STRONGLY UNIMODAL SEQUENCES AND HECKE-TYPE IDENTITIES

SU-PING CUI, HAI-XING DU, AND NANCY S.S. GU

Abstract. A strongly unimodal sequence of size n is a sequence of integers $\{a_j\}_{j=1}^{s}$ satisfying the following conditions:

$$0 < a_1 < a_2 < \cdots < a_k > a_{k+1} > \cdots > a_s > 0 \quad \text{and} \quad a_1 + a_2 + \cdots + a_s = n,$$

for a certain index k, and we usually define its rank as $s - 2k + 1$. Let $u(m, n)$ be the number of strongly unimodal sequences of size n with rank m, and the generating function for $u(m, n)$ is written as

$$\mathcal{U}(z; q) := \sum_{m,n} u(m, n) z^m q^n.$$

Recently, Chen and Garvan established some Hecke-type identities for the third order mock theta function $\psi(q)$ and $U(q)$, which are the specializations of $\mathcal{U}(z; q)$, as advocated by $\psi(q) = \mathcal{U}(\pm i; q)$ and $U(q) = \mathcal{U}(1; q)$. Meanwhile, they inquired whether these Hecke-type identities could be proved via the Bailey pair machinery. In this paper, we not only answer the inquiry of Chen and Garvan in the affirmative, but offer more instances in a broader setting, with, for example, some classical third order mock theta functions due to Ramanujan involved. Furthermore, we extend the Hecke-type identities into multiple series identities. Our work is built upon a handful of Bailey pairs and conjugate Bailey pairs.

1. Introduction

Throughout the paper, we always assume that q is a complex number such that $|q| < 1$ and the following standard q-series notation [28] will be utilized frequently. For positive integers n and m,

$$(a; q)_0 := 1, \quad (a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k),$$

$$(a; q)_{-n} := \frac{1}{(aq^{-n}; q)_n} = \frac{(-q/a)^n q^{\binom{n}{2}}}{(q/a; q)_n},$$

$$(a_1, a_2, \ldots, a_m; q)_n := (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n,$$

$$(a_1, a_2, \ldots, a_m; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty.$$

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We also need the (unilateral) basic hypergeometric series ${}_r\phi_s$ which is given by

$$ {}_r\phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, x\right) := \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, b_2, \ldots, b_s; q)_n} \left((-1)^n q^{\binom{n}{2}}\right)^{1+s-r} x^n. $$

Define

$$ j(x;q) := (x, q/x, q; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} x^n, \tag{1.1} $$

where the second equality is the Jacobi triple product identity [28, Eq. (1.6.1)]. For any positive integer m, let

$$ J_m := \prod_{i=1}^{\infty} (1 - q^{mi}). $$

In this paper, we mainly focus on Hecke-type double sums

$$ \sum_{(m,n)\in D} (-1)^{H(m,n)} q^{Q(m,n)+L(m,n)}, $$

where $H(m,n)$ and $L(m,n)$ are linear forms, $Q(m,n)$ is an indefinite quadratic form, and D is some subset of $\mathbb{Z} \times \mathbb{Z}$ such that $Q(m,n) \geq 0$ for all $(m,n) \in D$.

We follow the definition of strongly unimodal sequences due to Bryson, Ono, Pitman, and Rhoades [15]. A sequence of integers $\{a_j\}_{j=1}^{s}$ is considered as a strongly unimodal sequence of size n if it satisfies the following conditions:

$$ 0 < a_1 < a_2 < \cdots < a_k > a_{k+1} > \cdots > a_s > 0 \quad \text{and} \quad a_1 + a_2 + \cdots + a_s = n, $$

for some k. Let $u(n)$ denote the number of strongly unimodal sequences of size n. Additionally, the rank of a strongly unimodal sequence is defined as $s - 2k + 1$, which means that the number of terms after the maximum term minus the number of terms before the maximum term. Let $u(m,n)$ denote the number of strongly unimodal sequences of size n with rank m. Then the generating function for $u(m,n)$ is given by

$$ \mathcal{U}(z;q) := \sum_{m,n} u(m,n) z^m q^n = \sum_{n=0}^{\infty} (-zq;q)_n (-z^{-1}q;q)_n q^{n+1}. $$

For simplicity, set

$$ U(q) := \mathcal{U}(1;q) = \sum_{n=0}^{\infty} (-q;q)_n^2 q^{n+1}. $$

In [15], Bryson et al. established a connection between $U(-1;q)$ and a quantum modular form which is dual to a quantum modular form given by Zagier [45]. Meanwhile, they obtained a Hecke-type identity for $U(-1;q)$.

$$ U(-1;q) = \sum_{n>0} \sum_{6n\geq|6j+1|} (-1)^{j+1} q^{2n^2 - j(3j+1)/2} + 2 \sum_{n,m>0} \sum_{6n\geq|6j+1|} (-1)^{j+1} q^{2n^2+mn-j(3j+1)/2}. $$

Later, Hikami and Lovejoy [32] found the following Hecke-type identity for $\mathcal{U}(z;q)$.

$$(1+z)\mathcal{U}(z;q)=\frac{q}{J_1}\left(\sum_{n,r\geq 0}-\sum_{n,r<0}\right)(-1)^n z^{-r}q^{n(3n+5)/2+2nr+r(r+3)/2}.$$

Bryson et al. [15] also demonstrated that $U(\pm i;q)$ behaves like a mock modular form and observed that

$$\mathcal{U}(\pm i;q)=\psi(q):=\sum_{n=1}^{\infty}\frac{q^{n^2}}{(q;q^2)_n},\tag{1.2}$$

where $\psi(q)$ is a third order mock theta function given by Ramanujan [41]. Furthermore, they provided some interesting congruence results for $u(n)$ and $u(a,b;n)$ where $u(a,b;n)$ denotes the number of strongly unimodal sequences of size n with rank $\equiv a \pmod{b}$. Meanwhile, they proposed a congruence conjecture involving these two functions.

Beyond strongly unimodal sequences, Kim, Lim, and Lovejoy [33] introduced odd-balanced unimodal sequences. A sequence of integers $\{a_j\}_{j=1}^{s}$ satisfies

$$0<a_1\leq a_2\leq\cdots\leq a_{k-1}<a_k>a_{k+1}\geq\cdots\geq a_{s-1}\geq a_s \quad\text{and}\quad a_1+a_2+\cdots+a_s=n.$$

Besides, the peak a_k is even, odd parts to the right of the peak are the same as those to the left and even parts on both sides of the peak are distinct. Then such a sequence is called an odd-balanced unimodal sequence of size n. Let $v(n)$ denote the number of odd-balanced unimodal sequences of size $2n+2$ and let $v(m,n)$ denote the number of such sequences with rank m, where the definition of rank is consistent with that of $\mathcal{U}(z;q)$. Meanwhile, Kim et al. [33] gave the generating function for $v(m,n)$.

$$\mathcal{V}(z;q):=\sum_{m,n}v(m,n)z^m q^n=\sum_{n=0}^{\infty}\frac{(-zq,-z^{-1}q;q)_n q^n}{(q;q^2)_{n+1}}.\tag{1.3}$$

Setting $z=\pm i$ in the above equality gives

$$\mathcal{V}(\pm i;q)=\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^n}{(q;q^2)_{n+1}}=q^{-1}A(q),$$

where $A(q)$ is a second order mock theta function [38]. Similar to $U(z;q)$, the series $V(z;q)$ can also be expressed as Hecke-type double sums [33].

$$\left(1+\frac{1}{z}\right)\mathcal{V}(z;q)=\frac{J_2}{J_1^2}\left(\sum_{n,r\geq 0}-\sum_{n,r<0}\right)(-1)^n z^r q^{n^2+2n+(2n+1)r+r(r+1)/2}.\tag{1.4}$$

Recently, Bringmann and Lovejoy [14] initiated the study of odd strongly unimodal sequences whose numbers are odd. They derived

$$\sum_{m,n}ou^*(m,n)z^m q^n=\sum_{n=0}^{\infty}(-zq,-z^{-1}q;q^2)_n q^{2n+1},$$

where $ou^*(m,n)$ denotes the number of odd strongly unimodal sequences of size n with rank m. For convenience, we define

$$OU^*(z;q) := \sum_{n=0}^{\infty}(-zq,-z^{-1}q;q^2)_n q^{2n+1}. \tag{1.5}$$

Meanwhile, they [14] obtained the following Hecke-type identity.

$$OU^*(z;q) = \frac{q}{J_2}\left(\sum_{n,r\geq 0} - \sum_{n,r<0}\right)(-1)^n z^r q^{3n^2+3n+4nr+r^2+2r}.$$

For more on unimodal sequences, one can see [11, 12, 34, 35].

In [16], Chen and Garvan confirmed the conjecture posed by Bryson et al. [15] and also proved the other two conjectures related to odd-balanced unimodal sequences [33] and Andrews' spt-function [4]. Meanwhile, they established the following three Hecke-type identities.

$$\psi(q) = \frac{J_2}{J_1^2}\sum_{n=1}^{\infty}\sum_{m=1-n}^{n}(1-q^{2n})(-1)^{m-1}q^{n(3n-1)-2m^2+m}, \tag{1.6}$$

$$U(q) = \frac{1}{J_1}\sum_{n=1}^{\infty}\sum_{r=1}^{n}(1+q^{2n})(-1)^{r-1}q^{n(2n-1)-r(r-1)/2}, \tag{1.7}$$

$$U(q) = \frac{J_2}{J_1^2}\sum_{n=1}^{\infty}\sum_{m=1-n}^{n} sg(m)(1+q^{2n})(-1)^{n-1}q^{n(3n-1)-2m^2+m}, \tag{1.8}$$

where $sg(m) = 1$ if $m > 0$ and $sg(m) = -1$ otherwise. At the end of the paper, they proposed a question seeking a Bailey pair method to prove these identities.

The aim of this paper is to give a positive answer. Meanwhile, some analogous identities and generalizations with more parameters are obtained. In particular, we derive some Hecke-type identities for the third order mock theta functions due to Ramanujan. Furthermore, we extend some theorems into multiple series identities.

Mock theta functions which are a fascinating class of mathematical functions are applied to the fields of number theory and modular forms. These functions which were first introduced by Ramanujan [40] in 1920 have continuously attracted widespread attention from numerous scholars. In the literature, the Bailey pair machinery plays a very important role in deriving Hecke-type identities for mock theta functions.

Definition 1.1. The pair of sequences (α_n, β_n) is called a Bailey pair relative to (a, q) if (α_n, β_n) satisfies

$$\beta_n = \sum_{r=0}^{n}\frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}}. \tag{1.9}$$

In 1979, Andrews [1] gave the following identity

$$\alpha_n=\frac{(1-aq^{2n})(a;q)_n(-1)^nq^{\binom{n}{2}}}{(1-a)(q;q)_n}\sum_{j=0}^{n}(q^{-n},aq^n;q)_jq^j\beta_j, \tag{1.10}$$

where (α_n,β_n) forms a Bailey pair relative to (a,q).

Theorem 1.2. *([3]) (the Bailey lemma) Let ρ_1 and ρ_2 be nonzero complex numbers. If (α_n,β_n) is a Bailey pair relative to (a,q), then so is (α'_n,β'_n), where*

$$\alpha'_n=\frac{(\rho_1,\rho_2;q)_n(aq/\rho_1\rho_2)^n\alpha_n}{(aq/\rho_1,aq/\rho_2;q)_n}, \tag{1.11}$$

$$\beta'_n=\frac{1}{(aq/\rho_1,aq/\rho_2;q)_n}\sum_{j=0}^{n}\frac{(\rho_1,\rho_2;q)_j(aq/\rho_1\rho_2;q)_{n-j}(aq/\rho_1\rho_2)^j\beta_j}{(q;q)_{n-j}}. \tag{1.12}$$

Substituting (1.11) and (1.12) into (1.9), and then setting $n\to\infty$ yields that

$$\sum_{n=0}^{\infty}(\rho_1,\rho_2;q)_n(aq/\rho_1\rho_2)^n\beta_n=\frac{(aq/\rho_1,aq/\rho_2;q)_\infty}{(aq,aq/\rho_1\rho_2;q)_\infty}\sum_{n=0}^{\infty}\frac{(\rho_1,\rho_2;q)_n(aq/\rho_1\rho_2)^n\alpha_n}{(aq/\rho_1,aq/\rho_2;q)_n}. \tag{1.13}$$

Based on the Bailey lemma, Andrews [3] established the Hecke-type identities for the fifth and seventh order mock theta functions which play a crucial role in proving mock theta conjectures. Then Andrews and Hickerson [7] derived the Hecke-type identities for the sixth order mock theta functions. Later, in 2000, Choi [18, 19] discussed the tenth order mock theta functions. Subsequently, Berndt and Chan [10] used the Bailer pair method to obtain the Hecke-type identities for two sixth order mock theta functions. Moreover, Cui, Gu, and Hao [24] considered the second and eighth order mock theta functions. For additional research on mock theta functions, one can refer to [5,13,17,20–23,25,27,29–31,39,41,43,44].

Besides the third order mock theta function $\psi(q)$, recall the following two third order mock theta functions due to Ramanujan [41].

$$\nu(q):=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}, \tag{1.14}$$

$$\phi(q):=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q^2;q^2)_n}. \tag{1.15}$$

The Hecke-type identities for these three functions were widely studied. The following identities related to $\psi(q)$ were given by Andrews [5, Eq. (1.10)], Mortenson [39, Eq. (2.5)], and Chen and Wang [17, Eq. (4.37)], respectively.

$$1+\psi(q)=\frac{1}{J_1}\sum_{n=0}^{\infty}\sum_{j=0}^{n}(1-q^{6n+6})(-1)^nq^{2n^2+n-\binom{j+1}{2}},$$

$$1+2\psi(q)=\frac{1}{J_1}\sum_{n=0}^{\infty}\sum_{j=-n}^{n}(1+q^{2n+1})(-1)^nq^{2n^2+n-\binom{j+1}{2}},$$

$$\psi(q) = -\frac{1}{J_1}\sum_{n=1}^{\infty}\sum_{j=0}^{n-1}(1-q^{2n})(-1)^n q^{2n^2-n-\binom{j+1}{2}}.$$

In addition, Mortenson [39, Eq. (2.6)] found that

$$\nu(-q) = \frac{1}{J_1}\sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^n q^{2n^2+2n-\binom{j+1}{2}},$$

and Chen and Wang [17, Eq. (6.20), Eq. (6.30)] obtained that

$$\phi(q) = \frac{J_2}{J_1J_4}\left(1+2\sum_{n=1}^{\infty}(-1)^n q^{2n^2-n} - \sum_{n=1}^{\infty}\sum_{j=-n}^{n}(1-q^{2n})(-1)^j q^{3n^2-n-j^2}\right),$$

$$\nu(q) = \frac{J_4}{J_2^2}\sum_{n=0}^{\infty}\sum_{j=-n}^{n}(1-q^{2n+1})(-1)^j q^{3n^2+2n-j^2}.$$

The main results of this paper are stated as follows.

First, combining Theorem 1.2 and the Bailey pair obtained by Andrews and Hickerson [7], we derive the following result, which implies the Hecke-type identities for some third order mock theta functions. For any integer m, let a, x, and q be complex numbers with $|x| < 1$ and none of a and axq^{-m+1} of the form q^{-2k} for any nonnegative integer k. Then

$$\sum_{n=0}^{\infty}\frac{q^{n^2+mn}}{(-x;q^2)_{n+1}} = \sum_{n=0}^{\infty}(x^{-1}q^{m+1};q^2)_n(-x)^n \tag{1.16}$$

$$= \frac{(axq^{-m+1},-q^{m+1};q^2)_\infty}{(a,-x;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(1-aq^{4n})(x^{-1}q^{m+1};q^2)_n(-1)^n a^{2n}x^n q^{3n^2-2n-mn}}{(axq^{-m+1};q^2)_n}$$

$$\times\sum_{j=0}^{n}\frac{(1-aq^{4j-2})(a;q^2)_{j-1}(-aq^{-m-1};q^2)_j(-1)^j a^{-2j}q^{-2j^2+3j+mj}}{(q^2,-q^{m+1};q^2)_j}, \tag{1.17}$$

Corollary 1.3. *We have*

$$\nu(q) = \frac{J_1^2J_4^2}{J_2^5}\sum_{n=0}^{\infty}\sum_{j=-n}^{n}(1+q^{2n+1})(-1)^{n+j}q^{3n^2+2n-2j^2}, \tag{1.18}$$

$$\phi(q) = \frac{J_2}{J_4^2}\sum_{n=0}^{\infty}\sum_{j=-n}^{n}(1+q^{2n+1})(-1)^{n+j}q^{3n^2+2n-2j^2+j}, \tag{1.19}$$

$$\psi(q) = \frac{J_2}{J_1^2}\sum_{n=1}^{\infty}\sum_{j=-n+1}^{n}(1-q^{2n})(-1)^{j-1}q^{3n^2-n-2j^2+j} \tag{1.20}$$

$$= \frac{J_2}{J_1^2}\sum_{n=0}^{\infty}\sum_{j=-n+1}^{n}(-1)^j q^{3n^2+n-2j^2+j} - \frac{J_2}{J_1^2}\sum_{n=0}^{\infty}\sum_{j=-n}^{n+1}(-1)^j q^{3n^2+5n-2j^2+j+2}. \tag{1.21}$$

Remark: The identity (1.20) is Chen and Garvan's identity (1.6).

Theorem 1.4. *We have*

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n}}{(x,-q^2;q^2)_n} = \frac{J_1}{J_2^2} \sum_{r=1}^{\infty} \sum_{n=1}^{r} \frac{(1+q^{2n-1})(1+q^{2r})(xq^{-2n+2};q^4)_{n-1}(-1)^{n-1} q^{2r^2-r-n}}{(x;q^2)_{n-1}}. \tag{1.22}$$

Corollary 1.5. *We have*

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n}}{(-q;q)_{2n+1}} = \frac{J_1}{J_2^2} \sum_{r=1}^{\infty} \sum_{n=1}^{r} (1+q^{2r})(-1)^{n-1} q^{2r^2-r-\binom{n}{2}-1}, \tag{1.23}$$

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n}}{(-q;q)_{2n}} = \frac{J_1}{J_2^2} \sum_{r=1}^{\infty} \sum_{n=-r+1}^{r} sg(n)(1+q^{2r})(-1)^{n-1} q^{2r^2-r-\binom{n+1}{2}}, \tag{1.24}$$

where $sg(n) = 1$ if $n > 0$ and $sg(n) = -1$ otherwise.

Remark: Recall the following identity in [6, Eq. (1.2.1)]. For $|t| < 1$ and $|b| < 1$,

$$\sum_{n=0}^{\infty} \frac{(a;q^h)_n (b;q)_{hn}}{(q^h;q^h)_n (c;q)_{hn}} t^n = \frac{(at;q^h)_\infty (b;q)_\infty}{(t;q^h)_\infty (c;q)_\infty} \sum_{n=0}^{\infty} \frac{(t;q^h)_n (c/b;q)_n}{(q;q)_n (at;q^h)_n} b^n. \tag{1.25}$$

Setting $h = 2$, $a = 0$, $b = q$, $c = -q^2$, and $t = q^2$ in (1.25), we derive that

$$\sum_{n=0}^{\infty} (-q;q)_n^2 q^{n+1} = \frac{J_2^2}{J_1^2} \sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n+1}}{(-q;q)_{2n+1}}. \tag{1.26}$$

So, combining (1.23) and (1.26), and then exchanging n and r, we obtain Chen and Garvan's identity (1.7).

Theorem 1.6. *We have*

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n+1}}{(-q;q)_{2n+1}} = \frac{1}{J_2} \sum_{r=1}^{\infty} \sum_{n=-r+1}^{r} sg(n)(1+q^{2r})(-1)^{r-1} q^{3r^2-r-2n^2+n}, \tag{1.27}$$

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n+1}}{(-q^2;q)_{2n+1}} = -\frac{1}{J_2} \sum_{r=0}^{\infty} \sum_{n=-r}^{r} sg(n)(1-q^{4r+2})(-1)^{r-1} q^{3r^2+r-2n^2+n} + 1, \tag{1.28}$$

where $sg(n) = 1$ if $n > 0$ and $sg(n) = -1$ otherwise.

Remark: Combining (1.26) and (1.27), we deduce Chen and Garvan's identity (1.8).

Theorem 1.7. *We have*

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{2n}}{(q^2;q^4)_{n+1}} = \frac{1}{J_2} \sum_{r=1}^{\infty} \sum_{n=1}^{r} (1+q^{2r})(-1)^{r+n} q^{3r^2-r-2n^2+2n-2}, \tag{1.29}$$

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{2n}}{(q^4;q^4)_{n+1}} = \frac{1}{J_2} \sum_{r=1}^{\infty} \sum_{n=1}^{r} (1-q^{4r+2})(-1)^{r+n} q^{3r^2+r-2n^2-2}. \tag{1.30}$$

Remark: We can also obtain generalizations of (1.27)-(1.30), similar to Theorem 1.4. Since the expressions are too complicated, we only state the special cases in Theorems 1.6 and 1.7.

In view of (1.29), we derive the following Hecke-type identity for $OU^*(-1;q)$.

Corollary 1.8. *We have*

$$OU^*(-1;q)=\frac{J_1}{J_2^2}\sum_{r=1}^{\infty}\sum_{n=1}^{r}(1+q^r)(-1)^{r+n}q^{r(3r-1)/2-n^2+n}.$$

Theorem 1.9. *We have*

$$\sum_{n=0}^{\infty}\frac{(-q;q)_nq^n}{(q;q^2)_n}=\frac{J_2}{J_1^2}\sum_{r=0}^{\infty}\sum_{n=-\lfloor (r+1)/2\rfloor}^{\lfloor r/2\rfloor}sg'(n)(1-q^{4n+1})(-1)^{r+n}q^{r^2+2r-n^2-3n},$$

where $sg'(n)=1$ if $n\geq 0$ and $sg'(n)=-1$ otherwise.

Theorem 1.10. *We have*

$$\mathcal{V}(1;q)=\frac{J_2}{J_1^2}\sum_{r=1}^{\infty}\sum_{n=1}^{r}(1+q^{2r})(-1)^{n-1}q^{2r^2-r-n^2+n-1}.$$

Notice that the above identity is different from (1.4) with $z=1$.

Finally, we extend all the above theorems into infinite families. For the length of the paper, we merely provide the multiple series identities related to Theorems 1.4, 1.6, and 1.9 as examples.

Theorem 1.11. *For $k\geq 2$,*

$$\sum_{n=0}^{\infty}(q;q)_{2n}q^{2n}\sum_{n_1,\ldots,n_{k-1}=0}^{\infty}\frac{q^{2N_1^2+\cdots+2N_{k-1}^2+2N_1+\cdots+2N_{k-1}}}{(q^2;q^2)_{n-N_1}(q^2;q^2)_{n_1}\ldots(q^2;q^2)_{n_{k-2}}(q^4;q^4)_{n_{k-1}}(x;q^2)_{n_{k-1}}}$$
$$=\frac{J_1}{J_2^2}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(1+q^{2n+1})(1+q^{2n+2r+2})(xq^{-2n};q^4)_n(-1)^nq^{2kn(n+1)+2r^2+4nr+3r}}{(x;q^2)_n},$$

where $N_j=n_j+n_{j+1}+\cdots+n_{k-1}$.

Theorem 1.12. *For $k\geq 2$,*

$$\sum_{n=0}^{\infty}(q^2;q^2)_nq^{2n}\sum_{n_1,\ldots,n_{k-1}=0}^{\infty}\frac{(xq)^{2N_1+\cdots+2N_{k-1}}q^{2N_1^2+\cdots+2N_{k-1}^2}(q;q^2)_{n_{k-1}}}{(q^2;q^2)_{n-N_1}(q^2;q^2)_{n_1}\ldots(q^2;q^2)_{n_{k-1}}(-xq;q)_{2n_{k-1}+1}}$$
$$=\frac{1}{(x^2q^4;q^2)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(1-xq^{2n+1})(1-x^2q^{4n+4r+4})(x^2q^2;q^2)_{n+r}}{(1-x^2q^2)(q^2;q^2)_{n+r+1}}$$
$$\times(-1)^{r+n}x^{2(nk-n+r)}q^{(2k-1)n^2+2kn+3r^2+6nr+5r},$$

where $N_j=n_j+n_{j+1}+\cdots+n_{k-1}$.

Theorem 1.13. *For $k \geq 2$,*

$$\sum_{n=0}^{\infty}(q^2;q^2)_n q^n \sum_{n_1,\dots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}}}{(q;q)_{n-N_1}(q;q)_{n_1}\dots(q;q)_{n_{k-1}}(q;q^2)_{n_{k-1}}}$$
$$= \frac{J_2}{J_1^2}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}(1-q^{4n+1})(-1)^{n+r}q^{(4k-1)n^2+(2k-1)n+r^2+4nr+2r}$$
$$+\frac{J_2}{J_1^2}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}(1-q^{4n+3})(-1)^{n+r}q^{(4k-1)n^2+(6k-1)n+2k+r^2+4nr+4r},$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$.

This paper is organized as follows. In Section 2, some preliminaries are provided. In Section 3, we prove the main results.

2. Preliminaries

In this section, we collect some q-series identities. Meanwhile, we recall the Bailey transform and present some Bailey pairs and conjugate Bailey pairs.

Lemma 2.1. *([28, Appendix (II.3)]) (the q-binomial theorem) For $|z| < 1$,*

$$\sum_{j=0}^{\infty}\frac{(a;q)_j z^j}{(q;q)_j} = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{2.1}$$

Lemma 2.2. *([26, p. 15]) (the Rogers-Fine identity) For $|\tau| < 1$,*

$$\sum_{r=0}^{\infty}\frac{(\alpha;q)_r\tau^r}{(\beta;q)_r} = \sum_{r=0}^{\infty}\frac{(\alpha;q)_r(\alpha\tau q/\beta;q)_r\beta^r\tau^r q^{r^2-r}(1-\alpha\tau q^{2r})}{(\beta;q)_r(\tau;q)_{r+1}}. \tag{2.2}$$

Lemma 2.3. *([28, Appendix (III.1) and (III.2)]) (the Heine transformations) For $|z| < 1$ and $|b| < 1$,*

$${}_2\phi_1\left(\begin{matrix}a,\ b\\ c\end{matrix};q,z\right) = \frac{(b,az;q)_\infty}{(c,z;q)_\infty}{}_2\phi_1\left(\begin{matrix}c/b,\ z\\ az\end{matrix};q,b\right). \tag{2.3}$$

For $|z| < 1$ and $|c/b| < 1$,

$${}_2\phi_1\left(\begin{matrix}a,\ b\\ c\end{matrix};q,z\right) = \frac{(c/b,bz;q)_\infty}{(c,z;q)_\infty}{}_2\phi_1\left(\begin{matrix}abz/c,\ b\\ bz\end{matrix};q,\frac{c}{b}\right). \tag{2.4}$$

Lemma 2.4. *([28, Appendix (III.10)]) For $|de/abc| < 1$ and $|b| < 1$,*

$${}_3\phi_2\left(\begin{matrix}a,\ b,\ c\\ d,\ e\end{matrix};q,\frac{de}{abc}\right) = \frac{(b,de/ab,de/bc;q)_\infty}{(d,e,de/abc;q)_\infty}{}_3\phi_2\left(\begin{matrix}d/b,\ e/b,\ de/abc\\ de/ab,\ de/bc\end{matrix};q,b\right). \tag{2.5}$$

Lemma 2.5. *([28, Appendix (III.13)]) For any nonnegative integer n,*

$${}_3\phi_2\left(\begin{matrix}q^{-n},\ b,\ c\\ d,\ e\end{matrix};q,\frac{deq^n}{bc}\right) = \frac{(e/c;q)_n}{(e;q)_n}{}_3\phi_2\left(\begin{matrix}q^{-n},\ c,\ d/b\\ cq^{1-n}/e,\ d\end{matrix};q,q\right). \tag{2.6}$$

Lemma 2.6. *([17, Eq. (2.3)]) For any nonnegative integer n,*

$$ {}_3\phi_2\left(\begin{matrix} q^{-n}, \ aq^n, \ \beta \\ c, \ d \end{matrix}; q, q\right) = (-c)^n q^{n(n-1)/2} \frac{(aq/c;q)_n}{(c;q)_n} {}_3\phi_2\left(\begin{matrix} q^{-n}, \ aq^n, \ d/\beta \\ aq/c, \ d \end{matrix}; q, \frac{\beta q}{c}\right). \tag{2.7} $$

Next, in addition to the definition of Bailey pairs and the Bailey lemma mentioned in the introduction, we present some other facts related to Bailey pairs.

Definition 2.7. The pair of sequences (δ_n, γ_n) is called a conjugate Bailey pair relative to (a, q) if (δ_n, γ_n) satisfies

$$ \gamma_n = \sum_{r=n}^{\infty} \frac{\delta_r}{(q;q)_{r-n}(aq;q)_{r+n}}. $$

Lemma 2.8. *([8]) (the Bailey transform) If (α_n, β_n) is a Bailey pair relative to (a, q) and (δ_n, γ_n) is a conjugate Bailey pair relative to (a, q), then*

$$ \sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. $$

The following Bailey pair is given by Andrews and Hickerson [7].

Lemma 2.9. *([7, Theorem 2.3]) Let a, b, c, and q be complex numbers with $a \neq 1$, $b \neq 0$, $c \neq 0$, $q \neq 0$, and none of a/b, a/c, qb, and qc of the form q^{-k} with $k \geq 0$. For $n \geq 0$, define*

$$ \begin{aligned} A_n &= A_n(a, b, c, q) \\ &= \frac{(1-aq^{2n})(a/b, a/c; q)_n (bc)^n q^{n^2}}{(1-a)(bq, cq; q)_n} \sum_{j=0}^{n} \frac{(-1)^j (1-aq^{2j-1})(a;q)_{j-1}(b, c; q)_j}{q^{\binom{j}{2}} (bc)^j (q, a/b, a/c; q)_j}, \\ B_n &= B_n(a, b, c, q) = \frac{1}{(bq, cq; q)_n}. \end{aligned} $$

Then the pair of sequences (A_n, B_n) forms a Bailey pair relative to (a, q).

Recall the Bailer pairs introduced by Lovejoy [37] and Slater [42].

Lemma 2.10. *([37, Eqs. (23) and (24)]) The following pair of sequences (α_n, β_n) forms a Bailey pair relative to (x^2q^2, q^2), where*

$$ \begin{aligned} \alpha_n &= \frac{(1-xq^{2n+1})(x^2q^2;q^2)_n (-1)^n q^{n^2}}{(1-x^2q^2)(q^2;q^2)_n}, \\ \beta_n &= \frac{(q;q^2)_n}{(q^2;q^2)_n (-xq;q)_{2n+1}}. \end{aligned} $$

Lemma 2.11. *([42, Eq. (4.1)]) The following pair of sequences (α_n, β_n) forms a Bailey pair relative to (a, q), where*

$$ \alpha_n = \frac{(1-aq^{2n})(a, b, c; q)_n (-1)^n (aq/bc)^n q^{\binom{n}{2}}}{(1-a)(q, aq/b, aq/c; q)_n}, $$

$$\beta_n = \frac{(aq/bc;q)_n}{(q,aq/b,aq/c;q)_n}.$$

In addition, we also need the following Bailey pairs and conjugate Bailey pairs.

Lemma 2.12. *The following pair of sequences (α_n, β_n) forms a Bailey pair relative to (q^2, q^2), where*

$$\alpha_n = \frac{(1-q^{4n+2})(xq^{-2n};q^4)_n(-1)^n q^{2n^2}}{(1-q^2)(x;q^2)_n},$$

$$\beta_n = \frac{1}{(q^4;q^4)_n(x;q^2)_n}. \tag{2.8}$$

Proof. According to (1.10) with $q \to q^2$ and $a = q^2$, and then using (2.8), we deduce that

$$\alpha_n = \frac{(1-q^{4n+2})(-1)^n q^{n^2-n}}{1-q^2} \sum_{j=0}^{n} \frac{(q^{-2n}, q^{2n+2}; q^2)_j q^{2j}}{(q^4;q^4)_j(x;q^2)_j}. \tag{2.9}$$

In (2.7), replace q by q^2, and then set $a = q^2$, $c = -q^2$, and $d = x$. After letting $\beta \to 0$, we derive that

$$\sum_{j=0}^{n} \frac{(q^{-2n}, q^{2n+2}; q^2)_j q^{2j}}{(q^4;q^4)_j(x;q^2)_j} = q^{n^2+n} \sum_{j=0}^{n} \frac{(q^{-2n}, q^{2n+2}; q^2)_j x^j q^{j^2-j}}{(q^4;q^4)_j(x;q^2)_j}. \tag{2.10}$$

Then replacing q by q^2, and setting $b = q^{2n+2}$, $d = -q^2$, $e = x$, and $c \to \infty$ in (2.6), we obtain that

$$\begin{aligned}\sum_{j=0}^{n} \frac{(q^{-2n}, q^{2n+2}; q^2)_j x^j q^{j^2-j}}{(q^4;q^4)_j(x;q^2)_j} &= \frac{1}{(x;q^2)_n} \sum_{j=0}^{n} \frac{(q^{-4n};q^4)_j (xq^{2n})^j}{(q^4;q^4)_j} \\ &= \frac{(xq^{-2n};q^4)_n}{(x;q^2)_n},\end{aligned} \tag{2.11}$$

where the last step follows from the following identity [28, Appendix (II.4)].

$$\sum_{j=0}^{n} \frac{(q^{-n};q)_j z^j}{(q;q)_j} = (zq^{-n};q)_n.$$

Next, combining (2.10) and (2.11) yields that

$$\sum_{j=0}^{n} \frac{(q^{-2n}, q^{2n+2}; q^2)_j q^{2j}}{(q^4;q^4)_j(x;q^2)_j} = q^{n^2+n} \frac{(xq^{-2n};q^4)_n}{(x;q^2)_n}. \tag{2.12}$$

Finally, substituting (2.12) into (2.9), we prove the lemma. □

Lemma 2.13. *The following pair of sequences (δ_n, γ_n) forms a conjugate Bailey pair relative to (q^2, q^2), where*

$$\delta_n = (q;q)_{2n} q^{2n}, \tag{2.13}$$

$$\gamma_n = \frac{(1-q^2)q^{2n}J_1}{(1-q^{2n+1})J_2^2}\sum_{r=0}^{\infty}(1+q^{2r+2n+2})q^{2r^2+4nr+3r}. \tag{2.14}$$

Proof. Combining Definition 2.7 and (2.13) yields that

$$\begin{aligned}\gamma_n &= \sum_{r=n}^{\infty}\frac{\delta_r}{(q^2;q^2)_{r-n}(q^4;q^2)_{r+n}}\\ &= \sum_{r=0}^{\infty}\frac{\delta_{r+n}}{(q^2;q^2)_r(q^4;q^2)_{r+2n}}\\ &= \sum_{r=0}^{\infty}\frac{(q;q)_{2r+2n}q^{2r+2n}}{(q^2;q^2)_r(q^4;q^2)_{r+2n}}\\ &= \frac{(q;q)_{2n}q^{2n}}{(q^4;q^2)_{2n}}\sum_{r=0}^{\infty}\frac{(q^{2n+1};q)_{2r}q^{2r}}{(q^2;q^2)_r(q^{4n+4};q^2)_r}\\ &= \frac{(1-q^2)q^{2n}J_1}{(1-q^{2n+1})J_2^2}\sum_{r=0}^{\infty}\frac{(q^{2n+2};q^2)_r q^{2nr+2r}}{(q^{2n+3};q^2)_r},\end{aligned} \tag{2.15}$$

where we obtain the last step by invoking (2.3) with q replaced by q^2, and a, b, c, and z replaced by q^{2n+1}, q^{2n+2}, q^{4n+4}, and q^2, respectively.

Next in (2.2), replacing q by q^2, and then setting $\alpha = q^{2n+2}$, $\beta = q^{2n+3}$, and $\tau = q^{2n+2}$, we deduce that

$$\sum_{r=0}^{\infty}\frac{(q^{2n+2};q^2)_r q^{2nr+2r}}{(q^{2n+3};q^2)_r} = \sum_{r=0}^{\infty}(1+q^{2r+2n+2})q^{2r^2+4nr+3r}. \tag{2.16}$$

Thus, substituting (2.16) into (2.15) yields (2.14). Here we complete the proof. □

Lemma 2.14. *The following pair of sequences (γ_n, δ_n) forms a conjugate Bailey pair relative to (x^2q^2, q^2), where*

$$\delta_n = (q^2;q^2)_n q^{2n}, \tag{2.17}$$

$$\gamma_n = \frac{q^{2n}}{(1-q^{2n+2})(x^2q^4;q^2)_\infty}\sum_{r=0}^{\infty}\frac{(x^2q^{2n+2};q^2)_r(1-x^2q^{4r+4n+4})(-1)^r x^{2r}q^{3r^2+6nr+5r}}{(q^{2n+4};q^2)_r}.$$

Proof. According to Definition 2.7 and (2.17), we have

$$\begin{aligned}\gamma_n &= \sum_{r=n}^{\infty}\frac{(q^2;q^2)_r q^{2r}}{(q^2;q^2)_{r-n}(x^2q^4;q^2)_{r+n}}\\ &= \frac{(q^2;q^2)_n q^{2n}}{(x^2q^4;q^2)_{2n}}\sum_{r=0}^{\infty}\frac{(q^{2n+2};q^2)_r q^{2r}}{(q^2,x^2q^{4n+4};q^2)_r}\\ &= \frac{q^{2n}}{(1-q^{2n+2})(x^2q^4;q^2)_\infty}\sum_{r=0}^{\infty}\frac{(-1)^r(x^2q^{4n+4})^r q^{r^2-r}}{(q^{2n+4};q^2)_r},\end{aligned} \tag{2.18}$$

where the last step follows from (2.3) with $q \to q^2$, $a = q^{2n+2}$, $c = x^2q^{4n+4}$, $z = q^2$, and $b \to 0$. Then substituting (2.2) with $q \to q^2$, $\tau = x^2q^{4n+4}/\alpha$, $\beta = q^{2n+4}$, and $\alpha \to \infty$ into (2.18), we complete the proof. $\square$

Lemma 2.15. *The following pair of sequences* (α_n, β_n) *forms a Bailey pair relative to* (q, q)*, where*

$$\alpha_{2n} = \frac{(1-q^{4n+1})(-1)^n q^{3n^2-n}}{1-q}, \tag{2.19}$$

$$\alpha_{2n+1} = \frac{(1-q^{4n+3})(-1)^n q^{3n^2+3n+1}}{1-q}, \tag{2.20}$$

$$\beta_n = \frac{1}{(q;q)_n(q;q^2)_n}. \tag{2.21}$$

Proof. According to (1.10) with $a = q$, and then using (2.21), we deduce that

$$\alpha_n = \frac{(1-q^{2n+1})(-1)^n q^{\binom{n}{2}}}{1-q} \sum_{j=0}^{n} \frac{(q^{-n}, q^{n+1}; q)_j q^j}{(q;q)_j(q;q^2)_j}. \tag{2.22}$$

In (2.7), setting $a = q$, $c = q^{1/2}$, $d = -q^{1/2}$, and $\beta \to 0$, we derive that

$$\sum_{j=0}^{n} \frac{(q^{-n}, q^{n+1}; q)_j q^j}{(q;q)_j(q;q^2)_j} = (-1)^n q^{n^2/2} \frac{(q^{3/2};q)_n}{(q^{1/2};q)_n} \sum_{j=0}^{n} \frac{(q^{-n}, q^{n+1}; q)_j q^{(j^2+j)/2}}{(q, q^{3/2}, -q^{1/2}; q)_j}. \tag{2.23}$$

Next, letting $a = q^{-n}$, $b = q^{n+1}$, $d = q^{3/2}$, $e = -q^{1/2}$, and $c \to \infty$ in (2.5), we obtain that

$$\begin{aligned}
&\sum_{j=0}^{n} \frac{(q^{-n}, q^{n+1}; q)_j q^{(j^2+j)/2}}{(q, q^{3/2}, -q^{1/2}; q)_j} \\
&= \frac{(1-q^{1/2})(q^{n+1}, -q; q)_\infty}{(q;q^2)_\infty} \sum_{j=0}^{\infty} \frac{(q^{-n+1/2}, -q^{-n-1/2}; q)_j q^{nj+j}}{(q^2;q^2)_j} \\
&= \frac{(1-q^{1/2})J_2^2}{(1-q^{-n-1/2})(q;q)_n J_1} \sum_{j=0}^{\infty} \frac{(q^{-2n-1};q^2)_j (1-q^{j-n-1/2}) q^{nj+j}}{(q^2;q^2)_j} \\
&= -\frac{(1-q^{1/2})q^{n+1/2}J_2^2}{(1-q^{n+1/2})(q;q)_n J_1} \left(\sum_{j=0}^{\infty} \frac{(q^{-2n-1};q^2)_j q^{nj+j}}{(q^2;q^2)_j} - q^{-n-1/2} \sum_{j=0}^{\infty} \frac{(q^{-2n-1};q^2)_j q^{nj+2j}}{(q^2;q^2)_j} \right) \\
&= -\frac{(1-q^{1/2})q^{n+1/2}J_2^2}{(1-q^{n+1/2})(q;q)_n J_1} \left(\frac{(q^{-n};q^2)_\infty}{(q^{n+1};q^2)_\infty} - q^{-n-1/2} \frac{(q^{-n+1};q^2)_\infty}{(q^{n+2};q^2)_\infty} \right),
\end{aligned} \tag{2.24}$$

where the last step follows from the q-binomial theorem (2.1). Define

$$L_n := \frac{(q^{-n};q^2)_\infty}{(q^{n+1};q^2)_\infty} - q^{-n-1/2} \frac{(q^{-n+1};q^2)_\infty}{(q^{n+2};q^2)_\infty}. \tag{2.25}$$

Then combining (2.22), (2.23), (2.24), and (2.25), we deduce that

$$\alpha_n = -\frac{(1-q^{2n+1})q^{n^2+n/2+1/2}J_2^2}{(1-q)(q;q)_n J_1} L_n. \tag{2.26}$$

According to the parity of n, we consider the following two cases for L_n. For even n, replacing n by $2n$ in (2.25), we have

$$L_{2n} = -q^{-2n-1/2}\frac{(q^{-2n+1};q^2)_\infty}{(q^{2n+2};q^2)_\infty} = \frac{(q;q)_{2n}(-1)^{n+1}q^{-n^2-2n-1/2}J_1}{J_2^2}. \tag{2.27}$$

So, combining (2.26) and (2.27), we prove (2.19). Similarly, for odd n, replacing n by $2n+1$ in (2.25) yields that

$$L_{2n+1} = \frac{(q^{-2n-1};q^2)_\infty}{(q^{2n+2};q^2)_\infty} = \frac{(q;q)_{2n+1}(-1)^{n+1}q^{-n^2-2n-1}J_1}{J_2^2}. \tag{2.28}$$

Then combining (2.26) and (2.28), we establish (2.20). Therefore, we complete the proof. □

Lemma 2.16. *The following pair of sequences* (δ_n, γ_n) *forms a conjugate Bailey pair relative to* (q, q)*, where*

$$\delta_n = (q^2;q^2)_n q^n, \tag{2.29}$$

$$\gamma_n = \frac{(1-q)q^n J_2}{J_1^2}\sum_{r=0}^{\infty}(-1)^r q^{r^2+2nr+2r}. \tag{2.30}$$

Proof. From Definition 2.7 and (2.29), it can be seen that

$$\begin{aligned}\gamma_n &= \sum_{r=n}^{\infty}\frac{(q^2;q^2)_r q^r}{(q;q)_{r-n}(q^2;q)_{r+n}}\\ &= \frac{(q^2;q^2)_n q^n}{(q^2;q)_{2n}}\sum_{r=0}^{\infty}\frac{(q^{2n+2};q^2)_r q^r}{(q;q)_r(q^{2n+2};q)_r}\\ &= \frac{(1-q)q^n J_2}{(1-q^{n+1})J_1^2}\sum_{r=0}^{\infty}\frac{(-q^{n+1};q)_r(-1)^r q^{nr+r}}{(q^{n+2};q)_r},\end{aligned} \tag{2.31}$$

where we obtain the last step by utilizing (2.3) with a, b, c, and z replaced by q^{n+1}, $-q^{n+1}$, q^{2n+2}, and q, respectively.

Then in (2.2), setting $\alpha = -q^{n+1}$, $\beta = q^{n+2}$, and $\tau = -q^{n+1}$, we deduce that

$$\sum_{r=0}^{\infty}\frac{(-q^{n+1};q)_r(-1)^r q^{nr+r}}{(q^{n+2};q)_r} = (1-q^{n+1})\sum_{r=0}^{\infty}(-1)^r q^{r^2+2nr+2r}. \tag{2.32}$$

Thus, substituting (2.32) into (2.31) yields (2.30). Here we complete the proof. □

Remark: Notice that the identities (1.6) and (1.8) given by Lovejoy in [36] imply the cases $x=q$ and $x=1$ in Lemma 2.14, respectively. Similarly, Lemmas 2.13 and 2.16 can be derived from the identities (1.7) and (1.9) in [36], respectively.

To derive the infinite sequence of Bailey pairs, we require the following result, which can also be obtained by setting $\rho_1,\rho_2\to\infty$ in (1.11) and (1.12).

Lemma 2.17. *([2]) If (α_n,β_n) is a Bailey pair relative to (a,q). Then also (α'_n,β'_n), where*

$$\alpha'_n=a^nq^{n^2}\alpha_n,$$
$$\beta'_n=\sum_{j=0}^{\infty}\frac{a^jq^{j^2}}{(q;q)_{n-j}}\beta_j.$$

Iterating the above lemma yields a Bailey chain. So, combining Lemma 2.12 and the above lemma, we produce the following infinite sequence of Bailey pairs.

Lemma 2.18. *For $k\geq 2$, the following pair of sequences $(\alpha_n^{(k)},\beta_n^{(k)})$ forms a Bailey pair relative to (q^2,q^2), which is*

$$\alpha_n^{(k)}=\frac{(1-q^{4n+2})(xq^{-2n};q^4)_n(-1)^nq^{2kn^2+2(k-1)n}}{(1-q^2)(x;q^2)_n},$$
$$\beta_n^{(k)}=\sum_{n_1,\dots,n_{k-1}=0}^{\infty}\frac{q^{2N_1^2+\cdots+2N_{k-1}^2+2N_1+\cdots+2N_{k-1}}}{(q^2;q^2)_{n-N_1}(q^2;q^2)_{n_1}\dots(q^2;q^2)_{n_{k-2}}(q^4;q^4)_{n_{k-1}}(x;q^2)_{n_{k-1}}},$$

where $N_j=n_j+n_{j+1}+\cdots+n_{k-1}$.

Proof. We prove the lemma by induction on k. For $k=2$, substituting the Bailey pair in Lemma 2.12 into Lemma 2.17 yields that

$$\alpha_n^{(2)}=\frac{(1-q^{4n+2})(xq^{-2n};q^4)_n(-1)^nq^{4n^2+2n}}{(1-q^2)(x;q^2)_n},$$
$$\beta_n^{(2)}=\sum_{j=0}^{\infty}\frac{q^{2j^2+2j}}{(q^2;q^2)_{n-j}(q^4;q^4)_j(x:q^2)_j},$$

which form a new Bailey pair relative to (q^2,q^2), as desired.

Then we assume that $(\alpha_n^{(k-1)},\beta_n^{(k-1)})$ is a Bailey pair relative to (q^2,q^2), giving

$$\alpha_n^{(k-1)}=\frac{(1-q^{4n+2})(xq^{-2n};q^4)_n(-1)^nq^{2(k-1)n^2+2(k-2)n}}{(1-q^2)(x;q^2)_n},$$
$$\beta_n^{(k-1)}=\sum_{n'_1,\dots,n'_{k-2}=0}^{\infty}\frac{q^{2N_1'^2+\cdots+2N_{k-2}'^2+2N'_1+\cdots+2N'_{k-2}}}{(q^2;q^2)_{n-N'_1}(q^2;q^2)_{n'_1}\dots(q^2;q^2)_{n'_{k-3}}(q^4;q^4)_{n'_{k-2}}(x;q^2)_{n'_{k-2}}},$$

where $N'_j=n'_j+n'_{j+1}+\cdots+n'_{k-2}$.

Substituting $(\alpha_n^{(k-1)}, \beta_n^{(k-1)})$ into Lemma 2.17 enables us to obtain a new Bailey pair $(\alpha_n^{(k)}, \beta_n^{(k)})$ relative to (q^2, q^2), where

$$\begin{aligned}
\alpha_n^{(k)} &= \frac{(1-q^{4n+2})(xq^{-2n};q^4)_n(-1)^n q^{2kn^2+2(k-1)n}}{(1-q^2)(x;q^2)_n},\\
\beta_n^{(k)} &= \sum_{n'_{k-1}=0}^{\infty} \frac{q^{2n'^2_{k-1}+2n'_{k-1}}}{(q^2;q^2)_{n-n'_{k-1}}} \beta^{(k-1)}_{n'_{k-1}}\\
&= \sum_{n'_1,\ldots,n'_{k-1}=0}^{\infty} \frac{q^{2n'^2_{k-1}+2n'_{k-1}+2N'^2_1+\cdots+2N'^2_{k-2}+2N'_1+\cdots+2N'_{k-2}}}{(q^2;q^2)_{n-n'_{k-1}}(q^2;q^2)_{n'_{k-1}-N'_1}(q^2;q^2)_{n'_1}\ldots(q^2;q^2)_{n'_{k-3}}(q^4;q^4)_{n'_{k-2}}(x;q^2)_{n'_{k-2}}}.
\end{aligned} \tag{2.33}$$

Next, setting $n_1 = n'_{k-1} - N'_1,\ n_2 = n'_1,\ \ldots,\ n_{k-1} = n'_{k-2}$, and $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$ for $1 \le j \le k-1$ in (2.33) leads to $N_2 = N'_1,\ N_3 = N'_2,\ \ldots,\ N_{k-1} = N'_{k-2}$, and $N_1 = n'_{k-1}$. Hence, we complete the proof. $\square$

Similarly, based on Lemma 2.17, the following two lemmas can be obtained by Lemmas 2.10 and 2.15, respectively. Hence, we omit the proofs.

Lemma 2.19. *For $k \ge 2$, the following pair of sequences $(\alpha_n^{(k)}, \beta_n^{(k)})$ forms a Bailey pair relative to (x^2q^2, q^2), which is*

$$\begin{aligned}
\alpha_n^{(k)} &= \frac{(1-xq^{2n+1})(x^2q^2;q^2)_n(-1)^n(xq)^{2(k-1)n}q^{(2k-1)n^2}}{(1-x^2q^2)(q^2;q^2)_n},\\
\beta_n^{(k)} &= \sum_{n_1,\ldots,n_{k-1}=0}^{\infty} \frac{(q;q^2)_{n_{k-1}}(xq)^{2N_1+\cdots+2N_{k-1}}q^{2N_1^2+\cdots+2N_{k-1}^2}}{(q^2;q^2)_{n-N_1}(q^2;q^2)_{n_1}\ldots(q^2;q^2)_{n_{k-1}}(-xq;q)_{2n_{k-1}+1}},
\end{aligned}$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$.

Lemma 2.20. *For $k \ge 2$, the following pair of sequences $(\alpha_n^{(k)}, \beta_n^{(k)})$ forms a Bailey pair relative to (q, q), which is*

$$\begin{aligned}
\alpha_{2n}^{(k)} &= (-1)^n q^{(4k-1)n^2+(2k-3)n}\frac{1-q^{4n+1}}{1-q},\\
\alpha_{2n+1}^{(k)} &= (-1)^n q^{(4k-1)n^2+3(2k-1)n+2k-1}\frac{1-q^{4n+3}}{1-q},\\
\beta_n^{(k)} &= \sum_{n_1,\ldots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}}}{(q;q)_{n-N_1}(q;q)_{n_1}\ldots(q;q)_{n_{k-1}}(q;q^2)_{n_{k-1}}},
\end{aligned}$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$.

3. Proofs of the main results

In this section, we prove the main results.

Proof of (1.16) and (1.17). Replacing q by q^2 and setting $b = q^2$, $c = -xq^2$, $z = -q^{m+1}/a$, and then letting $a \to \infty$ in (2.4), we obtain that

$$\sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(-xq^2;q^2)_n} = (1+x)\sum_{n=0}^{\infty}(x^{-1}q^{m+1};q^2)_n(-x)^n.$$

So, (1.16) holds.

In Lemma 2.9, replacing q by q^2 and then setting $b = -aq^{-m-1}$ and $c \to 0$, we find that

$$\begin{aligned} A_n &= \frac{(1-aq^{4n})(-q^{m+1};q^2)_n a^{2n} q^{3n^2-2n-mn}}{(1-a)(-aq^{-m+1};q^2)_n} \\ &\quad \times \sum_{j=0}^{n} \frac{(1-aq^{4j-2})(a;q^2)_{j-1}(-aq^{-m-1};q^2)_j(-1)^j a^{-2j} q^{-2j^2+3j+mj}}{(q^2,-q^{m+1};q^2)_j}, \end{aligned} \tag{3.1}$$

$$B_n = \frac{1}{(-aq^{-m+1};q^2)_n}. \tag{3.2}$$

Then substituting (3.1) and (3.2) into (1.13) and setting $q \to q^2$, $\rho_1 = x^{-1}q^{m+1}$, and $\rho_2 = -aq^{-m+1}$, we obtain (1.17). Therefore, we complete the proof. □

Proof of Corollary 1.3. To obtain (1.18), we set $x = q$, $m = 1$, and $a = q^2$ in (1.17). So,

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q^2)_{n+1}} = \frac{J_1^2 J_4^2}{J_2^5} \sum_{n=0}^{\infty}(1+q^{2n+1})(-1)^n q^{3n^2+2n}\left(1+2\sum_{j=1}^{n}(-1)^j q^{-2j^2}\right). \tag{3.3}$$

Observe that

$$1+2\sum_{j=1}^{n}(-1)^j q^{-2j^2} = \sum_{j=-n}^{n}(-1)^j q^{-2j^2}.$$

Substituting the above identity into (3.3), and then using (1.14), we complete the proof of (1.18).

Next, setting $x = q^2$, $m = 2$, and $a = q^2$ in (1.17) yields that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(-q^2;q^2)_{n+1}} &= \frac{J_2}{(1+q)J_4^2} \sum_{n=0}^{\infty}(1+q^{2n+1})(-1)^n q^{3n^2+2n} \\ &\quad \times \left(1+(1+q^{-1})\sum_{j=1}^{n} \frac{(1+q)(1+q^{2j})(-1)^j q^{-2j^2+j}}{(1+q^{2j-1})(1+q^{2j+1})}\right). \end{aligned} \tag{3.4}$$

Notice that

$$\sum_{j=1}^{n} \frac{(1+q)(1+q^{2j})(-1)^j q^{-2j^2+j}}{(1+q^{2j-1})(1+q^{2j+1})}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \frac{(-1)^j q^{-2j^2+j}}{1+q^{2j-1}} + \sum_{j=1}^{n} \frac{(-1)^j q^{-2j^2+j+1}}{1+q^{2j+1}} \\
&= \sum_{j=0}^{n-1} \frac{(-1)^{j+1} q^{-2j^2-3j-1}}{1+q^{2j+1}} + \sum_{j=1}^{n} \frac{(-1)^j q^{-2j^2+j+1}}{1+q^{2j+1}} \\
&= \sum_{j=1}^{n-1} (1-q^{2j+1})(-1)^{j+1} q^{-2j^2-3j-1} + \frac{(-1)^n q^{-2n^2+n+1}}{1+q^{2n+1}} - \frac{q^{-1}}{1+q} \\
&= \sum_{j=2}^{n} (-1)^j q^{-2j^2+j} + \sum_{j=1}^{n-1} (-1)^j q^{-2j^2-j} + \frac{(-1)^n q^{-2n^2+n+1}}{1+q^{2n+1}} - \frac{q^{-1}}{1+q} \\
&= \sum_{j=-n+1}^{n} (-1)^j q^{-2j^2+j} - 1 + q^{-1} + \frac{(-1)^n q^{-2n^2+n+1}}{1+q^{2n+1}} - \frac{q^{-1}}{1+q} \\
&= \sum_{j=-n+1}^{n} (-1)^j q^{-2j^2+j} + \frac{(-1)^n q^{-2n^2+n+1}}{1+q^{2n+1}} - \frac{q}{1+q}. \qquad (3.5)
\end{aligned}
$$

So, substituting (3.5) into (3.4), we obtain that

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(-q^2;q^2)_{n+1}} &= \frac{J_2}{J_4^2} \sum_{n=0}^{\infty} \sum_{j=-n+1}^{n} (1+q^{2n+1})(-1)^{n+j} q^{3n^2+2n-2j^2+j-1} + \frac{J_2}{J_4^2} \sum_{n=0}^{\infty} q^{n^2+3n} \\
&= \frac{J_2}{J_4^2} \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (1+q^{2n+1})(-1)^{n+j} q^{3n^2+2n-2j^2+j-1} \\
&\quad - \frac{J_2}{J_4^2} \sum_{n=0}^{\infty} (1+q^{2n+1}) q^{n^2+n-1} + \frac{J_2}{J_4^2} \sum_{n=0}^{\infty} q^{n^2+3n} \\
&= \frac{J_2}{J_4^2} \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (1+q^{2n+1})(-1)^{n+j} q^{3n^2+2n-2j^2+j-1} - \frac{J_2}{J_4^2} \sum_{n=0}^{\infty} q^{n^2+n-1}.
\end{aligned}
$$

Then multiplying q on both sides, and replacing n by $n-1$ on the left-hand side of the above identity, we have

$$
\sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n} = \frac{J_2}{J_4^2} \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (1+q^{2n+1})(-1)^{n+j} q^{3n^2+2n-2j^2+j} - \frac{J_2}{J_4^2} \sum_{n=0}^{\infty} q^{n^2+n}. \qquad (3.6)
$$

Notice that [9, Eq. (1.3.14)]

$$
\sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{J_2^2}{J_1}. \qquad (3.7)
$$

Thus, substituting (3.7) into (3.6), and then employing (1.15), we prove (1.19).

Setting $x=-q, m=2$, and $a=q^4$ in (1.17) yields that

$$\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q;q^2)_{n+1}}=\frac{(1-q)J_2}{J_1^2}\sum_{n=0}^{\infty}(1-q^{2n+2})q^{3n^2+5n}\times\left(1+\frac{1}{1-q}\sum_{j=1}^{n}(1-q^{2j+1})(-1)^jq^{-2j^2-3j}\right). \tag{3.8}$$

Observe that

$$\begin{aligned}&\frac{1}{1-q}\sum_{j=1}^{n}(1-q^{2j+1})(-1)^jq^{-2j^2-3j}\\&=\frac{1}{1-q}\left(\sum_{j=1}^{n}(-1)^jq^{-2j^2-3j}-\sum_{j=1}^{n}(-1)^jq^{-2j^2-j+1}\right)\\&=\frac{1}{1-q}\left(\sum_{j=2}^{n+1}(-1)^{j-1}q^{-2j^2+j+1}+\sum_{j=-n}^{-1}(-1)^{j-1}q^{-2j^2+j+1}\right)\\&=\frac{1}{1-q}\left(\sum_{j=-n}^{n+1}(-1)^{j-1}q^{-2j^2+j+1}+q-1\right)\\&=\frac{1}{1-q}\sum_{j=-n}^{n+1}(-1)^{j-1}q^{-2j^2+j+1}-1.\end{aligned} \tag{3.9}$$

Then substituting (3.9) into (3.8), we get

$$\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q;q^2)_{n+1}}=\frac{J_2}{J_1^2}\sum_{n=0}^{\infty}\sum_{j=-n}^{n+1}(1-q^{2n+2})(-1)^{j-1}q^{3n^2+5n-2j^2+j+1}.$$

Multiplying q and replacing n by $n-1$ on both sides, and then using (1.2), we derive (1.20).

Finally, setting $x=-q$, $m=2$, and $a=q^2$ in (1.17), and then employing (1.2) and (3.5), we deduce that

$$\begin{aligned}\psi(q)&=\frac{J_2}{J_1^2}\sum_{n=0}^{\infty}\sum_{j=-n+1}^{n}(1-q^{4n+2})(-1)^jq^{3n^2+n-2j^2+j}+\frac{J_2}{J_1^2}\sum_{n=0}^{\infty}(1-q^{2n+1})(-1)^nq^{n^2+2n+1}\\&=\frac{J_2}{J_1^2}\sum_{n=0}^{\infty}\sum_{j=-n+1}^{n}(-1)^jq^{3n^2+n-2j^2+j}-\frac{J_2}{J_1^2}\left(\sum_{n=0}^{\infty}\sum_{j=-n+1}^{n}(-1)^jq^{3n^2+5n-2j^2+j+2}\right.\\&\quad\left.+\sum_{n=0}^{\infty}(-1)^{n+1}q^{n^2+2n+1}+\sum_{n=0}^{\infty}(-1)^nq^{n^2+4n+2}\right),\end{aligned}$$

which implies (1.21). Here we complete the proof. □

Proof of Theorem 1.4. Combining Lemmas 2.8, 2.12, and 2.13, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(q;q^2)_nq^{2n}}{(x,-q^2;q^2)_n}\\&=\frac{J_1}{J_2^2}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(1+q^{2n+1})(1+q^{2r+2n+2})(xq^{-2n};q^4)_n(-1)^nq^{2r^2+4nr+3r+2n^2+2n}}{(x;q^2)_n}\\&=\frac{J_1}{J_2^2}\sum_{n=0}^{\infty}\sum_{r=n}^{\infty}\frac{(1+q^{2n+1})(1+q^{2r+2})(xq^{-2n};q^4)_n(-1)^nq^{2r^2+3r-n}}{(x;q^2)_n}\\&=\frac{J_1}{J_2^2}\sum_{r=0}^{\infty}\sum_{n=0}^{r}\frac{(1+q^{2n+1})(1+q^{2r+2})(xq^{-2n};q^4)_n(-1)^nq^{2r^2+3r-n}}{(x;q^2)_n}\\&=\frac{J_1}{J_2^2}\sum_{r=1}^{\infty}\sum_{n=0}^{r-1}\frac{(1+q^{2n+1})(1+q^{2r})(xq^{-2n};q^4)_n(-1)^nq^{2r^2-r-n-1}}{(x;q^2)_n}.\end{aligned}$$

Then replacing n by $n-1$ on the right-hand side of the above identity, we complete the proof. □

Proof of Corollary 1.5. First, we simplify $(xq^{-2n};q^4)_n$. Depending on the parity of n, we discuss two cases. For even n, replacing n by $2n$ yields that

$$\begin{aligned}(xq^{-4n};q^4)_{2n}&=(xq^{-4n},x;q^4)_n\\&=(x,x^{-1}q^4;q^4)_n(-1)^nx^nq^{-2n^2-2n}.\end{aligned}\tag{3.10}$$

For odd n, replacing n by $2n+1$, we have

$$(xq^{-4n-2};q^4)_{2n+1}=(xq^2;q^4)_n(x^{-1}q^2;q^4)_{n+1}(-1)^{n+1}x^{n+1}q^{-2n^2-4n-2}.\tag{3.11}$$

So, setting $x=-q^3$ in $(xq^{-2n};q^4)_n$, and then using (3.10) and (3.11), we deduce that

$$\begin{aligned}(-q^{-4n+3};q^4)_{2n}&=(-q;q^2)_{2n}q^{-\binom{2n}{2}},\\(-q^{-4n+1};q^4)_{2n+1}&=(-q;q^2)_{2n+1}q^{-\binom{2n+1}{2}}.\end{aligned}$$

According to the above two identities, we conclude that

$$(-q^{-2n+3};q^4)_n=(-q;q^2)_nq^{-\binom{n}{2}}.\tag{3.12}$$

Similarly, setting $x=-q$ in (3.10) and (3.11), we have

$$(-q^{-2n+1};q^4)_n=(-q;q^2)_nq^{-\binom{n+1}{2}}.\tag{3.13}$$

Next, setting $x=-q^3$ in (1.22) yields that

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(q;q^2)_nq^{2n}}{(-q^3,-q^2;q^2)_n}\\&=\frac{J_1}{J_2^2}\sum_{r=1}^{\infty}\sum_{n=1}^{r}\frac{(1+q^{2n-1})(1+q^{2r})(-q^{-2n+5};q^4)_{n-1}(-1)^{n-1}q^{2r^2-r-n}}{(-q^3;q^2)_{n-1}}\end{aligned}$$

$$= \frac{J_1}{J_2^2}\sum_{r=1}^{\infty}\sum_{n=1}^{r}\frac{(1+q^{2n-1})(1+q^{2r})(-q;q^2)_{n-1}(-1)^{n-1}q^{2r^2-r-\binom{n}{2}-1}}{(-q^3;q^2)_{n-1}}$$
$$= \frac{(1+q)J_1}{J_2^2}\sum_{r=1}^{\infty}\sum_{n=1}^{r}(1+q^{2r})(-1)^{n-1}q^{2r^2-r-\binom{n}{2}-1},$$

where the second equality follows from (3.12). Then dividing both sides of the above identity by $(1+q)$, we prove (1.23).

To prove (1.24), we employ the same method by setting $x=-q$ in (1.22) and using (3.13). So,

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n q^{2n}}{(-q;q)_{2n}} = \frac{J_1}{J_2^2}\sum_{r=1}^{\infty}\sum_{n=1}^{r}\frac{(1+q^{2n-1})(1+q^{2r})(-q^{-2n+3};q^4)_{n-1}(-1)^{n-1}q^{2r^2-r-n}}{(-q;q^2)_{n-1}}$$
$$= \frac{J_1}{J_2^2}\sum_{r=1}^{\infty}\sum_{n=1}^{r}(1+q^{2n-1})(1+q^{2r})(-1)^{n-1}q^{2r^2-r-\binom{n+1}{2}}. \tag{3.14}$$

Notice that

$$\sum_{n=1}^{r}(1+q^{2n-1})(-1)^{n-1}q^{-\binom{n+1}{2}} = \sum_{n=1}^{r}(-1)^{n-1}q^{-n(n+1)/2}+\sum_{n=1}^{r}(-1)^{n-1}q^{-n^2/2+3n/2-1}$$
$$= \sum_{n=1}^{r}(-1)^{n-1}q^{-n(n+1)/2}-\sum_{n=1}^{r}(-1)^{n-1}q^{-n(n-1)/2}$$
$$= \sum_{n=-r+1}^{r} sg(n)(-1)^{n-1}q^{-n(n+1)/2},$$

where $sg(n)=1$ if $n>0$ and $sg(n)=-1$ otherwise. Then substituting the above identity into (3.14) yields (1.24). Therefore, we complete the proof. □

Proof of Theorem 1.6. Applying the Bailey transform in Lemma 2.8 and combining the Bailey pair in Lemma 2.10 and the conjugate Bailey pair in Lemma 2.14 with $x=1$, we arrive at

$$q\sum_{n=0}^{\infty}\beta_n\delta_n = \sum_{n=0}^{\infty}\frac{(q;q^2)_n q^{2n+1}}{(-q;q)_{2n+1}}$$
$$= q\sum_{n=0}^{\infty}\alpha_n\gamma_n$$
$$= \frac{1}{J_2}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}(1-q^{2n+1})(1+q^{2r+2n+2})(-1)^{r+n}q^{3r^2+6nr+5r+n^2+2n+1}$$
$$= \frac{1}{J_2}\sum_{n=0}^{\infty}\sum_{r=n}^{\infty}(1-q^{2n+1})(1+q^{2r+2})(-1)^{r}q^{3r^2+5r-2n^2-3n+1}$$

$$
\begin{aligned}
&= \frac{1}{J_2}\sum_{r=0}^{\infty}\sum_{n=0}^{r}(1-q^{2n+1})(1+q^{2r+2})(-1)^r q^{3r^2+5r-2n^2-3n+1} \\
&= \frac{1}{J_2}\sum_{r=1}^{\infty}\sum_{n=1}^{r}(1-q^{2n-1})(1+q^{2r})(-1)^{r-1} q^{3r^2-r-2n^2+n}. \qquad (3.15)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\sum_{n=1}^{r}(1-q^{2n-1})q^{-2n^2+n} &= \sum_{n=1}^{r} q^{-2n^2+n} - \sum_{n=1}^{r} q^{-2n^2+3n-1} \\
&= \sum_{n=1}^{r} q^{-2n^2+n} - \sum_{n=0}^{r-1} q^{-2n^2-n} \\
&= \sum_{n=1}^{r} q^{-2n^2+n} - \sum_{n=-r+1}^{0} q^{-2n^2+n} \\
&= \sum_{n=-r+1}^{r} sg(n) q^{-2n^2+n}, \qquad (3.16)
\end{aligned}
$$

where $sg(n)=1$ if $n>0$ and $sg(n)=-1$ otherwise. Combining (3.15) and (3.16), we prove (1.27).

Next, substituting the Bailey pair in Lemma 2.10 and the conjugate Bailey pair in Lemma 2.14 with $x=q$ into Lemma 2.8, we derive that

$$
\sum_{n=0}^{\infty}\frac{(q;q^2)_n q^{2n+1}}{(-q^2;q)_{2n+1}} = \frac{1}{J_2}\sum_{r=1}^{\infty}\sum_{n=1}^{r}(1-q^{2n})(1-q^{4r+2})(-1)^{r-1}q^{3r^2+r-2n^2-n}. \qquad (3.17)
$$

Notice that

$$
\begin{aligned}
\sum_{n=1}^{r}(1-q^{2n})q^{-2n^2-n} &= \sum_{n=1}^{r} q^{-2n^2-n} - \sum_{n=1}^{r} q^{-2n^2+n} \\
&= \sum_{n=-r}^{0} q^{-2n^2+n} - \sum_{n=1}^{r} q^{-2n^2+n} - 1 \\
&= -\sum_{n=-r}^{r} sg(n) q^{-2n^2+n} - 1.
\end{aligned}
$$

Then substituting the above identity into (3.17), we obtain that

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{(q;q^2)_n q^{2n+1}}{(-q^2;q)_{2n+1}} = &-\frac{1}{J_2}\sum_{r=1}^{\infty}\sum_{n=-r}^{r} sg(n)(1-q^{4r+2})(-1)^{r-1}q^{3r^2+r-2n^2+n} \\
&-\frac{1}{J_2}\sum_{r=1}^{\infty}(1-q^{4r+2})(-1)^{r-1}q^{3r^2+r}.
\end{aligned}
$$

Since

$$\begin{aligned}\sum_{r=1}^{\infty}(1-q^{4r+2})(-1)^{r-1}q^{3r^2+r} &= \sum_{r=1}^{\infty}(-1)^{r-1}q^{3r^2+r}-\sum_{r=1}^{\infty}(-1)^{r-1}q^{3r^2+5r+2}\\ &= \sum_{r=1}^{\infty}(-1)^{r-1}q^{3r^2+r}+\sum_{r=2}^{\infty}(-1)^{r-1}q^{3r^2-r}\\ &= \sum_{r=-\infty}^{\infty}(-1)^{r-1}q^{3r^2+r}+(1-q^2)\\ &= -J_2+(1-q^2),\end{aligned}$$

where we use (1.1) to derive the last step, we have

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_nq^{2n+1}}{(-q^2;q)_{2n+1}}=-\frac{1}{J_2}\sum_{r=1}^{\infty}\sum_{n=-r}^{r}sg(n)(1-q^{4r+2})(-1)^{r-1}q^{3r^2+r-2n^2+n}+1-\frac{1-q^2}{J_2},$$

which implies (1.28). Here we complete the proof. □

Proof of Theorem 1.7. In Lemma 2.11, replace q by q^2, and then let $a=q^2$, $b=-q$, and $c=q$. The resulting Bailey pair relative to (q^2,q^2) is

$$\begin{aligned}\alpha_n &= q^{n^2+n},\\ \beta_n &= \frac{(1-q^2)(-q^2;q^2)_n}{(q^2;q^2)_n(q^2;q^4)_{n+1}}.\end{aligned}$$

Moreover, invoking the conjugate Bailey pair in Lemma 2.14 with $x=1$, we establish that

$$\begin{aligned}\delta_n &= (q^2;q^2)_nq^{2n},\\ \gamma_n &= \frac{(1-q^2)q^{2n}}{J_2}\sum_{r=0}^{\infty}(1+q^{2r+2n+2})(-1)^rq^{3r^2+6nr+5r}.\end{aligned}$$

Then substituting the above Bailey pair and conjugate Bailey pair into Lemma 2.8, we derive that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_nq^{2n}}{(q^2;q^4)_{n+1}} &= \frac{1}{J_2}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}(1+q^{2r+2n+2})(-1)^rq^{3r^2+6nr+5r+n^2+3n}\\ &= \frac{1}{J_2}\sum_{n=0}^{\infty}\sum_{r=n}^{\infty}(1+q^{2r+2})(-1)^{r+n}q^{3r^2+5r-2n^2-2n}\\ &= \frac{1}{J_2}\sum_{r=0}^{\infty}\sum_{n=0}^{r}(1+q^{2r+2})(-1)^{r+n}q^{3r^2+5r-2n^2-2n}.\end{aligned}$$

Replacing r by $r-1$ and then n by $n-1$ on the right-hand side of the above identity, we obtain (1.29).

Finally, combining Lemma 2.11 with $q\to q^2$, $a=q^4$, $b=-q^2$, and $c=q^2$, Lemma 2.14 with $x=q$, and Lemma 2.8, we derive (1.30). Therefore, we complete the proof. □

Proof of Corollary 1.8. We start with $z=-1$ in (1.5), giving

$$OU^*(-1;q)=\sum_{n=0}^{\infty}(q;q^2)_n^2q^{2n+1}. \tag{3.18}$$

Then setting $a=q$, $b=q$, $c=0$, $h=2$, and $t=q^2$ in (1.25), we obtain that

$$\sum_{n=0}^{\infty}(q;q^2)_n^2q^{2n+1}=\frac{J_1^2}{J_2^2}\sum_{n=0}^{\infty}\frac{(-q;q)_nq^{n+1}}{(q;q^2)_{n+1}}. \tag{3.19}$$

Next, combining (1.29) with q replaced by $q^{1/2}$, (3.18), and (3.19), we complete the proof. □

Proof of Theorem 1.9. Applying the Bailey transform in Lemma 2.8, and combining the Bailey pair in Lemma 2.15 and the conjugate Bailey pair in Lemma 2.16, we arrive at

$$\begin{aligned}\sum_{n=0}^{\infty}\beta_n\delta_n&=\sum_{n=0}^{\infty}\frac{(-q;q)_nq^n}{(q;q^2)_n}\\
&=\sum_{n=0}^{\infty}\alpha_{2n}\gamma_{2n}+\sum_{n=0}^{\infty}\alpha_{2n+1}\gamma_{2n+1}\\
&=\frac{J_2}{J_1^2}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}(1-q^{4n+1})(-1)^{r+n}q^{r^2+4nr+2r+3n^2+n}\\
&\quad+\frac{J_2}{J_1^2}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}(1-q^{4n+3})(-1)^{r+n}q^{r^2+4nr+4r+3n^2+5n+2}\\
&=\frac{J_2}{J_1^2}\sum_{n=0}^{\infty}\sum_{r=2n}^{\infty}(1-q^{4n+1})(-1)^{r+n}q^{r^2+2r-n^2-3n}\\
&\quad+\frac{J_2}{J_1^2}\sum_{n=0}^{\infty}\sum_{r=2n}^{\infty}(1-q^{4n+3})(-1)^{r+n}q^{r^2+4r-n^2-3n+2}\\
&=\frac{J_2}{J_1^2}\sum_{r=0}^{\infty}\sum_{n=0}^{\lfloor r/2\rfloor}(1-q^{4n+1})(-1)^{r+n}q^{r^2+2r-n^2-3n}\\
&\quad+\frac{J_2}{J_1^2}\sum_{r=0}^{\infty}\sum_{n=0}^{\lfloor r/2\rfloor}(1-q^{4n+3})(-1)^{r+n}q^{r^2+4r-n^2-3n+2}.\end{aligned} \tag{3.20}$$

Then letting $r\to r-1$ and $n\to -n-1$ in the second term on the right-hand side of (3.20) and simplifying, we have

$$\frac{J_1^2}{J_2}\sum_{n=0}^{\infty}\frac{(-q;q)_nq^n}{(q;q^2)_n}$$

$$
\begin{aligned}
&= \sum_{r=0}^{\infty} \sum_{n=0}^{\lfloor r/2 \rfloor} (1-q^{4n+1})(-1)^{r+n} q^{r^2+2r-n^2-3n} \\
&\quad + \sum_{r=1}^{\infty} \sum_{n=-\lfloor (r+1)/2 \rfloor}^{-1} (1-q^{-4n-1})(-1)^{r+n} q^{r^2+2r-n^2+n+1} \\
&= \sum_{r=0}^{\infty} \sum_{n=0}^{\lfloor r/2 \rfloor} (-1)^{r+n} q^{r^2+2r-n^2-3n} - \sum_{r=0}^{\infty} \sum_{n=0}^{\lfloor r/2 \rfloor} (-1)^{r+n} q^{r^2+2r-n^2+n+1} \\
&\quad + \sum_{r=1}^{\infty} \sum_{n=-\lfloor (r+1)/2 \rfloor}^{-1} (-1)^{r+n} q^{r^2+2r-n^2+n+1} - \sum_{r=1}^{\infty} \sum_{n=-\lfloor (r+1)/2 \rfloor}^{-1} (-1)^{r+n} q^{r^2+2r-n^2-3n}. \qquad (3.21)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\sum_{r=0}^{\infty} \sum_{n=0}^{\lfloor r/2 \rfloor} (-1)^{r+n} q^{r^2+2r-n^2-3n} - \sum_{r=1}^{\infty} \sum_{n=-\lfloor (r+1)/2 \rfloor}^{-1} (-1)^{r+n} q^{r^2+2r-n^2-3n} \\
&= \sum_{r=0}^{\infty} \sum_{n=-\lfloor (r+1)/2 \rfloor}^{\lfloor r/2 \rfloor} sg'(n)(-1)^{r+n} q^{r^2+2r-n^2-3n} \qquad (3.22)
\end{aligned}
$$

and

$$
\begin{aligned}
&-\sum_{r=0}^{\infty} \sum_{n=0}^{\lfloor r/2 \rfloor} (-1)^{r+n} q^{r^2+2r-n^2+n+1} + \sum_{r=1}^{\infty} \sum_{n=-\lfloor (r+1)/2 \rfloor}^{-1} (-1)^{r+n} q^{r^2+2r-n^2+n+1} \\
&= -\sum_{r=0}^{\infty} \sum_{n=-\lfloor (r+1)/2 \rfloor}^{\lfloor r/2 \rfloor} sg'(n)(-1)^{r+n} q^{r^2+2r-n^2+n+1}, \qquad (3.23)
\end{aligned}
$$

where $sg'(n) = 1$ if $n \geq 0$ and $sg'(n) = -1$ otherwise. So, substituting (3.22) and (3.23) into (3.21), we complete the proof. □

Proof of Theorem 1.10. First, letting $z = 1$ in (1.3) and then setting $a = q$, $b = q$, $c = -q^2$, $h = 2$, and $t = q^2$ in (1.25), we have

$$
\mathcal{V}(1;q) = \sum_{n=0}^{\infty} \frac{(-q;q)_n^2 q^n}{(q;q^2)_{n+1}} = \frac{J_2^3}{J_1^3} \sum_{n=0}^{\infty} \frac{(q;q^2)_n^2 q^{2n}}{(-q;q)_{2n+1}}. \qquad (3.24)
$$

Then replacing q by q^2 and setting $a = q^2$, $b = -q^2$, and $c = -q$ in Lemma 2.11, we derive the following Bailey pair relative to $(q^2;q^2)$, where

$$
\alpha_n = \frac{(1-q^{2n+1})(-1)^n q^{n^2}}{1-q},
$$

$$
\beta_n = \frac{(q;q^2)_n}{(q^2,-q^2,-q^3;q^2)_n}.
$$

Thus, by the above Bailey pair and Lemmas 2.8 and 2.13, we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty}\beta_n\delta_n &= \sum_{n=0}^{\infty}\frac{(q;q^2)_n^2q^{2n}}{(-q^2;q)_{2n}}\\ &= \sum_{n=0}^{\infty}\alpha_n\gamma_n\\ &= \frac{(1+q)J_1}{J_2^2}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}(1+q^{2n+2r+2})(-1)^nq^{n^2+2n+2r^2+4nr+3r}\\ &= \frac{(1+q)J_1}{J_2^2}\sum_{n=0}^{\infty}\sum_{r=n}^{\infty}(1+q^{2r+2})(-1)^nq^{2r^2+3r-n^2-n}\\ &= \frac{(1+q)J_1}{J_2^2}\sum_{r=0}^{\infty}\sum_{n=0}^{r}(1+q^{2r+2})(-1)^nq^{2r^2+3r-n^2-n}\\ &= \frac{(1+q)J_1}{J_2^2}\sum_{r=1}^{\infty}\sum_{n=1}^{r}(1+q^{2r})(-1)^{n-1}q^{2r^2-r-n^2+n-1}, \qquad (3.25)\end{aligned}$$

where the last step is followed by shifting r to $r-1$ and n to $n-1$. Hence, combining (3.24) and (3.25), we complete the proof.

Proof of Theorem 1.11. Substituting Lemmas 2.13 and 2.18 into Lemma 2.8, we complete the proof. □

Similarly, applying Lemmas 2.8, 2.14, and 2.19, we prove Theorem 1.12. In view of Lemmas 2.8, 2.16, and 2.20, we prove Theorem 1.13.

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References

[1] G.E. Andrews, An introduction to Ramanujan's "lost" notebook, Amer. Math. Monthly 86 (1979) 89–108.
[2] G.E. Andrews, Multiple series Rogers–Ramanujan identities, Pacific J. Math. 114 (1984) 267–283.
[3] G.E. Andrews, The fifth and seventh order mock theta functions, Trans. Amer. Math. Soc. 293 (1986) 113–134.
[4] G.E. Andrews, The number of smallest parts in the partitions of n, J. Reine Angew. Math. 624 (2008) 133–142.
[5] G.E. Andrews, q-orthogonal polynomials, Rogers–Ramanujan identities, and mock theta functions, Tr. Mat. Inst. Steklova 276 (2012) 27–38.
[6] G.E. Andrews, B.C. Berndt, Ramanujan's Lost Notebook, Part II, Springer, New York, 2009.
[7] G.E. Andrews, D. Hickerson, Ramanujan's "lost" notebook VII: The sixth order mock theta functions, Adv. Math. 89 (1991) 60–105.
[8] W.N. Bailey, Identities of the Rogers–Ramanujan type, Proc. Lond. Math. Soc. (2) 50 (1948) 1–10.

[9] B.C. Berndt, Number theory in the Spirit of Ramanujan, American Mathematical Society, Providence, RI, 2006.
[10] B.C. Berndt, S.H. Chan, Sixth order mock theta functions, Adv. Math. 216 (2007) 771–786.
[11] W. Bridges, K. Bringmann, Statistics for unimodal sequences, Adv. Math. 401 (2022) 108288.
[12] W. Bridges, K. Bringmann, Taylor coefficients of false Jacobi forms and ranks of unimodal sequences, Adv. Math. 426 (2023) 109074.
[13] K. Bringmann, K. Hikami, J. Lovejoy, On the modularity of the unified WRT invariants of certain Seifert manifolds, Adv. in Appl. Math. 46 (2011) 86–93.
[14] K. Bringmann, J. Lovejoy, Odd unimodal sequence, arXiv preprint, arXiv:2308.11556,2023.
[15] J. Bryson, K. Ono, S. Pitman, R.C. Rhoades, Unimodal sequences and quantum and mock modular forms, Proc. Natl. Acad. Sci. USA (40) 109 (2012) 16063–16067.
[16] R. Chen, F.G. Garvan, A proof of the mod 4 unimodal sequence conjectures and related mock theta functions, Adv. Math. 398 (2022) 108235.
[17] D. Chen, L. Wang, Representations of mock theta functions, Adv. Math. 365 (2020) 107037.
[18] Y.-S. Choi, Tenth order mock theta functions in Ramanujan's lost notebook, Invent. Math. 136 (1999) 497–569.
[19] Y.-S. Choi, Tenth order mock theta functions in Ramanujan's lost notebook. II, Adv. Math. 156 (2000) 180–285.
[20] Y.-S. Choi, Tenth order mock theta functions in Ramanujan's lost notebook. IV, Trans. Amer. Math. Soc. 354 (2002) 705–733.
[21] Y.-S. Choi, Tenth order mock theta functions in Ramanujan's lost notebook. III, Proc. Lond. Math. Soc. (3) 94 (2007) 26–52.
[22] S.-P. Cui, H.-X. Du, N.S.S. Gu, Hecke-type double sums and mock theta functions, Colloq. Math. 172 (2023) 191–215.
[23] S.-P. Cui, N.S.S. Gu, Some new mock theta functions, Adv. in Appl. Math. 131 (2021) 102267.
[24] S.-P. Cui, N.S.S. Gu, L.-J. Hao, On second and eighth order mock theta functions, Ramanujan J. 50 (2019) 393–422.
[25] S.-P. Cui, N.S.S. Gu, Q.-H. Hou, C.-Y. Su, Three-parameter mock theta functions, J. Math. Anal. Appl. 515 (2022) 126459.
[26] N.J. Fine, Basic Hypergeometric Series and Applications, American Mathematical Society, Providence, RI, 1988.
[27] F.G. Garvan, Universal mock theta functions and two-variable Hecke–Rogers identities, Ramanujan J. 36 (2015) 267–296.
[28] G. Gasper, M. Rahman, Basic Hypergeometric Series, Second Ed., Cambridge University Press, Cambridge, 2004.
[29] B. Gordon, R.J. McIntosh, Some eighth order mock theta functions, J. London Math. Soc. (2) 62 (2000) 321–335.
[30] B. Gordon, R.J. McIntosh, A survey of classical mock theta functions, in "Partitions, q-series, and modular forms", Dev. Math. 23 (2012) 95–144.
[31] D.R. Hickerson, E. Mortenson, Hecke-type double sums, Appell–Lerch sums, and mock theta functions, I, Proc. Lond. Math. Soc. (3) 109 (2014) 382–422.
[32] K. Hikami, J. Lovejoy, Torus knots and quantum modular forms, Res. Math. Sci. 2 (2015) Art. 2, 15pp.
[33] B. Kim, S. Lim, J. Lovejoy, Odd-balanced unimodal sequences and related functions: parity, mock modularity and quantum modularity, Proc. Amer. Math. Soc. 144 (2016) 3687–3700.
[34] B. Kim, S. Lim, J. Lovejoy, The rank of a unimodal sequence and a partial theta identity of Ramanujan, Int. J. Number Theory 10 (2014) 1081–1098.
[35] B. Kim, S. Lim, J. Lovejoy, Ramanujan-type partial theta identities and rank differences for special unimodal sequences, Ann. Comb. 19 (2015) 705–733.

[36] J. Lovejoy, Ramanujan-type partial theta identities and conjugate Bailey pairs, Ramanujan J. 29 (2012) 51–67.
[37] J. Lovejoy, Bailey pairs and strange identities, J. Korean Math. Soc. 59 (2022) 1015–1045.
[38] R.J. McIntosh, Second order mock theta functions, Canad. Math. Bull. 50 (2007) 284–290.
[39] E. Mortenson, On the three third order mock theta functions and Hecke-type double sums, Ramanujan J. 30 (2013) 279–308.
[40] S. Ramanujan, Collected Papers, Cambridge Univ. Press, 1927; Reprinted, Chelsea, New York, 1962.
[41] S. Ramanujan, The Lost Notebook and Other Unpublished Papers, Narosa, New Delhi, 1988.
[42] L.J. Slater, A new proof of Rogers's transformations of infinite series, Proc. London Math. Soc. (2) 53 (1951) 460–475.
[43] G.N. Watson, The final problem: An account of the mock theta functions, J. London Math. Soc. 11 (1936) 55–80.
[44] G.N. Watson, The mock theta functions (2), Proc. London Math. Soc. (2) 42 (1936) 274–304.
[45] D. Zagier, Vassiliev invariants and a strange identity related to the Dedekind eta-function, Topology 40 (2001) 945–960.

(S.-P. Cui) School of Mathematical Sciences, Qufu Normal University, Qufu 273165, P.R. China

E-mail address: jiayoucui@163.com

(H.-X. Du) Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, P.R. China

E-mail address: haixing_du@163.com

(N.S.S. Gu) Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, P.R. China

E-mail address: gu@nankai.edu.cn