] obtained that for any $n \geq 0$,

$$c\phi_8(2n+1) \equiv 0 \pmod{64}. \tag{4.23}$$

The congruences (4.22) and (4.23) imply that (4.21) holds for the cases $N = 0$ and $N = 1$. Therefore, we will consider the cases $N \geq 2$.

From (2.22) we find that

$$f_{4N+4}(z) = \left\{A(q^2)L_{4,0,4,0} + 4q\psi^3(q^2)\big(L_{4,1,4,2} + L_{4,-1,4,-2}\big) + 2q^2B(q^2)\psi(q^8)L_{4,2,4,4}\right\}^{N+1},$$

from which we obtain that, modulo 32,

$$\begin{aligned}
c_{4N+4,0}(q) &= \mathrm{CT}_z\{f_{4N+4}(z)\} \\
&\equiv A^{N+1}(q^2) \sum_{\substack{n_1,n_2,\ldots,n_{N+1}=-\infty \\ n_1+n_2+\cdots+n_{N+1}=0}}^{\infty} q^{4n_1^2+4n_2^2+\cdots+4n_{N+1}^2} \\
&\quad + 2N(N+1)q^4A^{N-1}(q^2)B^2(q^2)\psi^2(q^8) \\
&\quad \times \sum_{\substack{n_1,n_2,\ldots,n_{N+1}=-\infty \\ n_1+n_2+\cdots+n_{N+1}+1=0}}^{\infty} q^{4n_1^2+\cdots+4n_{N-1}^2+4n_N^2+4n_N+4n_{N+1}^2+4n_{N+1}} \\
&\quad + 16\binom{N+1}{4}q^8A^{N-3}(q^2)B^4(q^2)\psi^4(q^8) \\
&\quad \times \sum_{\substack{n_1,n_2,\ldots,n_{N+1}=-\infty \\ n_1+n_2+\cdots+n_{N+1}+2=0}}^{\infty} q^{4n_1^2+\cdots+4n_{N-3}^2+4n_{N-2}^2+4n_{N-2}+\cdots+4n_{N+1}^2+4n_{N+1}},
\end{aligned}$$

where we have $\binom{N+1}{4} = 0$ if $N = 2$. Thanks to (1.18),

$$\begin{aligned}
\mathrm{C\Phi}_{4N+4}(q) &\equiv \frac{1}{J_1^{4N+4}}A^{N+1}(q) \sum_{\substack{n_1,n_2,\ldots,n_{N+1}=-\infty \\ n_1+n_2+\cdots+n_{N+1}=0}}^{\infty} q^{2n_1^2+2n_2^2+\cdots+2n_{N+1}^2} \\
&\quad + 2N(N+1)q^2\frac{1}{J_1^{4N+4}}A^{N-1}(q)B^2(q)\psi^2(q^4) \\
&\quad \times \sum_{\substack{n_1,n_2,\ldots,n_{N+1}=-\infty \\ n_1+n_2+\cdots+n_{N+1}+1=0}}^{\infty} q^{2n_1^2+\cdots+2n_{N-1}^2+2n_N^2+2n_N+2n_{N+1}^2+2n_{N+1}} \\
&\quad + 16\binom{N+1}{4}q^4\frac{1}{J_1^{4N+4}}A^{N-3}(q)B^4(q)\psi^4(q^4)
\end{aligned}$$

$$\times \sum_{\substack{n_1,n_2,\ldots,n_{N+1}=-\infty \\ n_1+n_2+\cdots+n_{N+1}+2=0}}^{\infty} q^{2n_1^2+\cdots+2n_{N-3}^2+2n_{N-2}^2+2n_{N-2}+\cdots+2n_{N+1}^2+2n_{N+1}}. \quad (4.24)$$

We also need the following 2-dissection identity due to Yao and Xia [48, Eq. (2.10)]:

$$\frac{1}{J_1^4} = \frac{J_4^{14}}{J_2^{14}J_8^4} + 4q\frac{J_4^2J_8^4}{J_2^{10}}. \quad (4.25)$$

Moreover, it follows from (2.4), (3.5) and (3.6) that

$$A(q) = \varphi^3(q^2) + 12q\psi(q^2)^2\psi(q^4), \quad (4.26)$$

$$B(q) = 3\varphi(q^2)^2 + 4q\psi(q^4)^2. \quad (4.27)$$

Substituting (4.25)–(4.27) into (4.24), and taking all the terms of the form q^{2n+1}, after simplification, we obtain that, modulo 32,

$$\sum_{n=0}^{\infty} c\phi_{4N+4}(2n+1)q^n \equiv 16(N+1)\frac{J_4^3}{J_2^{N+1}} \sum_{\substack{n_1,n_2,\ldots,n_{N+1}=-\infty \\ n_1+n_2+\cdots+n_{N+1}=0}}^{\infty} q^{n_1^2+n_2^2+\cdots+n_{N+1}^2}$$

$$\equiv 16(N+1)\frac{J_4^3}{J_2^{N+1}} \sum_{n_1,n_2,\ldots,n_N=-\infty}^{\infty} q^{\sum_{i=1}^N 2n_i^2+\sum_{1\le i<j\le N} 2n_in_j}.$$

This implies that (4.21) holds. □

5. Final remarks

The entire project began in an attempt to find the integral expressions of the generating functions of k-colored generalized Frobenius partitions. Based on general symmetric and recurrence relations for certain bivariable quadratic forms, we manage to establish many expressions of $\mathrm{C}\Phi_k(q)$ with integer coefficients. Some of these were derived by other scholars in the previous study, others are new. As an immediate consequence, we prove three infinite families of congruences satisfied by $c\phi_k(n)$, where k is allowed to grow arbitrary large. We conclude this paper with several remarks.

First, the congruence (4.22), the expressions (3.10), (3.18) and (3.21), together with the numerical evidence suggest the following congruence family, which can be viewed as an improvement of (4.21).

Conjecture 5.1. For any $N \ge 0$ and $n \ge 0$,

$$c\phi_{4N+4}(4n+3) \equiv 0 \pmod{256}. \quad (5.1)$$

Unfortunately, it seems that the method of constant terms used in the present paper will run into serious limitations beyond the modulus 32 in (5.1). One main obstacle is that we need to further analyse congruence properties of certain multiple q-series sums. Thus, a different approach may be necessary. Further, it would be interesting to find more instances similar to (4.6), (4.12) and (5.1).

Second, utilizing the theory of modular forms, Jiang, Rolen and Woodbury [23] also strived to find new ways to represent the generating functions $\mathrm{C}\Phi_k(q)$ as linear combinations of Dedekind's eta functions and Klein forms. Moreover, they [23, p. 5, Corollary] proved that there exists an algorithm to compute $\mathrm{C}\Phi_k(q)$ as sums of products of q-Pochhammer symbol, given by (1.1). It is not clear whether the coefficients of linear combinations in these expressions are integers.

Finally, as mentioned in the introduction, Chan, Wang and Yang [7] did not give integral expressions of $\mathrm{C}\Phi_k(q)$ with $k \in \{10, 12, 14, 15, 17\}$. Actually, Kolitsch [31, Theorem 1] proved a very neat identity, that is, for any $n \geq 0$,

$$\overline{c\phi}_{10}(n+1) = 5\overline{c\phi}_2(5n+3). \tag{5.2}$$

Utilizing the theory of modular forms, Paule and Radu [36] established the following generating function of $c\phi_2(5n+3)$:

$$\begin{aligned}\sum_{n=0}^{\infty} c\phi_2(5n+3)q^n = {} & 20\frac{J_5^2J_{10}^5}{J_1^6J_{20}^2} + 25q\frac{J_2^2J_{10}^3J_{20}^2}{J_4^4J_5^4} + 300q\frac{J_2J_5J_{10}^8}{J_1^9J_{20}^2} + 200q^2\frac{J_2^3J_{10}^6J_{20}^2}{J_1^3J_4^4J_5^5} \\ & + 300q^2\frac{J_2^2J_5^2J_{10}^3J_{20}^2}{J_1^6J_4^4} + 1000q^2\frac{J_2^2J_{10}^{11}}{J_1^{12}J_{20}^2} + 400q^3\frac{J_2^4J_{10}^9J_{20}^2}{J_1^6J_4^4J_5^6} \\ & + 2500q^3\frac{J_2^3J_5J_{10}^6J_{20}^2}{J_1^9J_4^4} + 5000q^4\frac{J_2^4J_{10}^9J_{20}^2}{J_1^{12}J_4^4}.\end{aligned} \tag{5.3}$$

According to (3.15), (5.2), (5.3) and the generating functions $\mathrm{C}\Phi_2(q)$ and $\mathrm{C}\Phi_5(q)$, one can derive an integral expression of $\mathrm{C}\Phi_{10}(q)$. It should be admitted that there are less terms than (3.16). However, our approach is elementary, unified and general. Following a similar strategy as deriving (2.33), we can prove the corresponding expression of $f_7(z)$. Based on this bivariable identity, one can derive an integral expression of $\mathrm{C}\Phi_{14}(q)$ after some tedious computations. Adopting a similar manipulation, one can also derive the integral expressions of $\mathrm{C}\Phi_{15}(q)$ and $\mathrm{C}\Phi_{17}(q)$. However, there are many terms containing the factor $\overline{J}_{a,b}$ in the expressions of $\mathrm{C}\Phi_{15}(q)$ and $\mathrm{C}\Phi_{17}(q)$, these terms can not be further simplified as in derivation of (3.21).

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