Compatible Powers of Hamilton Cycles in Dense Graphs

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Abstract

The notion of incompatibility system was first proposed by Krivelevich, Lee and Sudakov to formulate the robustness of Hamiltonicity of Dirac graphs. Given a graph $G = (V, E)$, an *incompatibility system* $\mathcal{F}$ over G is a family $\mathcal{F} = \{F_v\}_{v \in V}$ such that for every $v \in V$, F_v is a family of edge pairs $\{e, e'\} \in \binom{E(G)}{2}$ with $e \cap e' = \{v\}$. Moreover, for an integer $k \in \mathbb{N}$, we say $\mathcal{F}$ is *k-bounded* if for every vertex v and its incident edge e, there are at most k pairs in F_v containing e. Krivelevich, Lee and Sudakov proved that there is an universal constant $\mu > 0$ such that for every Dirac graph G and every μn-bounded incompatibility system $\mathcal{F}$ over G, there exists a Hamilton cycle $C \subseteq G$ where every pair of adjacent edges e, e' of C satisfies $\{e, e'\} \notin F_v$ for $\{v\} = e \cap e'$. This resolves a conjecture posed by Häggkvist in 1988 and such a Hamilton cycle is called *compatible* (with respect to $\mathcal{F}$). We study high powers of Hamilton cycles in this context and show that for every $\gamma > 0$ and $k \in \mathbb{N}$, there exists a constant $\mu > 0$ such that for sufficiently large $n \in \mathbb{N}$ and every μn-bounded incompatibility system over an n-vertex graph G with $\delta(G) \geq (\frac{k}{k+1} + \gamma)n$, there exists a compatible k-th power of a Hamilton cycle in G. Moreover, we give a μn-bounded construction which has minimum degree $\frac{k}{k+1}n + \Omega(n)$ and contains no compatible k-th power of a Hamilton cycle.

Keywords: Incompatibility system; Compatible subgraph; Power of Hamilton cycle.

1 Introduction

The classical Dirac's theorem [8] asserts that every graph of order $n \geq 3$ and minimum degree at least $\frac{n}{2}$ contains a *Hamilton cycle*, that is, a cycle passing through every vertex in the graph. Hamilton cycle is a very important and extensively studied notion in graph theory. Also, Dirac's theorem is a cornerstone result in extremal graph theory, and it has been generalized in several directions (see e.g. [7, 10, 17, 20, 24]). One fruitful area is to establish the existence, under certain (minimum) degree conditions, of more general spanning graphs than a Hamilton cycle. A remarkable direction is Pósa and Seymour's conjecture on the existence of powers of Hamilton cycles. For $k \in \mathbb{N}$, the k*-th power* of a Hamilton cycle is defined as a graph on the same vertex set whose edges join distinct vertices at distance at most k in the Hamilton cycle.

Conjecture 1.1. [9, 25] Let G be a graph on n vertices. If $\delta(G) \geq \frac{k}{k+1}n$, then G contains the k-th power of a Hamilton cycle.

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After two decades and several papers on this question, Komlós, Sárközy and Szemerédi [15] resolved this conjecture for large n. In this paper we are interested in a 'robust' version of Conjecture 1.1 under the incompatibility system, suggested by Krivelevich, Lee and Sudakov [19].

Definition 1.2. Let G be a graph with vertex set V. An *incompatibility system* $\mathcal{F}$ over G is a family $\mathcal{F} = \{F_v\}_{v \in V}$ such that for every $v \in V$, F_v is a family of edge pairs in $\{\{e, e'\} \in \binom{E(G)}{2} : e \cap e' = \{v\}\}$.

(1) For every two edges e, e' incident to a vertex v, if $\{e, e'\} \in F_v$, then we say that e and e' are *incompatible* at v. Otherwise, they are *compatible*. A subgraph $H \subseteq G$ is *compatible* if all pairs of adjacent edges are compatible.

(2) For a positive integer Δ, an incompatibility system $\mathcal{F}$ is Δ-*bounded* if for any vertex v and every edge e incident to v, there are at most Δ other edges incident to v that are incompatible with e.

(3) Given constants $\mu, \delta > 0$ and $n \in \mathbb{N}$, an (n, δ, μ)-*incompatibility system* $(G, \mathcal{F})$ consists of an n-vertex graph G with $\delta(G) \geq \delta n$ and a μn-bounded incompatibility system $\mathcal{F}$ over G.

The definition of an incompatibility system is motivated by two concepts in graph theory. First, it is a generalization of *locally ℓ-bounded edge-coloring*, which is an edge-coloring with each color appearing at most ℓ times at any given vertex. In particular, a locally ℓ-bounded edge-coloring induces an $(\ell - 1)$-bounded incompatibility system, where every two adjacent edges of the same color are incompatible. Finding *properly colored subgraphs* (any adjacent edges have different colors) in locally ℓ-bounded edge-colorings of graphs has received considerable attention (see e.g. [1–3,5,22,26]). Note that in incompatibility systems, the corresponding concept to properly colored subgraphs is compatible subgraphs. Moreover, the incompatibility system is also a generalization of the *transition system* introduced by Kotzig [16] in 1968, which is indeed a 1-bounded incompatibility system.

The notion of an incompatibility system appears to provide a new and interesting take on the robustness of graph properties. We refer the reader to a survey of Sudakov [27] where various measures of robustness and relevant results are collected. Krivelevich, Lee and Sudakov [19] first studied the existence of compatible Hamilton cycles in Dirac graphs, which can be viewed as a robust version of Dirac's theorem under the incompatibility system.

Theorem 1.3. [19] There exists a constant $\mu > 0$ such that for large enough n, every $(n, \frac{1}{2}, \mu)$-incompatibility system contains a compatible Hamilton cycle.

They further studied compatible Hamilton cycles in random graphs in [18]. In this paper we explore the degree condition forcing the existence of compatible high powers of Hamilton cycles and prove the following robust version of Conjecture 1.1.

Theorem 1.4 (Main Theorem)**.** For every $\gamma > 0$ and $k \in \mathbb{N}$ with $k \geq 2$, there exists a constant $\mu > 0$ such that for sufficiently large n, every $(n, \frac{k}{k+1} + \gamma, \mu)$-incompatibility system $(G, \mathcal{F})$ contains a compatible k-th power of a Hamilton cycle.

In particular, the term γn in the minimum degree condition of Theorem 1.4 cannot be omitted by the constructions obtained in our previous work [12]. We will give a simple construction for completeness in Section 1.1.

Note that the incompatibility system is a 'local' concept, that is, conflicting edges only appear in adjacent edges. There is an analogous 'global' concept, called a *system of conflicts* (see [6]), in which conflicting edges may not be adjacent. In such a system, the line of research is to find subgraphs with no conflicting edges. It is an interesting problem to prove an analogue of Theorem 1.4 in systems of conflicts.

1.1 A space barrier

We shall give an $(n, \frac{k}{k+1} + \frac{\mu}{2}, \mu)$-incompatibility system $(G, \mathcal{F})$ that contains no compatible k-th power of a Hamilton cycle. Let $0 < \mu < \frac{1}{2(k+1)}, n \in (k+1)\mathbb{N}$ and G_0 be an n-vertex complete $(k+1)$-partite graph with parts $V_1, V_2, \ldots, V_{k+1}$ satisfying $|V_1| = \frac{n}{k+1} + 1, |V_2| = \frac{n}{k+1} - 1$ and $|V_i| = \frac{n}{k+1}$ for every $i \in \{3, \ldots, k+1\}$. Inside every part V_i of G_0, we add a bipartite spanning subgraph with minimum degree at least $\frac{\mu n}{2} + 1$ and maximum degree at most μn. Denote by G the resulting graph. Hence, $\delta(G) \geq \left(1 - \frac{1}{k+1} + \frac{\mu}{2}\right) n$ and for every $i \in [k+1]$, $G[V_i]$ is a triangle-free graph with $\delta(G[V_i]) \geq \frac{\mu n}{2} + 1$ and $\Delta(G[V_i]) \leq \mu n$. Now we define an incompatibility system $\mathcal{F}$ over G. For every two different parts V_i, V_j of G, let v be any vertex in V_i and u, w be any two different vertices in V_j. If uw is an edge in $G[V_j]$, then let vu and vw be incompatible at v. Since $\Delta(G[V_j]) \leq \mu n$, the resulting system $\mathcal{F}$ is μn-bounded. Furthermore, suppose for contradiction that $(G, \mathcal{F})$ contains a compatible k-th power of a Hamilton cycle. As $n \in (k+1)\mathbb{N}$, $(G, \mathcal{F})$ also contains a compatible K_{k+1}-factor, say $\mathcal{K}$. Then since $|V_1| = \frac{n}{k+1} + 1 = |\mathcal{K}| + 1$ and each $G[V_i]$ is a triangle-free graph, by the Pigeonhole Principle, there exists a compatible copy K of K_{k+1} in $\mathcal{K}$ which intersects V_1 in exactly two vertices, say u_1 and w_1. As $k \geq 2$, K also intersects another part except V_1, say V_j for some $j \neq 1$, and choose $v_1 \in V(K) \cap V_j$. Then $u_1 v_1 w_1$ is a compatible triangle, a contradiction.

The paper is organised as follows. In Section 2, we set up some basic notation and crucial lemmas. Then we present the proof of Theorem 1.4 in Section 2.4. Sections 3, 4 and 5 are devoted to proving Lemmas 2.3, 2.4 and 2.5, respectively.

2 Notation and preliminaries

Let P_n be a path of order n. We use P_n^k to denote the k-th power of P_n, where we often call P_n the *base* path. Given a copy of P_n, say $v_1 v_2 \ldots v_n$, we call the k-tuples $(v_k, \ldots, v_1)$ and $(v_{n-k+1}, \ldots, v_n)$ the *ends* of P_n. More often, given a k-tuple $\mathbf{e} = (u_1, u_2, \ldots, u_k)$, we write $\overleftarrow{\mathbf{e}} := (u_k, u_{k-1}, \ldots, u_1)$. For two vertex-disjoint paths P and Q, $P \sim Q$ means that we connect P and Q, i.e. the path PQ.

Definition 2.1. Let G be an n-vertex graph and $\mathcal{F}$ be an incompatibility system over G. For every k-tuple $\mathbf{e} = (u_1, u_2, \ldots, u_k)$ such that $\{u_1, u_2, \ldots, u_k\}$ induces a compatible copy of K_k, we say $\mathbf{f} = (v_1, v_2, \ldots, v_k)$ is a *mate* of $\mathbf{e}$ if $u_1 \ldots u_k v_1 \ldots v_k$ forms a compatible copy of P_{2k}^k in $(G, \mathcal{F})$. Denote by $M(\mathbf{e})$ the number of mates of $\mathbf{e}$ in $(G, \mathcal{F})$.

2.1 Absorption strategy and main tools

Our proof uses the absorption method, introduced by Rödl, Ruciński and Szemerédi [23], and here we shall follow the strategy in the work of Levitt, Sárközy and Szemerédi [21] for embedding powers of Hamilton cycles. The first major task is to find an absorbing structure in the host graph which can "absorb" any (small) set of left-over vertices. Here we adopt a much weaker absorbing strategy in which we only need to absorb subsets of a fixed vertex set, namely the *reservoir* defined in Section 2.4. A similar idea has previously appeared in a recent work of Chang–Han–Thoma [4]. We will see that this weaker version is easier to handle. In particular, we will introduce a slightly stronger notion of absorbers to aid our proof.

Definition 2.2 (β-absorber for a vertex v)**.** Let $k \in \mathbb{N}$, and G be an n-vertex graph and $\mathcal{F}$ is an incompatibility system over G. For every $v \subseteq V$ and $\beta > 0$, we say that a compatible copy A of P^k_{2k} is an *absorber* for v if $V(A) \cup \{v\}$ induces a compatible copy of P^k_{2k+1} which shares the same ends with A. In addition, if both ends have at least βn^k mates in G, then we call A a β-*absorber* for v.

The first result ensures that every vertex has many β-absorbers as above.

Lemma 2.3. Let $k \in \mathbb{N}$. For any $\gamma > 0$, there exist $\beta_1, \beta_2, \mu > 0$ such that for every sufficiently large n, if $(G, \mathcal{F})$ is an $(n, \frac{k}{k+1} + \gamma, \mu)$-incompatibility system, then every vertex of G has at least $\beta_1 n$ vertex-disjoint β_2-absorbers.

In [21], a key step of the proof is to build a k-th power of a short path connecting every two fixed copies of K_k. We are attempting this approach which boils down to building a compatible k-th power of a short path connecting every two fixed compatible copies of K_k. However, in our context, it is still unclear whether a fixed compatible copy of K_k has a mate or not. The worst-case scenario would be that an end of a compatible k-th power of a path (that is, a compatible copy of K_k) has no mate and so it can not be extended to longer ones. The bulk of the work in our paper is to overcome this obstacle. Hence the second major task is to cover almost all vertices using a constant number of compatible k-th power of paths in a 'robust manner' that every end of them has enough mates so as to be extended further.

Lemma 2.4 (Almost cover)**.** Let $k \in \mathbb{N}$. For any $\gamma, \tau > 0$, there exist $\mu, \lambda, \beta > 0$, such that for every sufficiently large n, if $(G, \mathcal{F})$ is an $(n, \frac{k}{k+1} + \gamma, \mu)$-incompatibility system, then there exists a family of vertex-disjoint compatible k-th power of paths each of length at least λn covering all but at most τn vertices of G, where each end has at least βn^k mates.

The following result is used for connecting two ends (compatible copies of vertex-ordered K_r) via a compatible k-th power of a path provided that both of them have polynomially many mates.

Lemma 2.5 (Connecting ends)**.** For any $\beta, \gamma > 0$, there exist $\mu > 0$ and $L = L(\gamma) \in \mathbb{N}$ such that for every sufficiently large n the following is true. Let $(G, \mathcal{F})$ be an $(n, \frac{k}{k+1} + \gamma, \mu)$-incompatibility system, and $W \subseteq V(G)$ with $|W| < \min\{\frac{\gamma}{2}n, \frac{\beta}{2}n\}$. Then for every two disjoint k-tuples of vertices $\mathbf{e}_1, \mathbf{e}_2$ each with a family $\mathcal{M}_i$ of at least βn^k mates ($i \in [2]$), there exists a compatible k-th power of a path Q of length at most L, which has $\overleftarrow{\mathbf{e}}_1, \overleftarrow{\mathbf{e}}_2$ as ends and all other vertices in $G - W$.

As such, Lemma 2.5 allows us to connect two compatible k-th power of paths into a longer one.

Lemma 2.6 (Connecting paths)**.** For any $\beta, \gamma > 0$, there exist $\mu > 0$ and $L = L(\gamma) \in \mathbb{N}$ such that the following holds for sufficiently large n. Let $(G, \mathcal{F})$ be an $(n, \frac{k}{k+1} + \gamma, \mu)$-incompatibility system, and $W \subseteq V(G)$ with $|W| < \min\{\frac{\gamma}{4}n, \frac{\beta}{4}n\}$. Suppose G has two vertices u, v and vertex-disjoint absorbers A_u, A_v on base paths P_1 and P_2, respectively, such that each P_i has an end $\mathbf{e}_i$ with $M(\mathbf{e}_i) \geq \beta n^k, i = 1, 2$. Write

$$P_1 = a_1 a_2 \dots a_k u_1 u_2 \dots u_k,\ \mathbf{e}_1 = (u_1, \dots, u_k),$$

$$P_2 = b_1 b_2 \dots b_k v_1 v_2 \dots v_k,\ \mathbf{e}_2 = (v_1, \dots, v_k).$$

Then there exists a compatible k-th power of a path Q of length at most L in $G - W$, such that

(A) the k-th power of the paths $a_1 a_2 \dots a_k u_1 u_2 \dots u_k \sim Q \sim v_k \dots v_1 b_k \dots b_2 b_1$ and

$$a_1 a_2 \dots a_k \sim u \sim u_1 u_2 \dots u_k \sim Q \sim v_k \dots v_1 \sim v \sim b_k \dots b_2 b_1$$

are all compatible in $(G, \mathcal{F})$.

Proof. For any $\beta, \gamma > 0$, we choose

$$\frac{1}{n} \ll \mu \ll \beta, \gamma, \frac{1}{k} \text{ and additionally } \frac{1}{L} \ll \gamma.$$

We fix $(G, \mathcal{F})$ to be an $(n, \frac{k}{k+1} + \gamma, \mu)$-incompatibility system, and $W \subseteq V(G)$ with $|W| < \min\{\frac{\beta}{4}n, \frac{\gamma}{4}n\}$, and $P_1, P_2, \mathbf{e}_1, \mathbf{e}_2$ given as above such that $M(\mathbf{e}_i) \geq \beta n^k, i = 1, 2$. We say a mate $\mathbf{f} = (f_1, \dots, f_k)$ of $\mathbf{e}_1$ (similarly for $\mathbf{e}_2$) is *good* for P_1 if the k-th power of the paths $a_1 a_2 \dots a_k u_1 u_2 \dots u_k \sim f_1 f_2 \dots f_k$ and $a_1 a_2 \dots a_k \sim u \sim u_1 u_2 \dots u_k \sim f_1 f_2 \dots f_k$ are compatible, and otherwise it is *bad*. Thus if $\mathbf{f}$ is bad for P_1, then there exist $x \in \{a_1, a_2, \dots, a_k, u\}$, u_i and f_j for some $i, j \in [k]$ such that the edges $u_i f_j$ and $x u_i$ are incompatible (at u_i). Therefore as $(G, \mathcal{F})$ is μn-bounded, the number of bad mates $\mathbf{f}$ for P_1 (resp. for P_2) is at most $(k^2 \mu n) n^{k-1}$. For $i \in [2]$, we define the family

$$\mathcal{M}_i = \{\mathbf{f} : \mathbf{f} \text{ is a good mate of } \mathbf{e}_i\}.$$

Then by the choice of $\mu \ll \beta, \gamma, \frac{1}{k}$, it is easy to see that $|\mathcal{M}_i| \geq \frac{\beta}{2} n^k$ for $i \in [2]$ and by applying Lemma 2.5 to G with $(\gamma/2, \beta/2, W \cup V(P_1) \cup V(P_2) \cup \{u, v\})$ in place of (γ, β, W), we can obtain disjoint good mates $\mathbf{f}_i \in \mathcal{M}_i$ for $i \in [2]$ and a compatible k-th power of a path Q of length at most L whose ends are $\overleftarrow{\mathbf{f}}_1$ and $\overleftarrow{\mathbf{f}}_2$. It is easy to check that the k-th power of the paths

$$a_1 a_2 \dots a_k u_1 u_2 \dots u_k \sim Q \sim v_k \dots v_1 b_k \dots b_2 b_1,$$

$$a_1 a_2 \dots a_k \sim u \sim u_1 u_2 \dots u_k \sim Q \sim v_k \dots v_1 \sim v \sim b_k \dots b_2 b_1$$

are all compatible in $(G, \mathcal{F})$. □

To this end, it is worth to remark that Lemma 2.6 enables us to complete the absorption of the left-over vertices. Moreover, instead of β-absorbers, given any two vertex-disjoint compatible copies of P_{2k}^k each of which has an end $\mathbf{e}_i$ with $M(\mathbf{e}_i) \geq \beta n^k, i \in [2]$, we can connect them into a compatible k-th power of a longer path using almost the same argument. The other situations regarding the presence of u or v follow as well and we omit this in the statement of Lemma 2.6.

2.2 Regularity

An important ingredient in our proofs is Szemerédi's Regularity Lemma, and we first give the crucial notion of ε-regular pairs.

Definition 2.7. (Regular pair). Given a graph G and disjoint vertex subsets $X, Y \subseteq V(G)$, the *density* of the pair (X, Y) is defined as $d(X, Y) := \frac{e(X,Y)}{|X||Y|}$, where $e(X, Y) := e(G[X, Y])$. For $\varepsilon > 0$, the pair (X, Y) is ε*-regular* if for every $A \subseteq X, B \subseteq Y$ with $|A| \geq \varepsilon|X|, |B| \geq \varepsilon|Y|$, we have $|d(A, B) - d(X, Y)| < \varepsilon$. Moreover, if $d(X, Y) \geq d$ for some $d > 0$, then we say that (X, Y) is (ε, d)*-regular*.

Fact 2.8. Let (X, Y) be an (ε, d)-regular pair, and $B \subseteq Y$ with $|B| \geq \varepsilon|Y|$. Then all but at most $\varepsilon|X|$ vertices in X have degree at least $(d - \varepsilon)|B|$ in B.

Fact 2.9. (Slicing lemma, [14]). Let (X, Y) be an (ε, d)-regular pair. Then for any $\varepsilon \leq \eta \leq 1$ and $X' \subseteq X, Y' \subseteq Y$ with $|X'| \geq \eta|X|, |Y'| \geq \eta|Y|$, the pair (X', Y') is an (ε', d')-regular pair with $\varepsilon' = \max\{\varepsilon/\eta, 2\varepsilon\}$ and $d' = d - \varepsilon$.

Definition 2.10. (Regular partition). For a graph $G = (V, E)$ and $\varepsilon, d > 0$, a partition $V = V_0 \cup V_1 \cup \ldots \cup V_k$ is (ε, d)*-regular*, if $|V_0| \leq \varepsilon|V|$, $|V_1| = |V_2| = \ldots = |V_k| \leq \lceil \varepsilon|V| \rceil$ and all but at most εk^2 pairs (V_i, V_j) with $1 \leq i < j \leq k$ are (ε, d)-regular. We usually call $V_1, \ldots, V_k$ *clusters* and call V_0 the *exceptional set*.

Lemma 2.11. (Degree form of the Regularity Lemma, [14]). For every $\varepsilon > 0$, there is an $M = M(\varepsilon)$ such that if $G = (V, E)$ is any graph and $d \in (0, 1]$ is any real number, then there is an (ε, d)-regular partition $V = V_0 \cup V_1 \cup \ldots \cup V_k$ with $|V_i| = m$ for each $i \in [k]$, and a spanning subgraph $G' \subseteq G$ with the following properties:

- $1/\varepsilon \leq k \leq M$;
- $d_{G'}(v) > d_G(v) - (d + \varepsilon)|V|$ for all $v \in V$;
- $e(G'[V_i]) = 0$ for all $i \geq 1$;
- all pairs $(V_i, V_j)(1 \leq i < j \leq k)$ are ε-regular in G' with density 0 or at least d.

Definition 2.12. (Reduced graph). Given an arbitrary graph $G = (V, E)$, a partition $V = V_1 \cup \ldots \cup V_k$, and two parameters $\varepsilon, d > 0$, the *reduced graph* $R = R(\varepsilon, d)$ of G is defined as follows: $V(R) = [k]$, and $ij \in E(R)$ if and only if (V_i, V_j) is (ε, d)-regular.

Note that we usually apply Lemma 2.11 on a graph $G = (V, E)$ with parameters $\varepsilon, d > 0$, then obtain an (ε, d)-regular partition $V = V_0 \cup V_1 \cup \ldots \cup V_k$ and a subgraph G'. After that we consider the reduced graph R of $G' \setminus V_0$ with partition $V_1 \cup \ldots \cup V_k$. By Lemma 2.11,

$$\delta(R) \geq \frac{\delta(G) - (d + \varepsilon)|V| - |V_0|}{m} \geq \frac{\delta(G) - (d + 2\varepsilon)|V|}{m}.$$

In particular, if $\delta(G) \geq c|V|$, then $\delta(R) \geq (c - d - 2\varepsilon)|R|$.

We also need a 'compatible' variant of the graph counting lemma as follows.

Lemma 2.13. [12] For constant $d, \eta > 0$ and positive integers $r, h_1, \ldots, h_r$ with $\sum_{i=1}^{r} h_i =: h$, there exist positive constants $\varepsilon^* = \varepsilon^*(r, d, h), c = c(r, d, h)$ and $\mu = \mu(r, d, h, \eta)$ such that the following holds for sufficiently large n. Let $(G,\mathcal{F})$ be a μn-bounded incompatibility system with $|G|=n$ and $U_1,\dots,U_r$ be pairwise vertex-disjoint sets in $V(G)$ with $|U_i|\geq \eta n$, $i\in[r]$ and every pair (U_i,U_j) being (ε^*,d)-regular. Then there exist at least $c\prod_{i=1}^{r}|U_i|^{h_i}$ compatible copies of $K_r(h_1,\dots,h_r)$ in $G[U_1,\dots,U_r]$, each containing exactly h_i vertices in U_i for every $i\in[r]$, where $K_r(h_1,\dots,h_r)$ is the complete r-partite graph with each part of size h_i.

Corollary 2.14. For any $d,\eta>0$, integer $h\geq 1$ and a graph H with $V(H)=\{u_1,\dots,u_h\}$, there exist positive constants $\varepsilon^*=\varepsilon^*(h,d), c=c(h,d)$ and $\mu=\mu(h,d,\eta)$ such that the following holds for sufficiently large n. Let $(G,\mathcal{F})$ be a μn-bounded incompatibility system with $|G|=n$, and $U_1,\dots,U_h$ be pairwise vertex-disjoint sets in $V(G)$ with $|U_i|\geq \eta n$, $i\in[h]$, and (U_i,U_j) are (ε^*,d)-regular if $u_iu_j\in H$, where $\{i,j\}\subseteq[h]$. Then G contains $c\prod_{i=1}^{h}|U_i|$ compatible copies of H, and all the corresponding vertices of u_i are in U_i, where $i\in[h]$.

2.3 Probabilistic tools

We give two well-known concentration inequalities for random variables in this section.

Lemma 2.15. (Chernoff's inequality, [13], Corollary 2.3). Let $X\sim Bin(n,p)$, Then for every $0<a<3/2$, we have

$$\mathbb{P}(|X-\mathbb{E}X|>a\mathbb{E}X)<2e^{-a^2\mathbb{E}X/3}.$$

Lemma 2.16. (Janson's inequality, [13], Theorem 2.14). Let $p\in[0,1]$, G be a graph and R be a random vertex subset obtained by including every vertex of G independently with probability p. Let $\mathcal{F}\subseteq 2^{V(G)}$ be a collection of vertex subsets of G. Given a vertex subset F in $\mathcal{F}$, we denote by I_F the indicator random variable for the event that F is contained in R. Let $X=\sum_{F\in\mathcal{F}}I_F$, $\lambda=\mathbb{E}[X]$ and

$$\overline{\Delta}=\sum_{(F,F')\in\mathcal{F}^2:F\cap F'\neq\emptyset}\mathbb{E}[I_FI_{F'}].$$

Then, for every $\varepsilon\in(0,1)$, we have

$$\mathbb{P}(X\leq(1-\varepsilon)\lambda)\leq\exp\big(-\tfrac{\varepsilon^2\lambda^2}{2\overline{\Delta}}\big).$$

2.4 Putting things together

Proof of Theorem 1.4. For any $\gamma>0$, we choose

$$\frac{1}{n}\ll\mu\ll\lambda,\beta_3\ll\tau\ll p\ll\frac{1}{L},\beta_1,\beta_2\ll\gamma,\frac{1}{k},$$

and let $(G,\mathcal{F})$ be an $(n,\frac{k}{k+1}+\gamma,\mu)$-incompatibility system. By Lemma 2.3, there exist $\beta_1,\beta_2>0$ such that every vertex of G has $\beta_1 n$ vertex-disjoint β_2-absorbers. Let R be a random set of vertices obtained by including every vertex of G independently with probability p. We call R the *reservoir*.

Claim 2.17 (Reservoir)**.** The following properties hold with high probability.

(B1) $\frac{1}{2}pn\leq|R|\leq\frac{3}{2}pn$;

(B2) $d_R(v) \geq (\frac{k}{k+1} + \frac{\gamma}{2})|R|$ for each $v \in V(G)$;

(B3) Every vertex $v \in V(G)$ has at least $\frac{p^{2k}}{2}|\mathcal{A}(v)|$ vertex-disjoint β_2-absorbers in R, where $\mathcal{A}(v)$ is a maximum family of vertex-disjoint β_2-absorbers of v in $(G, \mathcal{F})$;

(B4) Given any fixed constant $\beta > 0$, every k-tuple $\mathbf{e}$ with $M(\mathbf{e}) \geq \beta n^k$ has at least $\frac{\beta n^k p^k}{2}$ mates in R, where $M(\mathbf{e})$ is the number of the mates of $\mathbf{e}$ in $(G, \mathcal{F})$.

Proof. (1) Note that $\mathbb{E}[|R|] = pn$. By Lemma 2.15, we have $\mathbb{P}(||R| - pn| > \frac{pn}{2}) < \exp(-\frac{1}{12}\mathbb{E}[|R|]) = \exp(-pn/12)$.

(2) Choose $\delta \ll \gamma$. We have $\mathbb{P}(|R| > (1+\delta)pn) < \exp(-\delta^2 pn/3)$ by Lemma 2.15. On the other hand, for a given vertex $v \in V(G)$, $\mathbb{E}[d_R(v)] \geq (\frac{k}{k+1}+\gamma)pn$. So $\mathbb{P}\big(d_R(v) < (\frac{k}{k+1}+\gamma)pn(1-\delta)\big) < \exp\big(-\delta^2(\frac{k}{k+1}+\gamma)pn/3\big)$ by Lemma 2.15. Hence

$$\begin{aligned}
\mathbb{P}\big(d_R(v) < (\frac{k}{k+1} + \frac{\gamma}{2})|R|\big) &< \mathbb{P}\big(d_R(v) < (\frac{k}{k+1} + \gamma)\frac{1-\delta}{1+\delta}|R|\big) \\
&\leq \mathbb{P}\big(d_R(v) < (\frac{k}{k+1} + \gamma)pn(1-\delta)\big) + \mathbb{P}(|R| > (1+\delta)pn) \\
&< \exp\big(- \delta^2(\frac{k}{k+1} + \gamma)pn/3\big) + \exp(-\delta^2 pn/3).
\end{aligned}$$

(3) For every vertex $v \in V(G)$, let X_v be the number of vertex-disjoint β_2-absorbers from $\mathcal{A}(v)$ that lie in R. Then $\mathbb{E}[X_v] = p^{2k}|\mathcal{A}(v)|$. Note that $|\mathcal{A}(v)| \geq \beta_1 n$. Hence $\mathbb{P}(X_v < \frac{p^{2k}}{2}|\mathcal{A}(v)|) = \mathbb{P}(X_v < \mathbb{E}[X_v]/2) < \exp(-\frac{\mathbb{E}[X_v]}{12}) < \exp(-\frac{p^{2k}}{12}|\mathcal{A}(v)|) < \exp(-p^{2k}\beta_1 n/12)$ by Lemma 2.15.

(4) For a given k-tuple $\mathbf{e}$ with $M(\mathbf{e}) \geq \beta n^k$, let $\mathcal{F}$ be the set of mates for $\mathbf{e}$. Then $|\mathcal{F}| = M(\mathbf{e}) \geq \beta n^k$. We choose a subfamily $\mathcal{F}'$ of $\mathcal{F}$ with exactly βn^k members. For each $F \in \mathcal{F}'$, let I_F be the indicator random variable with

$$I_F = \begin{cases} 1 & \text{if } F \text{ is contained in } R, \\ 0 & \text{otherwise,} \end{cases}$$

and let $X = \sum_{F \in \mathcal{F}'} I_F$. We have $\mathbb{E}[X] = \beta n^k p^k$. Recall that $\overline{\Delta} = \sum_{(F,F') \in \mathcal{F}'^2 : F \cap F' \neq \emptyset} \mathbb{E}[I_F I_{F'}]$. Note that $\mathbb{E}[I_F I_{F'}] = \mathbb{E}[I_F]$ when $F = F'$. Then

$$\sum_{(F,F') \in \mathcal{F}'^2 : F = F'} \mathbb{E}[I_F I_{F'}] = \sum_{F \in \mathcal{F}'} \mathbb{E}[I_F] = \mathbb{E}[X].$$

Hence $\overline{\Delta} \leq \mathbb{E}[X] + \sum_{s=1}^{k} \binom{k}{s}|\mathcal{F}'| \cdot n^{k-s} \cdot p^{2k-s}$ and so

$$\begin{aligned}
\overline{\Delta} - \mathbb{E}[X] &\leq \sum_{s=1}^{k-1} \binom{k}{s} |\mathcal{F}'| \cdot n^{k-s} \cdot p^{2k-s} \\
&= \sum_{s=1}^{k-1} \binom{k}{s} \beta (np)^{2k-s} \\
&\leq k^k \beta (np)^{2k-1}.
\end{aligned}$$

By Lemma 2.16 we have that $\mathbb{P}(X \leq \mathbb{E}[X]/2) \leq \exp\Big(- \frac{(\mathbb{E}[X])^2}{8\overline{\Delta}}\Big) \leq \exp\Big(- \frac{\beta np}{8k^k}\Big)$.

By the union bound, we obtain that with high probability, **(B1)**–**(B4)** hold. □

Connecting β_2-absorbers. We choose the reservoir R as above. Note that by Lemma 2.3 each vertex has at least $\beta_1 n$ vertex-disjoint β_2-absorbers. Then by the choice of $p \ll \beta_1, \beta_2$ and Claim 2.17 (**B1**), we can greedily pick a collection $\mathcal{C} = \{A_v : v \in R\}$ of vertex-disjoint β_2-absorbers A_v (for v) from $V(G) \setminus R$. Moreover, $\delta(G-R) \geq (\frac{k}{k+1} + \frac{\gamma}{2})n$ and every end of A_v has at least $\beta_2 n^k - |R|n^{k-1} \geq \frac{\beta_2}{2}n^k$ mates in $G-R$. Let $\mathcal{S}$ be an arbitrary sequence of all these copies $A_v, v \in R$. By repeatedly applying Lemma 2.6, we can iteratively connect two consecutive copies from $\mathcal{S}$ into a compatible k-th power of a path, say P_R, via a collection of $|R|-1$ vertex-disjoint compatible k-th power of paths of length at most L in $G-R$. Indeed, during the process, suppose we have two consecutive absorbers A_u, A_v to be connected and let $\mathcal{Q}$ be the family of vertex-disjoint k-th power of paths used for previous connections along the sequence. Since $p \ll \frac{1}{L}, \beta_2 \ll \gamma, \frac{1}{k}$ and thus $|R| + |V(\mathcal{Q})| + |V(\mathcal{S})| \leq (L+1+2k) \cdot \frac{3}{2}pn \leq \min\{\frac{\gamma}{8}n, \frac{\beta_2}{8}n\}$, we can apply Lemma 2.6 to $G-R$ with $(\frac{\gamma}{2}, \frac{\beta_2}{2}, R \cup V(\mathcal{Q}) \cup V(\mathcal{S}))$ in place of (γ, β, W) and obtain a compatible k-th power of a path Q_{uv} satisfying (**A**).

Almost cover. Let $G' = G[V(G) \setminus (R \cup V(P_R))]$. Then $\delta(G') \geq \delta(G) - (L+2k)\frac{3}{2}pn \geq (\frac{k}{k+1} + \frac{\gamma}{2})n$. By Lemma 2.4, G' contains a family of vertex-disjoint compatible k-th power of paths $P_1, \ldots, P_s$ for some $s \leq \frac{1}{\lambda}$, each of length at least λn, that cover all but at most τn vertices of G', and all the ends of them have at least $\beta_3 n^k$ mates in $V(G)$ and thus at least $\frac{\beta_3 p^k}{2}n^k$ mates in R by Claim 2.17 (**B4**). By repeatedly applying Lemma 2.6 to R with $(\frac{\gamma}{2}, \frac{\beta_3 p^k}{2})$ in place of (γ, β) as above, we can iteratively connect $P_1, \ldots, P_s$ into a compatible k-th power of a path, say $P_{G'}$, via vertex-disjoint k-th power of paths of length at most L in R. This can be done because in the process, we need to avoid a set of at most $(s-1)L < \min\{\frac{\gamma}{8}|R|, \frac{\beta_3 p^k}{8}|R|\}$ vertices that have been used in previous connections.

Connecting three paths. Let T the set of at most τn vertices not covered by the paths $P_1, \ldots, P_s$ and write $T = \{v_1, v_2, \ldots, v_t\}$, where $t = |T| \leq \tau n$. Note that $S := R \cap V(P_{G'})$ has size $|S| \leq sL \leq \frac{L}{\lambda}$. Thus by Lemma 2.3 and Claim 2.17 (**B3**), every vertex in T has at least $\beta_1 p^{2k} n/2 - |S| \geq \beta_1 p^{2k} n/4$ vertex-disjoint β_2-absorbers in $R \setminus V(P_{G'})$. We assign $v_1 \in T$ a β_2-absorber in $R \setminus V(P_{G'})$, say A_1, and denote by A_1' the corresponding compatible copy of P_{2k+1}^k induced by $A_1 \cup \{v_1\}$, each end of which has at least $\beta_2 n^k$ mates in G. Also, by applying Lemma 2.6 to R twice, we can connect $P_R, P_{G'}, A_1'$ in order and obtain a compatible k-th power of a path, say P_{comb}. This is because we only need to avoid a constant number of vertices used in previous connections. Note that both ends of P_{comb} have at least $\beta_2 n^k$ mates in G.

Connecting the rest. Note that $S_1 := R \cap V(P_{comb})$ has $|S_1| \leq (s+2)L$. Then every vertex in T has at least $\beta_1 p^{2k} n/2 - |S_1| \geq \beta_1 p^{2k} n/4$ vertex-disjoint β_2-absorbers in $R \setminus V(P_{G'})$. Now, since $|T| \leq \tau n$ and $\tau \ll p, \beta_1$, we can greedily assign every vertex v_i in $T \setminus \{v_1\}$ a β_2-absorber in $R \setminus S_1$, say A_i, such that these absorbers are vertex-disjoint. Similarly for every $i \in [2,t]$, we denote by A_i' the corresponding compatible copy of P_{2k+1}^k induced by $A_i \cup \{v_i\}$, each end of which has at least $\frac{\beta_2 p^k}{2}n^k$ mates in R.

Next we shall connect $P_{comb}, A_2', \ldots, A_t'$ into a compatible k-th power of a cycle. Recall that $t \leq \tau n$, $d_R(v) \geq (\frac{k}{k+1} + \frac{\gamma}{2})|R|$ for each $v \in V(G)$ and every end of $P_{comb}, A_2', \ldots, A_t'$ has at least $\frac{\beta_2 p^k}{2}n^k$ mates in R. Again, by applying Lemma 2.6 to R we iteratively connect $P_{comb}, A_2', \ldots, A_t'$ into a compatible k-th power of a cycle via a collection of t vertex-disjoint compatible k-th power of paths of length at most L. In fact, in each step $i, i \in [t]$, since $\tau \ll p, \beta_2, \gamma$ and thus the number of vertices in R covered in previous steps is at most $|S_1| + iL \leq 2L\tau n < \min\{\frac{\gamma}{8}|R|, \frac{\beta_2 p^k}{8}|R|\}$, by Lemma 2.6 one can obtain a path as desired. Let C be the resulting cycle. Finally, using property (**A**), we absorb the leftover vertices in $V(G)\setminus V(C)$ by the absorbers A_v in $\mathcal{C}$ and obtain a compatible k-th power of a Hamilton cycle, as desired. □

3 Absorbing lemma

Proof of Lemma 2.3. Given $\gamma > 0$, we choose

$$\frac{1}{n} \ll \mu, \beta_1, \beta_2 \ll \varepsilon, c \ll d \ll \gamma, \frac{1}{k}.$$

For every v, we use G_v to denote $G[N(v)]$. Firstly we apply Lemma 2.11 on G_v with density d and obtain an (ε, d)-regular partition $V(G_v) = V_0 \cup V_1 \cup \ldots \cup V_r$ for some $1/\varepsilon \le r \le M$ and $|V_i| = m \ge \frac{(1-\varepsilon)|G_v|}{r}$ for each $i \in [r]$. Let $R = R(\varepsilon, d)$ be the reduced graph for this partition. Note that

$$\begin{aligned}\delta(G_v) &\ge (\frac{k}{k+1} + \gamma)n - (n - |G_v|)\\ &\ge |G_v| - \frac{1}{k+1}n + \gamma n\\ &\ge (1 - \frac{1}{k})|G_v| + \gamma n.\end{aligned}$$

The last inequality follows since $|G_v| \ge (\frac{k}{k+1} + \gamma)n$. As $\varepsilon \ll d \ll \gamma$, we have $\delta(R) \ge \big(1 - \frac{1}{k} + \gamma - d - 2\varepsilon\big)r \ge \big(\frac{k-1}{k} + \frac{\gamma}{2}\big)r$. It follows that every k vertices in R have at least γr common neighbors. Hence we can greedily find a copy of P_{2k}^k in R, denoted by H. Let $V(H) = \{u_1, \ldots, u_{2k}\}$, and $V_1, \ldots, V_{2k}$ be the corresponding clusters. By Definition 2.12, (V_i, V_j) is (ε, d)-regular if $u_i u_j$ is an edge of H in R. By Corollary 2.14, there exists $c > 0$ such that $G_v[V_1 \cup \ldots \cup V_{2k}]$ contains at least $cm^{2k} \ge c' n^{2k}$ compatible copies of P_{2k}^k, where c' is a constant depending on ε and c.

We claim that for any $0 < \beta_2 < c'/2$, there exists a family $\mathcal{P}$ of at least $\frac{c'}{2} n^{2k}$ compatible copies of P_{2k}^k such that every end of them has at least $\beta_2 n^k$ mates in G. For otherwise, let $x < \frac{c'}{2} n^{2k}$ be the number of compatible copies of P_{2k}^k such that every end of them has at least $\beta_2 n^k$ mates. Then the number of k-tuples which have at least $\beta_2 n^k$ mates is at least $\frac{x}{n^k}$, and so the number of k-tuples which have less than $\beta_2 n^k$ mates is at most $n^k - \frac{x}{n^k}$. Hence the total number of compatible copies of P_{2k}^k in $G_v[V_1 \cup \ldots \cup V_{2k}]$ is at most

$$x + (n^k - \frac{x}{n^k}) \cdot \beta_2 n^k < (\frac{c'}{2} + \beta_2 - \frac{c'}{2}\beta_2) n^{2k} < c' n^{2k},$$

a contradiction.

Among those copies in $\mathcal{P}$, at least $(\frac{c'}{2} - 2\mu)n^{2k}$ of them are β_2-absorbers for v. Indeed, since $(G, \mathcal{F})$ is an $(n, \frac{k}{k+1} + \gamma, \mu)$-incompatibility system, there are at most μn^2 incompatible pairs vv_1, vv_2 and for each such pair vv_1, vv_2, there are at most n^{2k-2} copies of P_{2k}^k containing v_1 and v_2. Moreover, at most $\mu n^2 \cdot n^{2k-2}$ copies contain an edge $v_1 v_2$ which is incompatible with vv_1. Hence we obtain at least $(\frac{c'}{2} - 2\mu)n^{2k}$ β_2-absorbers for v. As $\mu, \beta_1 \ll \varepsilon, c$, we can greedily find $\beta_1 n$ vertex-disjoint β_2-absorbers for v.

4 Almost cover

Proof of Lemma 2.4. Given $\gamma, \tau > 0$ and $k \in \mathbb{N}$, we choose

$$\frac{1}{n} \ll \mu, \beta, \lambda \ll \varepsilon, c \ll \eta, d \ll \gamma, \tau, \frac{1}{k}$$

and fix $(G, \mathcal{F})$ to be an $(n, \frac{k}{k+1} + \gamma, \mu)$-incompatibility system. We apply Lemma 2.11 on G to obtain an (ε, d)-regular partition $V(G) = V_0 \cup V_1 \cup \ldots \cup V_r$ for some $1/\varepsilon \le r \le M$ and $|V_i| = m \ge \frac{(1-\varepsilon)n}{r}$ for each $i \in [r]$. Let $R = R(\varepsilon, d)$ be the reduced graph for this partition. Since $\varepsilon \ll d \ll \gamma$, we have $\delta(R) \ge \left(\frac{k}{k+1} + \gamma - d - 2\varepsilon\right)r \ge \left(\frac{k}{k+1} + \frac{\gamma}{2}\right)r$. By the Hajnal–Szemerédi theorem [11], R has a K_{k+1}-tiling $\mathcal{K} = \{K^{(1)}, K^{(2)}, \ldots, K^{\lfloor \frac{r}{k+1} \rfloor}\}$ covering all but at most k vertices. Next, we shall pick vertex-disjoint compatible k-th power of long paths within the corresponding $k+1$ clusters for every $K^{(i)} \in \mathcal{K}$. Without loss of generality, we may take $K^{(1)}$ for instance and assume that the corresponding clusters of $K^{(1)}$ are $V_1, \ldots, V_{k+1}$. It suffices to prove the following result.

Claim 4.1. For any collection of subsets $U_i \subseteq V_i, i \in [k+1]$ with $|U_i| \ge \frac{\eta}{2}m$, there exists, in $G[U_1, \ldots, U_{k+1}]$, a compatible copy of P_s^k which satisfies the following properties:

(C1) $s \in (k+1)\mathbb{N}$ and $s \ge \lambda n$;

(C2) both ends have at least βn^k mates.

Thus taking this for granted, we greedily pick a family of vertex-disjoint compatible copies of P_s^k, which altogether leave less than $\frac{\eta}{2}m$ vertices in each V_i ($i \in [k+1]$) uncovered. Applying this for every $K^{(i)} \in \mathcal{K}$, we obtain a desired family of vertex-disjoint compatible copies of P_s^k and the number of vertices uncovered is at most

$$km + \frac{\eta}{2}m \cdot r + \varepsilon n < \tau n.$$

Proof of Claim 4.1. By Corollary 2.14 applied with $h = k+1$, there exists $c = c(k, d-\varepsilon) > 0$ such that the $(k+1)$-partite graph $G[U_1, \ldots, U_{k+1}]$ contains at least $c(\frac{\eta}{2}m)^{k+1} \ge c_1 n^{k+1}$ compatible copies of K_{k+1} for a constant $c_1 > 0$. We construct an auxiliary $(k+1)$-uniform $(k+1)$-partite hypergraph H on $V(H) := U_1 \cup \ldots \cup U_{k+1}$ with

$$E(H) := \left\{\{v_1, v_2, \ldots, v_{k+1}\} : v_i \in U_i,\ i \in [k+1], \text{ and } v_1, \ldots, v_{k+1} \text{ induce a compatible copy of } K_{k+1}\right\}.$$

Thus H has at least $c_1 n^{k+1}$ hyperedges. Now we claim that H has a subgraph H' such that every k-set is contained in either at least $\frac{c_1}{2}n$ hyperedges or zero hyperedge. In fact, we can proceed by iteratively removing the edges from H as follows. If a k-set S is contained in less than $\frac{c_1}{2}n$ hyperedges in the current hypergraph, then we remove all hyperedges containing S. Note that in the process, we remove at most $\frac{c_1}{2}n \cdot m^k \cdot (k+1) \le \frac{c_1}{2}n^{k+1}$ hyperedges in total. The process terminates at a nonempty hypergraph H' as desired.

Now we pick a longest compatible k-th power of a path, denoted as $P = u_1 \ldots u_{s'}$ in $G[U_1, \ldots, U_{k+1}]$, such that for every $i \in [s'-k]$, $\{u_i, \ldots, u_{i+k}\}$ is a hyperedge of H'. Suppose for contradiction that $s' < 2\lambda n$. Note that in H', the k-set $S = \{u_{s'-k+1}, \ldots, u_{s'}\}$ has a set of at least $\frac{c_1}{2}n - s' \ge \frac{c_1}{4}n$ neighbors in $V(H') \setminus V(P)$, denoted as A. In particular, as $\mathcal{F}$ is μn-bounded, there are at most $k^2\mu n$ vertices v such that for some $i \in \{s'-k+1,\dots,s'\}$ and $j \in \{i-k,\dots,i-1\}$, the edges u_iv and u_iu_j are not compatible at u_i. By the choice of $\mu \ll \varepsilon$ and thus $|A| > k^2\mu n$, we can pick a vertex from A to extend P as required, a contradiction. Thus by consecutively removing vertices from one end of P, we can obtain a compatible k-th power of a subpath on s vertices with $s \in (k+1)\mathbb{N}$ and $s \geq \lambda n$.

It remains to prove **(C2)**. In fact, we can extend every k-set from a hyperedge to at least $(\frac{c_1}{4}n - k^2\mu n)^k \geq \beta n^k$ compatible copies of P^k_{2k} as above. This is because in each step, we have at least $\frac{c_1}{4}n - k^2\mu n$ choices for the next vertex. This completes the entire proof. ■

□

5 Connecting two ends

We will make use of a result of Komlós–Sárközy–Szemerédi [15].

Lemma 5.1. [15] For every $\gamma > 0$ and $k \in \mathbb{N}$, there exists $L \in \mathbb{N}$ such that the following holds for sufficiently large $n \in \mathbb{N}$. Let R be an n-vertex graph with $\delta(R) \geq (\frac{k}{k+1} + \gamma)n$ and $\mathbf{e}_1, \mathbf{e}_2$ be two disjoint k-tuples of vertices, each of which induces a copy of K_k. Then there exists a k-th power of a path P of length at most L, whose ends are $\overleftarrow{\mathbf{e}}_1$ and $\overleftarrow{\mathbf{e}}_2$.

Proof of Lemma 2.5. For any $\beta, \gamma > 0$, we choose

$$\frac{1}{n} \ll \mu \ll \varepsilon, c \ll d \ll \beta, \gamma, \frac{1}{k}, \text{ and additionally } \frac{1}{L} \ll \gamma.$$

We fix $(G, \mathcal{F})$ to be an $(n, \frac{k}{k+1} + \gamma, \mu)$-incompatibility system, and $W \subseteq V(G)$ with $|W| < \min\{\frac{\beta}{2}n, \frac{\gamma}{2}n\}$, and $\mathbf{e}_i, \mathcal{M}_i$ with $|\mathcal{M}_i| \geq \beta n^k, i = 1, 2$. Without loss of generality, we write

$$\mathbf{e}_1 = (u_1, \dots, u_k) \text{ and } \mathbf{e}_2 = (v_1, \dots, v_k).$$

Let $V' = V(G) \setminus (W \cup V(\mathbf{e}_1) \cup V(\mathbf{e}_2))$. For $i \in [2]$, we define the family

$$\mathcal{H}_i = \{\mathbf{f} \in \mathcal{M}_i : \mathbf{f} \text{ lies inside } V'\}.$$

Then by the choice of $|W| < \min\{\frac{\gamma}{2}n, \frac{\beta}{2}n\}$, it is easy to see that $|\mathcal{H}_i| \geq \frac{\beta}{4}n^k$ for $i \in [2]$. We then uniformly and randomly partition V' into $2k$ parts of nearly equal size, denoted as $U_1, U_2, \dots, U_{2k}$. Let $\mathcal{X}_1$ (or $\mathcal{X}_2$) be the family of mates $\mathbf{f} = (f_1, \dots, f_k)$ in $\mathcal{H}_1$ (resp. in $\mathcal{H}_2$) with $f_i \in U_i$ for $i \in [k]$ (resp. $f_i \in U_{k+i}$ for $i \in [k]$). It follows that $\mathbb{E}(|\mathcal{X}_i|) \geq \frac{\beta}{4(2k)^k}n^k$. Then by a standard application of Janson's inequality[1], there exists a partition such that $|\mathcal{X}_i| \geq \frac{1}{2}\mathbb{E}(|\mathcal{X}_i|) \geq \frac{\beta}{8(2k)^k}n^k$ for every $i \in [2]$. Let $U'_i = V(\mathcal{X}_1) \cap U_i$ and $U'_{k+i} = V(\mathcal{X}_2) \cap U_{k+i}$ for every $i \in [k]$. Then it is easy to observe the following properties:

(D1) $|U'_i| \geq \frac{\beta}{8}\frac{n}{2k}$ for every $i \in [2k]$;

(D2) Combined with $\mathbf{e}_1$, every copy of K_k in the resulting k-partite graph $G[U'_1, U'_2, \dots, U'_k]$ can form a copy of P^k_{2k}. The same assertion also holds for $\mathbf{e}_2$ and $G[U'_{k+1}, U'_{k+2}, \dots, U'_{2k}]$.

[1] Similar to the proof of Claim 2.17.

(D3) Moreover, for every $i \in [k]$ and $x \in U'_i$, the edges $xu_i, xu_{i+1}, \ldots, xu_k$ are pairwise compatible at x; similarly, for every $j \in [k]$ and $y \in U'_{k+j}$, the edges $yv_j, yv_{j+1}, \ldots, yv_k$ are pairwise compatible at y.

It is worth to remark that (D3) follows from the existence of a mate $\mathbf{f}$ for $\mathbf{e}_1$ (resp. for $\mathbf{e}_2$) whose i-th coordinate is x.

Let $G' = G[V']$ and $n' = |V'|$. Then $\delta(G') \geq \delta(G) - |W| - 2k \geq (\frac{k}{k+1} + \frac{\gamma}{3})n$. We apply Lemma 2.11 to G' to refine the current partition $\{U'_1, U'_2, \ldots, U'_{2k}, V' \setminus \bigcup_{i=1}^{2k} U'_i\}$. Denote the resulting (ε, d)-regular partition by $\mathcal{P} = \{V_0, V_1, \ldots, V_r\}$ for some $1/\varepsilon \leq r \leq M_\varepsilon$, where $|V_0| \leq \varepsilon n$ and all clusters V_i with $i \in [r]$ have the same size, denoted as m. Let $R = R(\varepsilon, d)$ be the reduced graph for this partition. Then as $\varepsilon \ll d \ll \gamma$, we obtain that $\delta(R) \geq \left(\frac{k}{k+1} + \frac{\gamma}{3} - d - 2\varepsilon\right)r \geq \left(\frac{k}{k+1} + \frac{\gamma}{4}\right)r$.

We claim that there exists a copy of K_k in R, such that each of the corresponding k clusters comes from a different part U'_i where $i \in [k]$. For otherwise, by removing the edges (of G') between irregular pairs, pairs with density less than d, and pairs incident to V_0, we obtain a subgraph of the k-partite graph $G[U'_1, U'_2, \ldots, U'_k]$ which does not contain any copy of K_k. Thus the total number of copies of K_k in $G[U'_1, U'_2, \ldots, U'_k]$ is at most

$$(\varepsilon r^2 \cdot m^2 + r^2 \cdot d \cdot m^2 + \varepsilon n' \cdot n)n'^{k-2} \leq (2\varepsilon + d)n^k < \frac{\beta}{8(2k)^k} n^k \leq |\mathcal{X}_1|,$$

yielding a contradiction. Similarly, we can find another copy of K_k in R, such that each of the corresponding k clusters comes from a different part U_i where $i \in [k+1, 2k]$. Without loss of generality, we may then assume that $V_i \subseteq U'_i$ for every $i \in [2k]$ and $\mathbf{f}_1 := (V_1, V_2, \ldots, V_k)$, $\mathbf{f}_2 := (V_{k+1}, V_{k+2}, \ldots, V_{2k})$ induce two copies of K_k in R.

By Lemma 5.1 applied to R, there exists a copy of P_ℓ^k with $2k \leq \ell \leq L$, whose ends are $\overleftarrow{\mathbf{f}}_1$ and $\overleftarrow{\mathbf{f}}_2$. Suppose $V_1, \ldots, V_\ell$ are the corresponding clusters in its base path. Applying Corollary 2.14 to G' with $h = \ell, H = P_\ell^k$, $U_i = V_i$ for every $i \in [\ell]$ and combining the choice of $\mu \ll \varepsilon \ll d$, we obtain a family $\mathcal{K}$ of at least cm^ℓ compatible copies of P_ℓ^k (in G') for some $c = c(\ell, d) > 0$. Moreover, every two ends of such a copy in $\mathcal{K}$ respect the orderings of $\overleftarrow{\mathbf{f}}_1$ and $\overleftarrow{\mathbf{f}}_2$, respectively. Recaping property (D2), we shall pick a copy of P_ℓ^k in $\mathcal{K}$ such that we can extend it to a desired compatible copy of $P_{\ell+2k}^k$ whose ends are $\overleftarrow{\mathbf{e}}_1$ and $\overleftarrow{\mathbf{e}}_2$, which completes the entire proof.

Recall that $\mathbf{e}_1 = (u_1, \ldots, u_k)$ and $\mathbf{e}_2 = (v_1, \ldots, v_k)$. For any fixed copy of P_ℓ^k in $\mathcal{K}$, its base path is denoted as $Q = w_1 w_2 \ldots w_\ell$ such that the path $u_1 \ldots u_k w_1 w_2 \ldots w_\ell v_k \ldots v_1$ forms a copy of $P_{\ell+2k}^k$. Note that $w_i \in U'_i$ and $w_{\ell-i+1} \in U'_{k+i}$ for every $i \in [k]$. Combining (D3) and the fact of $\ell \geq 2k$, we obtain that if Q fails to connect the ends $\overleftarrow{\mathbf{e}}_1$ and $\overleftarrow{\mathbf{e}}_2$ into a compatible copy of $P_{\ell+2k}^k$, then one of the following conditions holds:

(E1) There exist u_i ($i \in [k]$) and a pair $\{w_s, w_t\}$ with $s, t \in [\ell]$ such that either $\{u_i w_s, u_i w_t\}$, or $\{u_i w_s, w_s w_t\}$ forms an incompatible pair in $(G, \mathcal{F})$.

(E2) There exist v_j ($j \in [k]$) and a pair $\{w_s, w_t\}$ with $s, t \in [\ell]$ such that either $\{v_j w_s, v_j w_t\}$, or $\{v_j w_s, w_s w_t\}$ forms an incompatible pair in $(G, \mathcal{F})$.

Note that for any such u_i (or v_j) as in (E1) (resp. (E2)), the number of pair (w_s, w_t) is at most $2\mu n^2$. Hence the number of copies of P_ℓ^k in $\mathcal{K}$ satisfying (E1) or (E2) is at most

$$2k(2\mu n^2 + 2\mu n^2)n^{\ell-2} < cm^\ell \leq |\mathcal{K}|,$$

where the first inequality follows since $\mu \ll \varepsilon, c, \frac{1}{k}$. Thus we can pick from $\mathcal{K}$ a compatible copy of P_ℓ^k with a base path Q as desired. This completes the entire proof. □

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References

[1] N. Alon and G. Gutin. Properly colored Hamilton cycles in edge-colored complete graphs. *Random Structures & Algorithms*, 11(2):179–186, 1997.

[2] S. Antoniuk, N. Kamčev, and A. Ruciński. Properly colored Hamilton cycles in Dirac-type hypergraphs. *The Electronic Journal of Combinatorics*, 30(1):P1.44, 2023.

[3] B. Bollobás and P. Erdős. Alternating Hamiltonian cycles. *Israel Journal of Mathematics*, 23(2):126–131, 1976.

[4] Y. Chang, J. Han, and L. Thoma. On powers of tight Hamilton cycles in randomly perturbed hypergraphs. *Random Structures & Algorithms*, 63(3):591–609, 2023.

[5] C. Chen and D. E. Daykin. Graphs with Hamiltonian cycles having adjacent lines different colors. *Journal of Combinatorial Theory, Series B*, 21(2):135–139, 1976.

[6] M. Coulson and G. Perarnau. Rainbow matchings in Dirac bipartite graphs. *Random Structures & Algorithms*, 55(2):271–289, 2019.

[7] B. Cuckler and J. Kahn. Hamiltonian cycles in Dirac graphs. *Combinatorica*, 29(3):299–326, 2009.

[8] G. A. Dirac. Some theorems on abstract graphs. *Proceedings of the London Mathematical Society*, 3(1):69–81, 1952.

[9] P. Erdős. On extremal problems of graphs and generalized graphs. *Israel Journal of Mathematics*, 2:183–190, 1964.

[10] A. Ferber, M. Krivelevich, and B. Sudakov. Counting and packing Hamilton cycles in dense graphs and oriented graphs. *Journal of Combinatorial Theory, Series B*, 122:196–220, 2017.

[11] A. Hajnal and E. Szemerédi. Proof of a conjecture of P. Erdős. *Combinatorial Theory and Its Applications*, 2:601–623, 1970.

[12] J. Hu, H. Li, Y. Wang, and D. Yang. Graph tilings in incompatibility systems. *SIAM Journal on Discrete Mathematics*, 37(3):1930–1951, 2023.

[13] S. Janson, T. Łuczak, and A. Rucinski. *Random graphs*. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000.

[14] J. Komlós, G. Sárközy, and E. Szemerédi. Proof of the Alon–Yuster conjecture. *Discrete Mathematics*, 235(1-3):255–269, 2001.

[15] J. Komlós, G. N. Sárközy, and E. Szemerédi. On the Pósa–Seymour conjecture. *Journal of Graph Theory*, 29(3):167–176, 1998.

[16] A. Kotzig. Moves without forbidden transitions in a graph. *Matematický časopis*, 18(1):76–80, 1968.

[17] M. Krivelevich, C. Lee, and B. Sudakov. Robust Hamiltonicity of Dirac graphs. *Transactions of the American Mathematical Society*, 366(6):3095–3130, 2014.

[18] M. Krivelevich, C. Lee, and B. Sudakov. Compatible Hamilton cycles in random graphs. *Random Structures & Algorithms*, 49(3):533–557, 2016.

[19] M. Krivelevich, C. Lee, and B. Sudakov. Compatible Hamilton cycles in Dirac graphs. *Combinatorica*, 37(4):697–732, 2017.

[20] D. Kühn, J. Lapinskas, and D. Osthus. Optimal packings of Hamilton cycles in graphs of high minimum degree. *Combinatorics, Probability and Computing*, 22(3):394–416, 2013.

[21] I. Levitt, G. N. Sárközy, and E. Szemerédi. How to avoid using the Regularity Lemma: Pósa's conjecture revisited. *Discrete Mathematics*, 310(3):630–641, 2010.

[22] A. Lo. Properly coloured Hamiltonian cycles in edge-coloured complete graphs. *Combinatorica*, 36(4):471–492, 2016.

[23] V. Rödl, A. Ruciński, and E. Szemerédi. Perfect matchings in large uniform hypergraphs with large minimum collective degree. *Journal of Combinatorial Theory, Series A*, 116(3):613–636, 2009.

[24] G. N. Sárközy, S. M. Selkow, and E. Szemerédi. On the number of Hamiltonian cycles in Dirac graphs. *Discrete Mathematics*, 265(1-3):237–250, 2003.

[25] P. Seymour. Problem section. In *Combinatorics: Proceedings of the British Combinatorial Conference*, volume 1974, pages 201–202, 1973.

[26] J. Shearer. A property of the colored complete graph. *Discrete Mathematics*, 25(2):175–178, 1979.

[27] B. Sudakov. Robustness of graph properties. In *Surveys in combinatorics 2017*, volume 440 of *London Math. Soc. Lecture Note Ser.*, pages 372–408. Cambridge Univ. Press, Cambridge, 2017.