A CLASSIFICATION RESULT ABOUT BASIC 2-ARC-TRANSITIVE GRAPHS

JING JIAN LI, ZAI PING LU, RUO YU SONG, AND XIAO QIAN ZHANG

Abstract. A connected graph $\Gamma = (V, E)$ is called a basic 2-arc-transitive graph if its full automorphism group has a 2-arc-transitive subgroup G, and every minimal normal subgroup of G has at most two orbits on V. In 1993, Praeger proved that every finite 2-arc-transitive connected graph is a cover of some basic 2-arc-transitive graph, and proposed the classification problem of finite basic 2-arc-transitive graphs. In this paper, a classification is given for basic 2-arc-transitive non-bipartite graphs of order $r^a s^b$ and basic 2-arc-transitive bipartite graphs of order $2r^a s^b$, where r and s are distinct primes.

Keywords. 2-arc-transitive graph, quasiprimitive group, almost simple group.

1. Introduction

All graphs considered in this paper are assumed to be finite, simple and undirected.

Let $\Gamma = (V, E)$ be a graph with vertex set V and edge set E. An arc of Γ is an ordered pair of adjacent vertices, and a 2-arc is a triple (α, β, γ) of vertices with $\{\alpha, \beta\}$, $\{\beta, \gamma\} \in E$ and $\alpha \neq \gamma$. Denote by $\mathrm{Aut}(\Gamma)$ the full automorphism group of the graph Γ, and call every subgroup of $\mathrm{Aut}(\Gamma)$ an (automorphism) group of Γ. A group G of Γ is said to be vertex-transitive, edge-transitive, arc-transitive or 2-arc-transitive if G acts transitively on the vertices, edges, arcs or 2-arcs of Γ, respectively. A graph is said to be vertex-transitive, edge-transitive, arc-transitive or 2-arc-transitive if it admits a vertex-transitive, edge-transitive, arc-transitive or 2-arc-transitive group, respectively.

A connected graph $\Gamma = (V, E)$ with at least 3 vertices is called a basic 2-arc-transitive graph if it has a 2-arc-transitive group G such that every minimal normal subgroup of G has at most two orbits on the vertex set V. Praeger [24, 25] observed that every connected 2-arc-transitive graph is a cover of some basic 2-arc-transitive graph, and proposed the following problem.

Problem 1.1 ([25], Problem 1.2)**.** *Classify all finite basic 2-arc-transitive graphs.*

Let $\Gamma = (V, E)$ be a basic 2-arc-transitive graph with respect to a group G. Put $G^* = \langle G_\alpha, G_\beta \rangle$ for an edge $\{\alpha, \beta\} \in E$. It is well-known that $|G : G^*| \leqslant 2$, and Γ is bipartite if and only if $|G : G^*| = 2$, refer to [30, Exercise 3.8]. Praeger [24, 25] proved that either Γ is a complete bipartite graph, or G^* is a quasiprimitive group of type HA, AS, PA or TW (see [26] for the notation) on each G^*-orbit of vertices except for one case when Γ is bipartite. Inspired by Praeger's work, a lot of remarkable progresses have been made on classification or characterization of basic 2-arc-transitive

2010 Mathematics Subject Classification. 05C25, 20B25.

Supported by the National Natural Science Foundation of China (12331013, 12161141006, 11971248), the Fundamental Research Funds for the Central Universities, the NSF of Guangxi (2021GXNSFAA220116) and the special foundation for Guangxi Ba Gui Scholars.

graphs. For example, a construction of the graphs associated with quasiprimitive groups of type TW is given in [1], the graphs associated with quasiprimitive groups of type HA are classified in [15], the graphs associated with Suzuki simple groups, Ree simple groups and 2-dimensional projective linear groups are classified in [8, 9, 14] respectively. Besides, one can read out the basic 2-arc-transitive graphs of prime power order from [17]. In this paper, we focus on those basic 2-arc-transitive graphs of some given orders.

Let a, b be positive integers, r and s be distinct primes. Let $\Gamma = (V, E)$ be a basic 2-arc-transitive graph of valency k with respect to a group G. Assume that Γ is either non-bipartite and of order $r^a s^b$ or bipartite and of order $2r^a s^b$. It is easy to see that Γ is not a cycle. For the case where k is an odd prime, the graph Γ is determined in [22]. Recently, for an arbitrary valency k, it is shown in [23] that either Γ is a complete bipartite graph or G is an almost simple group. This allows us to give a classification of such graphs Γ. In Sections 4-6 of this paper, we prove a classification result stated as follows.

Theorem 1.2. *Let a and b be positive integers, r and s be distinct primes. Let $\Gamma = (V, E)$ be a basic 2-arc-transitive graph with respect to a group G, let $\{\alpha, \beta\} \in E$ and $G^* = \langle G_\alpha, G_\beta \rangle$. Assume that G is an almost simple group with socle T, and G^* has an orbit on V of length $r^a s^b$. Then Γ is isomorphic to one of the following graphs:*

1. *the complete graph $\mathsf{K}_{r^a s^b}$ and its standard double cover;*
2. *the Odd graph O_4 of valency 4 and its standard double cover;*
3. *the point-hyperplane incidence graph and non-incidence graph of the projective geometry $\mathrm{PG}(n-1, q)$, where $n \geqslant 3$ and $r^a s^b = \frac{q^n - 1}{q-1}$;*
4. *the incidence graph of the generalized quadrangle $\mathrm{GQ}(4, 2^{2^i})$, where $i \geqslant 1$;*
5. *the graphs in Examples 3.1, 3.2, 3.4-3.10, the standard double covers of the graph in Examples 3.1 and 3.2, and the graphs described as in Tables 4.2 and 6.7;*
6. *$T = \mathrm{PSL}_2(p^{2^i})$, $T_\alpha = \mathbb{Z}_p^{2^i}{:}\mathbb{Z}_{\frac{p^{2^i}-1}{2^{i_0+1}}}$, and Γ is of valency p^{2^i}, where p is an odd prime, $1 \leqslant i_0 < i$ and $r^a s^b = (p^{2^i} + 1)2^{i_0}$.*

Remark 1.3. We have no idea how to give a precise list for the graphs satisfying (6) of Theorem 1.2. The reader is referred to [14, Section 6] for the vertex-stabilizers and existence of such graphs.

2. Local structures and normal subgroups

Let $\Gamma = (V, E)$ be a connected graph, $G \leqslant \mathrm{Aut}(\Gamma)$ and $\{\alpha, \beta\} \in E$. Denote by $G_\alpha^{\Gamma(\alpha)}$ the permutation group induced by G_α on $\Gamma(\alpha)$, the neighborhood of α in Γ. Let $G_\alpha^{[1]}$ be the kernel of G_α acting on $\Gamma(\alpha)$. Then $G_\alpha^{\Gamma(\alpha)} \cong G_\alpha / G_\alpha^{[1]}$.

If G is arc-transitive on Γ then there is some $x \in G$ such that $(\alpha, \beta)^x = (\beta, \alpha)$, and the next simple fact follows, see [22, Lemma 2.1] for example.

Lemma 2.1. *Let $\Gamma = (V, E)$ be a connected graph, and $\{\alpha, \beta\} \in E$. If G is an arc-transitive group of Γ, then $|G_{\{\alpha,\beta\}} : G_{\alpha\beta}| = 2$ and $\langle G_\alpha, G_{\{\alpha,\beta\}} \rangle = G$; in particular, $|\mathbf{N}_G(G_{\alpha\beta}) : G_{\alpha\beta}|$ is even and $\langle \mathbf{N}_G(G_{\alpha\beta}), G_\alpha \rangle = G$.*

Assume next that G is 2-arc-transitive on Γ. Then $G_\alpha^{\Gamma(\alpha)}$ is a 2-transitive group. In particular, the socle $\mathrm{soc}(G_\alpha^{\Gamma(\alpha)})$ is either a nonabelian simple group or an elementary abelian group of prime power order. Since G is 2-arc-transitive on Γ, the arc-stabilizer $G_{\alpha\beta}$ acts transitively on $\Gamma(\beta)\setminus\{\alpha\}$. Let K be the kernel of $G_{\alpha\beta}$ acting on $\Gamma(\beta)\setminus\{\alpha\}$. Since $G_\beta^{[1]} \leqslant G_{\alpha\beta} \leqslant G_\beta$ and $G_\beta^{[1]}$ fixes $\Gamma(\beta)\setminus\{\alpha\}$ point-wise, we have $G_\beta^{[1]} \leqslant K \leqslant G_\beta^{[1]}$, and so $K = G_\beta^{[1]}$. Since $G_\alpha^{[1]}$ is normal in $G_{\alpha\beta}$, we have

$$(G_\alpha^{[1]})^{\Gamma(\beta)} \trianglelefteq G_{\alpha\beta}^{\Gamma(\beta)} \cong G_{\alpha\beta}/G_\beta^{[1]}. \tag{2.1}$$

Take $x \in G$ with $(\alpha,\beta)^x = (\beta,\alpha)$. Then $G_\beta = G_\alpha^x$ and $\Gamma(\beta) = \Gamma(\alpha)^x$. It follows that

$$(G_\alpha^{[1]})^{\Gamma(\beta)} \trianglelefteq (G_\beta^{\Gamma(\beta)})_\alpha = G_{\alpha\beta}^{\Gamma(\beta)} \cong G_{\alpha\beta}^{\Gamma(\alpha)} = (G_\alpha^{\Gamma(\alpha)})_\beta. \tag{2.2}$$

Thus, if $G_\alpha^{\Gamma(\alpha)}$ is solvable then $(G_\alpha^{\Gamma(\alpha)})_\beta$ is solvable, and hence $(G_\alpha^{[1]})^{\Gamma(\beta)}$ is solvable. Note the kernel of $G_\alpha^{[1]}$ acting on $\Gamma(\beta)$ is equal to the edge-kernel $G_{\alpha\beta}^{[1]} := G_\alpha^{[1]} \cap G_\beta^{[1]}$. We have $(G_\alpha^{[1]})^{\Gamma(\beta)} \cong G_\alpha^{[1]}/G_{\alpha\beta}^{[1]}$. It is well-known that $G_{\alpha\beta}^{[1]}$ has order a prime power, see [11]. Then $(G_\alpha^{[1]})^{\Gamma(\beta)}$ is solvable if and only if $G_\alpha^{[1]}$ is solvable. Recalling that $G_\alpha^{\Gamma(\alpha)} \cong G_\alpha/G_\alpha^{[1]}$, if $G_\alpha^{\Gamma(\alpha)}$ is solvable then G_α is solvable. Noting that $G_\alpha = G_{\alpha\beta}^{[1]}.(G_\alpha^{[1]})^{\Gamma(\beta)}.G_\alpha^{\Gamma(\alpha)}$. It follows that every insolvable composition factor of G_α occurs as a composition factor of $(G_\alpha^{[1]})^{\Gamma(\beta)}$ or $G_\alpha^{\Gamma(\alpha)}$.

Recall that $G_\alpha^{\Gamma(\alpha)}$ is 2-transitive and $(G_\alpha^{[1]})^{\Gamma(\beta)}$ is isomorphic to a normal subgroup of $(G_\alpha^{\Gamma(\alpha)})_\beta$. By [21, Corollary 2.5], $(G_\alpha^{\Gamma(\alpha)})_\beta$ has at most one insolvable composition factor. Checking one by one the finite 2-transitive groups given in [4, pages 195-197, Tables 7.3 and 7.4], we have the following lemma.

Lemma 2.2. *Let $\Gamma = (V,E)$ be a connected graph, $G \leqslant \mathrm{Aut}(\Gamma)$ and $\{\alpha,\beta\} \in E$. Assume that G is 2-arc-transitive on Γ and G_α is insolvable. Then $G_\alpha^{\Gamma(\alpha)}$ has a unique insolvable composition factor and $G_\alpha^{[1]}$ has at most one insolvable composition factor. If further $G_\alpha^{[1]}$ is insolvable then one of the following holds:*

(1) *$G_\alpha^{\Gamma(\alpha)}$ is an almost simple 2-transitive group, G_α has two nonisomorphic insolvable composition factors;*
(2) *$G_\alpha^{\Gamma(\alpha)}$ is an affine 2-transitive group, G_α has two isomorphic insolvable composition factors.*

The next result on 2-arc-transitive graphs is formulated from [28, 29, 30].

Theorem 2.3. *Let $\Gamma = (V,E)$ be a connected graph of valency $k \geqslant 3$, and let G be a 2-arc-transitive group of Γ. Assume that $G_{\alpha\beta}^{[1]} \neq 1$ for $\{\alpha,\beta\} \in E$. Then $G_{\alpha\beta}^{[1]}$ is a p-group for some prime p, $\mathrm{PSL}_d(p^f) \trianglelefteq G_\alpha^{\Gamma(\alpha)}$, $k = \frac{p^{fd}-1}{p^f-1}$, and either $d \geqslant 3$ or one of the following holds:*

(1) *$G_\alpha = [p^{2f}]{:}(c.\mathrm{PGL}(2,p^f)).[o]$, where $c = \frac{p^f-1}{(3,p^f-1)}$ and $o \mid (3,p^f-1)f$;*
(2) *$p = 2$ and $G_\alpha = [2^{3f}]{:}\mathrm{GL}(2,2^f).e$, where $e \mid f$;*
(3) *$p = 3$ and $G_\alpha = [3^{5f}]{:}\mathrm{GL}(2,3^f).e$, where $e \mid f$.*

In particular, G_α is solvable if and only if $(k,d,p^f) = (3,2,2)$ or $(4,2,3)$.

Assume that G is a 2-arc-transitive group of Γ, and $N \trianglelefteq G$ with $N_\alpha \neq 1$. Then N_α acts transitively on $\Gamma(\alpha)$, and N is edge-transitive on Γ, see [19, Lemma 2.5] for example. Note that $N_\alpha^{\Gamma(\alpha)}$ is a transitive normal subgroup of the 2-transitive group $G_\alpha^{\Gamma(\alpha)}$. It forces that $\mathrm{soc}(N_\alpha^{\Gamma(\alpha)}) = \mathrm{soc}(G_\alpha^{\Gamma(\alpha)})$. For the case where $G_\alpha^{\Gamma(\alpha)}$ is almost simple, it follows from [4, page 197, Table 7.4] that $\mathrm{soc}(N_\alpha^{\Gamma(\alpha)})$ is a 2-transitive normal subgroup of $G_\alpha^{\Gamma(\alpha)}$ unless $\mathrm{soc}(G_\alpha^{\Gamma(\alpha)}) = \mathrm{PSL}(2,8)$ acting on 28 points.

Lemma 2.4. *Let $\Gamma = (V, E)$ be a connected graph, and $\{\alpha, \beta\} \in E$. Assume that G is a 2-arc-transitive group of Γ and $G_\alpha^{\Gamma(\alpha)}$ is an affine 2-transitive group of degree $|\Gamma(\alpha)| = p^e$. Then one of the following holds:*

(1) $\mathbf{O}_p(G_\alpha) \cong \mathrm{soc}(G_\alpha^{\Gamma(\alpha)}) = \mathbb{Z}_p^e$, *and* $G_\alpha = \mathbf{O}_p(G_\alpha){:}G_{\alpha\beta}$*;*
(2) $p^e = 3$, *and* G_α *is one of* S_4 *and* $2 \times \mathrm{S}_4$*;*
(3) $p^e = 4$, *and* G_α *is one of* $3^2{:}\mathrm{GL}_2(3)$ *and* $[3^5]{:}\mathrm{GL}_2(3)$*.*

If further $N \trianglelefteq G$ and $N_\alpha \neq 1$ then either N_α is solvable, or one of the following holds:

(4) N_α *acts 2-transitively on* $\Gamma(\alpha)$*;*
(5) N_α *acts primitively on* $\Gamma(\alpha)$, $\mathbb{Z}_p^2{:}\mathrm{SL}_2(5) \trianglelefteq N_\alpha^{\Gamma(\alpha)} \trianglelefteq G_\alpha^{\Gamma(\alpha)} \trianglelefteq \mathbb{Z}_p^2{:}(\mathbb{Z}_{p-1} \circ \mathrm{SL}_2(5))$, $|\Gamma(\alpha)| = p^2$, *where* $p \in \{19, 29, 59\}$*.*

Proof. By [21, Proposition 3.4], either part (1) holds or $p^e \in \{3,4\}$ and $G_{\alpha\beta}^{[1]} \neq 1$. For the latter case, part (2) or (3) follows from Theorem 2.3 depending on whether $G_{\alpha\beta}^{[1]}$ is a 2-group or a 3-group, see also [2, page 126] and [20, Lemma 2.6].

Let $N \trianglelefteq G$ with $N_\alpha \neq 1$. By [21, Corollary 2.5], either one of (4) and (5) holds or $(N_\alpha^{\Gamma(\alpha)})_\beta$ is solvable. Assume next that the latter case occurs. Then $N_\alpha^{\Gamma(\alpha)}$ is solvable as $N_\alpha^{\Gamma(\alpha)} = \mathrm{soc}(G_\alpha^{\Gamma(\alpha)}){:}(N_\alpha^{\Gamma(\alpha)})_\beta$. Since G is arc-transitive on Γ, taking $x \in G$ with $(\alpha,\beta)^x = (\beta,\alpha)$, we have $N_{\alpha\beta}^x = (G_{\alpha\beta} \cap N)^x = G_{\alpha\beta} \cap N = N_{\alpha\beta}$. Since $\Gamma(\alpha)^x = \Gamma(\beta)$, we have $N_{\alpha\beta}^{\Gamma(\beta)} \cong N_{\alpha\beta}^{\Gamma(\alpha)} = (N_\alpha^{\Gamma(\alpha)})_\beta$. Noting that $(N_\alpha^{[1]})^{\Gamma(\beta)} \trianglelefteq N_{\alpha\beta}^{\Gamma(\beta)}$, it follows that $(N_\alpha^{[1]})^{\Gamma(\beta)}$ is solvable, and then N_α is solvable as $N_\alpha = N_{\alpha\beta}^{[1]}.(N_\alpha^{[1]})^{\Gamma(\beta)}.N_\alpha^{\Gamma(\alpha)}$. This completes the proof. □

Lemma 2.5. *Let $\Gamma = (V, E)$ be a connected graph, and let G be a 2-arc-transitive group of Γ. Let $N \trianglelefteq G$ and $\{\alpha, \beta\} \in E$. Assume that N_α is the dihedral group D_{2n} of order $2n$, where $n \geqslant 3$. Then $n = p$ or $2p$, $|\Gamma(\alpha)| = p$ and $G_\alpha^{\Gamma(\alpha)} = \mathrm{AGL}_1(p)$, where p is an odd prime.*

Proof. Let K be the unique cyclic subgroup of N_α with order n. Then K is normal in G_α. Thus either $K \leqslant G_\alpha^{[1]}$ or K acts transitively on $\Gamma(\alpha)$. Suppose that $K \leqslant G_\alpha^{[1]}$. Then $N_\alpha^{\Gamma(\alpha)} \cong \mathbb{Z}_2$. Since $N_\alpha^{\Gamma(\alpha)}$ is transitive on $\Gamma(\alpha)$, it follows that Γ is a cycle. Then $N_\alpha \trianglelefteq G_\alpha \cong \mathbb{Z}_2$, a contradiction.

Now K acts transitively on $\Gamma(\alpha)$. Then $\mathrm{soc}(G_\alpha^{\Gamma(\alpha)}) = \mathrm{soc}(N_\alpha^{\Gamma(\alpha)}) = \mathrm{soc}(K^{\Gamma(\alpha)}) \cong \mathbb{Z}_p$, where $p = |\Gamma(\alpha)|$ is an odd prime. Moreover, $G_\alpha^{\Gamma(\alpha)} = \mathrm{AGL}_1(p)$, and $N_{\alpha\beta} \cong \mathrm{D}_{\frac{2n}{p}}$. (Here, $\mathrm{D}_2 = \mathbb{Z}_2$ and $\mathrm{D}_4 = \mathbb{Z}_2^2$.) Suppose that $n > 2p$. Noting that $N_\alpha \cong N_\beta \cong \mathrm{D}_{2n}$, it follows that $N_{\alpha\beta}$ contains the unique cyclic subgroup L of both N_α and N_β with order $\frac{n}{p}$. Clearly, L is characteristic in both N_α and N_β, and then L is normal in both G_α and G_β. Thus $L \trianglelefteq \langle G_\alpha, G_\beta\rangle = G^*$. Since G^* is edge-transitive, L fixes every edge of Γ. By the connectedness of Γ, we have $L = 1$; however $|L| = \frac{n}{p} \geqslant 3$, a contradiction. Then the lemma follows. □

3. Constructions of 2-arc-transitive graphs

The purpose of this section is to construct some graphs involved in Theorem 1.2 as the coset graphs of almost simple groups or the orbital graphs of rank three primitive groups.

3.1. **Orbital graphs.** For a graph $\Gamma = (V, E)$, $\alpha \in V$ and a group G of Γ, if G is arc-transitive then $\Gamma(\alpha)$ is a self-paired suborbit of G (as a transitive subgroup of the symmetric group $\mathrm{Sym}(V)$). Conversely, if G is a transitive subgroup of $\mathrm{Sym}(V)$ with a self-paired suborbit $\Delta(\alpha)$ at α, then we have an arc-transitive graph on V, called an orbital graph of G, which has arc set $\{(\alpha, \beta)^g \mid \beta \in \Delta(\alpha), g \in G\}$, the orbital of G on V with respect to $\Delta(\alpha)$. Note, the orbital graph is independent of the choice of α, and if G_α acts 2-transitively on $\Delta(\alpha)$ then G is a 2-arc-transitive group of the graph.

We next construct some 2-arc-transitive graphs from primitive groups of rank 3. Note that the primitivity insures the connectedness of the resulting graphs.

Example 3.1. Let G be an almost simple group with socle $T = \mathrm{M}_{22}$. Then G is contained in the automorphism group of the Steiner system $\mathrm{S}(3, 6, 22)$. Let V be the set of 77 hexads of $\mathrm{S}(3, 6, 22)$. Then G is a primitive group on V of rank 3 with suborbits of length 1, 16 and 60. Define a graph Γ on V such $\alpha, \beta \in V$ are adjacent if and only if α and β are disjoint hexads. By the information given in the Atlas [6], every hexad α is disjoint from 16 others, $T_\alpha \cong 2^4{:}\mathrm{A}_6$, $T_{\alpha\beta} \cong \mathrm{A}_6$ and $T_{\{\alpha,\beta\}} \cong \mathrm{M}_{10}$ for a hexad β disjoint from α. Then Γ is a connected 2-arc-transitive graph of order 77 and valency 16. □

Example 3.2. Let G be an almost simple group with socle $T = \mathrm{PSU}_4(q)$, where $q = 2^{2^i}$ for some integer $i \geqslant 1$. Consider the action of G on the set V of $(q+1)^2(q^2-q+1)$ isotropic lines. Then G is a primitive group of rank 3 with suborbits of length 1, q^4 and $q(q^2+1)$. For an isotropic line α, the suborbit at α of length q^4 consists of all isotropic lines disjoint from α. Define a graph Γ on V such $\alpha, \beta \in V$ are adjacent if and only if α and β are disjoint isotropic lines. Then Γ is a connected 2-arc-transitive graph of order $(q+1)^2(q^2-q+1)$ and valency q^4. □

3.2. **Coset graphs.** It is well-known that, for a group G, every transitive action of G on a nonempty set is equivalent to the action induced by the multiplication of G on the right cosets of some subgroup in G. This fact leads to an efficient construction for 2-arc-transitive graphs as follows, refer to [12, 13, 22].

Let G be a finite group, and $H, K < G$ with $|K : (K \cap H)| = 2$ such that H is core-free in G, i.e., $\cap_{g \in G} H^g = 1$, and $K \cap H$ is maximal and non-normal in H. Define a graph $\mathrm{Cos}(G, H, K)$ on $[G : H] := \{Hg \mid g \in G\}$ such that $\{Hx, Hy\}$ is an edge if and only if $yx^{-1} \in HKH \setminus H$. It is easily shown that G is an arc-transitive group of $\mathrm{Cos}(G, H, K)$, where G acts on $[G : H]$ by right multiplication. Taking $x \in K \setminus H$, we have that $HKH = H \cup HxH$, and the subgroups H, $H \cap K$ and K serve as the stabilizers of the vertex H, arc (H, Hx) and edge $\{H, Hx\}$, respectively.

By [8, Theorem 2.1] and [22, Lemma 2.2], a characterization for 2-arc-transitive graphs is given as follows.

Theorem 3.3. *Let $\Gamma = (V, E)$ be a connected regular graph of valency $k \geqslant 3$, and $G \leqslant \mathrm{Aut}(\Gamma)$. Then G is 2-arc-transitive on Γ if and only if $\Gamma \cong \mathrm{Cos}(G, H, K)$, where $H, K < G$ satisfy the following conditions:*

(I) *H is core-free in G, $G = \langle H, K\rangle$, $|K : (K \cap H)| = 2$ and $|H : (K \cap H)| = k$;*
(II) *H acts 2-transitively on the set $[H : (K \cap H)]$ by right multiplication.*

Let G be a finite group. In order to determine whether or not there exist connected graphs admitting G as a 2-arc-transitive group, it suffices to find all feasible pairs (H, K) satisfying the above conditions (I) and (II). In practice, up to isomorphism of graphs, the vertex-stabilizer H is chosen up to the conjugation under $\mathrm{Aut}(G)$, and the edge-stabilizer K is chosen from the normalizer $\mathbf{N}_G(L)$ with $|K : L| = 2$, while L is an intended arc-stabilizer $K \cap H$ chosen up to the conjugation under $\mathbf{N}_{\mathrm{Aut}(G)}(H)$. Nevertheless, for a given pair (G, H), different edge-stabilizers K might give isomorphic graphs.

Example 3.4. Let $G = \mathrm{PSL}_3(3).2$ with socle $T = \mathrm{PSL}_3(3)$. Then G has a unique conjugacy class of subgroups isomorphic to $3^2{:}2\mathrm{A}_4 \cong \mathrm{ASL}_2(3)$, confirmed by GAP [10]. Let $H = 3^2{:}2\mathrm{A}_4 < T$ and $L = 2\mathrm{A}_4 < H$. Then $\mathbf{N}_G(L) = 2\mathrm{S}_4{:}2 = 2\mathrm{A}_4.2^2$, and $\mathbf{N}_G(L)$ has two subgroups K_1, K_2 with $K_1 \cong K_2 \cong 2\mathrm{S}_4$ and $K_1, K_2 \not\leq T$, confirmed by GAP. Thus we have two connected bipartite graphs $\mathrm{Cos}(G, H, K_i)$, $i \in \{1, 2\}$, which have valency 9, order 52 and admit G as a 2-arc-transitive group. □

Example 3.5. Let $G = \mathrm{PGL}_2(7)$ and $T = \mathrm{PSL}_2(7)$. Then T has two conjugacy classes of subgroups A_4, which are merged into one class in G. Take $\mathrm{A}_4 \cong H < T$, and $\mathbb{Z}_3 \cong L < H$. Then $\mathbf{N}_G(L) \cong \mathrm{D}_{12}$, which contains three subgroups of order 6, one of them, say $K \cong \mathbb{Z}_6$, together with H generate G, the other two are dihedral and each of them together with H generate a maximal subgroup S_4 of T. Thus we get a 2-arc-transitive graph $\mathrm{Cos}(G, H, K)$, which has order 28 and valency 4. □

Example 3.6. Let $G = \mathrm{PGL}_2(11)$ and $T = \mathrm{PSL}_2(11)$. Then T has a unique conjugacy class of subgroups A_4 and G has a unique conjugacy class of maximal subgroups S_4. Let $H_1 = \mathrm{A}_4 < T$, $L_1 = \mathbb{Z}_3 < H_1 < H_2 = \mathrm{S}_4 < G$ and $L_2 = \mathrm{S}_3 < H_2$. Then the following hold, confirmed by GAP.

(1) $\mathbf{N}_T(L_1) = \mathrm{D}_{12}$ contains three subgroups of order 6, one of them is cyclic, say K_{11}, which together with H_1 generate T, the other two are dihedral and each of them together with H_1 generate a maximal subgroup A_5 of T;
(2) $\mathbf{N}_G(L_1) = \mathrm{D}_{24}$ contains five subgroups of order 6, three of them are contained in T, each of the other two, say K_{12} and K_{13}, together with H_1 generate G;
(3) $K_2 := \mathbf{N}_G(L_2) = \mathrm{D}_{12} \not\leq T$ and $K_2 \cap T = L_2$.

Thus we get four 2-arc-transitive tetravalent graphs $\mathrm{Cos}(T, H_1, K_{11})$, $\mathrm{Cos}(G, H_2, K_2)$, $\mathrm{Cos}(G, H_1, K_{12})$ and $\mathrm{Cos}(G, H_1, K_{13})$. Note, the first two graphs are isomorphic and of order 55, the last two graphs are bipartite and of order 110. □

Example 3.7. Let $T = \mathrm{PSL}_2(23)$. Then T has two conjugacy classes of maximal subgroups S_4, which are merged into one class in $\mathrm{PGL}_2(23)$. Take $\mathrm{S}_4 \cong H < T$, and $\mathrm{S}_3 \cong L < H$. Then $\mathbf{N}_T(L) \cong \mathrm{D}_{12}$ and $\langle H, \mathbf{N}_T(L)\rangle = T$, confirmed by GAP. Thus $\mathrm{Cos}(T, H, \mathbf{N}_T(L))$ is a 2-arc-transitive graph of order 253 and valency 4. □

Example 3.8. The alternating group A_9 has two conjugacy classes of subgroups $2^3{:}\mathrm{PSL}_3(2)$, which are merged into one class in S_9. Take $2^3{:}\mathrm{PSL}_3(2) \cong H < \mathrm{A}_8 < \mathrm{A}_9$. Consider the natural action of S_9 on $\Omega = \{1, 2, \ldots, 9\}$. We may assume that H fixes the point 9 and acts 2-transitively on $\Omega_1 = \Omega \setminus \{9\}$. Then, up to conjugation, H has two subgroups $\mathrm{PSL}_3(2)$, one of them is 2-transitive on Ω_1, the other one say L fixes 2 and is 2-transitive on $\Omega_1 \setminus \{2\}$. Moreover, $\mathbf{N}_{\mathrm{S}_9}(L) = \mathbf{N}_{\mathrm{S}_7 \times \langle (2\,9)\rangle}(L) = L \times \langle (2\,9)\rangle$.

Then $\langle H, \mathbf{N}_{S_9}(L)\rangle$ is 2-transitive on Ω and contains a transposition $(2\,9)$. This yields that $\langle H, \mathbf{N}_{S_9}(L)\rangle = S_9$, see [7, page 77, Theorem 3.3A]. Thus $\mathrm{Cos}(S_9, H, \mathbf{N}_{S_9}(L))$ is a 2-arc-transitive bipartite graph of order $2\cdot 3^3\cdot 5$ and valency 8. □

Example 3.9. Let $T = \mathrm{PSL}_n(q)$ with $n \geqslant 3$ and q a prime power, and let $M \cong q^{n-1}{:}\mathrm{SL}_{n-1}(q).\frac{q-1}{(n,q-1)}$ be a point-stabilizer of T acting on the projective geometry $\mathrm{PG}(n-1,q)$. Take $x \in \mathrm{Aut}(T)$ such that M^x is a hyperplane-stabilizer of T acting on $\mathrm{PG}(n-1,q)$ and $M^{x^2} = M$. Identifying T with its inner automorphism group, set $X = T\langle x\rangle$. Let $q^{n-1}{:}\mathrm{SL}_{n-1}(q).o \cong H_0 \leqslant M$. Then H_0 is a characteristic subgroup of M, and so $x^2 \in \mathbf{N}_X(H_0)$. Set $H = H_0\langle x^2\rangle$ and $K = (H_0 \cap H_0^x)\langle x\rangle$. Then X is a 2-arc-transitive group of $\mathrm{Cos}(X,H,K)$, and one of the following holds:

(1) $|H : (H\cap K)| = q^{n-1}$;
(2) $|H : (H\cap K)| = \frac{q^{n-1}-1}{q-1}$.

Note, if we take $H_0 = M$ then the resulting graph is just the point-hyperplane incidence or non-incidence graph of $\mathrm{PG}(n-1,q)$. We can get examples for Theorem 1.2 with further limitations on q and n, refer to [18, Table 4.1] and [22, Remark 5.2]. □

Example 3.10. Let $G = T.2$ be an almost simple group with socle $T = \mathrm{P}\Omega_8^+(2)$. By the Atlas [6], T has three conjugacy classes of maximal subgroups $2^6{:}\mathrm{PSL}_4(2)$. One of them consists of the stabilizers of isotropic points. The other two classes are merged into one class in G and consist of respectively the stabilizers of T acting on two orbits of maximal isotropic subspaces. Let H be a stabilizer of some maximal isotropic subspace in T. Then all subgroups $[2^9]{:}\mathrm{PSL}_3(2)$ of H are self-normalized (confirmed by GAP) in T, and such subgroups form two classes in G according to whether or not they have normalizer $[2^9]{:}(\mathrm{PSL}_3(2)\times 2)$. Let $H > L \cong [2^9]{:}\mathrm{PSL}_3(2)$ with $\mathbf{N}_G(L) \cong [2^9]{:}(\mathrm{PSL}_3(2)\times 2)$. Then $\mathrm{Cos}(G,H,\mathbf{N}_G(L))$ is a 2-arc-transitive graph of order 270 and valency 15. □

3.3. **Standard double covers.** For a graph $\Gamma = (V,E)$, the standard double cover $\Gamma^{(2)}$ of Γ is a bipartite graph with vertex set $V\times\mathbb{Z}_2$ such that $\{(\alpha,0),(\beta,1)\}$ is an edge if and only if $\{\alpha,\beta\}\in E$. It is well-known that $\Gamma^{(2)}$ is connected if and only if Γ is connected and non-bipartite.

Define a map $\tau : V\times\mathbb{Z}_2 \to V\times\mathbb{Z}_2$, $(\alpha,i)\mapsto(\alpha,i+1)$. It is easily shown that $\tau\in\mathrm{Aut}(\Gamma^{(2)})$, and τ interchanges $V\times\{0\}$ and $V\times\{1\}$. Moreover, if $G\leqslant\mathrm{Aut}(\Gamma)$ then G can be viewed as a subgroup of $\mathrm{Aut}(\Gamma^{(2)})$ by

$$(\alpha,i)^g = (\alpha^g,i),\ \alpha\in V,\ g\in G.$$

Thus, if G is a 2-arc-transitive group of Γ then $G\times\langle\tau\rangle$ is a 2-arc-transitive group of $\Gamma^{(2)}$. Further, the following lemma is easily shown.

Lemma 3.11. *Let $\Gamma = (V,E)$ be a graph, and $G_0 < G \leqslant \mathrm{Aut}(\Gamma)$ with $|G : G_0| = 2$. If G_0 is 2-arc-transitive then $G_0\langle x\tau\rangle$ is a 2-arc-transitive group of $\Gamma^{(2)}$, where $x\in G\setminus G_0$.*

Example 3.12. Let $\Omega = \{1,2,3,\ldots,2k-1\}$ for integer $k\geqslant 3$. The Odd graph O_k is defined on the set of $(k-1)$-subsets of Ω with two vertices adjacent if and only if they are disjoint $(k-1)$-subsets. The graph O_k has valency k and automorphism group S_{2k-1}, while A_{2k-1} is a 2-arc-transitive group of O_k. By Lemma 3.11, $\mathsf{O}_k^{(2)}$ has a 2-arc-transitive group isomorphic to S_{2k-1}. Note, up to isomorphism, the graph $\mathsf{O}_k^{(2)}$ is also constructed on the $(k-1)$-subsets and k-subsets of Ω by the inclusion of sets.

Lemma 3.13. *Let $\Gamma = (V, E)$ be a connected bipartite graph of valency k, and let G be a 2-arc-transitive group of Γ. Assume that G^* acts faithfully and equivalently on both parts of Γ. If G^* has a unique suborbit of length k on one part of Γ, then Γ is isomorphic to the standard double cover of some graph which admits G^* as a 2-arc-transitive group.*

Proof. Assume that G^* has a unique suborbit of length k on one part of Γ, say U. Let Σ be the orbital graph of G^* with respect to this suborbit. Take $\alpha, \beta \in V$ with $\alpha \in U$ and $\beta \in V \setminus U$ and $G_\alpha = G_\beta$. Since Γ is bipartite, we have $\Gamma(\beta) \subseteq U$, and so $\Gamma(\beta)$ is a G_β-orbit on U as G_β acts transitively on $\Gamma(\beta)$. Then both $\Gamma(\beta)$ and $\Sigma(\alpha)$ are G_α-orbits on U. Since G^* has a unique suborbit of length k on U, we have $\Gamma(\beta) = \Sigma(\alpha)$. Thus G_α acts 2-transitively on $\Sigma(\alpha)$, and G^* is a 2-arc-transitive group of Σ. Define a map $\phi : V \to U \times \mathbb{Z}_2$ by

$$\alpha^x \mapsto (\alpha^x, 0),\ \beta^x \mapsto (\alpha^x, 1),\ x \in G^*.$$

It is easy to show that ϕ is a graph isomorphism, and the lemma follows. □

4. Graphs with prime valency or solvable stabilizers

Throughout the rest of this paper, r and s are distinct primes, a and b are positive integers, $\Gamma = (V, E)$ denotes a basic 2-arc-transitive graph of valency k with respect to an almost simple group G, $T = \mathrm{soc}(G)$, $G^* = \langle G_\alpha, G_\beta \rangle$, and $|G^* : G_\alpha| = r^a s^b$, where $\{\alpha, \beta\} \in E$. Since a cycle has solvable automorphism group, we have $k \geqslant 3$.

Clearly, $T \leqslant G^*$, and T is transitive on each G^*-orbit on V. Then $|T : T_\alpha| = |G^* : G_\alpha|$, and $G^* = TG_\alpha$, yielding that $G^*/T \cong G_\alpha/T_\alpha$. Since G is almost simple, G/T is solvable, and so G^*/T is solvable. Then G_α/T_α is solvable, and thus G_α is solvable if and only if T_α is solvable. Moreover, since T has a subgroup T_α of index $r^a s^b$, T is one of the simple groups given in [18, Tables 3.1,3,2,4.1-4.5, 5.1] and [22, Remark 5.2]. In this and the following two sections, we will prove Theorem 1.2 in three cases:

(A) either T_α is solvable or k is an odd prime, see Theorems 4.1, 4.2 and 4.4;
(B) $G_\alpha^{\Gamma(\alpha)}$ is an insolvable affine 2-transitive group, see Theorem 5.4;
(C) $G_\alpha^{\Gamma(\alpha)}$ is an almost simple 2-transitive group, and k is not prime, see Theorems 6.1 and 6.6.

In this section we first deal with the case (A).

Theorem 4.1. *Assume that T_α is solvable. Then one of the following holds:*

(1) *either $k = 4$ or k is a prime;*
(2) *Γ is a complete graph or the standard double cover of a complete graph;*
(3) *Γ is a bipartite graph described as in Example 3.4;*
(4) *$T = \mathrm{PSL}_2(p^{2^i})$, $p^{2^i}{:}\frac{p^{2^i}-1}{2^{i_0+1}} = T_\alpha \leqslant G_\alpha \leqslant \mathrm{A\Gamma L}_1(p^{2^i})$, and Γ is of valency p^{2^i}, where p is an odd prime and $1 \leqslant i_0 < i$.*

Proof. Noting that G_α is solvable, $G_\alpha^{\Gamma(\alpha)}$ is a solvable 2-transitive group, and $k = p^e$ for some prime p and integer $e \geqslant 1$. If $e = 1$ or $k = 4$ then the result holds.

Assume that $e > 1$ and $k = p^e > 4$. First, by Lemma 2.5, T_α is not a dihedral group. Since $G_\alpha^{\Gamma(\alpha)}$ is solvable, we can read out $G_\alpha^{\Gamma(\alpha)}$ from [4, page 195, Table 7.3], see also [21, Theorem 2.4]. Then one of the following holds:

(i) $k = p^e$, $(G_\alpha^{\Gamma(\alpha)})_\beta \leqslant (p^e-1){:}e$;
(ii) $k = 3^4$, $(G_\alpha^{\Gamma(\alpha)})_\beta \leqslant 2^{1+4}{:}5{:}4$;
(iii) $k = 5^2$, $(G_\alpha^{\Gamma(\alpha)})_\beta = \mathrm{Q}_8{:}3$ or $\mathrm{Q}_8.6$;
(iv) $k = 7^2$, $(G_\alpha^{\Gamma(\alpha)})_\beta = \mathrm{Q}_8{:}\mathrm{S}_3$, $3 \times (\mathrm{Q}_8.2)$ or $3 \times (\mathrm{Q}_8{:}\mathrm{S}_3)$;
(v) $k = 11^2$, $(G_\alpha^{\Gamma(\alpha)})_\beta = 5 \times (\mathrm{Q}_8{:}3)$ or $5 \times (\mathrm{Q}_8{:}\mathrm{S}_3)$;
(vi) $k = 23^2$, $(G_\alpha^{\Gamma(\alpha)})_\beta = 11 \times (\mathrm{Q}_8{:}\mathrm{S}_3)$.

Moreover, by Lemma 2.4, $\mathbf{O}_p(G_\alpha) \cong \mathrm{soc}(G_\alpha^{\Gamma(\alpha)}) = \mathbb{Z}_p^e$ and $G_\alpha = \mathbf{O}_p(G_\alpha){:}G_{\alpha\beta}$. Recalling that $\mathrm{soc}(T_\alpha^{\Gamma(\alpha)}) = \mathrm{soc}(G_\alpha^{\Gamma(\alpha)})$, we have $T_\alpha = \mathbf{O}_p(G_\alpha){:}T_{\alpha\beta}$ and $\mathbf{O}_p(T_\alpha) = \mathbf{O}_p(G_\alpha)$.

Since $k > 4$, we have $G_{\alpha\beta}^{[1]} = 1$ by Theorem 2.3, and thus $G_\alpha^{[1]}$ is isomorphic to a normal subgroup of $(G_\alpha^{\Gamma(\alpha)})_\beta$. In particular, $|G_\alpha|$ has the form of $p^e m$, where m is a divisor of $|(G_\alpha^{\Gamma(\alpha)})_\beta|^2$. Then either $\mathbf{O}_p(G_\alpha)$ is a Sylow subgroup of G_α and of T_α, or p is a divisor of e and G_α has a Sylow p-subgroup of order at most p^{2e}. Further, recalling that $G_\alpha/T_\alpha \cong G^*/T$, it follows that $|G_\alpha| = |T_\alpha||G^*:T|$, which is divisible by p^e-1. With these limitations, we read out the pair (T, T_α) from [18, Tables 3.1,3.2,4.1-4.5, 5.1] and [22, Remark 5.2]. Then the pair (T, T_α) is described as in Table 4.1.

T	T_α	$r^a s^b$	$k=p^e$	
A_6	$3^2{:}2, 3^2{:}4$	$2^2\cdot 5, 2\cdot 5$	3^2	
M_{11}	$\mathrm{M}_9{:}2$	$5\cdot 11$	3^2	
$\mathrm{PSL}_2(p^e)$	$p^e{:}m$	$(p^e+1)l$	p^e	$ml = \frac{p^e-1}{(2,p-1)}, e>1, p^e>4$
$\mathrm{PSL}_3(3)$	$3^2{:}2\mathrm{A}_4$	$2\cdot 13$	3^2	
$\mathrm{PSU}_3(4)$	$5^2:\mathrm{S}_3$	$2^5\cdot 13$	5^2	$\|G^*:T\| = \|G_\alpha:T_\alpha\| = 4$

Table 4.1. Candidates of T for $k > 4$ with solvable stabilizers

We first exclude the groups M_{11} and $\mathrm{PSU}_3(4)$. If $T = \mathrm{M}_{11}$ then $T = G$, and hence Γ has odd order and odd valency, which is impossible. Suppose that $T = \mathrm{PSU}_3(4)$. Then T_α is maximal in T, $G = \mathrm{PSU}_3(4).4$ and $G_\alpha \cong 5^2{:}(4\times \mathrm{S}_3)$. In particular, $G_{\alpha\beta} \cong 4 \times \mathrm{S}_3$. Checking the maximal subgroups of G in the Atlas [6], we conclude that $\mathbf{N}_G(G_{\alpha\beta}) \leqslant G_\alpha$, which contradicts Lemma 2.1.

Assume that $T = \mathrm{PSL}_3(3)$. If $T_\alpha \neq G_\alpha$ then $G = \mathrm{PSL}_3(3).2$ and $G_\alpha \cong 3^2{:}2\mathrm{S}_4 \leqslant T$, a contradiction. Thus $T_\alpha = G_\alpha$, and either $G = T$ or Γ is bipartite and $G^* = T$. Suppose that $G = T$. Then $G_\alpha \cong 3^2{:}2\mathrm{A}_4$, $G_{\alpha\beta} \cong 2\mathrm{A}_4$ and $\mathbf{N}_G(G_{\alpha\beta}) \cong 2\mathrm{S}_4$, where $\beta \in \Gamma(\alpha)$. It follows that G_α and $\mathbf{N}_G(G_{\alpha\beta})$ are both contained in a subgroup $3^2{:}2\mathrm{S}_4$ of G, a contradiction. Then Γ is a bipartite graph described as in Example 3.4.

Assume that $T = \mathrm{PSL}_2(p^e)$, $T_\alpha = p^e{:}m$, $r^a s^b = (p^e+1)l$ and $ml = \frac{p^e-1}{(2,p-1)}$, where $e > 1$ and $p^e > 4$. If $l = 1$ then T is 2-transitive on each T-orbit on V, and thus Γ is a complete graph or the standard double cover of a complete graph. Now let $l > 1$. Suppose that $p = 2$. Then $(l, 2^e+1) = 1$, since $(2^e+1)l = r^a s^b$, we conclude that $e = 2^j$ for some $j > 1$. It follows that $|G^*:T|$ is a divisor of 2^j. Then $|G_\alpha| = |T_\alpha||G^*:T| = 2^e 2^{j_0}\frac{2^e-1}{l}$ for some $j_0 \leqslant j$, and thus $|G_\alpha|$ is indivisible by 2^e-1, a contradiction. Therefore, p is an odd prime. Since $e > 1$, it is easily shown that p^e+1 is not a power of 2. Noting that $(p^e+1, p^e-1) = 2$ and $(p^e+1)l = r^a s^b$, it follows that $l = 2^{i_0}$ and $e = 2^i$, where $i_0, i \geqslant 1$. In particular, $|G^*:T|$ is a divisor of 2^i. Setting $|G^*:T| = 2^{i_1}$ for some $i_1 \leqslant i$, we have $m = \frac{p^e-1}{2^{i_0+1}}$ and $|G_\alpha| = |T_\alpha||G^*:T| = p^e 2^{i_1}\frac{p^e-1}{2^{i_0+1}}$. Since $|G_\alpha|$ has a divisor p^e-1, we have $i_0+1 \leqslant i_1 \leqslant i$, and so $i_0 < i$. Checking the maximal subgroups of G, it follows from [3, page 377, Table 8.1] that $G_\alpha \leqslant \mathrm{A\Gamma L}_1(p^{2^i})$, desired as in part (4) of this theorem.

Finally, let $T = \mathrm{A}_6$. Noting that $\mathrm{A}_6 \cong \mathrm{PSL}_2(9)$, by the above argument, we have $T_\alpha \ncong 3^2{:}2$. Then $T_\alpha \cong 3^2{:}4$, and thus Γ is either the complete graph K_{10} or the standard double cover of K_{10}. This completes the proof. □

For the case where k is a prime, we can read out the 2-arc-transitive members from the graphs given in [22].

Theorem 4.2. *If k is an odd prime then one of the following holds:*

1. Γ *is either a complete graph or the standard double cover of a complete graph;*
2. Γ *is the point-hyperplane incidence graph of the projective geometry* $\mathrm{PG}(n-1, p^e)$*, where p is a prime and* $(n, p^e) = (3, 4)$, $(4, p)$ *or* $(6, 2)$*;*
3. Γ *is the incidence graph of the generalized quadrangle* $\mathrm{GQ}(4, 2^{2^i})$*, where* $i \geqslant 1$*;*
4. $\Gamma \cong \mathrm{Cos}(G, H, K)$ *with* (G, H, K) *listed in Table* 4.2*.*

G	H	K	$r^a s^b$	k	Remark
$\mathrm{A}_5, \mathrm{S}_5$	$\mathrm{S}_3, \mathrm{D}_{12}$	$\mathbb{Z}_2^2, \mathrm{D}_8$	$2 \cdot 5$	3	F010
S_5	S_3	$\mathbb{Z}_2^2$	$2 \cdot 5$	3	F020A
S_5	S_3	$\mathbb{Z}_4$	$2 \cdot 5$	3	F020B, bipartite
$\mathrm{PGL}_2(7), \mathrm{PSL}_2(7)$	$\mathrm{D}_{12}, \mathrm{S}_3$	$\mathrm{D}_8, \mathbb{Z}_4$	$2^2 \cdot 7$	3	F028
$\mathrm{PGL}_2(7)$	S_3	$\mathbb{Z}_2^2$	$2^2 \cdot 7$	3	F056B
$\mathrm{PGL}_2(7)$	S_3	$\mathbb{Z}_4$	$2^2 \cdot 7$	3	F056C, bipartite
$\mathrm{PGL}_2(9), \mathrm{M}_{10}, \mathrm{P\Gamma L}_2(9)$	$\mathrm{S}_4, \mathrm{S}_4, 2 \times \mathrm{S}_4$	$\mathrm{D}_{16}, \mathrm{QD}_{16}, [2^5]$	$3 \cdot 5$	3	F030, bipartite
$\mathrm{PGL}_2(11)$	D_{12}	D_8	$5 \cdot 11$	3	F110, bipartite
$\mathrm{PGL}_2(13)$	D_{12}	D_8	$7 \cdot 13$	3	F182D, bipartite
$\mathrm{PGL}_2(23)$	S_4	D_{16}	$11 \cdot 23$	3	F506B, bipartite
$\mathrm{PGL}_2(25), \mathrm{P\Gamma L}_2(25)$	$\mathrm{S}_4, 2 \times \mathrm{S}_4$	$\mathrm{D}_{16}, [2^5]$	$5^2 \cdot 13$	3	F650B*, bipartite,
$\mathrm{P\Gamma L}_2(9), \mathrm{M}_{10}$	$10{:}4, 5{:}4$	$[2^4], \mathrm{Q}_8$	$2^2 \cdot 3^2$	5	
S_6	$5{:}4$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$2^2 \cdot 3^2$	5	
$\mathrm{P\Gamma L}_2(9)$	$5{:}4$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$2^2 \cdot 3^2$	5	bipartite
$\mathrm{M}_{12}.2$	$11{:}10$	D_{20}	$2^6 \cdot 3^3$	11	
$\mathrm{PSL}_2(31)$	A_5	S_4	$2^3 \cdot 31$	5	
$\mathrm{PGL}_2(p)$	A_5	S_4	$\frac{p(p^2-1)}{120}$	5	bipartite $p \in \{19, 29, 59, 61\}$
$\mathrm{PSL}_5(2).2$	$2^6{:}(\mathrm{S}_3 \times \mathrm{PSL}_3(2))$	$[2^8](\mathrm{S}_3 \times \mathrm{S}_3){:}2$	$5 \cdot 31$	7	bipartite
$\mathrm{PSU}_3(3).2$	$\mathrm{PSL}_3(2)$	$\mathrm{S}_4 \times 2$	$2^2 \cdot 3^2$	7	bipartite
$\mathrm{PSU}_3(5).2, \mathrm{PSU}_3(5)$	$\mathrm{S}_7, \mathrm{A}_7$	$\mathrm{A}_6.2^2, \mathrm{M}_{10}$	$5^2 \cdot 2$	7	Hoffman-Singleton
$\mathrm{PSU}_3(5).2$	A_7	M_{10}	$5^2 \cdot 2$	7	bipartite,
$\mathrm{PSU}_3(4).4$	$13{:}12$	$\mathrm{S}_3 \times 4$	$2^6 \cdot 5^2$	13	bipartite
$\mathrm{PSU}_5(2).2$	$\mathrm{PSL}_2(11)$	S_5	$2^8 \cdot 3^4$	11	bipartite
$\mathrm{M}_{12}.2$	$\mathrm{PSL}_2(11)$	S_5	$2^4 \cdot 3^2$	11	bipartite, $H \nleqslant \mathrm{M}_{11}$
M_{12}	$\mathrm{PSL}_2(11)$	S_5	$2^4 \cdot 3^2$	11	$H < \mathrm{M}_{11}$

TABLE 4.2. Some coset graphs of prime valency

Remark 4.3. (i) *The names for cubic graphs in Table* 4.2 *follow from* [5]*, the graphs* F506B *and* F650B* *in Table* 4.2 *were missed in* [22, Theorem 1.2 (1)].

(ii) *Note the isomorphisms of graphs:* F010 *and Petersen graph* O_3, F020B *and* $\mathsf{O}_3^{(2)}$*, and* F030*, Tutte's* 8*-cage and the incidence graph of* $\mathrm{GQ}(4, 2)$.

(iii) *The graph in Table* 4.2 *associated with* $\mathrm{PSL}_5(2).2$ *is in fact the line-plane incidence graph of* $\mathrm{PG}(4, 2)$*. This graph was missed in both* [19, Theorem 1.1] *and* [22, Theorem 1.2].

Theorem 4.4. *If $k = 4$ then one of the following holds:*

(1) Γ *is either the Odd graph* O_4 *with* $G \in \{\mathrm{A}_7, \mathrm{S}_7\}$ *or the standard double cover* $\mathsf{O}_4^{(2)}$ *with* $G = \mathrm{S}_7$*;*

(2) Γ *is isomorphic to one of the three graphs in Examples* 3.5*,* 3.6 *and* 3.7*.*

Proof. Assume that $k = 4$. Then G_α is one of A_4, S_4, $3 \times \mathrm{A}_4$, $(3 \times \mathrm{A}_4){:}2$, $\mathrm{S}_3 \times \mathrm{S}_4$, $3^2{:}\mathrm{GL}_2(3)$ and $[3^5]{:}\mathrm{GL}_2(3)$, refer to [20, Lemma 2.6]. Combining Lemma 2.4, it is easily shown that either $\mathbf{O}_2(T_\alpha) = \mathbf{O}_2(G_\alpha) \cong \mathbb{Z}_2^2$ or $\mathbf{O}_3(G_\alpha) \geqslant \mathbf{O}_3(T_\alpha) \neq 1$. Further, noting that $G_\alpha/T_\alpha \cong G^*/T \leqslant G/T$, it follows that $|G : T|$ has a divisor $|\mathbf{O}_3(G_\alpha) : \mathbf{O}_3(T_\alpha)|$. Then, checking one by one the simple groups given in [18, Tables 3.1,3,2,4.1-4.5, 5.1] and [22, Remark 5.2], all possible pairs (T, T_α) are described as in Table 4.3

T	T_α	$r^a s^b$	G
A_6	S_4	$3 \cdot 5$	$\mathrm{A}_6, \mathrm{S}_6$(If Γ bipartite)
A_7	$(3 \times \mathrm{A}_4){:}2$	$5 \cdot 7$	$\mathrm{A}_7, \mathrm{S}_7$
$\mathrm{PSL}_2(7)$	A_4	$2 \cdot 7$	$\mathrm{PSL}_2(7), \mathrm{PGL}_2(7)$(If Γ bipartite)
$\mathrm{PSL}_2(11)$	A_4	$7 \cdot 13$	$\mathrm{PSL}_2(11), \mathrm{PGL}_2(11)$
$\mathrm{PSL}_2(23)$	S_4	$11 \cdot 23$	$\mathrm{PSL}_2(23), \mathrm{PGL}_2(23)$(If Γ bipartite)
$\mathrm{PSL}_2(25)$	S_4	$5^2 \cdot 13$	$\mathrm{PSL}_2(25), \mathrm{P\Sigma L}_2(25)$(If Γ bipartite)

TABLE 4.3. Candidates of T for $k = 4$

Suppose that $T = \mathrm{A}_6$. Checking the subgroups of every almost simple group with socle A_6, since $|G : G_\alpha| = 15$ or 30, we conclude that $T_\alpha = G_\alpha$, and $G = \mathrm{A}_6$ or S_6. Computation by GAP shows that G_α and $\mathbf{N}_G(G_{\alpha\beta})$ are contained in a maximal subgroup S_4 or $2 \times \mathrm{S}_4$ of G, which contradicts Lemma 2.1.

Suppose $T = \mathrm{PSL}_2(25)$ and $T_\alpha \cong \mathrm{S}_4$. Then $G_\alpha = T_\alpha$, and $G_{\alpha\beta} \cong \mathrm{S}_3$. Take a maximal subgroup $M_1 \cong \mathrm{S}_5$ of T and a maximal subgroup $M_2 \cong \mathrm{S}_5 \times 2$ of $\mathrm{P\Sigma L}_2(25)$ such that $G_\alpha \leqslant M_1 \leqslant M_2$. Computation by GAP shows that either $G = \mathrm{PSL}_2(5)$ and $\mathrm{D}_{12} \cong \mathbf{N}_G(G_{\alpha\beta}) \leqslant M_1$, or $G = \mathrm{P\Sigma L}_2(25)$ and $2 \times \mathrm{D}_{12} \cong \mathbf{N}_G(G_{\alpha\beta}) \leqslant M_2$. Then $G \neq \langle G_\alpha, \mathbf{N}_G(G_{\alpha\beta})\rangle$, a contradiction.

Suppose that $T = \mathrm{A}_7$. Then the action of T on each of its orbits is equivalent to the action on 3-subsets induced by the natural action of A_7 of degree 7. Thus the resulting graph is either the Odd graph O_4 with $G \in \{\mathrm{A}_7, \mathrm{S}_7\}$ or the standard double cover $\mathsf{O}_4^{(2)}$ with $G = \mathrm{S}_7$.

Suppose that $T = \mathrm{PSL}_2(7)$ and $T_\alpha \cong \mathrm{A}_4$. Then $G_\alpha = T_\alpha$, and $G_{\alpha\beta} \cong \mathbb{Z}_3$. If $G = T$ then $\mathbf{N}_G(G_{\alpha\beta}) \cong \mathrm{S}_3$ and $\langle G_\alpha, \mathbf{N}_G(G_{\alpha\beta})\rangle \cong \mathrm{S}_4$, a contradiction. Thus $G = \mathrm{PGL}_2(7)$, and Γ is isomorphic to the graph given in Example 3.5.

Suppose that $T = \mathrm{PSL}_2(11)$ and $T_\alpha \cong \mathrm{A}_4$. Then either $G_\alpha = T_\alpha$, or $G = \mathrm{PGL}_2(11)$ and $G_\alpha \cong \mathrm{S}_4$. We have $G_{\alpha\beta} \cong \mathbb{Z}_3$ or S_3, respectively. Thus Γ is isomorphic to of one the graphs given in Example 3.6.

Finally, let $T = \mathrm{PSL}_2(23)$ and $T_\alpha \cong \mathrm{S}_4$. Then $G_\alpha = T_\alpha$, and $G_{\alpha\beta} \cong \mathrm{S}_3$. Computation by GAP shows that $\mathbf{N}_T(G_{\alpha\beta}) = \mathbf{N}_{\mathrm{PGL}_2(23)}(G_{\alpha\beta}) \cong \mathrm{D}_{12}$. Noting that G_α is maximal in T, we have $\langle G_\alpha, \mathbf{N}_T(G_{\alpha\beta})\rangle = T$. Then Γ is isomorphic to the graph given in Example 3.7. $\square$

5. Graphs with affine stabilizers

In this section, we deal with the case (B), see Section 4. Assume that $G_\alpha^{\Gamma(\alpha)}$ is an insolvable affine 2-transitive group of degree $|\Gamma(\alpha)| = p^e > 5$, where p is a prime and $e > 1$. Let $\{\alpha, \beta\} \in E$. By Lemma 2.4, since T_α is insolvable, we have

(1) $\mathbf{O}_p(G_\alpha) = \mathbf{O}_p(T_\alpha) \cong \mathrm{soc}(G_\alpha^{\Gamma(\alpha)}) = \mathrm{soc}(T_\alpha^{\Gamma(\alpha)}) = \mathbb{Z}_p^e$; and
(2) $G_\alpha = \mathbf{O}_p(T_\alpha){:}G_{\alpha\beta}$, $T_\alpha = \mathbf{O}_p(T_\alpha){:}T_{\alpha\beta}$; and either
  (i) $T_\alpha^{\Gamma(\alpha)}$ is 2-transitive, in particular, $|\mathbf{O}_p(T_\alpha)| - 1$ is a divisor of $|T_{\alpha\beta}|$; or
  (ii) $e = 2$, $p \in \{19, 29, 59\}$, and $\mathbb{Z}_p^2{:}\mathrm{SL}_2(5) \trianglelefteq T_\alpha^{\Gamma(\alpha)} \trianglelefteq G_\alpha^{\Gamma(\alpha)} \trianglelefteq \mathbb{Z}_p^2{:}(\mathbb{Z}_{p-1} \circ \mathrm{SL}_2(5))$.

Checking one by one the simple groups given in [18, Tables 3.1,3,2,4.1-4.5, 5.1] and [22, Remark 5.2], we conclude that all possible T_α are described as follows:

(a1) $(T, T_\alpha, r^a s^b, p^e)$ is one of $(\mathrm{A}_8, 2^3{:}\mathrm{PSL}_3(2), 3\cdot 5, 2^3)$, $(\mathrm{A}_9, 2^3{:}\mathrm{PSL}_3(2), 3^3\cdot 5, 2^3)$, $(\mathrm{M}_{22}, 2^4{:}\mathrm{A}_6, 7\cdot 11, 2^4)$, $(\mathrm{M}_{23}, 2^4{:}\mathrm{A}_7, 11\cdot 23, 2^4)$, $(\mathrm{PSL}_4(3), 2^4{:}\mathrm{S}_5, 3^5\cdot 13, 2^4)$, $(\mathrm{PSU}_4(3), 2^4{:}\mathrm{A}_6, 3^4\cdot 7, 2^4)$ and $(\mathrm{G}_2(3), 2^3{:}\mathrm{PSL}_3(2), 3^5\cdot 13, 2^3)$;
(a2) $T = \mathrm{PSL}_n(p^f)$ and $T_\alpha \cong p^{(n-1)f}{:}(\mathrm{SL}_{n-1}(p^f).o)$, where $\mathrm{SL}_{n-1}(p^f)$ is insolvable, and o is a divisor of $\frac{p^f-1}{(n,q-1)}$;
(a3) $T = \mathrm{PSU}_4(2^{2^i})$ and $T_\alpha \cong 2^{2^{i+2}}{:}\mathrm{SL}_2(2^{2^{i+1}}){:}(2^{2^i}-1)$, where $i \geqslant 1$.

Lemma 5.1. *Assume that* (a1) *holds. Then* $T \neq \mathrm{M}_{23}$, $\mathrm{PSL}_4(3)$, $\mathrm{PSU}_4(3)$ *or* $\mathrm{G}_2(3)$.

Proof. Suppose that $T = \mathrm{M}_{23}$ and $T_\alpha \cong 2^4{:}\mathrm{A}_7$. Then $G = T$ and $G_{\alpha\beta} \cong \mathrm{A}_7$. Checking the maximal subgroups of M_{23} containing $\mathbf{N}_G(G_{\alpha\beta})$, we conclude that $\mathbf{N}_G(G_{\alpha\beta}) = G_{\alpha\beta}$, a contradiction.

Suppose that $T = \mathrm{PSL}_4(3)$ and $T_\alpha \cong 2^4{:}\mathrm{S}_5$. Assume that Γ is not bipartite. Then T is a 2-arc-transitive group of Γ, and thus $T = \langle T_\alpha, \mathbf{N}_T(T_{\alpha\beta})\rangle$. Noting that $T_{\alpha\beta} \cong \mathrm{S}_5$ and T_α is contained in a maximal subgroup $M \cong \mathrm{PSU}_4(2){:}2$ of T, computation by GAP shows that $2 \times \mathrm{S}_5 \cong \mathbf{N}_T(T_{\alpha\beta}) < M$. Thus $\langle T_\alpha, \mathbf{N}_T(T_{\alpha\beta})\rangle \leqslant M$, a contradiction. Therefore, Γ is bipartite, and $G^* = \langle G_\alpha, G_\beta\rangle$ has index 2 in G. Note that $G_\alpha/T_\alpha \cong G^*/T$. Assume that $T \neq G^*$. Then $G^* = T.2$. Since $|G^* : G_\alpha| = |T : T_\alpha| = 3^5 \cdot 13$, by the information given in the Atlas [6], we conclude that $G_\alpha \cong 2^4{:}\mathrm{S}_5 \times 2$. Then $\mathbf{O}_2(G_\alpha) \cong \mathbb{Z}_2^5$, a contradiction. Thus we have $G^* = T$, $|G : T| = 2$, $G_\alpha = T_\alpha$ and $G_{\alpha\beta} = T_{\alpha\beta}$. Checking the maximal subgroups of G containing $\mathbf{N}_T(T_{\alpha\beta})$, we conclude that either $\mathbf{N}_G(T_{\alpha\beta}) = \mathbf{N}_T(T_{\alpha\beta}) < M$ or $\mathbf{N}_G(T_{\alpha\beta}) = \mathbf{N}_T(T_{\alpha\beta}) < M \times \langle g\rangle$, where $g \in G \setminus T$. For both cases, we have $\langle G_\alpha, \mathbf{N}_G(T_{\alpha\beta})\rangle \neq G$, a contradiction.

Suppose that $T = \mathrm{PSU}_4(3)$ and $T_\alpha \cong 2^4{:}\mathrm{A}_6$. Then T_α is maximal in T, and $T_{\alpha\beta} \cong \mathrm{A}_6$. Confirmed by GAP, we have $\mathbf{N}_T(T_{\alpha\beta}) = T_{\alpha\beta}$, and then $G > T$. Pick a maximal subgroup M of G with $G_\alpha \leqslant M \nleqslant T$. Then $M \cong 2^4{:}\mathrm{S}_6$, $2^5{:}\mathrm{A}_6$ or $2^5{:}\mathrm{S}_6$. Recall that G_α/T_α is solvable. It is easily shown that $T_{\alpha\beta}$ is characteristic in $G_{\alpha\beta}$, and thus $T_{\alpha\beta} \trianglelefteq \mathbf{N}_G(G_{\alpha\beta})$. Then $\mathbf{N}_G(G_{\alpha\beta}) \leqslant \mathbf{N}_G(T_{\alpha\beta})$. Assume that $\mathbf{N}_G(T_{\alpha\beta}) \nleqslant M$. Then $\mathbf{N}_G(T_{\alpha\beta}) = T_{\alpha\beta}.[2o]$, where $o = |G : T|$. Thus $o|T| = |G| \geqslant |T\mathbf{N}_G(T_{\alpha\beta})| = \frac{|T||\mathbf{N}_G(T_{\alpha\beta})|}{|\mathbf{N}_T(T_{\alpha\beta})|} = 2o|T|$, yielding $1 \geqslant 2$, a contradiction. Then $\mathbf{N}_G(G_{\alpha\beta}) \leqslant \mathbf{N}_G(T_{\alpha\beta}) \leqslant M$, and thus $\langle G_\alpha, \mathbf{N}_G(G_{\alpha\beta})\rangle \leqslant M \neq G$, a contradiction.

Finally, for $T = \mathrm{G}_2(3)$, computation by GAP shows that both G_α and $\mathbf{N}_G(G_{\alpha\beta})$ are contained in a same maximal subgroup $2^3{:}\mathrm{PSL}_3(2)$ or $2^3{:}\mathrm{PSL}_3(2){:}2$ of G, which is not the case. Then the lemma follows. □

Lemma 5.2. *If* (a1) *holds then one of the following holds:*

(1) $G = \mathrm{S}_8$ *and* Γ *is the point-line non-incidence graph of* $\mathrm{PG}(3,2)$*;*
(2) Γ *is isomorphic to one of the graphs in Examples* 3.1 *and* 3.8*;*
(3) $G = \mathrm{M}_{22}.2$ *and* Γ *is the standard double cover of the graph in Example* 3.1*.*

Proof. Assume that $T = \mathrm{A}_8$ and $T_\alpha \cong 2^3{:}\mathrm{PSL}_3(2)$. Then $T_\alpha \cong T_\alpha^{\Gamma(\alpha)} = \mathrm{AGL}_3(2) = G_\alpha^{\Gamma(\alpha)}$, yielding $G_\alpha = T_\alpha$. Then $G_{\alpha\beta} = T_{\alpha\beta} \cong \mathrm{PSL}_3(2)$, and T as a permutation group on each T-orbit is permutation isomorphic to $\mathrm{PSL}_4(2)$ acting on the points or lines of the projective geometry $\mathrm{PG}(3,2)$. Checking the maximal subgroups of T containing $\mathrm{PSL}_3(2)$, we have $\mathbf{N}_T(T_{\alpha\beta}) < T_\alpha$. Thus $G \neq T$, and $G = \mathrm{S}_8$. Then the resulting graph Γ is the point-line non-incidence graph of $\mathrm{PG}(3,2)$.

Assume that $T = \mathrm{M}_{22}$ and $T_\alpha \cong 2^4{:}\mathrm{A}_6$. Then the action of T on each of its orbits is equivalent to that on the 77 hexads. Thus, if Γ is not bipartite then $G = \mathrm{M}_{22}$ or $\mathrm{M}_{22}.2$ and Γ is the graph in Example 3.1, if Γ is bipartite then $G = \mathrm{M}_{22}.2$ and Γ is the standard double cover of the graph in Example 3.1.

Assume that $T = \mathrm{A}_9$ and $T_\alpha \cong 2^3{:}\mathrm{PSL}_3(2)$. Similarly as above, $G_\alpha = T_\alpha$ and $G_{\alpha\beta} = T_{\alpha\beta} \cong \mathrm{PSL}_3(2)$. Confirmed by GAP, we have $\mathbf{N}_T(T_{\alpha\beta}) = T_{\alpha\beta}$. It follows that $G = \mathrm{S}_9$. Further, checking the maximal subgroups of S_9, we conclude that T_α is contained in a subgroup A_8. Thus Γ is isomorphic to the graph in Example 3.8. □

For a finite group X, denote by $X^{(\infty)}$ the intersection of all subgroups appearing the derived series of X.

Lemma 5.3. *If* (a2) *holds then* Γ *is isomorphic to one of the graphs in Example* 3.9*.*

Proof. Assume that $T = \mathrm{PSL}_n(p^f)$ and $T_\alpha \cong p^{(n-1)f}{:}(\mathrm{SL}_{n-1}(p^f).o)$. Then $T_\alpha^{(\infty)} \cong p^{(n-1)f}{:}\mathrm{SL}_{n-1}(p^f)$. Pick a maximal subgroup M of T with $T_\alpha \leqslant M$. Then M is a stabilizer of T acting on the point set or hyperplane set of the projective geometry $\mathrm{PG}(n-1,p^f)$, $M^{(\infty)} = T_\alpha^{(\infty)}$ and $M/T_\alpha^{(\infty)} \cong \mathbb{Z}_{\frac{p^f-1}{(n,p^f-1)}}$. In particular, T_α is characteristic in M. Without loss of generality, we let M be the stabilizer of some point of $\mathrm{PG}(n-1,p^f)$.

Pick $x \in G$ with $(\alpha,\beta)^x = (\beta,\alpha)$, and set $X = T\langle x\rangle$. Then X is a 2-arc-transitive group of Γ, and so $\Gamma \cong \mathrm{Cos}(X, X_\alpha, X_{\{\alpha,\beta\}})$ and $X = \langle X_\alpha, X_{\{\alpha,\beta\}}\rangle$. Clearly, $x^2 \in \mathrm{P\Gamma L}_n(p^f)$. Then M^{x^2} is a point-stabilizer of T acting on $\mathrm{PG}(n-1,p^f)$. Noting that $x^2 \in X_\alpha$, we have $x^2 \in \mathbf{N}_X(T_\alpha)$. Thus $T_\alpha \leqslant M \cap M^{x^2}$, forcing $M = M^{x^2}$. Suppose that $M = M^x$. Then $M\langle x\rangle$ is maximal in X, and $X_\alpha \leqslant \mathbf{N}_X(T_\alpha) \leqslant \mathbf{N}_X(T_\alpha^{(\infty)}) = M\langle x\rangle$. Thus both X_α and $X_{\{\alpha,\beta\}}$ are contained in $M\langle x\rangle$, a contradiction. Therefore, $M \neq M^x$.

Since $T_\alpha \trianglelefteq M$, we have $T_\alpha(M\cap M^x) \leqslant M$, and so $|T_\alpha(M\cap M^x) : (M\cap M^x)| = |T_\alpha : (T_\alpha \cap M^x)|$. Noting that $T_\beta \leqslant M^x$, it follows that $|T_\alpha(M\cap M^x) : (M\cap M^x)|$ is a divisor of $|\Gamma(\alpha)| = |T_\alpha : T_{\alpha\beta}|$. Suppose that M^x is a point-stabilizer of T acting on $\mathrm{PG}(n-1,p^f)$. Then $|T_\alpha(M\cap M^x) : (M\cap M^x)|$ is divisible by $\frac{p^f(p^{(n-1)f}-1)}{p^f-1}$. Thus $|\Gamma(\alpha)|$ has a divisor $\frac{p^f(p^{(n-1)f}-1)}{p^f-1}$, which is impossible. Therefore, M^x is a hyperplane-stabilizer of T acting on $\mathrm{PG}(n-1,p^f)$. Noting $\mathbf{N}_T(T_\alpha) = M$ and $\mathbf{N}_T(T_\beta) = M^x$, it follows that T_α and T_β are not conjugate in T. Then Γ is bipartite.

Finally, since $|T_\alpha(M\cap M^x) : (M\cap M^x)|$ is a divisor of $|\Gamma(\alpha)| = p^{(n-1)f}$, we conclude that the hyperplane fixed by M^x does not contain the point fixed by M. Thus Γ is isomorphic to one of the graphs in Example 3.9 (1), and the lemma holds. □

Theorem 5.4. *If $G_\alpha^{\Gamma(\alpha)}$ is an insolvable affine 2-transitive group then one of the following holds:*

(1) Γ *is isomorphic to one of the graphs in Examples* 3.1, 3.2, 3.8 *and* 3.9*; or*
(2) $G = \mathrm{M}_{22}.2$ *and* Γ *is the standard double cover of the graph in Example* 3.1*;*
(3) $G = \mathrm{PSU}_4(2^{2^i}).2^j$ *and* Γ *is the standard double cover of the graph in Example* 3.2*, where* $1 \leqslant j \leqslant i+1$.

Proof. Assume that $G_\alpha^{\Gamma(\alpha)}$ is an insolvable affine 2-transitive group. By Lemmas 5.2, 5.3 and the argument ahead of Lemma 5.1, only the case (a3) is left. Thus we assume that $T = \mathrm{PSU}_4(2^{2^i})$ and $T_\alpha \cong 2^{2^{i+2}}{:}\mathrm{SL}_2(2^{2^{i+1}}){:}(2^{2^i}-1)$, where $i \geqslant 1$. Then the action of T on each of its orbits is equivalent to that on the set of $(2^{2^i}+1)^2(2^{2^{i+1}}-2^{2^i}+1)$ isotropic lines. If Γ is not bipartite then Γ is described as in Example 3.2. If Γ is bipartite then, by Lemmas 3.11 and 3.13, we have part (3), and the result follows. □

6. Graphs with almost simple stabilizers

In this section, we deal with the case (C), see Section 4. Assume that $G_\alpha^{\Gamma(\alpha)}$ is almost simple, and $k = |\Gamma(\alpha)|$ is not a prime. Recall that $\mathrm{soc}(T_\alpha^{\Gamma(\alpha)}) = \mathrm{soc}(G_\alpha^{\Gamma(\alpha)})$.

6.1. **Graphs with non-trivial edge-kernel.** Assume that $G_{\alpha\beta}^{[1]} \neq 1$, where $\{\alpha,\beta\} \in E$. Then $\mathrm{soc}(T_\alpha^{\Gamma(\alpha)}) = \mathrm{soc}(G_\alpha^{\Gamma(\alpha)}) \cong \mathrm{PSL}_d(q)$ and $k = \frac{q^d-1}{q-1}$ by Theorem 2.3. Set $q = p^f$ for a prime p. By Theorem 2.3 and [27], since k is not a prime, $\mathbf{O}_p(G_\alpha)$ is described as Table 6.4.

$\mathbf{O}_p(G_\alpha)$	d	q	G_α
$[p^{2f}]$	2	p^f	$[p^{2f}]{:}(\frac{p^f-1}{(3,p^f-1)}.\mathrm{PGL}(2,p^f)).[o]$ with $o \mid (3,p^f-1)f$
$[2^{3f}]$	2	2^f	$[2^{3f}]{:}\mathrm{GL}(2,2^f).e$, $e \mid f, f>2$
$[3^{5f}]$	2	3^f	$[3^{5f}]{:}\mathrm{GL}(2,3^f).e$, $e \mid f, f>1$
$p^{d(d-1)f}$	$\geqslant 3$	p^f	$\mathrm{SL}_{d-1}(q){\times}\mathrm{SL}_d(q) \trianglelefteq G_\alpha/\mathbf{O}_p(G_\alpha)$, $(d,q)\neq(3,2)$
p^{df}	$\geqslant 3$	p^f	$o.\mathrm{PSL}_d(q) \trianglelefteq G_\alpha/\mathbf{O}_p(G_\alpha)$, $o \mid q-1, (d,q)\neq(3,2)$
$p^{\frac{d(d-1)f}{2}}$	$\geqslant 3$	p^f	$o.\mathrm{PSL}_d(q) \trianglelefteq G_\alpha/\mathbf{O}_p(G_\alpha)$, $o \mid q-1, (d,q)\neq(3,2)$
$[q^{20}]$	3	2^f	$\mathrm{SL}_2(q){\times}\mathrm{SL}_3(q) \trianglelefteq G_\alpha/\mathbf{O}_p(G_\alpha)$, $q\neq 2$
3^6	3	3	$3^6{:}\mathrm{SL}_3(3)$
2^{d+1}	$d>3$	2	$2^{d+1}{:}\mathrm{SL}_d(2)$
$2^{11}, 2^{14}$	4	2	$2^{11}{:}\mathrm{SL}_4(2), 2^{14}{:}\mathrm{SL}_4(2)$

Table 6.4. Stabilizers with non-trivial edge-kernel

It is easily shown that $\mathbf{O}_p(T_\alpha) = T_\alpha \cap \mathbf{O}_p(G_\alpha)$. Then

$$|\mathbf{O}_p(G_\alpha)| = |\mathbf{O}_p(T_\alpha)||T_\alpha\mathbf{O}_p(G_\alpha) : T_\alpha|.$$

Recalling that $G_\alpha/T_\alpha = G^*/T$, it follows that $|\mathbf{O}_p(G_\alpha)|$ is a divisor of $|\mathbf{O}_p(T_\alpha)||G^*:T|$. Combining Table 6.4, we read out all possible pairs (T, T_α) from [18] and [22] as in Table 6.5. (Note, we deal with the alternating group A_8 as $\mathrm{PSL}_4(2)$.)

Theorem 6.1. *Assume that $G_\alpha^{\Gamma(\alpha)}$ is almost simple, $G_{\alpha\beta}^{[1]} \neq 1$ and $k = |\Gamma(\alpha)|$ is not a prime, where $\{\alpha,\beta\} \in E$. Then one of the following holds:*

(1) Γ *is isomorphic to one of the graphs in Examples* 3.9 *and* 3.10*;*

T	T_α	d	
$\mathrm{PSL}_n(p^f)$	$p^{(n-1)f}{:}(\mathrm{SL}_{n-1}(p^f).o)$	$n-1$	$o \mid \frac{p^f-1}{(n,p^f-1)}$, $(n,p^f)\neq(3,2),(3,3),(4,2)$
$\mathrm{PSp}_4(2^f)$	$2^{3f}{:}\mathrm{GL}_2(2^f)$	2	$f=2^i$ for $i\geqslant 2$
$\mathrm{P}\Omega_8^+(2)$	$2^6{:}\mathrm{PSL}_4(2)$	4	

Table 6.5. Candidates of T with non-trivial edge-kernel

(2) Γ *is the incidence graph of the generalized quadrangle* $\mathrm{GQ}(4,2^{2^i})$*, where* $i\geqslant 1$.

Proof. By the foregoing argument, we need only consider the simple groups listed in Table 6.5. For $T=\mathrm{PSL}_n(p^f)$, by a similar argument as in the proof of Lemma 5.3, T_α and T_β are contained respectively in the stabilizers of a point and a hyperplane in T, and then Γ is described as in (2) of Example 3.9.

Assume that $T=\mathrm{P}\Omega_8^+(2)$. Then $k=15$ and $|T:T_\alpha|=135$ is odd, and thus Γ is bipartite. By the Atlas [6], checking the maximal subgroups of almost simple groups with socle T, we have $G=T.2$. Then Γ is isomorphic to the graph in Example 3.10.

Now let $T=\mathrm{PSp}_4(2^f)$ and $T_\alpha\cong 2^{3f}{:}\mathrm{GL}_2(2^f)$. In this case, T_α is maximal in T, $k=2^f+1$ and $|T:T_\alpha|=(2^f+1)(2^{2f}+1)$, which are odd. Thus Γ is bipartite, each T-orbit on V may be identified with the point set or the line set of $\mathrm{GQ}(4,2^{2^i})$. In particular, T has rank three on each orbit with suborbits of length 1, 2^{3f} and $2^f(2^f+1)$. If the actions of T on both orbits are equivalent, then $k=2^{3f}$ or $2^f(2^f+1)$, a contradiction. Therefore, one T-orbit is the point set of $\mathrm{GQ}(4,2^{2^i})$, and the other one is the line set of $\mathrm{GQ}(4,2^{2^i})$. Then (2) of this theorem follows. □

6.2. **Graphs with trivial edge-kernel.** Assume that $G_{\alpha\beta}^{[1]}=1$, where $\{\alpha,\beta\}\in E$. Recall that $\mathrm{soc}(G_\alpha^{\Gamma(\alpha)})=\mathrm{soc}(T_\alpha^{\Gamma(\alpha)})$, $|T:T_\alpha|=r^as^b$ and $k=|\Gamma(\alpha)|$ is not a prime. Combining the classification of almost simple 2-transitive groups (see [4, page 197, Table 7.4]), we can read out all possible $\mathrm{soc}(G_\alpha^{\Gamma(\alpha)})$ from [18, Theorem 1.1] and [22, Remark 5.2]. Since $G_{\alpha\beta}^{[1]}=1$, by (2.1) and (2.2) given in Section 2, $G_\alpha^{[1]}$ is isomorphic to a normal subgroup of $(G_\alpha^{\Gamma(\alpha)})_\beta$. It follows that one of the following holds:

- (s1) $T_\alpha^{\Gamma(\alpha)}=G_\alpha^{\Gamma(\alpha)}\cong \mathrm{A}_7$, $k=15$, and $G_\alpha=T_\alpha\cong\mathrm{A}_7$ or $\mathrm{PSL}_2(7)\times\mathrm{A}_7$;
- (s2) $T_\alpha^{\Gamma(\alpha)}\cong\mathrm{M}_{11}$ with $k=12$, or $T_\alpha^{\Gamma(\alpha)}=\mathrm{M}_{22}$ with $k=22$;
- (s3) $\mathrm{soc}(G_\alpha^{\Gamma(\alpha)})=\mathrm{A}_k$, $k\geqslant 6$, and T_α or G_α is isomorphic to one of A_k, S_k, $\mathrm{A}_{k-1}\times\mathrm{A}_k$, $(\mathrm{A}_{k-1}\times\mathrm{A}_k).2$ and $\mathrm{S}_{k-1}\times\mathrm{S}_k$;
- (s4) $\mathrm{soc}(G_\alpha^{\Gamma(\alpha)})=\mathrm{PSU}_3(p^f)$, $k=p^{3f}+1$, and either $T_\alpha^{[1]}=1$ or $p^f\leqslant|\mathbf{O}_p(T_\alpha)|\leqslant p^{3f}$;
- (s5) $\mathrm{soc}(G_\alpha^{\Gamma(\alpha)})=\mathrm{PSp}_{2d}(2)$ with $d\geqslant 3$, $k=2^{2d-1}\pm 2^{d-1}$, and T_α is isomorphic to a normal subgroup of $\mathrm{P}\Omega_{2d}^{\pm}(2).2\times\mathrm{PSp}_{2d}(2)$;
- (s6) $\mathrm{soc}(G_\alpha^{\Gamma(\alpha)})=\mathrm{PSL}_d(p^f)$, $k=\frac{p^{df}-1}{p^f-1}$, and either $T_\alpha^{[1]}=1$ or $\mathbf{O}_p(T_\alpha)\cong\mathbb{Z}_p^{(d-1)f}$.

Then, checking the groups given in [18, Theorem 1.1] and [22, Remark 5.2], all possible pairs (T,T_α) are listed in Table 6.6.

Lemma 6.2. $T\neq\mathrm{PSU}_6(2)$ *or* $\mathrm{P}\Omega_7(3)$.

Proof. Suppose that $T=\mathrm{PSU}_6(2)$. Then $T_\alpha\cong\mathrm{M}_{22}\cong T_\beta$, $k=22$ and $T_{\alpha\beta}\cong\mathrm{PSL}_3(4)$. In particular, T_α is maximal in T, and thus T is a primitive group of degree $2^8\cdot 3^4$

T	T_α	$r^a s^b$	k	$T_{\alpha\beta}$
A_9	A_7	$2^3\cdot 3^2$	15	$\mathrm{PSL}_3(2)$
M_{22}	A_7	$2^4\cdot 11$	15	$\mathrm{PSL}_3(2)$
$\mathrm{PSp}_6(2)$	A_7	$2^6\cdot 3^2$	15	$\mathrm{PSL}_3(2)$
$\mathrm{PSU}_3(5)$	A_7	$2\cdot 5^2$	15	$\mathrm{PSL}_3(2)$
$\mathrm{PSU}_4(3)$	A_7	$2^4\cdot 3^4$	15	$\mathrm{PSL}_3(2)$
M_{12}	M_{11}	$2^2\cdot 3$	12	$\mathrm{PSL}_2(11)$
HS	M_{22}	$2^2\cdot 5^2$	22	$\mathrm{PSL}_3(4)$
McL	M_{22}	$3^4\cdot 5^2$	22	$\mathrm{PSL}_3(4)$
$\mathrm{PSU}_6(2)$	M_{22}	$2^8\cdot 3^4$	22	$\mathrm{PSL}_3(4)$
M_{11}	A_6	$2\cdot 11$	6	A_5
A_n	A_{n-1}	n	$n-1$	A_{n-2}
A_n	S_{n-2}	$\frac{n(n-1)}{2}$	$n-2$	S_{n-3}
A_n	A_{n-2}	$n(n-1)$	$n-2$	A_{n-3}
$\mathrm{PSL}_3(4)$	A_6	$2^3\cdot 7$	6	A_5
$\mathrm{PSp}_4(3)$	S_6	$2^2\cdot 3^2$	6	S_5
$\mathrm{PSp}_4(3)$	A_6	$2^3\cdot 3^2$	6	A_5
$\mathrm{PSp}_6(2)$	S_8	$2^2\cdot 3^2$	8	S_7
$\mathrm{PSp}_6(2)$	A_8	$2^3\cdot 3^2$	8	A_7
J_2	$\mathrm{PSU}_3(3)$	$2^2\cdot 5^2$	28	$[3^3]{:}8$
HS	$\mathrm{PSU}_3(5).2$	$2^4\cdot 11$	126	$[5^3]{:}8{:}2$
HS	$\mathrm{PSU}_3(5)$	$2^5\cdot 11$	126	$[5^3]{:}8$
$\mathrm{G}_2(3)$	$\mathrm{PSU}_3(3){:}2$	$3^3\cdot 13$	28	$[3^3]{:}8{:}2$
$\mathrm{P\Omega}_7(3)$	$\mathrm{PSp}_6(2)$	$3^5\cdot 13$	$28, 36$	$\mathrm{PSU}_4(2){:}2, \mathrm{S}_8$
$\mathrm{P\Omega}^+_{2m}(2)$	$\mathrm{PSp}_{2m-2}(2)$	$2^{m-1}\cdot(2^m-1)$	$2^{2m-3}\pm 2^{m-2}$	$\mathrm{P\Omega}^\pm_{2m-2}(2).2$, prime $m=2^i+1\geqslant 5$
$\mathrm{P\Omega}^-_{2m}(2)$	$\mathrm{PSp}_{2m-2}(2)$	$2^{m-1}\cdot(2^m+1)$	$2^{2m-3}\pm 2^{m-2}$	$\mathrm{P\Omega}^\pm_{2m-2}(2).2, m=2^i\geqslant 4$
A_6	$\mathrm{PSL}_2(5)$	$2\cdot 3$	6	D_{10}
A_7	$\mathrm{PGL}_2(5)$	$3\cdot 7$	6	$5{:}4$
A_7	$\mathrm{PSL}_2(7)$	$3\cdot 5$	8	$7{:}3$
A_8	$\mathrm{PSL}_2(7)$	$3^4\cdot 5$	8	$7{:}3$
A_8	$\mathrm{PSL}_2(9)$	$2^3\cdot 7$	10	$3^2{:}4$
A_8	S_6	$2^2\cdot 7$	10	$3^2{:}\mathrm{D}_8$
M_{11}	$\mathrm{PSL}_2(11)$	$2^2\cdot 3$	12	$11{:}5$
M_{11}	$\mathrm{PSL}_2(9)$	$2\cdot 11$	10	$3^2{:}4$
M_{12}	$\mathrm{PSL}_2(11)$	$2^4\cdot 3^2$	12	$11{:}5$
M_{22}	$\mathrm{PSL}_3(4)$	$2\cdot 11$	21	$2^4{:}\mathrm{A}_5$
$\mathrm{PSL}_2(16)$	$\mathrm{PSL}_2(5)$	$2^2\cdot 17$	6	D_{10}
$\mathrm{PSL}_2(19)$	$\mathrm{PSL}_2(5)$	$3\cdot 19$	6	D_{10}
$\mathrm{PSL}_2(25)$	$\mathrm{PGL}_2(5)$	$5\cdot 13$	6	$5:4$
$\mathrm{PSL}_2(29)$	$\mathrm{PSL}_2(5)$	$7\cdot 29$	6	D_{10}
$\mathrm{PSL}_2(31)$	$\mathrm{PSL}_2(5)$	$2^3\cdot 31$	6	D_{10}
$\mathrm{PSL}_2(59)$	$\mathrm{PSL}_2(5)$	$29\cdot 59$	6	D_{10}
$\mathrm{PSL}_2(61)$	$\mathrm{PSL}_2(5)$	$31\cdot 61$	6	D_{10}
$\mathrm{PSL}_3(4)$	$\mathrm{PSL}_2(9)$	$2^3\cdot 7$	10	$3^2{:}4$
$\mathrm{PSL}_2(2^{2^{i+1}})$	$\mathrm{PSL}_2(2^{2^i})$	$2^{2^i}(2^{2^{i+1}}+1)$	$2^{2^i}+1$	$2^{2^i}{:}(2^{2^i}-1), i\geqslant 2$
$\mathrm{PSL}_n(2)$	$\mathrm{PSL}_{n-1}(2)$	$2^{n-1}(2^n-1)$	$2^{n-1}-1$	$2^{n-2}{:}\mathrm{PSL}_{n-2}(2),$ odd prime $n\geqslant 5$
$\mathrm{PSp}_4(3)$	$\mathrm{PGL}_2(5)$	$2^3\cdot 3^3$	6	$5{:}4$
$\mathrm{PSp}_4(3)$	$\mathrm{PSL}_2(5)$	$2^4\cdot 3^3$	6	D_{10}
$\mathrm{PSp}_4(3)$	$\mathrm{P\Sigma L}_2(9)$	$2^2\cdot 3^2$	10	$3^2{:}\mathrm{D}_8$
$\mathrm{PSp}_4(3)$	$\mathrm{PSL}_2(9)$	$2^3\cdot 3^2$	10	$3^2{:}4$
$\mathrm{PSp}_6(2)$	$\mathrm{PSL}_4(2)$	$2^3\cdot 3^2$	15	$2^3{:}\mathrm{PSL}_3(2)$
$\mathrm{PSU}_3(3)$	$\mathrm{PSL}_2(7)$	$2^2\cdot 3^2$	8	$7{:}3$
$\mathrm{PSU}_3(5)$	M_{10}	$5^2\cdot 7$	10	$3^2{:}\mathrm{Q}_8$
$\mathrm{PSU}_3(4)$	$5\times\mathrm{PSL}_2(5)$	$2^4\cdot 13$	6	$5\times\mathrm{D}_{10}$
$\mathrm{PSU}_4(3)$	$\mathrm{PSL}_3(4)$	$2\cdot 3^4$	21	$2^4{:}\mathrm{A}_5$
$\mathrm{PSU}_5(2)$	$\mathrm{PSL}_2(11)$	$2^8\cdot 3^4$	12	$11{:}5$
${}^2\mathrm{F}_4(2)'$	$\mathrm{PSL}_2(25)$	$2^8\cdot 3^2$	26	$5^2{:}12$
$\mathrm{G}_2(3)$	$\mathrm{PSL}_2(13)$	$2^4\cdot 3^5$	14	$13{:}6$

TABLE 6.6. Candidates of T with trivial edge-kernel

on each T-orbit. By the web version of [6], as a primitive group of degree $2^8\cdot 3^4$, the group T has no suborbit of length 22. This forces that Γ is bipartite, $G=\mathrm{PSU}_6(2).2$,

$G_\alpha = T_\alpha$ and $G_{\alpha\beta} = T_{\alpha\beta} \cong \mathrm{PSL}_3(4)$. Computation by Magma shows that either $\mathbf{N}_G(G_{\alpha\beta}) = G_{\alpha\beta}$, or both G_α and $G_{\alpha\beta}$ are contained in a maximal subgroup $\mathrm{M}_{22}.2$ of G, which contradicts Lemma 2.1.

Suppose that $T = \mathrm{P}\Omega_7(3)$. We have $G < \mathrm{PGL}_7(3)$, $G_\alpha = T_\alpha \cong \mathrm{PSp}_6(2)$, and $G_{\alpha\beta} \cong \mathrm{PSU}_4(2){:}2$ or S_8. Consulting the 7-dimensional representation of $\mathrm{PSp}_6(2)$ over the field of order 3, computation by GAP shows that $|\mathbf{N}_{\mathrm{PGL}_7(3)}(G_{\alpha\beta}) : G_{\alpha\beta}| = 3$ or 1. Thus $|\mathbf{N}_G(G_{\alpha\beta}) : G_{\alpha\beta}|$ is odd, which contradicts Lemma 2.1. □

Lemma 6.3. *$T \neq \mathrm{P}\Omega^{\pm}_{2m}(2)$ for $m \geqslant 4$.*

Proof. Suppose that $T = \mathrm{P}\Omega^{\pm}_{2m}(2)$ for some $m \geqslant 4$. Then each of T-orbits on V may be viewed as a copy of the set of $2^{m-1}(2^m \mp 1)$ non-isotropic points. Thus T_α has exactly three orbits on the T-orbit say U containing α, which have length 1, $2^{2m-2}-1$ and $2^{m-1}(2^{m-1} \mp 1)$ respectively, refer to [16]. Then Γ is either an orbital graph of T on U or the standard double cover of some orbital graph (see Lemma 3.13). It follows that $|\Gamma(\alpha)| = 2^{2m-2}-1$ or $2^{m-1}(2^{m-1} \mp 1)$, a contradiction. □

Lemma 6.4. *$T \neq \mathrm{PSL}_2(2^{2^{i+1}})$ or $\mathrm{PSL}_n(2)$, where $i \geqslant 2$, and $n \geqslant 5$ is a prime.*

Proof. First, the group $\mathrm{PSL}_2(2^{2^{i+1}})$ is excluded by [14, Proposition 3.1].

Suppose next that $T = \mathrm{PSL}_n(2)$ for some prime $n \geqslant 5$. In this case, $T_\alpha \cong \mathrm{PSL}_{n-1}(2)$. Consider the natural action of T on the n-dimensional vector space $\mathbb{F}_2^n$ over the field of order 2. It follows that T_α is the stabilizer of some decomposition of $\mathbb{F}_2^n$ into 1- and $(n-1)$-dimensional subspaces. Thus we identify every vertex in each T-orbit with an ordered pair $(\langle u\rangle, U)$, where u and U satisfy $\mathbb{F}_2^n = \langle u\rangle \oplus U$ and $\dim(U) = n-1$. Assume that the first entries of some adjacent pairs are equal. Then, since T is transitive on E, the first entries of every adjacent pairs are equal. It follows that Γ is the union of 2^n-1 copies of $\mathsf{K}_{2^{n-1}}$ or $\mathsf{K}^{(2)}_{2^{n-1}}$, which contradicts the connectedness of Γ. Similarly, if the second entries of some adjacent pairs are equal then we also have a contradiction.

For $\{\alpha,\beta\} \in E$, choosing a suitable base $u_1, u_2, \ldots, u_n$ of $\mathbb{F}_2^n$, we may let $\alpha = (\langle u_1\rangle, \langle u_2, u_3, \ldots, u_n\rangle)$, and $\beta = (\langle u_1+u_3\rangle, \langle u_1+u_2, u_3, \ldots, u_n\rangle)$ or $(\langle u_2+u_3\rangle, \langle u_1+u_2, u_3, \ldots, u_n\rangle)$. Then $T_{\alpha\beta} = T_\alpha \cap T_\beta$ is properly contained in the stabilizer of $(\langle u_3, \ldots, u_n\rangle, \langle u_2, u_3, \ldots, u_n\rangle)$ in $\mathrm{PSL}_{n-1}(2)$. Thus $|\Gamma(\alpha)| = |T_\alpha : T_{\alpha\beta}| > 2^{n-1}+1$, a contradiction. This completes the proof. □

Lemma 6.5. *Assume that $T = \mathrm{A}_n$ and $\mathrm{soc}(T_\alpha) \cong \mathrm{A}_k$ with $k = |\Gamma(\alpha)|$. Then Γ is either a complete graph or the standard double cover of a complete graph.*

Proof. By Table 6.6, $k = n-1$ or $n-2$. For $k = n-1$, it is easily shown $\Gamma = \mathsf{K}_n$ or $\mathsf{K}_n^{(2)}$. Suppose next that $k = n-2$. Then $T_\alpha \cong \mathrm{A}_{n-2}$ or S_{n-2}, and $T_{\alpha\beta} \cong \mathrm{A}_{n-3}$ or S_{n-3} respectively. Consider the natural action of S_n on an n-set Ω. Then T_α is contained in the stabilizer of a 2-subset Δ, and $T_{\alpha\beta}$ fixes a point $\delta \in \Omega \setminus \Delta$.

Assume that $T_\alpha \cong \mathrm{S}_{n-2}$. Then $G_\alpha = T_\alpha$ and $G_{\alpha\beta} = T_{\alpha\beta}$, and hence $G_{\alpha\beta}$ has exactly three orbits on Ω, say $\{\delta\}$, Δ and $\Omega \setminus (\Delta \cup \{\delta\})$. This implies that $\mathbf{N}_G(G_{\alpha\beta})$ fixes Δ set-wise. Then $\langle G_\alpha, \mathbf{N}_G(G_{\alpha\beta})\rangle \neq G$, which contradicts Lemma 2.1.

Assume that $T_\alpha \cong \mathrm{A}_{n-2}$. Then T_α fixes Δ point-wise, and $T_{\alpha\beta}$ fixes $\Delta \cup \{\delta\}$ point-wise. This implies that every 2-element in $\mathbf{N}_{\mathrm{S}_n}(T_{\alpha\beta})$ fixes at least one point in $\Delta \cup \{\delta\}$. Thus neither T nor S_n can be generated by T_α and a 2-element in $\mathbf{N}_{\mathrm{S}_n}(T_{\alpha\beta})$. By Lemma 2.1, $G = \langle G_\alpha, G_{\{\alpha,\beta\}}\rangle$, and so $G = \langle G_\alpha, x\rangle$ for some 2-element

x in $\mathbf{N}_G(G_{\alpha\beta})$. It follows that $G_\alpha \neq T_\alpha$, and hence $G_\alpha \cong \mathrm{S}_{n-2}$ and $G = \mathrm{S}_n$. This forces that T is 2-arc-transitive on Γ. Since Γ is connected, $\langle T_\alpha, x\rangle = T$ for some 2-element $x \in \mathbf{N}_T(T_{\alpha\beta})$, again a contradiction. This completes the proof. □

For each of the rest simple groups in Table 6.6, consulting the Atlas [6] and computation by GAP, we check all possible subgroups H of G with $H \cong G_\alpha$ up to the conjugation under $\mathrm{Aut}(T)$, work out the subgroups L of H with $L \cong G_{\alpha\beta}$ up to the conjugation under $\mathbf{N}_{\mathrm{Aut}(T)}(H)$, and then compute the normalizer $\mathbf{N}_G(L)$ for each L. Then, by Lemma 2.1 and Theorem 3.3, we can determine the graph Γ up to isomorphism. Thus we have the following result.

Theorem 6.6. *Assume that $G_\alpha^{\Gamma(\alpha)}$ is almost simple, and $k = |\Gamma(\alpha)|$ is not a prime. If $G_{\alpha\beta}^{[1]} = 1$ for $\{\alpha, \beta\} \in E$, then one of the following holds:*

(1) *Γ is either a complete graph or the standard double cover of a complete graph;*
(2) *$\Gamma \cong \mathrm{Cos}(G, H, K)$ with G, H and K listed in Table 6.7.*

G	H	r^as^b	k	K	Γ
$\mathrm{M}_{22}.2$	A_7	$2^4\cdot 11$	15	$\mathrm{PSL}_2(7).2$	bipartite
$\mathrm{PSU}_3(5).2$	A_7	$2\cdot 5^2$	15	$\mathrm{PSL}_2(7).2$	bipartite
$\mathrm{PSU}_4(3).2_3$	A_7	$2^4\cdot 3^4$	15	$\mathrm{PSL}_3(2).2_1$	bipartite
$\mathrm{M}_{12}.2$	M_{11}	$2^2\cdot 3$	12	$\mathrm{PSL}_2(11).2$	$\mathsf{K}_{12,12}$
$\mathrm{HS}.2^i$	$\mathrm{M}_{22}.2^i$	$2^2\cdot 5^2$	22	$\mathrm{PSL}_3(4).2^{i+1}$	Higman-Sims
$\mathrm{HS}.2$	M_{22}	$2^2\cdot 5^2$	22	$\mathrm{PSL}_3(4).2_3$	bipartite
$\mathrm{McL}.2$	M_{22}	$3^4\cdot 5^2$	22	$\mathrm{PSL}_3(4).2_3$	bipartite
$\mathrm{HS}.2$	$\mathrm{PSU}_3(5).2$	$2^4\cdot 11$	126	$[5^3]{:}[2^5]$	bipartite
S_6	$\mathrm{PSL}_2(5)$	$2\cdot 3$	6	5:4	$\mathsf{K}_{6,6}$
S_7	$\mathrm{PSL}_2(7)$	$3\cdot 5$	8	7:6	$\mathrm{PG}(3,2)$
$\mathrm{M}_{22}.2^i$	$\mathrm{PSL}_3(4).2^i$	$2\cdot 11$	21	$2^{4+i}{:}\mathrm{S}_5$	K_{22}
$\mathrm{M}_{22}.2$	$\mathrm{PSL}_3(4)$	$2\cdot 11$	21	$2^4{:}\mathrm{S}_5$	$\mathsf{K}_{22,22}$
$\mathrm{PSL}_2(19)$	$\mathrm{PSL}_2(5)$	$3\cdot 19$	6	D_{20}	
$\mathrm{PGL}_2(29)$	$\mathrm{PSL}_2(5)$	$7\cdot 29$	6	D_{20}	bipartite
$\mathrm{PGL}_2(31)$	$\mathrm{PSL}_2(5)$	$2^3\cdot 31$	6	D_{20}	bipartite
$\mathrm{PSL}_2(59)$	$\mathrm{PSL}_2(5)$	$29\cdot 59$	6	D_{20}	
$\mathrm{PSL}_2(61)$	$\mathrm{PSL}_2(5)$	$31\cdot 61$	6	D_{20}	
$\mathrm{PSL}_3(4).2^i$	$\mathrm{PSL}_2(9).2^i$	$2^3\cdot 7$	10	$3^2{:}[2^{3+i}]$	
$\mathrm{PSL}_3(4).2^i$	$\mathrm{PSL}_2(9).2^{i-1}$	$2^3\cdot 7$	10	$3^2{:}[2^{2+i}]$	bipartite
$\mathrm{PSU}_4(3).2^2$	$\mathrm{PSL}_3(4).2$	$2\cdot 3^4$	21	$2^5{:}\mathrm{S}_5$	bipartite

TABLE 6.7. Some coset graphs with trivial edge-kernel

Finally, Theorem 1.2 follows from Theorems 4.1, 4.2, 4.4, 5.4, 6.1 and 6.6.

REFERENCES

[1] R.W. Baddeley, Two-arc-transitive graphs and twisted wreath products, *J. Algebra Combin* **2** (1993), 215-237.
[2] N.L. Biggs, *Algebraic graph theory*, Cambridge University Press, Cambridge, 1974.
[3] J.N. Bray, D. F. Holt and C. M. Roney-Dougal, *The Maximal Subgroups of the Low-Dimensional Finite Classical Groups*, Cambridge University Press, New York, 2013.

[4] P.J. Cameron, *Permutation Groups*, Cambridge University Press, Cambridge, 1999.
[5] M. Conder and P. Dobcsányi, Trivalent symmetric graphs on up to 768 vertices, *J. Combin. Math. Combin. Comput.* **40** (2002), 41–63.
[6] J.H. Conway, R.T. Curtis, S.P. Noton, R.A. Parker and R.A. Wilson, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985. http://brauer.maths.qmul.ac.uk/Atlas/v3/
[7] D.J. Dixon and B. Mortimer, *Permutation Groups*, Springer-Verlag, New York, 1996.
[8] X.G. Fang and C.E. Praeger, Finite two-arc-transitive graphs admitting a Suzuki simple group, *Comm. Algebra* **27(8)** (1999), 3727-3754.
[9] X.G. Fang and C.E. Praeger, Finite two-arc-transitive graphs admitting a Ree simple group, *Comm. Algebra* **27(8)** (1999), 3755-3769.
[10] The GAP Group, GAP-Groups, Algorithms, and Programming, Version 4.11.1, 2021. http://www.gap-system.org
[11] A. Gardiner, Arc transitivity in graphs, *Quart. J. Math. Oxford*, **24** (1973), 399-407.
[12] M. Giudici and C.H. Li, On finite edge-primitive and edge-quasiprimitive graphs, *J. Combin. Theory Ser. B* **100** (2010), 275-298.
[13] H. Han, H.C. Liao and Z.P. Lu, On edge-primitive graphs with soluble edge-stabilizers, *J. Aust. Math. Soc.* **113** (2022), 160-187.
[14] A. Hassani, L.R. Nochefranca and C.E. Praeger, Two-arc transitive graphs admitting a two-dimensional projective linear group, *J. Group Theory* **2** (1999), 335-353.
[15] A.A. Ivanov and C.E. Praeger, On finite affine 2-arc-transitive graph, *Eur. J. Combin.* **14** (1993), 421-444.
[16] W.M. Kantor and R.A. Liebler, The rank 3 permutation representations of the finite classical groups, *Trans. Amer. Math. Soc.* **271** (1982), 1-70.
[17] C.H. Li, Finite s-arc transitive graphs of prime-power order, *Bull. London Math. Soc.* **33** (2001), 129-137.
[18] C.H. Li and X. Li, On permutation groups of degree a product of two prime-powers, *Comm. Algebra* **42** (2014), 4722-4743.
[19] C.H. Li, Z.P. Lu and G.X. Wang, Arc-transitive graphs of square-free order and small valency, *Discrete Math.* **339** (2016), 2907-2918.
[20] C.H. Li, Z.P. Lu and G.X. Wang, The vertex-transitive and edge-transitive tetravalent graphs of square-free order, *J. Algebraic Combin.* **42** (2015), 25–50.
[21] C.H. Li, Ákos Seress and S.J. Song, s-arc-transitive graphs and normal subgroups, *J. Algebra* **421** (2015), 331-348.
[22] J.J. Li, H.C. Liao, Z.P. Lu and W.Y. Zhu, Symmetric graphs of prime valency associated with some almost simple groups, *Discrete Math.* **344** (2021), 112547.
[23] Z.P. Lu and R.Y. Song, On basic 2-arc-transitive graphs, *J. Algebraic Combin.* **58** (2023), 1081–1093.
[24] C.E. Praeger, An O'Nan-Scott theorem for finite quasiprimitive permutation groups and an application to 2-arc transitive graphs, *J. London Math. Soc.* **47** (1993), 227-239.
[25] C.E. Praeger, On a reduction theorem for finite, bipartite 2-arc-transitive graphs, *Austral. J. Combin.* **7** (1993), 21-36.
[26] C.E. Praeger, Finite quasiprimitive graphs, *Surveys in combinatorics*, 1997. Proceedings of the 16th British combinatorial conference, London, UK, July 1997 (R. A. Bailey, ed.), *Lond. Math. Soc. Lect. Note Ser.*, no. 241, Cambridge University Press, 1997, pp. 65-85.
[27] V.I. Trofimov, Vertex stabilizers of locally projective groups of automorphisms of graphs: a summary, In: *Groups, Combinatorics and Geometry*, pp.313-326, World Sci. Publ. Co., Durham, 2001.
[28] R. Weiss, Groups with a (B, N)-pair and locally transitive graphs, *Nagoya Math. J.* **74** (1979), 1-21.
[29] R. Weiss, The nonexistence of 8-transitive graphs, *Combinatorica* **1** (1981), 309-311.
[30] R. Weiss, s-Transitive graphs, Algebraic methods in graph theory, *Colloq. Soc. Janos Bolyai* **25** (1981), 827–847.

J.J. Li, College of Mathematics and Information Science, Guangxi University, Nanning 530004, P. R. China

E-mail address: lijjhx@163.com

Z.P. Lu, Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, P. R. China

E-mail address: lu@nankai.edu.cn

R.Y. Song, Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, P. R. China

E-mail address: 851878536@qq.com

X.Q. Zhang, College of Mathematics and Information Science, Guangxi University, Nanning 530004, P. R. China

E-mail address: zhangxq@st.gxu.edu.cn