# EDGE-TRANSITIVE CUBIC GRAPHS
# OF TWICE SQUARE-FREE ORDER

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ABSTRACT. A graph is edge-transitive if its automorphism group acts transitively on the edge set. This paper presents a complete classification for connected edge-transitive cubic graphs of order $2n$, where $n$ is even and square-free. In particular, it is shown that such a graph is either symmetric or isomorphic to one of the following graphs: a semisymmetric graph of order 420, a semisymmetric graph of order 29260 and five families of semisymmetric graphs constructed from the simple group $\mathrm{PSL}_2(p)$.

KEYWORDS. Edge-transitive graph, symmetric graph, semisymmetric graph, coset graph, bi-coset graph.

## 1. INTRODUCTION

All graphs in this paper are finite, simple and undirected, and have no isolated vertex.

Let $\Gamma = (V, E)$ be a graph with vertex set $V$ and edge set $E$, and denote by $\mathrm{Aut}\Gamma$ the automorphism group of $\Gamma$. Let $G$ be a subgroup of $\mathrm{Aut}\Gamma$, written as $G \leqslant \mathrm{Aut}\Gamma$. Then $\Gamma$ is said to be $G$-vertex-transitive or $G$-edge-transitive if $G$ acts transitively on $V$ or $E$, respectively. If $\Gamma$ is $G$-edge-transitive but not $G$-vertex-transitive then $\Gamma$ is a bipartite graph with a bipartition given by the $G$-orbits on $V$; in this case, $\Gamma$ is called $G$-semisymmetric if further it is a regular graph. Recall that an arc in $\Gamma$ is an ordered pair of adjacent vertices. Then $\Gamma$ is said to be $G$-symmetric if $G$ acts transitively on the set of arcs. For a vertex $v \in V$, set $\Gamma(v) = \{v' \in V \mid \{v, v'\} \in E\}$ and $G_v = \{g \in G \mid v^g = v\}$, called the neighborhood and stabilizer of $v$ in $\Gamma$ and $G$, respectively. Clearly, if $\Gamma$ is either $G$-symmetric or $G$-semisymmetric then $G_v$ acts transitively on $\Gamma(v)$ for all $v \in V$.

A graph $\Gamma$ is called vertex-transitive, edge-transitive, symmetric and semisymmetric if it is $\mathrm{Aut}\Gamma$-vertex-transitive, $\mathrm{Aut}\Gamma$-edge-transitive, $\mathrm{Aut}\Gamma$-symmetric and $\mathrm{Aut}\Gamma$-semisymmetric, respectively. Clearly, symmetric graphs are both edge-transitive and vertex-transitive, and by [31, p.55, 7.31], the converse is also true for regular graphs of odd valency. In particular, edge-transitive cubic graphs (regular graphs of valency 3) are either symmetric or semisymmetric.

In this paper, we focus on connected edge-transitive cubic graphs. Interest in edge-transitive cubic graphs stems from the classical result on symmetric cubic graphs due to Tutte. In [29, 30], Tutte considered the automorphism groups of connected symmetric cubic graphs, and proved that the order of a vertex-stabilizer is a divisor of $2^4 \cdot 3$. Tutte's result was generalized by Goldschmidt in [16] where it is proved that the stabilizers of two adjacent vertices in a connected edge-transitive cubic graph are isomorphic to one of

fifteen pairs of groups; in particular, the order of a vertex-stabilizer is a divisor of $2^7 \cdot 3$. Following these two classical results, edge-transitive cubic graphs have been extensively studied from different perspectives over the decades, see [5, 6, 7, 8, 9, 12, 18, 24, 26, 27, 28] for example. In recent papers [21] and [23], connedcted edge-transitive cubic graphs of square-free order were classified. This motivates us to classify connected edge-transitive cubic graphs of order $2n$, where $n$ is even and square-free.

Let $\Gamma$ be an arbitrary connected edge-transitive cubic graph of order $2n$ with $n$ even and square-free. The group-theoretic structure of $\Gamma$ is investigated in Section 2, where it is proved that, with four exceptions for $\Gamma$, an edge-transitive group of $\Gamma$ has a unique insolvable minimal normal subgroup say $T$, which is isomorphic to $\mathrm{J}_1$ or $\mathrm{PSL}_2(p)$. In Section 3, we collect two group-theoretic constructions for edge-transitive graphs, and present some improvements on the automorphisms or isomorphisms of coset graphs and bi-coset graphs. Then $\Gamma$ is determined in Section 4 for the case where $T = \mathrm{J}_1$, followed by the classifications for $\mathrm{PSL}_2(p)$-symmetric $\Gamma$ and $\mathrm{PSL}_2(p)$-semisymmetric $\Gamma$ in Sections 5 and 6, respectively. Finally, the case where $\Gamma$ is not $\mathrm{PSL}_2(p)$-edge-transitive is settled in Section 7, and then our main result stated as follows is proved.

**Theorem 1.1.** *Assume that $\Gamma = (V, E)$ is a connected edge-transitive cubic graph of order $2n$, where $n$ is even and square-free. Let $p$ be the largest prime divisor of $n$, and choose $\varepsilon, \eta \in \{1, -1\}$ for those odd $p$ with $p+\varepsilon$ and $p+\eta$ divisible by 3 and 4, respectively. Let $\delta = 1$ if $p \equiv \pm 1 \pmod{10}$, or $\delta = 0$ otherwise.*

(1) *If $\Gamma$ is not bipartite then $\Gamma$ is isomorphic to either the complete graph $\mathsf{K}_4$ of order 4 or one of the graphs described as Table 1, where $v \in V$, $T = \mathrm{PSL}_2(p)$ and $\omega$ is the number of non-isomorphic graphs with isomorphic automorphism groups.*

(2) *If $\Gamma$ is bipartite then $\Gamma$ is isomorphic to one of the graphs described as Table 2, where $\{u, w\} \in E$, $T = \mathrm{PSL}_2(p)$ and $\nu$ is the number of non-isomorphic graphs with isomorphic automorphism groups.*

## 2. On the automorphism groups

In this and the following sections, $G$ is a finite group. Denote by $\mathrm{Aut}(G)$ the automorphism group of $G$. If $\alpha$ is a subset or an element of $G$, then we write $g^{-1}\alpha g$ to denote the conjugation of $\alpha$ under some $g \in G$. For subsets $X, Y \subseteq G$, we write $\mathbf{C}_X(Y) = \{x \in X \mid x^{-1}yx = y \text{ for all } y \in Y\}$ and $\mathbf{N}_X(Y) = \{x \in X \mid x^{-1}Yx = Y\}$, called the centralizer and normalizer of $Y$ in $X$, respectively.

In the following, $\Gamma = (V, E)$ is assumed to be a connected $G$-edge-transitive cubic graph. Note that $\Gamma$ is either $G$-symmetric or $G$-semisymmetric. Let $\{u, w\} \in E$. If $\Gamma$ is $G$-semisymmetric then $\Gamma$ is bipartite, and $G = \langle G_u, G_v \rangle$. Suppose that $\Gamma$ is $G$-symmetric. Then $\Gamma$ is $\langle G_u, G_v \rangle$-edge-transitive, and $|G : \langle G_u, G_v \rangle| \leqslant 2$, where the equality holds if and only if $\Gamma$ is bipartite, refer to [32, Exercise 3.8]. Clearly, if $|G : \langle G_u, G_v \rangle| = 2$ then $\Gamma$ is $\langle G_u, G_v \rangle$-semisymmetric. Thus, replacing $G$ by $\langle G_u, G_v \rangle$ if necessary, we assume further that

(C1) $\Gamma$ is either $G$-semisymmetric, or non-bipartite and $G$-symmetric, where $G \leqslant \mathrm{Aut}\Gamma$; and

(C2) $|V| = 2n$, where $n$ is even and square-free.

| | $G = \operatorname{Aut}\Gamma$ | $G_v$ | $\omega$ | Comments |
|---|---|---|---|---|
| 1 | $A_6$ | $S_3$ | $1$ | F60, cf. [6] |
| 2 | $\operatorname{PSL}_2(8)$ | $S_3$ | $1$ | F84, cf. [6] |
| 3 | $J_1$ | $S_3$ | $10$ | Example 3.5 |
| 4 | $\operatorname{PGL}_2(p)$ | $S_3$ | $\frac{p-\eta-6}{4}$ | Theorem 5.7 $p\equiv\pm3\,(\mathrm{mod}\ 8)$ |
| 5 | $\operatorname{PSL}_2(p)\times\mathbb{Z}_2$ | $S_3$ | $\frac{p+\eta-2\lvert\varepsilon+\eta\rvert}{4}-2\delta$ | Theorem 5.7 $p\equiv\pm3\,(\mathrm{mod}\ 8)$ |
| 6 | $\operatorname{PGL}_2(p)$ | $D_{12}$ | $1-\frac{\lvert\varepsilon+\eta\rvert}{2}$ | Theorem 5.12 $p\equiv\pm7\,(\mathrm{mod}\ 16)$ |
| 7 | $\operatorname{PSL}_2(p)\times\mathbb{Z}_2$ | $D_{12}$ | $\lvert\varepsilon+\eta\rvert$ | Theorem 5.12 $p\equiv\pm7\,(\mathrm{mod}\ 16)$ |
| 8 | $\operatorname{PSL}_2(p)$ | $S_3$ | $\frac{p+\eta-4\lvert\varepsilon+\eta\rvert}{8}-1-\delta$ | Theorem 5.12 $p\equiv\pm7\,(\mathrm{mod}\ 16)$ |
| 9 | $\operatorname{PSL}_2(p)$ | $D_{12}$ | $1$ | Theorem 5.13 $p\equiv\pm47\,(\mathrm{mod}\ 96)$ |
| 10 | $\operatorname{PSL}_2(p)$ | $S_4$ | $1$ | Theorem 5.14 $p\equiv\pm31\,(\mathrm{mod}\ 64)$ |
| 11 | $(\operatorname{PSL}_2(p)\times\mathbb{Z}_3){:}\mathbb{Z}_2$ | $S_3$ | $\frac{p-\eta-6}{4}$ | $T=\operatorname{PSL}_2(p),T_v=1$ $p\equiv\pm3\,(\mathrm{mod}\ 8)$ |
| 12 | $\operatorname{PSL}_2(p)\times S_3$ | $S_3$ | $\frac{p+\eta-2\lvert\varepsilon+\eta\rvert}{4}-2\delta$ | $T=\operatorname{PSL}_2(p),T_v=1$ $p\equiv\pm3\,(\mathrm{mod}\ 8)$ |

TABLE 1. Non-bipartite symmetric cubic graphs.

2.1. **Preliminaries.** Let $\{u,w\}\in E$. If $\Gamma$ is $G$-symmetric then $G_u$ and $G_w$ are conjugate in $G$ and, by [2, p.147, 18f], $G_u\cong\mathbb{Z}_3,S_3,D_{12},S_4$ or $\mathbb{Z}_2\times S_4$; in particular, $\lvert G_u\rvert$ is a divisor of 48. Suppose that $\Gamma$ is $G$-semisymmetric. Then $G$ has exactly two orbits on $V$, $G=\langle G_u,G_w\rangle$, and $G_{uw}$ is a Sylow 2-subgroup of $G_u$ (and $G_w$). The triple $(G_u,G_{uw},G_w)$ was determined by Goldschmidt in [16] where it is shown that $(G_u,G_{uw},G_w)$ is isomorphic to one of fifteen triples, see also [28, Table 3]. Then we have the following lemma.

**Lemma 2.1.** *Let* $\{u,w\}\in E$. *Then one of the following holds:*

(1) $G_u\cong G_w\cong\mathbb{Z}_3,S_3,D_{12},S_4$ *or* $\mathbb{Z}_2\times S_4$;
(2) $\Gamma$ *is* $G$*-semisymmetric,* $G_u\not\cong G_w$, *and either* $\lvert G_u\rvert=\lvert G_w\rvert=2^i\cdot3$ *with* $i\in\{5,6,7\}$ *or* $(G_u,G_w)$ *is isomorphic to one of* $(S_3,\mathbb{Z}_6),(D_{12},A_4),(D_{24},S_4),((\mathbb{Z}_2^2\times\mathbb{Z}_3).\mathbb{Z}_2,S_4),(\mathbb{Z}_2\times D_{12},\mathbb{Z}_2\times A_4)$ *and* $(D_8\times S_3,\mathbb{Z}_2\times S_4)$.

*In particular,*

(i) *if* $\lvert G_u\rvert>3$ *then* $G$ *contains at least two involutions; if* $\lvert G_u\rvert>12$ *then either* $(G_u,G_w)\cong(\mathbb{Z}_2\times D_{12},\mathbb{Z}_2\times A_4)$, *or* $G$ *contains nonabelian Sylow 2-subgroups;*
(ii) *if* $\Gamma$ *is* $G$*-symmetric then* $\lvert G\rvert$ *is a divisor of* $2^5\cdot3n$; *if* $\Gamma$ *is* $G$*-semisymmetric then* $\lvert G\rvert$ *is a divisor of* $2^8\cdot3n$.

Let $N$ be a normal subgroup of $G$, written as $N\trianglelefteq G$. Suppose that $N$ is intransitive on $V$. For $v\in V$, denote by $\bar{v}$ the $N$-orbit containing $v$. Put $\bar{V}=\{\bar{v}\mid v\in V\}$. The normal quotient graph $\Gamma_N$ of $\Gamma$ relative to $G$ and $N$ is defined on $\bar{V}$ with edge set

| | $G = \mathrm{Aut}\Gamma$ | $G_u$, $G_w$ | $\nu$ | Symmetric? | Comments |
|---|---|---|---|---|---|
| 1 | $S_7 \times \mathbb{Z}_2$ | $S_4 \times \mathbb{Z}_2$, $S_3 \times D_8$ | 1 | No | S420, cf. [8] |
| 2 | $J_1$ | $D_{12}$, $D_{12}$ | 1 | No | Example 3.11 |
| 3 | $\mathrm{PSL}_2(p) \times \mathbb{Z}_2$ | $D_{12}$, $D_{12}$ | $\frac{|\varepsilon+\eta|}{2}$ | No | Theorem 6.5 $p \equiv \pm 3 \,(\mathrm{mod}\ 8)$ |
| 4 | $\mathrm{PGL}_2(p) \times \mathbb{Z}_2$ | $D_{12}$, $D_{12}$ | 1 | Yes | Theorem 6.5 $p \equiv \pm 3 \,(\mathrm{mod}\ 8)$ |
| 5 | $\mathrm{PGL}_2(p)$ | $S_3$, $S_3$ | $\frac{p+\eta-4}{8}$ | Yes | Theorem 6.5 $p \equiv \pm 3 \,(\mathrm{mod}\ 8)$ |
| 6 | $\mathrm{PSL}_2(p) \times \mathbb{Z}_2$ | $S_3$, $S_3$ | $\frac{p+\eta-4}{8} - \delta$ | Yes | Theorem 6.5 $p \equiv \pm 3 \,(\mathrm{mod}\ 8)$ |
| 7 | $\mathrm{PSL}_2(p) \times \mathbb{Z}_2$ | $D_{24}$, $S_4$ | 1 | No | Theorem 6.7 $p \equiv \pm 23 \,(\mathrm{mod}\ 48)$ |
| 8 | $\mathrm{PSL}_2(p) \times \mathbb{Z}_2$ | $D_{12}$, $D_{12}$ | 1 | Yes | Theorem 6.7 $p \equiv \pm 23 \,(\mathrm{mod}\ 48)$ |
| 9 | $\mathrm{PSL}_2(p)$ | $D_{24}$, $S_4$ | 1 | No | Theorem 6.8 $p \equiv \pm 47 \,(\mathrm{mod}\ 96)$ |
| 10 | $\mathrm{PSL}_2(p) \times \mathbb{Z}_2$ | $S_4$, $S_4$ | 1 | Yes | Theorem 6.8 $p \equiv \pm 15 \,(\mathrm{mod}\ 32)$ |
| 11 | $\mathrm{PSL}_2(p) \times S_3$ | $D_{12}$, $D_{12}$ | 1 | No | $T = \mathrm{PSL}_2(p), T_u \cong S_3, T_w \cong \mathbb{Z}_2$ $p \equiv \pm 11 \,(\mathrm{mod}\ 24)$ |
| 12 | $\mathrm{PSL}_2(p) \times S_3$ | $D_{24}$, $S_4$ | 1 | No | $T = \mathrm{PSL}_2(p), T_u \cong D_{12}, T_w \cong \mathbb{Z}_2^2$ $p \equiv \pm 23 \,(\mathrm{mod}\ 48)$ |

Table 2. Bipartite edge-transitive cubic graphs.

$\bar{E} := \{\{\bar{u}, \bar{w}\} \mid \{u, w\} \in E\}$. Denote by $G^{\bar{V}}$ (by $\bar{G}$ for short) the permutation group induced by $G$ on $\bar{V}$. Recall that $N$ is said to be semiregular (on $V$) if all its orbits have length $|N|$, i.e., $N_v = 1$ for all $v \in V$. We have the following lemma, see [22, Lemma 2.6] for example.

**Lemma 2.2.** *Let $N \trianglelefteq G$. Assume that $N$ is intransitive on each $G$-orbit on $V$. Then $\Gamma_N$ is cubic and $\bar{G}$-edge-transitive, $N$ is semiregular on $V$, and $\bar{G} \cong G/N$.*

**Lemma 2.3.** *Let $N \trianglelefteq G$. Assume that $N$ is not semiregular on $V$. Then either $\Gamma$ is $N$-edge-transitive, or $\Gamma$ is bipartite and the following hold:*

(1) *$N$ acts transitively on one part say $U$ of $\Gamma$ and has three orbits on the other part;*
(2) *$|G : N|$ is divisible by 3, $|N|$ is indivisible by 9 and, for $u \in U$, the stabilizer $N_u$ is a 2-group and acts trivially on $\Gamma(u)$.*

*Proof.* Assume first that $N$ is transitive on each $G$-orbit on $V$. Then $|N : N_u| = |N : N_w| = 2n$ or $n$, in particular, $|N_u| = |N_w|$, where $u, w \in V$. Suppose that $N_u$ acts trivially on $\Gamma(u)$. Then, letting $w \in \Gamma(u)$, we have $N_u = N_w$. Since $N_w \trianglelefteq G_w$ and $G_w$ acts transitively on $\Gamma(w)$, we deduce that $N_w$ acts trivially on $\Gamma(w)$. It follows from the connectedness of $\Gamma$ that $N_u$ fixes $V$ point-wise, and so $N_u = 1$, a contradiction. Thus $N_u$ acts transitively on $\Gamma(u)$ for all $u \in V$, and hence $\Gamma$ is $N$-edge-transitive.

Assume now that $\Gamma$ is bipartite, and $N$ is not transitive on one part of $\Gamma$, say $W$. Since $N$ is not semiregular, by Lemma 2.2, $N$ is transitive on $U := V \setminus W$. By [15, Lemma 5.5], $N$ has three orbits on $W$ and, for $u \in U$, the stabilizer $N_u$ is contained in the kernel of $G_u$ acting on $\Gamma(u)$. It follows that $N_u$ is a 2-group, and $|G_u : N_u|$ is divisible by 3. Noting that $|G : G_u| = n = |N : N_u|$, we have that $|G : N| = |G_u : N_u|$, and $|N|$ is indivisible by 9. Then the lemma follows. $\qquad\square$

2.2. **The solvable case.** For a prime divisor $p$, denote by $\mathbf{O}_p(G)$ the maximal normal $p$-subgroup of $G$.

**Lemma 2.4.** *Either* $\Gamma \cong \mathsf{K}_4$*, or* $|\mathbf{O}_p(G)| \in \{1, p\}$ *for every prime divisor* $p$ *of* $|G|$*.*

*Proof.* Assume first that $p$ is an odd prime. Since each $G$-orbit on $V$ has even length $n$ or $2n$, we know that $\mathbf{O}_p(G)$ is intransitive on each $G$-orbit on $V$. By Lemma 2.2, $\mathbf{O}_p(G)$ has order a divisor of $2n$, yielding $|\mathbf{O}_p(G)| \in \{1, p\}$.

Now consider the case where $p = 2$. Assume that $\mathbf{O}_2(G)$ is not transitive on each $G$-orbit. By Lemma 2.2, $\mathbf{O}_2(G)$ is semiregular on $V$, and so $|\mathbf{O}_2(G)| \in \{1, 2, 4\}$. If $|\mathbf{O}_2(G)| = 4$ then we get a cubic graph $\Gamma_{\mathbf{O}_2(G)}$ of odd order, which is impossible. Thus $|\mathbf{O}_2(G)| \in \{1, 2\}$. Assume that $\mathbf{O}_2(G)$ is transitive on one of $G$-orbits, say $U$. Then $|U|$ is a divisor of $|\mathbf{O}_2(G)|$, which forces that either $|U| = n = 2$ or $|V| = |U| = 4$. It follows that $\Gamma \cong \mathsf{K}_4$. This completes the proof. $\qquad\square$

**Theorem 2.5.** *Assume that* $G$ *is solvable. Then* $\Gamma \cong \mathsf{K}_4$*.*

*Proof.* Let $F$ be the Fitting subgroup of $G$, i.e., the direct product of all $\mathbf{O}_p(G)$, where $p$ runs over the prime divisors of $|G|$. Since $G$ is solvable, every minimal normal subgroup of $G$ has prime power order, and so $F \neq 1$.

Suppose that $\Gamma \not\cong \mathsf{K}_4$. Then $2n = |V| > 4$ and, by Lemma 2.4, $F$ is cyclic and has order a divisor of $n$. In particular, $F$ is intransitive on $V$ as $|V| = 2n$. Let $B$ be an arbitrary $F$-orbit on $V$, and let $K$ be the kernel of $F$ acting on $B$. Since $F$ is cyclic, $K$ is characteristic in $G$, and so $K \trianglelefteq G$. If $G$ is transitive on $V$ then, since all $K$-orbits have equal length, $K$ acts trivially on $V$, and so $K = 1$. Assume that $G$ is intransitive on $V$. Then $G$ has exactly two orbits on $V$, say $U$ and $W$. Without loss of generality, let $B \subseteq U$. Then $K$ acts trivially on $U$. If $K \neq 1$ then it is easily shown that $\Gamma$ is isomorphic to the complete bipartite graph $\mathsf{K}_{3,3}$, and so $2n = 6$, which is not the case. Therefore, $F$ is faithful and hence regular on each of its orbits; in particular, $F$ is semiregular on $V$.

Assume that $F$ has two orbits on $V$. Then $\Gamma$ is bipartite and $|F| = n$. Let $L$ be the $2'$-Hall subgroup of $F$. Then $L$ is a normal subgroup of $G$. Clearly, $L$ is intransitive on both the $F$-orbits. By Lemma 2.2, the quotient graph $\Gamma_L$ has valency 3. However, $\Gamma_L$ is a bipartite graph of order 4, a contradiction.

Assume that $F$ has at least three orbits on $V$. In this case, it is easy to see that $F$ is intransitive on each $G$-orbit on $V$. Then, by Lemma 2.2, the quotient graph $\Gamma_F$ is cubic, and $G$ induces an edge-transitive subgroup of $\mathrm{Aut}\Gamma_F$, which is isomorphic to $G/F$. Since $G$ is solvable, $\mathbf{C}_G(F) \leq F$, refer to [1, p.158, (31.10)]. Thus $\mathbf{C}_G(F) = F$. Noting that $G$ induces a subgroup $\mathrm{Aut}(F)$ by conjugation, we have $G/F = \mathbf{N}_G(F)/\mathbf{C}_G(F) \lesssim \mathrm{Aut}(F)$. Since $F$ is cyclic, $\mathrm{Aut}(F)$ is abelian, and so does $G/F$. It follows that $\mathrm{Aut}\Gamma_F$ has an abelian edge-transitive subgroup. Then the only possibility is that $\Gamma_F \cong \mathsf{K}_{3,3}$ and $G/F \cong \mathbb{Z}_3^2$. In particular, $n = 3|F|$, and $\Gamma$ is bipartite. Let $L$ be the $2'$-Hall subgroup

of $F$. Then $L$ is normal in $G$ and intransitive on each of $F$-orbits. By Lemma 2.2, $G$ induces an edge-transitive subgroup of $\mathrm{Aut}\Gamma_L$, which is isomorphic to $G/L$. Noting that $F/L$ is a normal subgroup of $G/L$ of order 2, we have $G/L \cong \mathbb{Z}_2 \times \mathbb{Z}_3^2$. It follows that $\mathrm{Aut}\Gamma_L$ has an abelian edge-transitive subgroup, and thus $\Gamma_L \cong \mathsf{K}_{3,3}$, which is impossible as $\Gamma_L$ has order divisible by 4. Therefore, $\Gamma \cong \mathsf{K}_4$, and the result follows. $\qquad\square$

2.3. **The insolvable case.** In this subsection, the group $G$ is assumed to be insolvable. Denote by $\mathsf{rad}(G)$ the maximal solvable normal subgroup of $G$. Then $\mathsf{rad}(G)$ is a characteristic subgroup $G$. If $\mathsf{rad}(G)$ is transitive on one of $G$-orbits on $V$, then $G = \mathsf{rad}(G)G_v$ for some $v \in V$, which implies that $G$ is solvable, a contradiction. Then Lemma 2.2 is available for the triple $(\Gamma, G, \mathsf{rad}(G))$. For $v \in V$, denote by $\bar{v}$ the $\mathsf{rad}(G)$-orbit containing $v$. Put $\bar{V} = \{\bar{v} \mid v \in V\}$, and $\bar{G} = G^{\bar{V}}$. We have the following lemma.

**Lemma 2.6.** *Assume that $G$ is insolvable. Then $\Gamma_{\mathsf{rad}(G)}$ is a connected $\bar{G}$-edge-transitive cubic graph, $|\mathsf{rad}(G)|$ is a divisor of $n$, $|\bar{V}| = \frac{2n}{|\mathsf{rad}(G)|}$ and $\bar{G} \cong G/\mathsf{rad}(G)$.*

**Lemma 2.7.** *Assume that $G$ is insolvable. Then $\bar{G}$ has a unique minimal normal subgroup say $\bar{N}$, $\Gamma_{\mathsf{rad}(G)}$ is $\bar{N}$-edge-transitive, and $\bar{N}$ is isomorphic to one of the following simple groups: $\mathrm{A}_6$, $\mathrm{A}_7$, $\mathrm{J}_1$, $\mathrm{PSL}_2(8)$ and $\mathrm{PSL}_2(p)$, where $p \geqslant 5$ is a prime.*

*Proof.* Let $\bar{N}$ be a minimal normal subgroup of $\bar{G}$. Then $\bar{N}$ is insolvable, and $|\bar{N}|$ is a divisor of $2^8 \cdot 3n$. Note that $\bar{N}$ is a direct product of isomorphic nonabelian simple groups. If $\bar{N}$ is not simple then $|\bar{N}|$ has a divisor $r^2$ for some prime $r > 3$, and so $n$ is divisible by $r^2$, which contradicts the assumption that $n$ is square-free. Thus $\bar{N}$ is simple. If $|\mathsf{rad}(G)|$ is even then, noting that $\Gamma_{\mathsf{rad}(G)}$ has square-free order $|\bar{V}|$, our lemma follows from [21, Lemma 6.3] and [23, Lemma 4.3]; in this case, $\bar{N} \cong \mathrm{A}_6$, $\mathrm{A}_7$ or $\mathrm{PSL}_2(p)$. Thus, we assume next that $|\mathsf{rad}(G)|$ is an odd divisor of $n$.

If $\bar{N}$ is intransitive on each $\bar{G}$-orbit on $\bar{V}$ then, by Lemma 2.2, the quotient graph of $\Gamma_{\mathsf{rad}(G)}$ with respect to $\bar{N}$ is cubic and of order $|\bar{V}|/|\bar{N}|$; however, $|\bar{N}|$ is divisible by 4, and so $|\bar{V}|/|\bar{N}|$ is odd, a contradiction. Thus $\bar{N}$ is transitive on at least one of $\bar{G}$-orbits, say $\bar{U}$. Then $\bar{G} = \bar{N}\bar{G}_{\bar{u}}$ for some $\bar{u} \in \bar{U}$. Let $C = \mathbf{C}_{\bar{G}}(\bar{N})$. We have $\bar{N} \cap C = 1$, and so $C \cong \bar{N}C/\bar{N} \leqslant \bar{G}/\bar{N} \cong \bar{G}_{\bar{u}}/\bar{N}_{\bar{u}}$. It follows that $C$ is solvable, and so $C = 1$ as $\mathsf{rad}(\bar{G}) = 1$ and $C \trianglelefteq \bar{G}$. This says that $\bar{N}$ is the unique minimal normal subgroup of $\bar{G}$.

Note that $|\bar{N}|$ is not divisible by $2^{10}$, $3^3$ or $r^2$, where $r$ is an arbitrary prime with $r \geqslant 5$. Inspecting the orders of finite simple groups (refer to [19, Tables 5.1.A-C]), we deduce that $\bar{N}$ is isomorphic to one of the following groups: $\mathrm{A}_6$, $\mathrm{A}_7$, $\mathrm{A}_8$, $\mathrm{M}_{11}$, $\mathrm{M}_{22}$, $\mathrm{M}_{23}$, $\mathrm{J}_1$, $\mathrm{PSL}_3(4)$, $\mathrm{PSL}_2(2^f)$ and $\mathrm{PSL}_2(p)$, where $3 \leqslant f \leqslant 8$, and $p \geqslant 5$ is a prime.

Suppose that $\bar{N}$ is isomorphic to one of $\mathrm{A}_6$, $\mathrm{A}_7$, $\mathrm{PSL}_2(8)$, $\mathrm{A}_8$, $\mathrm{M}_{11}$, $\mathrm{M}_{22}$, $\mathrm{M}_{23}$, $\mathrm{PSL}_3(4)$ and $\mathrm{PSL}_2(2^6)$. Then $|\bar{N}|$ is divisible by 9. It follows from Lemma 2.3 that $\Gamma_{\mathsf{rad}(G)}$ is $\bar{N}$-edge-transitive. If $\bar{N} \cong \mathrm{PSL}_2(2^6)$ then $|\bar{N}_{\bar{v}}|$ is divisible by $2^4 \cdot 3$, by Lemma 2.1 (i), $\bar{N}$ has nonabelian Sylow 2-subgroups, which is impossible. Assume that $\bar{N} \cong \mathrm{M}_{22}$ or $\mathrm{M}_{23}$. Then $|\bar{N}_{\bar{u}}|$ is divisible by $2^5 \cdot 3$. By Lemma 2.1, $\Gamma_{\mathsf{rad}(G)}$ is $\bar{N}$-semisymmetric, and then $|\bar{N}_{\bar{u}}| = 2^6 \cdot 3$. Since $\Gamma_{\mathsf{rad}(G)}$ is connected, $\bar{N} = \langle L, R \rangle$, where $R$ and $L$ are the stabilizers of two adjacent vertices. For such a pair $(L, R)$, noting that $|L| = |R| = 2^6 \cdot 3$ and $|L \cap R| = 64$, computation with GAP [14] shows that either $|\langle L, R \rangle| = 1344$, or $\bar{N} \cong \mathrm{M}_{23}$ and $|\langle L, R \rangle| \in \{576, 1920, 40320\}$, and so $\bar{N} \neq \langle L, R \rangle$, a contradiction. Assume that $\bar{N} \cong \mathrm{PSL}_3(4)$, $\mathrm{A}_8$ or $\mathrm{M}_{11}$. Then $|\bar{V}| = 2\frac{n}{|\mathsf{rad}(G)|} = 420$, 420 or 660, respectively.

By [6, 8], up to graph isomorphisms, there exist one connected edge-transitive cubic graph of order 420, and two connected edge-transitive cubic graphs of order 660, which have automorphism groups of order 10080, 3960 and 3960 respectively. Then $|\bar{N}| > |\text{Aut}\Gamma_{\mathsf{rad}(G)}|$, a contradiction. Thus, in this case, $\Gamma_{\mathsf{rad}(G)}$ is $\bar{N}$-edge-transitive, and $\bar{N}$ is one of $A_6$, $A_7$ and $\text{PSL}_2(8)$.

Finally, suppose that $\bar{N} \cong J_1$, $\text{PSL}_2(2^4)$, $\text{PSL}_2(2^5)$, $\text{PSL}_2(2^7)$, $\text{PSL}_2(2^8)$ or $\text{PSL}_2(p)$. Recalling that $\mathbf{C}_{\bar{G}}(\bar{N}) = 1$, we know that $\bar{G}$ is almost simple, and $\bar{G} = \bar{N}.O$, where $O$ is a subgroup of the outer automorphism group of $\bar{N}$. Checking [19, Tables 5.1.A and 5.1.C], we conclude that $|O|$ is a divisor of 1, 4, 5, 7, 8 or 2, respectively. Then $|\bar{G} : \bar{N}| = |O|$ is indivisible by 3. Noting that $|\bar{G}_{\bar{v}} : \bar{N}_{\bar{v}}| = |\bar{N}\bar{G}_{\bar{v}} : \bar{N}|$, it follows that $|\bar{N}_{\bar{v}}|$ is divisible by 3 for all $\bar{v} \in \bar{V}$. By Lemma 2.3, $\Gamma_{\mathsf{rad}(G)}$ is $\bar{N}$-edge-transitive. If $\bar{N} \cong \text{PSL}_2(2^4)$ then $|\bar{V}| = 340$; however, by [6, 8], there exists no connected edge-transitive cubic graph of order 340. Suppose that $\bar{N} \cong \text{PSL}_2(2^f)$, where $f \in \{5, 7, 8\}$. Then $f - 2 \geqslant 3$, and $|\bar{N}_{\bar{v}}|$ is divisible by $2^{f-2} \cdot 3$. Noting that $\text{PSL}_2(2^f)$ has abelian Sylow 2-subgroups, by Lemma 2.1 (i), we conclude that $f = 5$, $\bar{N}_{\bar{v}} \cong \mathbb{Z}_2 \times D_{12}$ or $\mathbb{Z}_2 \times A_4$. This contradicts that $\text{PSL}_2(2^5)$ has no subgroup isomorphic to $\mathbb{Z}_2 \times D_{12}$ or $\mathbb{Z}_2 \times A_4$, see Lemma 5.1. Therefore, $\Gamma_{\mathsf{rad}(G)}$ is $\bar{N}$-edge-transitive, and $\bar{N} \cong J_1$ or $\text{PSL}_2(p)$. This completes the proof. $\square$

Denote by $G^{(\infty)}$ the intersection of all terms appearing in the derived series of $G$.

**Lemma 2.8.** *Assume that $G$ is insolvable. Let $T = G^{(\infty)}$. Then $T \cong A_6$, $A_7$, $J_1$, $\text{PSL}_2(8)$ or $\text{PSL}_2(p)$, $\mathsf{rad}(G) = \mathbf{C}_G(T)$ and $\Gamma$ is $\mathsf{rad}(G)T$-edge-transitive.*

*Proof.* By Lemma 2.7, $\bar{G}$ has a unique minimal normal subgroup $\bar{N} \cong A_6$, $A_7$, $J_1$, $\text{PSL}_2(8)$ or $\text{PSL}_2(p)$, and $\Gamma_{\mathsf{rad}(G)}$ is $\bar{N}$-edge-transitive. By the edge-transitivity of $\bar{N}$, we conclude that $\bar{N}$ is transitive on each of $\bar{G}$-orbits on $\bar{V}$. Then $\bar{G} = \bar{N}\bar{G}_{\bar{v}}$ for $\bar{v} \in \bar{V}$. Since $\bar{G}_{\bar{v}}$ is solvable, we have $\bar{N} = \bar{G}^{(\infty)}$. Noting that $\mathsf{rad}(G)T/\mathsf{rad}(G) = (G/\mathsf{rad}(G))^{(\infty)} \cong \bar{G}^{(\infty)} = \bar{N}$, it follows that $\mathsf{rad}(G)T$ is the primage of $\bar{N}$ in $G$. Then, considering $\Gamma_{\mathsf{rad}(G)}$ as a normal quotient of $\Gamma$ with respect $\mathsf{rad}(G)T$ and $\mathsf{rad}(G)$, it is easily shown that $\Gamma$ is $\mathsf{rad}(G)T$-edge-transitive.

Note that $T/(\mathsf{rad}(G) \cap T) \cong \mathsf{rad}(G)T/\mathsf{rad}(G) \cong \bar{N}$. Suppose that $\mathsf{rad}(G) \cap T = 1$. Then $T \cong \bar{N} \cong A_6$, $A_7$, $J_1$, $\text{PSL}_2(8)$ or $\text{PSL}_2(p)$. In addition, $\mathsf{rad}(G) \leqslant \mathbf{C}_G(T)$. Since $(\mathbf{C}_G(T))^{(\infty)} \leqslant G^{(\infty)} = T$ and $\mathbf{C}_G(T) \cap T = 1$, we have $(\mathbf{C}_G(T))^{(\infty)} = 1$, and so $\mathbf{C}_G(T)$ is a solvable normal subgroup of $G$. It follows that $\mathsf{rad}(G) = \mathbf{C}_G(T)$. Thus, to complete the proof, it suffices to show that $\mathsf{rad}(G) \cap T = 1$.

Clearly, $|\mathsf{rad}(G) \cap T|$ is square-free, and so $\text{Aut}(\mathsf{rad}(G) \cap T)$ is solvable. Note that $T$ induces a subgroup of $\text{Aut}(\mathsf{rad}(G) \cap T)$ by conjugation with kernel equal to $\mathbf{C}_T(\mathsf{rad}(G) \cap T)$. Since $T$ is simple, $\mathbf{C}_T(\mathsf{rad}(G) \cap T) = 1$ or $T$. If $\mathbf{C}_T(\mathsf{rad}(G) \cap T) = 1$ then $\text{Aut}(\mathsf{rad}(G) \cap T)$ has a subgroup isomorphic to $T$, and so $\text{Aut}(\mathsf{rad}(G) \cap T)$ is insolvable, a contradiction. We have $T = \mathbf{C}_T(\mathsf{rad}(G) \cap T)$, and thus $T$ is a covering group of the simple group $\bar{N}$ with center $\mathsf{rad}(G) \cap T$. Then $\mathsf{rad}(G) \cap T$ is a homomorphic image of the Schur multiplier of $\bar{N}$, refer to [1, p.168, (33.8)]. If $\bar{N} \cong \text{PSL}_2(8)$ or $J_1$ then $\bar{N}$ has Schur multiplier 1 (see [19, p. 173, Theorem 5.14]), and so $\mathsf{rad}(G) \cap T = 1$.

Next we suppose that $\mathsf{rad}(G) \cap T \neq 1$, and produce a contradiction. By the above argument, we have that $\bar{N} \cong A_6$, $A_7$ or $\text{PSL}_2(p)$, and $\bar{N}$ has Schur multiplier $\mathbb{Z}_6$, $\mathbb{Z}_6$ or $\mathbb{Z}_2$ respectively, refer to [19, p.173, Theorem 5.14]. For $\bar{N} \cong A_6$ or $A_7$, recalling that $|G|$ is indivisible by $3^3$, we have $\mathsf{rad}(G) \cap T \cong \mathbb{Z}_2$; in this case, computation with GAP

shows that $T$ contains a unique involution. If $\bar{N} \cong \mathrm{PSL}_2(p)$ then $\mathsf{rad}(G) \cap T \cong \mathbb{Z}_2$ and $T \cong \mathrm{SL}_2(p)$; in this case, $T$ also contains a unique involution.

Let $N = \mathsf{rad}(G)T$, the primage of $\bar{N}$ in $G$. Recall that $\Gamma$ is $N$-edge-transitive. Since $|\mathsf{rad}(G)|$ is square-free, $\mathsf{rad}(G)$ has a unique Hall $2'$-subgroup say $L$. Then $L \trianglelefteq N$, and $L$ is not transitive on each of $N$-orbits on $V$. Then, by Lemma 2.2, $\Gamma_L$ is a cubic graph, and $N$ induces an edge-transitive subgroup say $X$ of $\mathrm{Aut}\Gamma_L$ with kernel equal to $L$. By the choice of $L$, we have $\mathsf{rad}(G) = L \times (\mathsf{rad}(G) \cap T)$, and so $X \cong N/L = TL/L \cong T$. In particular, $|X|$ is divisible by 8, and so $X_\alpha$ has order divisible by 6, where $\alpha$ is an $L$-orbit. By Lemma 2.1 (i), $X$ contains at least two involutions, and hence so does $T$, a contradiction. Therefore, $\mathsf{rad}(G) \cap T = 1$. This completes the proof. $\qquad\square$

Assume that $G$ is insolvable. Let $M = \mathsf{rad}(G)$ and $T = G^{(\infty)}$. For $v \in V$, denote by $\bar{v}$ the $M$-orbit containing $v$. Put $\bar{V} = \{\bar{v} \mid v \in V\}$, and $\bar{T} = T^{\bar{V}}$. Then $MT = M \times T$ and $\bar{T} \cong MT/M \cong T$. Considering the set-wise stabilizers $T_{\bar{v}}$ and $(MT)_{\bar{v}}$ of $\bar{v}$ in $T$ and $MT$ respectively, we have $M(MT)_v = (MT)_{\bar{v}} = MT_{\bar{v}}$, and so

$$(2.1) \qquad\qquad T_{\bar{v}} \cong (MT)_v \cong (MT)_{\bar{v}}/M \cong \bar{T}_{\bar{v}}.$$

Choose a $G$-orbit on $V$, say $W$, such that $T$ is transitive on $W$. For $w \in W$, it is easily shown that $T_{\bar{w}}$ is transitive on $\bar{w}$. Noting that $M$ is regular on $\bar{w}$ and centralizes $T_{\bar{w}}$, it follows from [11, p.109, Theorem 4.2A] that

$$(2.2) \qquad\qquad T_w \trianglelefteq T_{\bar{w}}, \; M \cong T_{\bar{w}}/T_w.$$

In particular, since $|M|$ is square-free and $|T_{\bar{w}}| = 2^s \cdot 3$ for some integer $s$, we have

$$(2.3) \qquad\qquad |M| \in \{1, 2, 3, 6\}.$$

**Lemma 2.9.** *Assume that $G$ is insolvable. Let $M = \mathsf{rad}(G)$ and $T = G^{(\infty)}$. Then $\Gamma$ is $MT$-edge-transitive, and either $\Gamma$ is $T$-edge-transitive, or $|M| \in \{3, 6\}$ and one of the following holds:*

(1) *$\Gamma$ is bipartite, $T \in \{\mathrm{J}_1, \mathrm{PSL}_2(p)\}$, and $T$ is transitive on one part of $\Gamma$ and has three orbits on the other part;*
(2) *$T = \mathrm{PSL}_2(p)$ is regular on $V$, and $p \equiv \pm 3 \,(\mathrm{mod}\, 8)$.*

*Proof.* By Lemma 2.8, $\Gamma$ is $MT$-edge-transitive. Note that $|MT : T| = |M|$. If $T$ is not semiregular on $V$ then, applying Lemmas 2.3 and 2.8 to the triple $(\Gamma, MT, T)$, either $\Gamma$ is $T$-edge-transitive, or $|M| \in \{3, 6\}$ and (1) occurs.

Assume that $T$ is semiregular on $V$. Then $T$ has an odd number of orbits on $V$. Since there exists no cubic graph of odd order, by Lemma 2.2, we conclude that $T$ is transitive on $V$, and so $T$ is regular on $V$. In particular, $|T|$ is not divisible by 8 or 9, and so $T = \mathrm{PSL}_2(p)$ with $p \equiv \pm 3 \,(\mathrm{mod}\, 8)$, desired as in (2). $\qquad\square$

**Theorem 2.10.** *Let $A = \mathrm{Aut}\Gamma$, and $T = G^{(\infty)}$. Assume that $G$ is insolvable. Then*

(1) *either $T \in \{\mathrm{J}_1, \mathrm{PSL}_2(p)\}$ or one of the following holds:*
    (i) *$\Gamma \cong \mathsf{F60}$ and $\mathrm{Aut}\Gamma = \mathrm{A}_6$;*
    (ii) *$\Gamma \cong \mathsf{S420}$ and $\mathrm{Aut}\Gamma = \mathbb{Z}_2 \times \mathrm{S}_7$;*
    (iii) *$\Gamma \cong \mathsf{F84}$ and $\mathrm{Aut}\Gamma = \mathrm{PSL}_2(8)$;*
(2) *$A^{(\infty)} = T$, and either $|\mathsf{rad}(G)| = 2$ or $\mathsf{rad}(G) \trianglelefteq A$.*

*Proof.* By Lemma 2.8, $T \cong A_6$, $A_7$, $\mathrm{PSL}_2(8)$, $J_1$ or $\mathrm{PSL}_2(p)$, where $p \geqslant 5$ is a prime. Suppose that $T \cong A_6$, $A_7$ or $\mathrm{PSL}_2(8)$. Then $|T|$ has a divisor 9, and so $\Gamma$ is $T$-edge-transitive by Lemma 2.3. We have $|V| = 60$, 420 or 84, respectively. Employing [6, 8], we conclude that $\Gamma$ is desired as in (i), (ii) or (iii), and part (1) follows.

Let $X = \langle A_u, A_w \rangle$ for an edge $\{u, w\} \in E$. Then $|A : X| \leqslant 2$, where the equality holds if and only if $\Gamma$ is bipartite, refer to [32, Exercise 3.8]. In particular, $A^{(\infty)} = X^{(\infty)}$. Clearly, $G \leqslant X$, and $\Gamma$ is either non-bipartite or $X$-semisymmetric. Then, by Lemma 2.8, $A^{(\infty)} = X^{(\infty)} \cong A_6$, $A_7$, $\mathrm{PSL}_2(8)$, $J_1$ or $\mathrm{PSL}_2(p)$. By Lemma 2.3, we may choose a $G$-orbit $U$ such that $T$ acts transitively on it. Noting that $T = G^{(\infty)} \leqslant A^{(\infty)}$, we know that $U$ is also a $A^{(\infty)}$-orbit. In particular, $|T : T_u| = |U| = |A^{(\infty)} : (A^{(\infty)})_u|$, where $u \in U$. Then $|T|$ and $|A^{(\infty)}|$ have the same prime divisors no less than 5. It follows that $A^{(\infty)} = T$, desired as in (2).

Finally, by (2.3), $|\mathsf{rad}(X)|$ is a divisor of 6. Noting that $\mathsf{rad}(G) = \mathbf{C}_G(T) \leqslant \mathbf{C}_X(T) = \mathsf{rad}(X)$, if $|\mathsf{rad}(G)| \neq 2$ then $|\mathsf{rad}(G)| = 1$, 3 or 6, and so $\mathsf{rad}(G)$ is a characteristic subgroup of $\mathsf{rad}(X)$, yielding $\mathsf{rad}(G) \trianglelefteq A$. This completes the proof. $\square$

## 3. Coset graphs and bi-coset graphs

Let $G$ be a finite group. If $G$ is normal in some group $A$ then each $a \in A$ induces an automorphism $\mathsf{conj}(a)$ of $G$ by conjugation:

$$x^{\mathsf{conj}(a)} := a^{-1}xa, \ \forall x \in G.$$

For $X_1, \ldots, X_m \subseteq G$, we write

$$\mathbf{N}_G(X_1, \ldots, X_m) = \cap_{i=1}^m \mathbf{N}_G(X_i),$$
$$\mathbf{N}_G(\{X_1, \ldots, X_m\}) = \{g \in G \mid \{g^{-1}X_1 g, \ldots, g^{-1}X_m g\} = \{X_1, \ldots, X_m\}\},$$
$$\mathrm{Aut}(G, X_1, \ldots, X_m) = \{\sigma \in \mathrm{Aut}(G) \mid X_i^\sigma = X_i, 1 \leqslant i \leqslant m\},$$
$$\mathrm{Aut}(G, \{X_1, \ldots, X_m\}) = \{\sigma \in \mathrm{Aut}(G) \mid \{X_1^\sigma, \ldots, X_m^\sigma\} = \{X_1, \ldots, X_m\}\}.$$

3.1. **Coset actions.** Assume that $H$ is a core-free subgroup of $G$, that is, $H$ contains no nontrivial normal subgroup of $G$. Then $G$ acts faithfully and transitively on $[G : H] := \{Hx \mid x \in G\}$ by right multiplication:

(3.1) $$(Hx)^g := Hxg, \ \forall x, g \in G.$$

The resulting transitive subgroup of $\mathrm{Sym}([G : H])$ is still denoted by $G$ in the following.

Note that the group $\mathrm{Aut}(G, H)$ has a natural action on $[G : H]$ by

$$(Hx)^\sigma := Hx^\sigma, \ x \in G, \sigma \in \mathrm{Aut}(G, H).$$

For $\sigma \in \mathrm{Aut}(G, H)$, we denote by $\sigma_H$ the permutation induced by $\sigma$ on $[G : H]$. Clearly,

(3.2) $$\mathsf{conj}(h)_H = h, \ \forall h \in H.$$

The next lemma says that $\sigma \mapsto \sigma_H$ is an embedding from $\mathrm{Aut}(G, H)$ into $\mathrm{Sym}([G : H])$.

**Lemma 3.1.** $\mathrm{Aut}(G, H)$ *acts faithfully on* $[G : H]$.

*Proof.* Clearly, if $H = 1$ then the action of $\mathrm{Aut}(G, H)$ is faithful. Thus let $H \neq 1$. Pick $\sigma \in \mathrm{Aut}(G, H)$ such that $Hx^\sigma = Hx$, i.e., $x^\sigma x^{-1} \in H$, for all $x \in G$. For $x, y \in G$,

$$Hyx = H(yx)^\sigma = Hy^\sigma x^\sigma = Hyx^\sigma \Rightarrow yx^\sigma x^{-1}y^{-1} \in H.$$

Then, for each $x \in G$, the subgroup $H$ contains a normal subgroup $\langle yx^\sigma x^{-1}y^{-1} \mid y \in G \rangle$ of $G$. Since $H$ is core-free, we have $x^\sigma x^{-1} = 1$, i.e., $x^\sigma = x$ for all $x \in G$. Thus $\sigma = 1$, and the lemma follows. $\square$

If $g \in \mathbf{N}_G(H)$, then $g$ induces a permutation $\hat{g}$ on $[G : H]$ by

$$(3.3) \qquad\qquad\qquad (Hx)^{\hat{g}} := Hg^{-1}x, \ \forall x \in G.$$

In fact, $\hat{g}g = \mathsf{conj}(g)_H = g\hat{g}$, where $g$ acts on $[G : H]$ by the way described as in (3.1).

**Lemma 3.2.** $\mathbf{N}_G(H)/H \cong \mathbf{C}_{\mathrm{Sym}([G:H])}(G) = \{\hat{g} \mid g \in \mathbf{N}_G(H)\}$, *and* $\mathbf{N}_{\mathrm{Sym}([G:H])}(G) = G\{\sigma_H \mid \sigma \in \mathrm{Aut}(G, H)\}$.

*Proof.* The first part of this lemma follows directly from [11, p.108, Lemma 4.2A].

Let $N = \mathbf{N}_{\mathrm{Sym}([G:H])}(G)$, and $K$ be the point-stabilizer of $H$ in $N$. Then $G \leqslant N$ and, since $G$ is transitive on $[G : H]$, we have $N = GK$. Clearly, $\mathrm{Aut}(G, H) \cong \{\sigma_H \mid \sigma \in \mathrm{Aut}(G, H)\} \leqslant K$. For $t \in K$, considering the point-stabilizers of $H^t$ and $H$ in $G$, we have $t^{-1}Ht = H$, and so $\mathsf{conj}(t) \in \mathrm{Aut}(G, H)$. Thus we have a group homomorphism: $K \to \mathrm{Aut}(G, H)$, $t \mapsto \mathsf{conj}(t)$, and the kernel equals to $\mathbf{C}_K(G)$. Noting that $\mathbf{C}_K(G)$ is semiregular on $[G : H]$, we have $\mathbf{C}_K(G) = 1$. Thus $K$ is isomorphic to a subgroup of $\mathrm{Aut}(G, H)$, and so $|K| \leqslant |\mathrm{Aut}(G, H)|$. We have $K = \{\sigma_H \mid \sigma \in \mathrm{Aut}(G, H)\}$, and the lemma follows. $\square$

### 3.2. Coset graphs.
Let $G \neq 1$ be a finite group, and let $H$ be a core-free subgroup of $G$. Suppose that $H$ has a subgroup $K$ with index $k > 1$, and

(I) there exists $o \in \mathbf{N}_G(K) \setminus H$ such that $o^2 \in K$ and $H \cap o^{-1}Ho = K$.

The coset graph $\mathsf{Cos}(G, H, K, o)$ is defined on $[G : H]$ such that $Hx$ and $Hy$ are adjacent if and only if $yx^{-1} \in HoH$. Then $\mathsf{Cos}(G, H, K, o)$ is a well-defined $G$-symmetric graph of valency $k$. It is well-known that every connected symmetric graph of valency $k$ is isomorphic to a coset graph defined as above. The following facts are easily shown, see also [20] for example.

(II) $\mathsf{Cos}(G, H, K, o)$ is connected if and only if $G = \langle H, o \rangle$.

(III) If $\sigma \in \mathrm{Aut}(G)$ then $Hx \mapsto H^\sigma x^\sigma$ defines an isomorphism from $\mathsf{Cos}(G, H, K, o)$ to $\mathsf{Cos}(G, H^\sigma, K^\sigma, o^\sigma)$. In particular, if $\sigma \in \mathrm{Aut}(G, H)$ then $\sigma_H$ is an automorphism of $\mathsf{Cos}(G, H, K, o)$ if and only if $Ho^\sigma H = HoH$. (Note, for $h \in H$, we have $\mathsf{Cos}(G, H, K, o) = \mathsf{Cos}(G, H, h^{-1}Kh, h^{-1}oh)$.)

In view of (III), up to isomorphism of graphs, $H$, $K$ and $o$ may be chosen up to the conjugacy under $\mathrm{Aut}(G)$, $\mathrm{Aut}(G, H)$ and $\mathrm{Aut}(G, H, K)$, respectively.

**Lemma 3.3.** *Let $\Gamma = \mathsf{Cos}(G, H, K, o)$ and $\Sigma = \mathsf{Cos}(G, H, K, o')$. Suppose that both $\mathrm{Aut}\Gamma$ and $\mathrm{Aut}\Sigma$ have a unique subgroup isomorphic to $G$. Then $\Gamma \cong \Sigma$ if and only if $Ho^\sigma H = Ho'H$ for some $\sigma \in \mathrm{Aut}(G, H, K)$.*

*Proof.* The sufficiency of $\Gamma \cong \Sigma$ is immediate from the above (III). Now let $\lambda$ be an isomorphism from $\mathsf{Cos}(G, H, K, o)$ to $\mathsf{Cos}(G, H, K, o')$. Then $\mathrm{Aut}\Sigma = \lambda^{-1}\mathrm{Aut}\Gamma\lambda$. It follows that $G = \lambda^{-1}G\lambda$. Since $G$ is transitive on the arc sets of $\Gamma$ and $\Sigma$, without

loss of generality, we choose $\lambda$ with $(H, Ho)^\lambda = (H, Ho')$. Considering the stabilizers of $H$, $(H, Ho)$ and $(H, Ho')$ in $G$, we have $H = \lambda^{-1}H\lambda$ and $K = \lambda^{-1}K\lambda$. Then $\sigma := \mathsf{conj}(\lambda) \in \mathrm{Aut}(G, H, K)$. For $Hx \in [G : H]$, since $\lambda$ fixes the vertex $H$, we have

$$(Hx)^\lambda = H^{x\lambda} = H^{\lambda^{-1}x\lambda} = H(\lambda^{-1}x\lambda) = Hx^\sigma.$$

Considering the neighborhoods of $H$ in $\Gamma$ and $\Sigma$, we have

$$\{Ho'h \mid h \in H\} = \{Hoh \mid h \in H\}^\lambda = \{H\lambda^{-1}oh\lambda \mid h \in H\} = \{Ho^\sigma h^\sigma \lambda \mid h \in H\}.$$

This implies that $Ho'H = Ho^\sigma H$, and the lemma follows.                    $\square$

Using Lemma 3.2, the following lemma is easily shown.

**Lemma 3.4.** *Let* $\Gamma = \mathsf{Cos}(G, H, K, o)$, *and view* $G$ *as a subgroup of* $\mathrm{Aut}\Gamma$. *Then* $\mathbf{C}_{\mathrm{Aut}\Gamma}(G) = \{\hat{g} \mid g \in \mathbf{N}_G(H, HoH)\}$, *and* $\mathbf{N}_{\mathrm{Aut}\Gamma}(G) = G\{\sigma_H \mid \sigma \in \mathrm{Aut}(G, H, HoH)\}$.

**Example 3.5.** Let $T = \mathrm{J}_1$, the first Janko group. Computation with GAP [14] shows that, up to conjugacy, $\mathrm{J}_1$ has two subgroup isomorphic to $\mathrm{S}_3$, and only one of them say $H$ has a subgroup $K$ which has order 2 and satisfies the condition that $\mathbf{N}_T(K) \setminus K$ contains elements $o$ with $o^2 \in K$ and $\langle H, o \rangle = T$. Fix such a pair $(H, K)$. Then $\mathbf{N}_T(K) = \mathbb{Z}_2 \times \mathrm{A}_5$, and thus every desired $o$ should be an involution. Further computation shows that there exist exactly 20 desired involutions, which are conjugate in pairs under $\mathbf{N}_T(H, K)$ and produce 10 distinct double cosets $HoH$. Thus we get ten connected $T$-symmetric cubic graphs of order $4 \cdot 5 \cdot 7 \cdot 11 \cdot 19$. It is shown in Section 4 that these graphs are not isomorphic to each other.                    $\square$

3.3. **Bi-coset graphs.** Let $G$ be a finite group, and $L, R < G$ with $L \neq R$, $|L| = |R|$ and $L \cap R$ core-free in $G$. The bi-coset graph $\mathsf{BC}(G, L, R)$ is defined with bipartition $([G : L], [G : R])$ such that $Lx$ and $Ry$ are adjacent if and only if $yx^{-1} \in RL$, i.e., $xy^{-1} \in LR$. Then $\mathsf{BC}(G, L, R)$ is a well-defined regular graph of valency $|L : (L \cap R)|$, and $\mathsf{BC}(G, L, R) = \mathsf{BC}(G, R, L)$. View $G$ as a subgroup of $\mathrm{Aut}\mathsf{BC}(G, L, R)$, where $G$ acts on $[G : L]$ and $[G : R]$ by right multiplications:

$$(3.4) \qquad\qquad (Lx)^g := Lxg, \ (Ry)^g := Ryg, \ \forall g, x, y \in G.$$

Then $\mathsf{BC}(G, L, R)$ is $G$-semisymmetric. It is easily shown that $\mathsf{BC}(G, L, R)$ is connected if and only if $G = \langle L, R \rangle$. The reader is referred to [13, 25] for more information about bi-coset graphs.

Each $\sigma \in \mathrm{Aut}(G)$ defines an isomorphism from $\mathsf{BC}(G, L, R)$ to $\mathsf{BC}(G, L^\sigma, R^\sigma)$ by

$$(3.5) \qquad\qquad Lx \mapsto L^\sigma x^\sigma, \ Ry \mapsto R^\sigma y^\sigma, \ \forall x, y \in G.$$

Thus, up to isomorphism of graphs, the subgroups $L$ and $R$ may be chosen under $\mathrm{Aut}(G)$-conjugacy and $\mathrm{Aut}(G, L)$-conjugacy, respectively.

**Lemma 3.6.** *Assume that* $G = \langle L_1, R_1 \rangle = \langle L_2, R_2 \rangle$, *and* $\Gamma_i = \mathsf{BC}(G, L_i, R_i)$ *for* $i = 1, 2$.
  (1) *If* $\{L_1^\sigma, R_1^\sigma\} = \{L_2, R_2\}$ *for some* $\sigma \in \mathrm{Aut}(G)$ *then* $\Gamma_1 \cong \Gamma_2$.
  (2) *Suppose that both* $\mathrm{Aut}\Gamma_1$ *and* $\mathrm{Aut}\Gamma_2$ *have a unique subgroup isomorphic to* $G$. *If* $\Gamma_1 \cong \Gamma_2$ *then* $\{L_1^\sigma, R_1^\sigma\} = \{L_2, R_2\}$ *for some* $\sigma \in \mathrm{Aut}(G)$, *and* $\sigma$ *is chosen from* $\mathrm{Aut}(G, L_1)$ *for the case where* $L_1 = L_2$ *and either* $\Gamma_1$ *is symmetric or* $L_1$ *and* $R_1$ *are not conjugate under* $\mathrm{Aut}(G)$.

*Proof.* Part (1) of the lemma is pretty obvious. Suppose that both $\mathrm{Aut}\Gamma_1$ and $\mathrm{Aut}\Gamma_2$ have a unique subgroup isomorphic to $G$, and let $\lambda$ be an isomorphism from $\Gamma_1$ to $\Gamma_2$. Then $\mathrm{Aut}\Gamma_2 = \lambda^{-1}\mathrm{Aut}\Gamma_1\lambda$, and $G = \lambda^{-1}G\lambda$. Since $G$ acts transitively on the edge sets, we choose $\lambda$ such that $\{L_1, R_1\}^\lambda = \{L_2, R_2\}$. Let $\sigma$ be the automorphism of $G$ induced by $\lambda$. Considering the vertex-stabilizers of $L_1$, $L_2$, $R_1$ and $R_2$ in $G$, we deduce that

$$\{L_2, R_2\} = \{\lambda^{-1}L_1\lambda, \lambda^{-1}R_1\lambda\} = \{L_1^\sigma, R_1^\sigma\}.$$

Assume further that $L_1 = L_2$, and either $\Gamma_1$ is symmetric or $L_1$ and $R_1$ are not conjugate under $\mathrm{Aut}(G)$. It is easily shown that $\lambda$ may be chosen such that $(L_1, R_1)^\lambda = (L_1, R_2)$. This implies that $L_1^\sigma = L_1$ and $R_2 = R_1^\sigma$, and so part (2) of the lemma follows. $\qquad\square$

Note that $\mathrm{Aut}(G, \{L, R\})$ induces a subgroup of $\mathrm{Aut}\mathsf{BC}(G, L, R)$, see (3.5). Denote $\sigma_{\{L,R\}}$ the graph automorphism induced by $\sigma \in \mathrm{Aut}(G, \{L, R\})$. Clearly,

$$\mathsf{conj}(h)_{\{L,R\}} = h, \ \forall h \in L \cap R.$$

**Lemma 3.7.** $\mathrm{Aut}(G, \{L, R\})$ *acts faithfully on* $[G : L] \cup [G : R]$.

*Proof.* Let $K$ be the kernel of $\mathrm{Aut}(G, \{L, R\})$ acting on $[G : L] \cup [G : R]$. Then $K \leqslant \mathrm{Aut}(G, L, R)$. Let $\sigma \in K$ and $x \in G$. It is easily shown that both $L$ and $R$ contains a normal subgroup $\langle yx^\sigma x^{-1}y^{-1} \mid y \in G\rangle$ of $G$, see the proof of Lemma 3.1. Since $L \cap R$ is core-free in $G$, we have $x^\sigma x^{-1} = 1$. Thus $x^\sigma = x$ for all $x \in G$ and $\sigma \in K$. Then $K = 1$, and the lemma follows. $\qquad\square$

**Lemma 3.8.** *Let* $\Gamma = \mathsf{BC}(G, L, R)$ *and* $N = \mathbf{N}_{\mathrm{Aut}\Gamma}(G)$. *Then* $N = G\{\sigma_{\{L,R\}} \mid \sigma \in \mathrm{Aut}(G, \{L, R\})\}$.

*Proof.* Let $H$ be the edge-stabilizer of $\{L, R\}$ in $N$. We have $H \geqslant \{\sigma_{\{L,R\}} \mid \sigma \in \mathrm{Aut}(G, \{L, R\})\} \cong \mathrm{Aut}(G, \{L, R\})$ and, since $\Gamma$ is $G$-edge-transitive, $N = GH$. Considering the conjugation of $H$ on $G$, we have a homomorphism $\rho : H \to \mathrm{Aut}(G)$ with kernel equal to $\mathbf{C}_H(G)$. Note that $\Gamma$ has valency $|L : (L \cap R)| > 1$. It follows that $N$ acts faithfully on the edge set of $\Gamma$. Then $\mathbf{C}_H(G)$ is faithful and semiregular on the edge set of $\Gamma$. Thus $\mathbf{C}_H(G) = 1$, and $\rho$ is injective. In particular, $|H| = |\rho(H)|$.

Let $t \in H$. Then either $L^t = L$ and $R^t = R$, or $L^t = R$ and $R^t = L$. Now consider the vertex-stabilizers of $L$, $R$, $L^t$ and $R^t$ in $G$. If $L^t = L$ and $R^t = R$, then $L^{\rho(t)} = t^{-1}Lt = L$ and $R^{\rho(t)} = t^{-1}Rt = R$; if $L^t = R$ and $R^t = L$ then $L^{\rho(t)} = t^{-1}Lt = R$ and $R^{\rho(t)} = t^{-1}Rt = L$. For both cases, $\rho(t) \in \mathrm{Aut}(G, \{L, R\})$. Thus $|H| = |\rho(H)| \leqslant |\mathrm{Aut}(G, \{L, R\})| = |\{\sigma_{\{L,R\}} \mid \sigma \in \mathrm{Aut}(G, \{L, R\})\}|$. Recalling that $\{\sigma_{\{L,R\}} \mid \sigma \in \mathrm{Aut}(G, \{L, R\})\} \leqslant H$, it follows that $\{\sigma_{\{L,R\}} \mid \sigma \in \mathrm{Aut}(G, \{L, R\})\} = H$. Then the lemma follows. $\qquad\square$

For $g_1 \in \mathbf{N}_G(L)$ and $g_2 \in \mathbf{N}_G(R)$, define

$$\begin{aligned}\tilde{g}_1 : \ & [G : L] \cup [G : R] \to Lx \mapsto Lg_1^{-1}x, Ry \mapsto Ry;\\ \hat{g}_2 : \ & [G : L] \cup [G : R] \to Lx \mapsto Lx, Ry \mapsto Rg_2^{-1}y.\end{aligned}$$

Then

$$\mathbf{C}_{\mathrm{Sym}([G:L])\times\mathrm{Sym}([G:R])}(G) = \{\tilde{g}_1\hat{g}_2 \mid g_1 \in \mathbf{N}_G(L), g_2 \in \mathbf{N}_G(R)\}.$$

Further, we have the following lemma.

**Lemma 3.9.** *Let* $\Gamma = \mathsf{BC}(G, L, R)$. *If* $g_1 \in \mathbf{N}_G(L)$ *and* $g_2 \in \mathbf{N}_G(R)$, *then* $\tilde{g}_1\hat{g}_2 \in \mathbf{C}_{\mathrm{Aut}\Gamma}(G)$ *if and only if* $Rg_2^{-1}g_1L = RL$, *and* $\tilde{g}_1\hat{g}_2 = 1$ *if and only if* $g_1 \in L$ *and* $g_2 \in R$.

**Lemma 3.10.** *Let $\Gamma = (V, E)$ be a connected $G$-semisymmetric graph of valency $k > 1$. Then $\Gamma \cong \mathsf{BC}(G, L, R)$ for some $L, R < G$ with $|L| = |R|$, $k = |L : (L \cap R)|$, $G = \langle L, R \rangle$ and $L \cap R$ core-free in $G$.*

*Proof.* Clearly, for $v \in V$, the stabilizer $G_v$ acts transitively $\Gamma(v)$, and so $k = |G_v : (G_v \cap G_{v'})|$ for $v' \in \Gamma(v)$. Let $U$ and $W$ be the $G$-orbits on $V$, and fix an edge $\{u, w\} \in E$ with $u \in U$ and $w \in W$. Since $\Gamma$ is regular, we have $|G : G_u| = |U| = |W| = |G : G_w|$, and so $|G_u| = |G_w|$. Since $\Gamma$ is connected, $G = \langle G_u, G_w \rangle$. Since $\Gamma$ has valency $k > 1$, it is easily shown that $G$ acts faithfully on $E$. If $G_u \cap G_w$ contains a normal subgroup $N$ of $G$ then $N$ fixes $E$ point-wise, and so $N = 1$. Thus $G_u \cap G_w$ is core-free in $G$. Put $L = G_u$ and $R = G_w$. Noting that $U = \{u^x \mid x \in G\}$ and $W = \{w^y \mid y \in G\}$, define

$$\rho : U \cup W \to [G : L] \cup [G : R], \; u^x \mapsto Lx, \; w^y \mapsto Ry.$$

Then $\rho$ is a bijection and, for $u^x \in U$ and $w^y \in W$,

$$\{u^x, w^y\} \in E \Leftrightarrow w^{yx^{-1}} \in \Gamma(u) \Leftrightarrow yx^{-1} \in G_w G_u = RL.$$

Thus $\rho$ is an isomorphism from $\Gamma$ to $\mathsf{BC}(G, L, R)$, and the lemma follows. $\qquad\square$

**Example 3.11.** Let $T = \mathrm{J}_1$. Computation with GAP [14] shows that
  (i) $T$ has a unique conjugacy class of subgroups isomorphic to $\mathrm{D}_{12}$, and each subgroup $\mathrm{D}_{12}$ is self-normalized in $T$; and
  (ii) fixing a subgroup $L \cong \mathrm{D}_{12}$, there exist exactly 6 subgroups $R \cong \mathrm{D}_{12}$ with $|L \cap R| = 4$ and $\langle L, R \rangle = G$, which form two classes under the conjugation of $L$.

Thus, up to isomorphism of graphs, we get two connected $T$-semisymmetric cubic graphs, say $\Gamma_1 = \mathsf{BC}(T, L, R_1)$ and $\Gamma_2 = \mathsf{BC}(T, L, R_2)$ with the stabilizers of two adjacent vertices isomorphic to $\mathrm{D}_{12}$. We next show that $\Gamma_1 \cong \Gamma_2$.

Since $\mathbf{N}_T(L) = L$, there is a unique $o \in G$ with $R_1 = o^{-1}Lo$. Set $R = oLo^{-1}$. Then $\langle L, R \rangle = T$ and $|L \cap R| = 4$. Suppose that $R = x^{-1}R_1 x$ for some $x \in L$. We have $oLo^{-1} = x^{-1}o^{-1}Lox$, yielding $o^{-1} = ox$, and so $o^2 = x^{-1} \in L$. Then there exists a connected $T$-symmetric cubic graph $\mathsf{Cos}(T, L, L \cap L^o, o)$, which is impossible by [21, Lemma 6.3]. Therefore, $R$ and $R_1$ are not conjugate under $L$, and so we may choose $R_2 = oLo^{-1}$. Noting that $\{L, R_2\}^{\mathsf{conj}(o)} = \{L, R_1\}$, we have $\Gamma_1 \cong \Gamma_2$ by Lemma 3.6. $\quad\square$

## 4. The graphs arising from $\mathrm{J}_1$

In this section, we assume that $\Gamma = (V, E)$ is a connected edge-transitive cubic graph of order $2n$ with $n$ even and square-free. Assume further that $\mathrm{J}_1 \leqslant \mathrm{Aut}\Gamma$.

**Lemma 4.1.** *Suppose that $\Gamma$ is $\mathrm{J}_1$-edge-transitive. Then $\mathrm{Aut}\Gamma = \mathrm{J}_1$, and either*
  (1) *$\Gamma$ is isomorphic to one of ten non-isomorphic graphs in Example 3.5; or*
  (2) *$\Gamma$ is semisymmetric and isomorphic to the graph constructed in Example 3.11.*

*Proof.* Let $T = \mathrm{J}_1$. We discuss in two cases according whether $\Gamma$ is bipartite or not.

**Case 1**. Assume that $\Gamma$ is not bipartite. Then $\Gamma$ is $T$-symmetric, and $2n = |V| = |T : T_u|$ for $u \in V$. We have $|T_u| = 6$, and so $T_u \cong \mathrm{S}_3$ by Lemma 2.1. Then $\Gamma$ is isomorphic one of the ten coset graphs $\mathsf{Cos}(T, H, K, o)$ given as in Example 3.5. Let $A = \mathrm{Aut}\mathsf{Cos}(T, H, K, o)$. Then $T = A^{(\infty)}$ by Theorem 2.10. In particular, $\mathbf{N}_A(T) = \mathrm{Aut}\mathsf{Cos}(T, H, K, o)$. Note that every automorphism of $T$ is induced by the conjugation

of some element in $T$. Computation with GAP shows that $\mathrm{Aut}(T, H) \cong \mathrm{D}_{12}$, and if $\sigma \in \mathrm{Aut}(T, H)$ such that $H o^{\sigma} H = H o H$ then $\sigma = \mathsf{conj}(h)$ for some $h \in H$. We deduce from Lemma 3.4 that $\mathsf{AutCos}(T, H, K, o) = T$. Thus every graph in Example 3.5 has automorphism group $T$. By Lemma 3.3, these coset graphs are not isomorphic to each other, and part (1) if the lemma follows.

**Case 2**. Assume that $\Gamma$ is bipartite. Then $T$ is intransitive on $V$; otherwise, $T$ has a subgroup of index 2, and so $T$ is not simple, a contradiction. Thus $\Gamma$ is $T$-semisymmetric, and $n = |T : T_u|$ for $u \in V$. We have $|T_u| = 12$. By Lemma 2.1, we assume that $T_u \cong \mathrm{D}_{12}$ and $T_w \cong \mathrm{D}_{12}$ or $\mathrm{A}_4$, where $w \in \Gamma(u)$. If $T_u \not\cong T_w$ then computation with GAP shows that $|\langle T_u, T_w \rangle| = 660 \neq |T|$, which contradicts the fact that $\Gamma$ is connected. We have $T_u \cong T_w \cong \mathrm{D}_{12}$. By Lemma 3.10, $\Gamma$ is isomorphic to the bi-coset graph $\mathsf{BC}(T, L, R_1)$ given in Example 3.11. By Theorem 2.10, we have $T \trianglelefteq \mathsf{AutBC}(T, L, R_1)$. Computation with GAP shows that $\mathrm{Aut}(T, \{L, R_1\}) = \{\mathsf{conj}(h) \mid h \in L \cap R_1\}$. It follows from Lemma 3.8 that $\mathsf{AutBC}(T, L, R_1) = T$. Then $\Gamma$ is semisymmetric, and part (2) of the lemma follows.                                                                $\square$

**Theorem 4.2.** *Let $A = \mathrm{Aut}\Gamma$. Assume that $A^{(\infty)} = \mathrm{J}_1$. Then $\Gamma$ is $\mathrm{J}_1$-edge-transitive, and $\Gamma$ is described as in Lemma 4.1.*

*Proof.* By Lemma 4.1, it suffices to show that $\Gamma$ is $\mathrm{J}_1$-edge-transitive. We next suppose that $\Gamma$ is not $\mathrm{J}_1$-edge-transitive, and produce a contradiction. By Lemma 2.9, $\Gamma$ is bipartite, and $T := \mathrm{J}_1$ is transitive on one part of $\Gamma$ say $W$ and has three orbits on the other part $U$. Let $\{u, w\} \in E$ with $u \in U$ and $w \in W$. Then $n = |T : T_w|$ and $n = 3|T : T_u|$. It follows that $|T_w| = 4$ and $|T_u| = 12$.

Let $G = \langle A_u, A_w \rangle$ and $M = \mathsf{rad}(G)$. By Lemma 2.9, $|M| = 3$ or 6. Clearly, the quotient graph $\Gamma_M$ is bipartite. Then, by Lemma 2.7, $\Gamma_M$ is $\bar{T}$-semisymmetric. In addition, $|\bar{T} : \bar{T}_{\bar{v}}| = \frac{n}{|M|}$ is square-free, where $v \in V$. By Lemma 2.1 and inspecting the subgroups of $\mathrm{J}_1$, we conclude that $\bar{T}_{\bar{u}}$ and $\bar{T}_{\bar{w}}$ are isomorphic to $\mathrm{D}_{12}$ or $\mathrm{A}_4$. In particular, $\frac{n}{|M|}$ is even, and so $|M|$ is odd. We have $|M| = 3$. Recall that $\bar{T}_{\bar{w}} \cong T_{\bar{w}}$ and $M \cong T_{\bar{w}}/T_w$, see (2.1) and (2.2). This implies that $\bar{T}_{\bar{w}} \cong \mathrm{A}_4$, and so $\bar{T}_{\bar{u}} \cong \mathrm{D}_{12}$ by Lemma 2.1. However, since $|\bar{T} : \bar{T}_{\bar{v}}|$ is even and square-free, (2) of Lemma 4.1 is available for the pair $(\bar{T}, \Gamma_M)$, which leads to $\bar{T}_{\bar{w}} \cong \bar{T}_{\bar{u}} \cong \mathrm{D}_{12}$, a contradiction. This completes the proof.        $\square$

## 5. $\mathrm{PSL}_2(p)$-SYMMETRIC GRAPHS

In this section, $\Gamma = (V, E)$ is a connected $T$-symmetric cubic graph of order $2n$, where $T = \mathrm{PSL}_2(p)$ for some prime $p \geqslant 5$, and $n$ is even and square-free. Choose $\varepsilon, \eta \in \{1, -1\}$ with $p + \varepsilon$ and $p + \eta$ divisible by 3 and 4, respectively. Our discussion is based on the subgroup structure of $\mathrm{PSL}_2(p)$ and $\mathrm{PGL}_2(p)$. The reader is referred to [17, II.8.27] and [3, Theorem 3] for the subgroups of $\mathrm{PSL}_2(p)$, and to [4, Theorem 2] for the subgroups of $\mathrm{PGL}_2(p)$. For convenience, we list the subgroups of $\mathrm{PSL}_2(p)$ and $\mathrm{PGL}_2(p)$ in the following two lemmas.

**Lemma 5.1.** *Let $p \geqslant 5$ be a prime. Then the subgroups of $\mathrm{PSL}_2(p)$ are listed as follows.*

(1) *One conjugacy class of $\frac{p(p-\eta)}{2}$ cyclic subgroups $\mathbb{Z}_2$.*
(2) *One conjugacy class of $\frac{p(p\mp1)}{2}$ cyclic subgroups $\mathbb{Z}_d$, where $d \mid \frac{p\pm1}{2}$ and $d > 2$.*

(3) $\frac{p(p^2-1)}{24}$ *elementary abelian subgroups* $\mathbb{Z}_2^2$.

(4) $\frac{p(p^2-1)}{4d}$ *dihedral subgroups* $\mathrm{D}_{2d}$, *where* $d \mid \frac{p\pm 1}{2}$ *and* $d > 2$.

(5) *One conjugacy class of* $p+1$ *subgroups* $\mathbb{Z}_p{:}\mathbb{Z}_d$, *where* $d \mid \frac{p-1}{2}$ *and* $d \geqslant 1$.

(6) $\frac{p(p^2-1)}{24}$ *subgroups* $\mathrm{A}_4$.

(7) *Two conjugacy classes of subgroups* $\mathrm{S}_4$, *each consists of* $\frac{p(p^2-1)}{48}$ *subgroups, where* $p \equiv \pm 1 \,(\mathrm{mod}\ 8)$.

(8) *Two conjugacy classes of subgroups* $\mathrm{A}_5$, *each consists of* $\frac{p(p^2-1)}{120}$ *subgroups, where* $p \equiv \pm 1 \,(\mathrm{mod}\ 10)$.

*Moreover, isomorphic subgroups of* $\mathrm{PSL}_2(p)$ *are conjugate in* $\mathrm{PGL}_2(p)$.

**Lemma 5.2.** *Let* $p \geqslant 5$ *be a prime. Then the subgroups of* $\mathrm{PGL}_2(p)$ *are listed as follows.*

(1) *The subgroup* $\mathrm{PSL}_2(p)$.

(2) *Two conjugacy classes of cyclic subgroup* $\mathbb{Z}_2$, *one class consists of* $\frac{p(p-\eta)}{2}$ *subgroups which lie in* $\mathrm{PSL}_2(p)$, *and the other one consists of* $\frac{p(p+\eta)}{2}$ *subgroups.*

(3) *One conjugacy class of* $\frac{p(p\mp 1)}{2}$ *cyclic subgroups* $\mathbb{Z}_d$, *where* $d \mid p \pm 1$ *and* $d > 2$.

(4) *Two conjugacy classes of subgroups* $\mathbb{Z}_2^2$, *one class consists of* $\frac{p(p^2-1)}{24}$ *subgroups which lie in* $\mathrm{PSL}_2(p)$, *and the other one consists of* $\frac{p(p^2-1)}{8}$ *subgroups.*

(5) *Two conjugacy classes of subgroups* $\mathrm{D}_{2d}$, *one class consists of* $\frac{p(p^2-1)}{4d}$ *subgroups which lie in* $\mathrm{PSL}_2(p)$, *and the other one consists of* $\frac{p(p^2-1)}{4d}$ *subgroups, where* $d \mid \frac{p\pm 1}{2}$ *and* $d > 2$.

(6) *One conjugacy class of* $\frac{p(p^2-1)}{2d}$ *subgroups* $\mathrm{D}_{2d}$, *where* $d > 2$ *and* $\frac{p\pm 1}{d}$ *is an odd integer.*

(7) *One conjugacy class of* $p+1$ *subgroups* $\mathbb{Z}_p{:}\mathbb{Z}_d$, *where* $d \mid (p-1)$ *and* $d \geqslant 1$.

(8) *One conjugacy class of* $\frac{p(p^2-1)}{24}$ *subgroups* $\mathrm{A}_4$.

(9) *One conjugacy class of* $\frac{p(p^2-1)}{24}$ *subgroups* $\mathrm{S}_4$.

(10) *One conjugacy classes of* $\frac{p(p^2-1)}{60}$ *subgroups* $\mathrm{A}_5$, *where* $p \equiv \pm 1 \,(\mathrm{mod}\ 10)$.

By Lemma 2.1 and inspecting the subgroups of $\mathrm{PSL}_2(p)$, we have $T_v \cong \mathbb{Z}_3$, $\mathrm{S}_3$, $\mathrm{D}_{12}$ or $\mathrm{S}_4$, where $v \in V$. Then

(5.1) $$p \equiv 2^{i+2} \pm 1 \,(\mathrm{mod}\ 2^{i+3}) \text{ and } |T_v| = 2^i \cdot 3 \text{ for } 0 \leqslant i \leqslant 3.$$

We deduce from Lemmas 5.1 and 5.2 that $T$ contains at most two conjugacy classes of subgroups isomorphic to $T_v$, and these subgroups are all conjugate in $\mathrm{PGL}_2(p)$. Thus up to isomorphism of graphs, we fix two subgroups $K$, $H$ of $T$, and write

$$\Gamma \cong \mathsf{Cos}(T, H, K, o),$$

where $K < H \cong T_v$, $|H : K| = 3$ and $o \in \mathbf{N}_T(K)$ with $o^2 \in K$ and $\langle o, H \rangle = T$.

By Theorem 2.10, $T \trianglelefteq \mathrm{Aut}\Gamma$. Noting that $\mathrm{Aut}(T) = \{\mathsf{conj}(g) \mid g \in \mathrm{PGL}_2(p)\}$, we have

(5.2) $$\mathrm{Aut}\mathsf{Cos}(T, H, K, o) = T\{\mathsf{conj}(g)_H \mid g \in \mathbf{N}_{\mathrm{PGL}_2(p)}(H, HoH)\},$$

by Lemma 3.4. Recall that $\mathsf{conj}(g)_H = g\hat{g}$ for $g \in \mathbf{N}_T(H)$.

5.1. $|H| = 3$. Assume that $H \cong \mathbb{Z}_3$. Then $p \equiv \pm 3 \pmod 8$ by (5.1), $K = 1$, and $o$ is an involution. Let $S$ and $O$ be the sets of involutions $x \in T$ with $\langle x, H \rangle \neq T$ and $\langle x, H \rangle = T$, respectively. Then $|S| + |O| = \frac{p(p-\eta)}{2}$, see Lemma 5.1 (1).

**Lemma 5.3.**

$$
|S| = \begin{cases}
\frac{3p+3\varepsilon+|\varepsilon+\eta|}{2} & \text{if } p \not\equiv \pm 1 \,(\text{mod } 10), \varepsilon + \eta \neq -2, \\
\frac{7p-5}{2} & \text{if } p \not\equiv \pm 1 \,(\text{mod } 10), \varepsilon = \eta = -1, \\
\frac{7p+7\varepsilon+|\varepsilon+\eta|}{2} & \text{if } p \equiv \pm 1 \,(\text{mod } 10), \varepsilon + \eta \neq -2, \\
\frac{11p-9}{2} & \text{if } p \equiv \pm 1 \,(\text{mod } 10), \varepsilon = \eta = -1.
\end{cases}
$$

*Proof.* For an arbitrary $x \in S$, inspecting the subgroups of $\mathrm{PSL}_2(p)$, we deduce that $\langle x, H \rangle \cong \mathrm{S}_3$, $\mathbb{Z}_6$ (if $\varepsilon = \eta$), $\mathbb{Z}_p{:}\mathbb{Z}_6$ (if $\varepsilon = \eta = -1$), $\mathrm{A}_4$, or $\mathrm{A}_5$ (if $p \equiv \pm 1 \,(\text{mod } 10)$). Let $\Delta_1 = \{X < \mathrm{PSL}_2(p) \mid H < X \cong \mathrm{S}_3\}$, $\Delta_2 = \{X < \mathrm{PSL}_2(p) \mid H < X \cong \mathbb{Z}_6\}$ when $\varepsilon = \eta$, $\Delta_3 = \{X < \mathrm{PSL}_2(p) \mid H < X \cong \mathbb{Z}_p{:}\mathbb{Z}_6\}$ when $\varepsilon = \eta = -1$, $\Delta_4 = \{X < \mathrm{PSL}_2(p) \mid H < X \cong \mathrm{A}_4\}$, and $\Delta_5 = \{X < \mathrm{PSL}_2(p) \mid H < X \cong \mathrm{A}_5\}$ when $p \equiv \pm 1 \,(\text{mod } 10)$. Then $x \in S$ if and only if $x$ is an involution contained in one member of $\Delta_i$ for some $i$.

By Lemma 5.1, $\mathrm{PSL}_2(p)$ contains exactly $\frac{p(p-\varepsilon)}{2}$ subgroups $\mathbb{Z}_3$, $\frac{p(p^2-1)}{12}$ subgroups $\mathrm{S}_3$, $\frac{p(p-\varepsilon)}{2}$ subgroups $\mathbb{Z}_6$, $p - \varepsilon$ subgroups $\mathbb{Z}_p{:}\mathbb{Z}_6$, $\frac{p(p^2-1)}{24}$ subgroups $\mathrm{A}_4$, and $\frac{p(p^2-1)}{60}$ subgroups $\mathrm{A}_5$. Note that $\mathrm{S}_3$, $\mathbb{Z}_6$, $\mathbb{Z}_p{:}\mathbb{Z}_6$, $\mathrm{A}_4$ and $\mathrm{A}_5$ contain exactly 1, 1, $p$, 4 and 10 subgroups $\mathbb{Z}_3$, respectively. Enumerating the pairs $(Y, X)$ with $\mathbb{Z}_3 \cong Y < X \cong \mathrm{S}_3$, $\mathbb{Z}_6$, $\mathbb{Z}_p{:}\mathbb{Z}_6$, $\mathrm{A}_4$ or $\mathrm{A}_5$, we have

$$
\frac{p(p-\varepsilon)}{2}|\Delta_i| = \begin{cases}
\frac{p(p^2-1)}{12}, & i = 1; \\
\frac{p(p-\varepsilon)}{2}, & i = 2, \varepsilon = \eta; \\
p(p-\varepsilon), & i = 3, \varepsilon = \eta = -1; \\
4\frac{p(p^2-1)}{24}, & i = 4; \\
10\frac{p(p^2-1)}{60}, & i = 5.
\end{cases}
$$

It follows that $|\Delta_1| = \frac{p+\varepsilon}{6}$, $|\Delta_2| = 1$ if $\varepsilon = \eta$, $|\Delta_3| = 2$ if $\varepsilon = \eta = -1$, $|\Delta_4| = \frac{p+\varepsilon}{3}$, and $|\Delta_5| = \frac{p+\varepsilon}{3}$ if $p \equiv \pm 1 \,(\text{mod } 10)$.

Let $S_i$ be the set of involutions contained in the members of $\Delta_i$, where $1 \leqslant i \leqslant 5$. Then $x \in S$ if and only if $x \in S_i$ for some $i$. Note that none of $\mathrm{S}_3$, $\mathrm{A}_4$ and $\mathrm{A}_5$ contains elements of order 6, and $\mathrm{A}_4$ has no subgroup isomorphic to $\mathrm{S}_3$. It is easily shown that the following hold: $|S_1| = \frac{p+\varepsilon}{2}$; $|S_2| = 1$ and $(S_1 \cup S_4 \cup S_5) \cap S_2 = \emptyset$ when $\varepsilon = \eta$; $(S_1 \cup S_4 \cup S_5) \cap S_3 = \emptyset$ when $\varepsilon = \eta = -1$; $|S_4| = p + \varepsilon$ and $S_1 \cap S_4 = \emptyset$. Moreover, for $\varepsilon = \eta = -1$, putting $\Delta_2 = \{X\}$ and $\Delta_3 = \{X_1, X_2\}$, it is easily shown that $X_1 \cap X_2 = X$, this implies that $S_2 \subset S_3$ and $|S_3| = 2p - 1$.

Assume first that $p \not\equiv \pm 1 \,(\text{mod } 10)$. If $\varepsilon = \eta = 1$ then $S = S_1 \cup S_2 \cup S_4$, and so $|S| = \frac{p+1}{2} + 1 + p + 1 = \frac{3p+3\varepsilon+|\varepsilon+\eta|}{2}$. If $\varepsilon \neq \eta$, i.e., $\varepsilon + \eta = 0$ then $S = S_1 \cup S_4$, and so $|S| = \frac{p+\varepsilon}{2} + p + \varepsilon = \frac{3p+3\varepsilon+|\varepsilon+\eta|}{2}$. If $\varepsilon = \eta = -1$ then $S = S_1 \cup S_3 \cup S_4$, and so $|S| = \frac{p-1}{2} + 2p - 1 + p - 1 = \frac{7p-5}{2}$.

Assume next that $p \equiv \pm 1 \,(\text{mod } 10)$. In this case, each subgroup of $\mathrm{PSL}_2(p)$ which is isomorphic to $\mathrm{S}_3$ or $\mathrm{A}_4$ is contained in a subgroup isomorphic to $\mathrm{A}_5$. It follows that each member of $\Delta_1 \cup \Delta_4$ is a subgroup of some member of $\Delta_5$. Then one of the following holds: $S = S_5$ if $\varepsilon \neq \eta$; $S = S_2 \cup S_5$ if $\varepsilon = \eta = 1$; $S = S_3 \cup S_5$ if $\varepsilon = \eta = -1$. For a given subgroup of order 3 in $\mathrm{A}_5$, it is easily checked that $\mathrm{A}_5$ contains exactly

one subgroup which is isomorphic to $S_3$ and contains the subgroup of order 3, and two subgroups which are isomorphic to $A_4$ and contain the subgroup of order 3. From this observation, we deduce that each member of $\Delta_5$ contributes $15 - 3 - 2 \cdot 3 = 6$ involutions to $S_5 \setminus (S_1 \cup S_4)$. Thus $|S_5 \setminus (S_1 \cup S_4)| = 6\frac{p+\varepsilon}{3} = 2(p+\varepsilon)$. If $\varepsilon \neq \eta$ then $\varepsilon + \eta = 0$, and $|S| = |S_5| = |S_5 \setminus (S_1 \cup S_4)| + |S_1| + |S_4| = 2(p+\varepsilon) + \frac{p+\varepsilon}{2} + p + \varepsilon = \frac{7p+7\varepsilon}{2} = \frac{7p+7\varepsilon+|\varepsilon+\eta|}{2}$. If $\varepsilon = \eta = 1$ then $|S| = |S_2| + |S_5| = 1 + \frac{7p+7\varepsilon}{2} = \frac{7p+7\varepsilon+|\varepsilon+\eta|}{2}$. If $\varepsilon = \eta = -1$ then $|S| = |S_3| + |S_5| = 2p - 1 + \frac{7p+7\varepsilon}{2} = \frac{11p-9}{2}$. This completes the proof. $\qquad\square$

It is easy to see that $|S| < \frac{p(p-\eta)}{2}$. We have $|O| = \frac{p(p-\eta)}{2} - |S| > 0$. Clearly, $O$ is invariant under the conjugation of $\mathbf{N}_{\mathrm{PGL}_2(p)}(H)$. Noting that $\mathbf{N}_{\mathrm{PGL}_2(p)}(H) \cong \mathrm{D}_{2(p+\varepsilon)}$, we write

$$\mathbf{N}_{\mathrm{PGL}_2(p)}(H) = \langle a, b \rangle,$$

where $a$ has order $p + \varepsilon$ and $b$ is an involution not contained in $T$. Then

$$H \leqslant \langle a^2 \rangle < \langle a \rangle, \ \mathbf{N}_T(H) = \langle a^2, ab \rangle.$$

**Lemma 5.4.** (1) *If $o \in O$ then $\mathbf{C}_{\mathrm{PGL}_2(p)}(o) \cap \langle a \rangle = 1$.*
(2) *If $Ho_1H = Ho_2H$ for $o_1, o_2 \in O$, then $o_1$ and $o_2$ are conjugate under $\langle a \rangle$.*

*Proof.* Assume that $o \in O$ and $y \in \mathbf{C}_{\mathrm{PGL}_2(p)}(o) \cap \langle a \rangle$. Then $\mathrm{PSL}_2(p) = \langle o, H \rangle \leqslant \mathbf{C}_{\mathrm{PGL}_2(p)}(y)$, forcing that $y = 1$. Thus (1) of the lemma follows.

Assume that $Ho_1H = Ho_2H$ for some $o_1, o_2 \in O$. Then $o_2 = xo_1y$ for some $x, y \in H$. If $xy = 1$ then $x = y^{-1}$, and (2) follows. Suppose that $yx \neq 1$, and so $H = \langle yx \rangle$. Since $o_2$ is an involution, we have $xo_1yxo_1y = o_2^2 = 1$, yielding $o_1yxo_1 = x^{-1}y^{-1} = (yx)^{-1}$. Then $T = \langle o_1, H \rangle = \langle o_1, yx \rangle \cong S_3$, a contradiction. This completes the proof. $\qquad\square$

By (1) of Lemma 5.4, if $o \in O$ then either $\mathbf{N}_{\mathrm{PGL}_2(p)}(H) \cap \mathbf{C}_{\mathrm{PGL}_2(p)}(o) = 1$ or $o \in \mathbf{C}_{\mathrm{PGL}_2(p)}(a^i b)$ for some integer $i$. For the latter case, $o \in \mathbf{C}_T(a^i b)$ as $o \in T$. Define

$$O_1 = \{o \in O \mid \exists i \text{ s.t. } o \in \mathbf{C}_T(a^{2i+1}b)\},$$
$$O_2 = \{o \in O \mid \exists i \text{ s.t. } o \in \mathbf{C}_T(a^{2i}b)\}.$$

Clearly, $O_1 \cap O_2 = \emptyset$.

**Lemma 5.5.**
$$|O_1| = \begin{cases} \frac{(p+\varepsilon)(p+\eta-2|\varepsilon+\eta|)}{4} & \text{if } p \not\equiv \pm 1 \,(\mathrm{mod}\ 10), \\ \frac{(p+\varepsilon)(p+\eta-2|\varepsilon+\eta|-8)}{4} & \text{if } p \equiv \pm 1 \,(\mathrm{mod}\ 10). \end{cases}$$

*Proof.* Let $x \in \mathbf{C}_T(a^{2i+1}b) \setminus \{a^{2i+1}b\}$ be an involution. Then $x \in O_1$ if and only if $\langle x, H \rangle = T$, or equivalently, $\langle x, H, a^{2i+1}b \rangle = T$. Note that $\langle H, a^{2i+1}b \rangle \cong S_3$. Suppose that $\langle x, H, a^{2i+1}b \rangle \neq T$. Inspecting the subgroups of $T$, we deduce that either $\langle x, H, a^{2i+1}b \rangle \leqslant \mathbf{N}_T(H)$, or $p \equiv \pm 1 \,(\mathrm{mod}\ 10)$ and $\langle x, H, a^{2i+1}b \rangle \cong A_5$. The former case implies that $x$ lies in the center of $\mathbf{N}_T(H)$, and then $\varepsilon = \eta$, $x = a^{\frac{p+\varepsilon}{2}}$ or $a^{\frac{p+\varepsilon}{2}}a^{2i+1}b$. Assume that the latter case occurs. Enumerating the subgroups $A_5$ which contain a given subgroup $S_3$, we deduce that $\langle H, a^{2i+1}b \rangle$ is contained exactly in two subgroups $A_5$. It follows that there exist exactly four choices of $x$ with $\langle x, H, a^{2i+1}b \rangle \cong A_5$. Thus

$$|\mathbf{C}_T(a^{2i+1}b) \cap O_1| = \begin{cases} \frac{p+\eta-2|\varepsilon+\eta|}{2} & \text{if } p \not\equiv \pm 1 \,(\mathrm{mod}\ 10), \\ \frac{p+\eta-2|\varepsilon+\eta|-8}{2} & \text{if } p \equiv \pm 1 \,(\mathrm{mod}\ 10). \end{cases}$$

Assume that $o \in \mathbf{C}_T(a^{2i+1}b) \cap \mathbf{C}_T(a^{2j+1}b) \cap O_1$. Then $o \in \mathbf{C}_T(a^{2(i-j)})$. If $a^{2(i-j)} \neq 1$ then $o \in \mathbf{C}_T(a^{2(i-j)}) = \mathbf{N}_T(H)$, which is impossible as $\langle o, H \rangle = T$. Thus $a^{2(i-j)} = 1$, and so $a^{2i+1}b = a^{2j+1}b$. This says that every $o \in O_1$ centralizes exactly one of $\frac{p+\varepsilon}{2}$ involutions $a^{2i+1}b$. Then $|O_1|$ is desired as in the lemma. $\qquad\square$

**Lemma 5.6.** $|O_2| = \frac{(p+\varepsilon)(p-\eta-6)}{4}$.

*Proof.* Let $x \in \mathbf{C}_T(a^{2i}b)$ be an involution. Then $x \in O_2$ if and only if $\langle x, H \rangle = T$, or equivalently, $\langle x, H, a^{2i}b \rangle = \mathrm{PGL}_2(p)$. Note that $\langle H, a^{2i}b \rangle \cong \mathrm{S}_3$. Suppose that $\langle x, H, a^{2i}b \rangle \neq \mathrm{PGL}_2(p)$. Inspecting the subgroups of $\mathrm{PGL}_2(p)$, either $\langle x, H, a^{2i}b \rangle \leqslant \mathbf{N}_{\mathrm{PGL}_2(p)}(H)$, or $\langle x, H, a^{2i}b \rangle \cong \mathrm{S}_4$. The former case implies that either $\varepsilon = \eta$ and $x = a^{\frac{p+\varepsilon}{2}}$, or $\varepsilon \neq \eta$ and $x = a^{\frac{p+\varepsilon}{2}}a^{2i}b$. For $\langle x, H, a^{2i}b \rangle \cong \mathrm{S}_4$, enumerating the subgroups $\mathrm{S}_4$ which contain a given subgroup $\mathrm{S}_3$, we deduce that $\langle H, a^{2i}b \rangle$ is contained exactly in two subgroups $\mathrm{S}_4$. Noting that $\langle x, H, a^{2i}b \rangle \cap T \cong \mathrm{A}_4$, it follows that there exist exactly two choices of $x$ with $\langle x, H, a^{2i}b \rangle \cong \mathrm{S}_4$. Since $\mathbf{C}_T(a^{2i}b) \cong \mathrm{D}_{p-\eta}$, we have $|\mathbf{C}_T(a^{2i}b) \cap O_2| = \frac{p-\eta-6}{2}$. Similarly as in the proof of Lemma 5.5, it is easily shown that every $o \in O_2$ centralizes exactly one of $\frac{p+\varepsilon}{2}$ involutions $a^{2i}b$. Then $|O_2|$ is desired as in the lemma. $\qquad\square$

It is easy to check that $|O_1| + |O_2| = \frac{p(p-\eta)}{2} - |S| = |O|$, and so $O = O_1 \cup O_2$. Clearly, $O_1$ and $O_2$ are invariant under the conjugation of $\langle a \rangle$, and so each of them is the union of some $\langle a \rangle$-conjugacy classes. Selecting a representative $o$ from each $\langle a \rangle$-conjugacy class in $O$ such that $\mathbf{N}_{\mathrm{PGL}_2(p)}(H) \cap \mathbf{C}_{\mathrm{PGL}_2(p)}(o) = \langle ab \rangle$ or $\langle b \rangle$, we have a set $O_0$ of $\omega_0$ involutions, where

$$\omega_0 = \begin{cases} \frac{p-|\varepsilon+\eta|-3}{2} & \text{if } p \not\equiv \pm 1 \,(\mathrm{mod}\ 10), \\ \frac{p-|\varepsilon+\eta|-7}{2} & \text{if } p \equiv \pm 1 \,(\mathrm{mod}\ 10). \end{cases}$$

Then $O_0$ consists of $\omega_0 - \frac{p-\eta-6}{4}$ involutions from $O_1$, and $\frac{p-\eta-6}{4}$ involutions from $O_2$.

**Theorem 5.7.** *Assume that $H \cong \mathbb{Z}_3$. Then $\Gamma$ is isomorphic to one of $\omega_0$ non-isomorphic symmetric cubic graphs, $\frac{p-\eta-6}{4}$ of them have automorphism group $T\langle \mathrm{conj}(b)_H \rangle \cong \mathrm{PGL}_2(p)$, and the others have automorphism group $\langle \hat{ab} \rangle \times T$.*

*Proof.* By the foregoing argument, $\Gamma \cong \mathsf{Cos}(T, H, 1, o)$ for some $o \in O_0$.

Let $o \in O_0$. Then $\mathrm{Aut}\,\mathsf{Cos}(T, H, 1, o) \geqslant \langle \hat{ab} \rangle \times T$ or $T\langle \mathrm{conj}(b)_H \rangle$ depending on $o \in O_1$ or $o \in O_2$, respectively. Pick an arbitrary element $z \in \mathbf{N}_{\mathrm{PGL}_2(p)}(H) \backslash H$ with $Hz^{-1}ozH = HoH$. We have $z^{-1}oz = xoy$ for some $x, y \in H$, and so $xoyxoy = 1$, yielding $oyxo = (yx)^{-1}$. If $yx \neq 1$ then $T = \langle o, H \rangle = \langle o, yx \rangle \cong \mathrm{S}_3$, a contradiction. Then $yx = 1$, i.e, $y = x^{-1}$. Thus $z^{-1}oz = xoy = xox^{-1}$, and so $(zx)^{-1}ozx = o$. By the choice of $O_0$, we have $\langle zx \rangle = \mathbf{N}_{\mathrm{PGL}_2(p)}(H) \cap \mathbf{C}_{\mathrm{PGL}_2(p)}(o) = \langle ab \rangle$ or $\langle b \rangle$. It follows that $\mathbf{N}_{\mathrm{PGL}_2(p)}(H, HoH) = H\langle ab \rangle$ or $H\langle b \rangle$. Thus, by (5.2), $\mathrm{Aut}\,\mathsf{Cos}(T, H, 1, o) = \langle \hat{ab} \rangle \times T$ or $T\langle \mathrm{conj}(b)_H \rangle$.

By Lemma 5.4 and the choice of $O_0$, distinct elements in $O_0$ produce distinct coset graphs $\mathsf{Cos}(T, H, 1, o)$. Then, by Lemma 3.3, we have $\omega_0$ non-isomorphic symmetric cubic graphs $\mathsf{Cos}(T, H, 1, o)$. This completes the proof. $\qquad\square$

**5.2.** $|H| = 6$. Assume that $H \cong \mathrm{S}_3$. Then $p \equiv \pm 7 \,(\mathrm{mod}\ 16)$ by (5.1), $K \cong \mathbb{Z}_2$, and $o \in \mathbf{N}_T(K) = \mathbf{C}_T(K) \cong \mathrm{D}_{p+\eta}$. Since $o^2 \in K$, either $o$ is an involution or $o$ has order 4. Let

$$O = \{o \in \mathbf{C}_T(K) \mid o^2 \in K, \langle o, H \rangle = T\}.$$

**Lemma 5.8.** *$O$ contains two inverse elements of order $4$ and $|O| - 2$ involutions, and*

$$|O| = \begin{cases} \frac{p+\eta-2|\varepsilon+\eta|}{2} - 2 & \text{if } p \not\equiv \pm 1 \,(\mathrm{mod}\ 10), \\ \frac{p+\eta-2|\varepsilon+\eta|}{2} - 6 & \text{if } p \equiv \pm 1 \,(\mathrm{mod}\ 10). \end{cases}$$

*Proof.* Let $S = \{x \in \mathbf{C}_T(K) \setminus K \mid x^2 \in K, \langle o, H \rangle \neq T\}$. Then $|S| + |O| = \frac{p+\eta+4}{2}$, and $S \cup O$ consists of two inverse elements of order $4$ and $\frac{p+\eta}{2}$ involutions in $\mathbf{C}_T(K) \setminus K$.

Let $x \in S$. Then $\langle x, H \rangle \cong \mathrm{D}_m$, $\mathrm{S}_4$, or $\mathrm{A}_5$ (if $p \equiv \pm 1 \,(\mathrm{mod}\ 10)$), where $m > 6$ is a divisor of $p + \varepsilon$ and divisible by $6$. By the choice of $x$ and inspecting the elements of $\mathrm{D}_m$, $\mathrm{S}_4$ and $\mathrm{A}_5$, we deduce that $x$ is an involution. By Lemma 5.1, all subgroups $\mathrm{S}_3$ of $T$ are conjugate in $\mathrm{PGL}_2(p)$. Enumerating the maximal subgroups of $T$ which contain $H$, we deduce that $H$ is contained exactly in one subgroup $\mathrm{D}_{p+\varepsilon}$, two subgroups $\mathrm{S}_4$, and two subgroups $\mathrm{A}_5$ if $p \equiv \pm 1 \,(\mathrm{mod}\ 10)$. Let $L$ be a maximal subgroup of $T$ with $\langle x, H \rangle \leqslant L$. If $L \cong \mathrm{D}_{p+\varepsilon}$ then $|S \cap L| = |\varepsilon + \eta|$. If $L \cong \mathrm{S}_4$ or $\mathrm{A}_5$ then $|S \cap L| = 2$. We deduce that $|S| = |\varepsilon + \eta| + 8$ if $p \equiv \pm 1 \,(\mathrm{mod}\ 10)$, or $|S| = |\varepsilon + \eta| + 4$ otherwise. Then $|O|$ is given as in this lemma. Clearly, $S$ consists of involutions. Then the lemma follows. $\square$

Note that $Ko \subseteq O$ for $o \in O$. It follows that $O$ is the union of $\frac{|O|}{2}$ cosets of $K$.

**Lemma 5.9.** *Let $o, o' \in O$. Then $Ho'H = HoH$ if and only of $Ko = Ko'$.*

*Proof.* Clearly, if $Ko' = Ko$ then $Ho'H = HoH$. Conversely, suppose that $Ho'H = HoH$ for distinct $o, o' \in O$. If $o$ and $o'$ are of order $4$ then $K = \langle o^2 \rangle$ and $o' \in \{o, o^{-1}\}$, we have $Ko = Ko'$. Thus, without loss of generality, we assume that $o$ is an involution. Write $o = xo'y$ for some $x, y \in H$. Then $xo'yxo'y = o^2 = 1$, yielding $o'yxo' = (yx)^{-1}$.

If $yx$ has order $3$, then $o' \in \mathbf{N}_T(\langle yx \rangle) = \mathbf{N}_T(H)$, which contradicts that $\langle o', H \rangle = T$. Assume that $yx = 1$. Then $1 \neq y \notin K$, and $o = y^{-1}o'y \in \mathbf{C}_T(K) \cap \mathbf{C}_T(y^{-1}Ky)$. This implies that $o$ centralizes $\langle K, y^{-1}Ky \rangle = H$. We have $\langle o, H \rangle \neq T$, a contradiction. Thus $yx \neq 1$. It follows that $yx$ is an involution, and so $o' \in \mathbf{C}_T(yx)$. In addition, $yx \in K$ since, otherwise, $o'$ centralizes $\langle K, yx \rangle = H$, which will give a contradiction.

Now we have $K = \langle yx \rangle$. Then $o = xo'y = y^{-1}(yx)o'y \in \mathbf{C}_T(K) \cap \mathbf{C}_T(y^{-1}Ky)$, and so $o$ centralizes $\langle K, y^{-1}Ky \rangle$. If $y \notin K$ then $\langle K, y^{-1}Ky \rangle = H$, and so $o$ centralizes $T = \langle o, H \rangle$, a contradiction. Then $y \in K$, and $x \in K$. Thus $o = xo'y = yxo' \in Ko'$, yielding $Ko = Ko'$. This completes the proof. $\square$

Note that $\mathbf{N}_{\mathrm{PGL}_2(p)}(H) \cong \mathrm{D}_{12}$, which has center of order $2$. Let $c$ be the involution in the center of $\mathbf{N}_{\mathrm{PGL}_2(p)}(H)$. Clearly, $o \in \mathbf{C}_{\mathrm{PGL}_2(p)}(K)$. Then $\mathbf{N}_{\mathrm{PGL}_2(p)}(H, K) = \langle c \rangle \times K$, and $c \in T$ if and only if $\varepsilon = \eta$. Consider the conjugation of $\langle c \rangle$ on $\Omega := \{Ko \mid o \in O\}$.

**Lemma 5.10.** *The action of $\langle c \rangle$ on $\Omega$ produces $\frac{2+|\varepsilon+\eta|}{2}$ orbits of size $1$, and $\frac{|O|-|\varepsilon+\eta|-2}{4}$ orbits of size $2$.*

*Proof.* Pick an element $o_0 \in O$ of order $4$. Then $co_0c = o_0^{-1}$, $c$ fixes $Ko_0$, and $\langle o_0, c \rangle \cong \mathrm{D}_8$. It is easily shown that $\langle o_0, c \rangle \cap O = \{o_0, o_0^{-1}, o_0c, o_0^{-1}c\}$ or $\{o_0, o_0^{-1}\}$ depending on whether $\varepsilon = \eta$ or not. Note that $Ko_0 = Ko_0^{-1}$ and $Ko_0c = Ko_0^{-1}c$. It follows $\langle o_0, c \rangle$ contributes $\frac{2+|\varepsilon+\eta|}{2}$ fixed-points of $\langle c \rangle$ on $\Omega$.

Now assume that $Ko$ is fixed by $\langle c \rangle$, where $o \in O$. Then $Kcoc = Ko = Ko^{-1}$, yielding $coco \in K$, and so $co$ has order $2$ or $4$. Recall that $c, o \in \mathbf{C}_{\mathrm{PGL}_2(p)}(K) \setminus K$ and $\mathbf{C}_{\mathrm{PGL}_2(p)}(K) \cong \mathrm{D}_{2(p+\eta)}$. If $co$ has order $4$ then $co \in \{o_0, o_0^{-1}\}$, and so $o \in \langle c, o_0 \rangle$. Assume

that $co$ is an involution. Then either $co$ or $o = cco$ is contained in the cyclic subgroup of $\mathbf{C}_{\mathrm{PGL}_2(p)}(K)$ of index 2. This implies that either $co$ or $o$ lies in $\langle o_0 \rangle$, and hence $o \in \langle c, o_0 \rangle$. Therefore, $\langle c \rangle$ has exactly $\frac{2+|\varepsilon+\eta|}{2}$ fixed-points on $\Omega$. Since $\langle c \rangle \cong \mathbb{Z}_2$, every $\langle c \rangle$-orbit on $\Omega$ has length 1 or 2. Then the lemma follows. $\qquad\square$

Choosing a coset $Ko$ from each $\langle c \rangle$-orbit on $\Omega$ and a representative from $Ko$, we have a set $O_1$ of size

$$\omega_1 = \begin{cases} \frac{p+\eta}{8} & \text{if } p \not\equiv \pm 1 \,(\mathrm{mod}\ 10), \\ \frac{p+\eta}{8} - 1 & \text{if } p \equiv \pm 1 \,(\mathrm{mod}\ 10). \end{cases}$$

By the foregoing argument, the following statements hold:

(i) $\Gamma \cong \mathsf{Cos}(T, H, K, o)$ for some $o \in O_1$, and $HoH \neq Ho'H$ for distinct $o, o' \in O_1$;

(ii) $O_1$ contains a unique element of order 4, say $o_0$, and $\mathbf{N}_{\mathrm{PGL}_2(p)}(H, Ho_0H) \geqslant \langle c \rangle \times K = \mathbf{N}_{\mathrm{PGL}_2(p)}(H, K)$;

(iii) if $o \in O_1$ is an involution then $\mathbf{N}_{\mathrm{PGL}_2(p)}(H, K, HoH) = K$, except that $\varepsilon = \eta$, $Ko = Ko_0 c$, and $\mathbf{N}_{\mathrm{PGL}_2(p)}(H, Ho_0H) \geqslant \langle c \rangle \times K = \mathbf{N}_{\mathrm{PGL}_2(p)}(H, K)$.

**Lemma 5.11.** *Let $o \in O_1$. Then $\mathbf{N}_{\mathrm{PGL}_2(p)}(H, HoH) = K$, except that*

(1) $o = o_0$, *in this case,* $\mathbf{N}_{\mathrm{PGL}_2(p)}(H, HoH) = K \times \langle c \rangle$; *and*

(2) $\eta = \varepsilon$ *and* $Ko = Ko_0 c$, *in this case,* $\mathbf{N}_{\mathrm{PGL}_2(p)}(H, HoH) = K \times \langle c \rangle$.

*Proof.* Let $g$ be an arbitrary element in $\mathbf{N}_{\mathrm{PGL}_2(p)}(H, HoH) \setminus H$. Noting that $Hg^{-1}ogH = HoH$, by Lemma 5.9, $Ko = Kg^{-1}og$. Then $\langle Kg^{-1}og \rangle = \langle Ko \rangle = \langle o \rangle \times K$. This implies that $g^{-1}og \in \mathbf{C}_T(K)$, and so $o \in \mathbf{C}_T(gKg^{-1})$. Then $o$ centralizes $\langle K, gKg^{-1} \rangle$. Since $\langle o, H \rangle = T$ and $\langle K, gKg^{-1} \rangle \leqslant H$, we have $K = gKg^{-1}$, i.e., $g \in \mathbf{N}_{\mathrm{PGL}_2(p)}(K)$. Thus $g \in \mathbf{N}_{\mathrm{PGL}_2(p)}(H, K, HoH)$. Then the lemma follows from (ii) and (iii) listed as above. $\quad\square$

**Theorem 5.12.** *Assume that $H \cong \mathrm{S}_3$. Then $\Gamma$ is isomorphic to one of $\omega_1$ non-isomorphic symmetric cubic graphs, and $\mathrm{Aut}\Gamma = \mathrm{PSL}_2(p)$ except that*

(1) $\Gamma \cong \mathsf{Cos}(T, H, K, o_0)$, *and $\mathrm{Aut}\Gamma = \mathbb{Z}_2 \times \mathrm{PSL}_2(p)$ or $\mathrm{PGL}_2(p)$ depending on whether $\eta = \varepsilon$ or not; and*

(2) $\eta = \varepsilon$, $\Gamma \cong \mathsf{Cos}(T, H, K, o_0 c)$, *and $\mathrm{Aut}\Gamma = \mathbb{Z}_2 \times \mathrm{PSL}_2(p)$.*

*Proof.* Recall that $\Gamma \cong \mathsf{Cos}(T, H, K, o)$ for some $o \in O_1$. By (5.2) and Lemma 5.11, we deduce that $\mathrm{Aut}\Gamma$ is described as in this lemma. Then it suffices to show that if $\mathsf{Cos}(T, H, K, o) \cong \mathsf{Cos}(T, H, K, o')$ for $o, o' \in O_1$ then $o = o'$.

Suppose that $\mathsf{Cos}(T, H, K, o) \cong \mathsf{Cos}(T, H, K, o')$ for some $o, o' \in O_1$. By Lemma 5.11, we deduce from (5.2) that $A := \mathrm{Aut}\mathsf{Cos}(T, H, K, o) = \mathrm{Aut}\mathsf{Cos}(T, H, K, o')$. It follows from Lemma 3.3 that $Hg^{-1}ogH = Ho'H$ for some $g \in \mathbf{N}_{\mathrm{PGL}_2(p)}(H)$. By Lemma 5.9, $Kg^{-1}og = Ko'$, which forces that $g^{-1}og$ centralizes $K$. Then $o$ centralizes $\langle K, gKg^{-1} \rangle$. Noting that $\langle K, gKg^{-1} \rangle \leqslant H$ and $\langle o, H \rangle = T$, we have $K = gKg^{-1}$, and so $g \in \mathbf{N}_{\mathrm{PGL}_2(p)}(H, K)$. By the choice of $O_1$, we have $o = o'$, and the result follows. $\quad\square$

**5.3.** $|H| = 12$. Assume that $H \cong \mathrm{D}_{12}$. Then $p \equiv \pm 15 \,(\mathrm{mod}\ 32)$ by (5.1), and $\varepsilon = \eta$. This implies that $p \equiv \pm 47 \,(\mathrm{mod}\ 96)$. Since $K \cong \mathbb{Z}_2^2$, by [17, II.8.16], $\mathbf{N}_T(K) \cong \mathrm{S}_4$, and thus $o$ is either an involution or of order 4. Clearly, $o$ lies in some Sylow 2-subgroup of $\mathbf{N}_T(K)$.

**Theorem 5.13.** *Assume that $H \cong \mathrm{D}_{12}$. Then $\Gamma$ is isomorphic to a unique symmetric cubic graph, which has automorphism group $\mathrm{PSL}_2(p)$.*

*Proof.* By the choice of $\eta$, we know that $p + \eta$ is divisible by 4, and so $p - \eta$ is indivisible by 4. Noting that $(p + \eta)(p - \eta) = p^2 - 1 \equiv 0 \,(\mathrm{mod}\,32)$, we have $p \equiv -\eta \,(\mathrm{mod}\,16)$. Thus $p + \varepsilon = p + \eta$ is divisible by 16. We have $\mathbf{N}_T(H) \cong \mathrm{D}_{24}$ and $\mathbf{N}_T(H, K) \cong \mathrm{D}_8$. Let $P := \mathbf{N}_T(H, K)$, $P_0$ and $P_1$ be the three Sylow 2-subgroups of $\mathbf{N}_T(K)$. It is easily shown that there exists an involution $x \in P \setminus K$ such that $xP_0x = P_1$. Pick an involution $o_0 \in P_0 \setminus K$. Suppose that $\langle o_0, H \rangle \neq T$. Inspecting the subgroups of $\mathrm{PSL}_2(p)$, we deduce that $\langle o_0, H \rangle \leqslant \mathbf{N}_T(H)$. Then $o_0 \in \mathbf{N}_T(H, K) = P$, and so $P_0 = \langle o_0, K \rangle \leqslant P$, a contradiction. Thus $\langle o_0, H \rangle = T$. Recalling that $o \in P \cup P_0 \cup P_1$, since $\langle o, H \rangle = T$, we have $o \in P_0 \cup P_1$. Then $HoH = Ho_0H$ or $Hxo_0xH$. Since $x \in \mathbf{N}_T(H)$, we have $\mathsf{Cos}(T, H, K, o_0) \cong \mathsf{Cos}(T, H, K, xo_0x)$, and so $\Gamma \cong \Sigma := \mathsf{Cos}(T, H, K, o_0)$.

Choose a maximal subgroup $L$ of $\mathrm{PGL}_2(p)$ with $\mathbf{N}_{\mathrm{PGL}_2(p)}(H) \leqslant L$. Then $L \cong \mathrm{D}_{2(p+\varepsilon)}$, and $\mathbf{N}_{\mathrm{PGL}_2(p)}(H) = \mathbf{N}_L(H) \cong \mathrm{D}_{24}$. Recalling that $\mathbf{N}_T(H) \cong \mathrm{D}_{24}$, we have $\mathbf{N}_{\mathrm{PGL}_2(p)}(H) = \mathbf{N}_T(H)$. Then $\mathbf{N}_{\mathrm{PGL}_2(p)}(H) = HP = H\langle x \rangle$. By (5.2), we deduce that $\mathrm{Aut}\Sigma = T\langle \mathsf{conj}(x) \rangle$ or $T$ depending on whether $Hxo_0xH = Ho_0H$ or not.

Suppose that $\mathrm{Aut}\Sigma = T\langle \mathsf{conj}(x) \rangle$. Then $\mathrm{Aut}\Sigma = T \times \langle \hat{x} \rangle$, where $\hat{x}$ is defined as in (3.3). Let $M = \langle \hat{x} \rangle$, and consider the quotient graph $\Sigma_M$. Let $\bar{T}$ be the subgroup of $\mathrm{Aut}\Sigma_M$ induced by $T$. Then $\Sigma_M$ is a $\bar{T}$-symmetric cubic graph of square-free order $n$. Let $\bar{v}$ be the $M$-orbit on $[T : H]$ containing $v := H$. We have $n = |\bar{T} : \bar{T}_{\bar{v}}|$. Since $\bar{T} \cong \mathrm{PSL}_2(p)$ has order divisible by 16, it follows that $|\bar{T}_{\bar{v}}|$ is divisible by 8. By Lemma 2.1, $\bar{T}_{\bar{v}} \cong \mathrm{S}_4$, and so $T_{\bar{v}} \cong \mathrm{S}_4$ by (2.1). By (2.2), $T_v$ has index 2 in $T_{\bar{v}}$, forcing $T_v \cong \mathrm{A}_4$, which is impossible as $\Sigma$ is $T$-symmetric. Therefore, $\mathrm{Aut}\Sigma = T$, and our result follows. $\qquad\square$

5.4. $|H| = 24$. Assume that $H \cong \mathrm{S}_4$. Then $p \equiv \pm 31 \,(\mathrm{mod}\,64)$ by (5.1). In this case, $H$ is maximal in $T$, $K \cong \mathrm{D}_8$ and $\mathbf{N}_G(K) \cong \mathrm{D}_{16}$. Fix an involution $o_0 \in \mathbf{N}_G(K) \setminus K$. We have $\langle H, o_0 \rangle = T$, and $H\mathbf{N}_G(K)H = H \cup Ho_0H$. Then $\Gamma \cong \mathsf{Cos}(T, H, K, o_0)$. Checking the subgroups of $\mathrm{PGL}_2(p)$, we deduce that $\mathbf{N}_{\mathrm{PGL}_2(p)}(H) = H$, and so $\mathbf{N}_{\mathrm{PGL}_2(p)}(H, Ho_0H) = \mathbf{N}_T(H, Ho_0H) = H$. Then we have the following result.

**Theorem 5.14.** *Assume that $H \cong \mathrm{S}_4$. Then $\Gamma$ is isomorphic to a unique symmetric cubic graph, which has automorphism group $\mathrm{PSL}_2(p)$.*


# 6. $\mathrm{PSL}_2(p)$-SEMISYMMETRIC GRAPHS

In this section, $\Gamma = (V, E)$ is a connected $T$-semisymmetric cubic graph of order $2n$, where $T = \mathrm{PSL}_2(p)$ for some prime $p \geqslant 5$, and $n$ is even and square-free. Choose $\varepsilon, \eta \in \{1, -1\}$ with $p + \varepsilon$ and $p + \eta$ divisible by 3 and 4, respectively.

Let $\{u, w\} \in E$. By Lemma 2.1 and inspecting the subgroups of $\mathrm{PSL}_2(p)$, we may assume that $(T_u, T_w) \cong (\mathrm{S}_3, \mathrm{S}_3)$, $(\mathrm{D}_{12}, \mathrm{D}_{12})$, $(\mathrm{S}_4, \mathrm{S}_4)$, $(\mathrm{S}_3, \mathbb{Z}_6)$, $(\mathrm{D}_{12}, \mathrm{A}_4)$ or $(\mathrm{S}_4, \mathrm{D}_{24})$. By Lemma 3.10, $\Gamma \cong \mathsf{BC}(T, L, R)$, where $L \cong T_u$ and $R \cong T_w$. Note that $|T : L| = n$ is even and square-free. We have

(6.1) $\qquad p \equiv 2^{i+1} \pm 1 \,(\mathrm{mod}\,2^{i+2})$ and $|L| = 2^i \cdot 3$ for $1 \leqslant i \leqslant 3$.

In addition, $\eta = \varepsilon$ if $L$ or $R$ has a subgroup isomorphic to $\mathbb{Z}_6$.

It follows from Lemma 5.2 that $T$ contains at most two conjugacy classes of subgroup isomorphic to $L$, and these subgroups are conjugate in $\mathrm{PGL}_2(p)$. Then, up to isomorphism of graphs, we may fix a subgroup $L$. Note that $L \cap R$ is a Sylow 2-subgroup of

$L$, and $\mathsf{BC}(T, L, R) \cong \mathsf{BC}(T, L, h^{-1}Rh)$ for $h \in L$. Thus, fixing a Sylow 2-subgroup $P$ of $L$, one of our main tasks is to determine those subgroups $R$ with $|R| = |L|$, $L \cap R = P$ and $\langle L, R \rangle = T$. Put

$$\mathcal{R} = \{R < T \mid |R| = |L|, L \cap R = P\}.$$

**Lemma 6.1.** *Let $L \cong R < T$. Then $R \in \mathcal{R}$ if and only if $R = z^{-1}Lz$ for some $z \in \mathbf{N}_{\mathrm{PGL}_2(p)}(P) \setminus \mathbf{N}_{\mathrm{PGL}_2(p)}(L, P)$.*

*Proof.* The sufficiency is trivial. Now assume that $L \cong R \in \mathcal{R}$. By Lemma 5.2, $L$ and $R$ are conjugate in $\mathrm{PGL}_2(p)$. Then $R = x^{-1}Lx$ for some $x \in \mathrm{PGL}_2(p)$. We have $P, xPx^{-1} \leqslant L$, and so $xPx^{-1} = y^{-1}Py$ for some $y \in L$. Then $yx \in \mathbf{N}_{\mathrm{PGL}_2(p)}(P)$, and so $x = y^{-1}z$ for some $z \in \mathbf{N}_{\mathrm{PGL}_2(p)}(P)$. Thus $R = x^{-1}Lx = z^{-1}Lz$. Since $L \cap R = P \neq L$, we know that $L$ is not normalized by $z$, and so $z \in \mathbf{N}_{\mathrm{PGL}_2(p)}(P) \setminus \mathbf{N}_{\mathrm{PGL}_2(p)}(L, P)$. Then the lemma follows. $\square$

**6.1.** $|L| = 6$. Assume that $L \cong \mathrm{S}_3$. Then $p \equiv \pm 3 \,(\mathrm{mod}\ 8)$ by (6.1), $\mathbf{N}_{\mathrm{PGL}_2(p)}(L) \cong \mathrm{D}_{12}$, $P \cong \mathbb{Z}_2$ and $\mathbf{N}_{\mathrm{PGL}_2(p)}(P) = \mathbf{C}_{\mathrm{PGL}_2(p)}(P) \cong \mathrm{D}_{2(p+\eta)}$. Clearly, the center of $\mathbf{N}_{\mathrm{PGL}_2(p)}(L)$ has order 2 and is contained in $\mathbf{C}_{\mathrm{PGL}_2(p)}(P)$. Write

$$\mathbf{C}_{\mathrm{PGL}_2(p)}(P) = \langle a, c \rangle,$$

where $a$ has order $p + \eta$ and $c$ generates the center of $\mathbf{N}_{\mathrm{PGL}_2(p)}(L)$. Then

$$P = \langle a^{\frac{p+\eta}{2}} \rangle, \quad \mathbf{N}_{\mathrm{PGL}_2(p)}(L, P) = \langle c, a^{\frac{p+\eta}{2}} \rangle \cong \mathbb{Z}_2^2.$$

In addition, $c \in T$ if and only if $\varepsilon = \eta$.

**Lemma 6.2.** *If $\varepsilon \neq \eta$ then $\mathcal{R} = \{a^{-i}La^i \mid 1 \leqslant i < \frac{p+\eta}{2}\}$, if $\varepsilon = \eta$ then $\mathcal{R} = \{\langle a^{\frac{p+\eta}{6}} \rangle\} \cup \{a^{-i}La^i \mid 1 \leqslant i < \frac{p+\eta}{2}\}$.*

*Proof.* Recalling that $P = \langle a^{\frac{p+\eta}{2}} \rangle$, we have $P < a^{-i}La^i$ for an arbitrary integer $i$. If $i \equiv j \,(\mathrm{mod}\ \frac{p+\eta}{2})$ then it is easily shown that $a^{-i}La^i = a^{-j}La^j$. Conversely, suppose that $a^{-i}La^i = a^{-j}La^j$ for some integers $i$ and $j$. Then $a^{i-j} \in \mathbf{N}_{\mathrm{PGL}_2(p)}(L) \cap \mathbf{N}_{\mathrm{PGL}_2(p)}(P) = \mathbf{N}_{\mathrm{PGL}_2(p)}(L, P) = \langle c, P \rangle$. This implies that $a^{i-j} \in P$, and so $i \equiv j \,(\mathrm{mod}\ \frac{p+\eta}{2})$. By Lemma 6.1, all members $\mathrm{S}_3$ of $\mathcal{R}$ are contained in $\{a^{-i}La^i \mid 1 \leqslant i < \frac{p+\eta}{2}\}$.

Assume that $R \in \mathcal{R}$ and $R \not\cong \mathrm{S}_3$. Then $R \cong \mathbb{Z}_6$, and so $R < \mathbf{C}_{\mathrm{PGL}_2(p)}(P) = \langle a, c \rangle \cong \mathrm{D}_{2(p+\eta)}$. In particular, $p + \eta$ is divisible by 3, and so $\varepsilon = \eta$. Note that $\mathrm{D}_{2(p+\eta)}$ has a unique subgroup $\mathbb{Z}_6$, which is generated by $a^{\frac{p+\eta}{6}}$. Then the lemma follows. $\square$

**Lemma 6.3.** *Let $R_i = a^{-i}La^i$ for $1 \leqslant i < \frac{p+\eta}{2}$, and $R_0 = \langle a^{\frac{p+\eta}{6}} \rangle$ if further $\varepsilon = \eta$. Then*

(1) $\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R_0\}) = \mathbf{N}_{\mathrm{PGL}_2(p)}(L, R_0) = \langle a^{\frac{p+\eta}{2}}, c \rangle < T$, *in this case, $\varepsilon = \eta$;*

(2) $\mathbf{N}_{\mathrm{PGL}_2(p)}(L, R_i) = P$ *and* $\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R_i\}) = \langle a^{\frac{p+\eta}{2}}, a^i c \rangle$, *where $i \neq \frac{p+\eta}{4}$ and $1 \leqslant i < \frac{p+\eta}{2}$.*

(3) $\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R_{\frac{p+\eta}{4}}\}) = \langle a^{\frac{p+\eta}{4}}, c \rangle$, *and* $\mathbf{N}_{\mathrm{PGL}_2(p)}(L, R_{\frac{p+\eta}{4}}) = \langle a^{\frac{p+\eta}{2}}, c \rangle$.

*Proof.* Clearly, $|\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R\}) : \mathbf{N}_{\mathrm{PGL}_2(p)}(L, R)| \leqslant 2$, and if the equality holds then $R \cong \mathrm{S}_3$. In particular, since $L \not\cong R_0$, we have $\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R_0\}) = \mathbf{N}_{\mathrm{PGL}_2(p)}(L, R_0)$. Recall that $\mathbf{N}_{\mathrm{PGL}_2(p)}(L) = L \times \langle c \rangle$. If $\varepsilon = \eta$ then $c \in T$ and, noting that $\mathbf{N}_{\mathrm{PGL}_2(p)}(R_0) = \mathbf{C}_{\mathrm{PGL}_2(p)}(P)$, we have $\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R_0\}) = \mathbf{N}_{\mathrm{PGL}_2(p)}(L, R_0) = \langle a^{\frac{p+\eta}{2}}, c \rangle$, desired as in (1).

Now let $R = R_i$, where $1 \leqslant i < \frac{p+\eta}{2}$. Note that $P \leqslant \mathbf{N}_{\mathrm{PGL}_2(p)}(L, R) \leqslant \mathbf{N}_{\mathrm{PGL}_2(p)}(L, P) = \langle a^{\frac{p+\eta}{2}}, c \rangle \cong \mathbb{Z}_2^2$. If $R = R_{\frac{p+\eta}{4}}$ then $cRc = ca^{-\frac{p+\eta}{4}}La^{\frac{p+\eta}{4}}c = a^{\frac{p+\eta}{4}}La^{-\frac{p+\eta}{4}} = a^{-\frac{p+\eta}{4}}La^{\frac{p+\eta}{4}} = R$, and so $\mathbf{N}_{\mathrm{PGL}_2(p)}(L, R) = \langle a^{\frac{p+\eta}{2}}, c \rangle$. Suppose that $\mathbf{N}_{\mathrm{PGL}_2(p)}(L, R) = \langle a^{\frac{p+\eta}{2}}, c \rangle$. Then $a^{-i}La^i = R = cRc = ca^{-i}La^ic = a^iLa^{-i}$, and so $a^{-2i}La^{2i} = L$. This implies that $2i \equiv 0 \,(\mathrm{mod} \, \frac{p+\eta}{2})$, yielding $i = \frac{p+\eta}{4}$. Thus $\mathbf{N}_{\mathrm{PGL}_2(p)}(L, R) = \langle a^{\frac{p+\eta}{2}}, c \rangle$ if and only if $R = R_{\frac{p+\eta}{4}}$. Noting that $\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R\}) = \mathbf{N}_{\mathrm{PGL}_2(p)}(L, R)\langle a^i c \rangle$, we obtain (2) or (3). Then the lemma follows. $\square$

**Lemma 6.4.** *Let $R \in \mathcal{R}$. Then either $\langle L, R \rangle = T$, or $p \equiv \pm 1 \,(\mathrm{mod} \, 10)$ and $\langle L, R \rangle \cong \mathrm{A}_5$. For the latter case, $R = a^{-i}La^i$ or $a^{-(\frac{p+\eta}{2}-i)}La^{\frac{p+\eta}{2}-i}$ for a unique $i$ with $1 < i < \frac{p+\eta}{2}$, $i \neq \frac{p+\eta}{4}$ and $a^i c \in T$; in particular, $i$ is odd or even depending on whether $\varepsilon = \eta$ or not.*

*Proof.* Assume that $\langle L, R \rangle \neq T$. Inspecting the subgroups of $\mathrm{PSL}_2(p)$, we deduce that either $\langle L, R \rangle$ is isomorphic to a subgroup of $\mathrm{D}_{p+\varepsilon}$, or $p \equiv \pm 1 \,(\mathrm{mod} \, 10)$ and $\langle L, R \rangle \cong \mathrm{A}_5$. For the former case, noting that $\mathrm{D}_{p+\varepsilon}$ has a unique subgroup of order 3, we have $|L \cap R| \geqslant 3$, a contradiction. Then the latter case occurs; in particular, $L$ and $R$ are conjugate in $T$. It is easily shown that for each subgroup of $\mathrm{A}_5$ that isomorphic to $\mathrm{S}_3$, there exists a unique subgroup isomorphic to $\mathrm{S}_3$ such that their intersection is a subgroup of order 2. Then $R$ is uniquely determined by $L$ in $\langle L, R \rangle$. Enumerating the subgroups $\mathrm{A}_5$ of $T$ which contain $L$, it follows that $L$ is contained exactly in two subgroups $\mathrm{A}_5$. Then $R$ has exactly two choices.

Fix an $R \in \mathcal{R}$ with $\langle L, R \rangle \cong \mathrm{A}_5$. Then $cRc \in \mathcal{R}$ and $\langle L, cRc \rangle \cong \mathrm{A}_5$. Write $R = a^{-i}La^i$, where $1 \leqslant i < \frac{p+\eta}{2}$. Then $cRc = a^{-(\frac{p+\eta}{2}-i)}La^{\frac{p+\eta}{2}-i}$. By (2) and (3) of Lemma 6.3, the involution $a^i c$ normalizes $\langle L, R \rangle$. Noting that $\mathrm{PGL}_2(p)$ has no proper subgroup isomorphic to $\mathrm{S}_5$ or $\mathbb{Z}_2 \times \mathrm{A}_5$, it follows that $a^i c \in \langle L, R \rangle < T$. Suppose that $i = \frac{p+\eta}{4}$. Noting that $a^{\frac{p+\eta}{4}} \notin T$, we have $c \notin T$. By (3) of Lemma 6.3, $c$ normalizes $\langle L, R \rangle$. Then $\langle L, R, c \rangle \cong \mathrm{S}_5$ or $\mathbb{Z}_2 \times \mathrm{A}_5$, which is impossible. Thus $i \neq \frac{p+\eta}{4}$, and the lemma follows. $\square$

Define
$$\nu_1 = \begin{cases} \frac{p+\eta+2|\varepsilon+\eta|}{4} & \text{if } p \not\equiv \pm 1 \,(\mathrm{mod} \, 10), \\ \frac{p+\eta+2|\varepsilon+\eta|}{4} - 1 & \text{if } p \equiv \pm 1 \,(\mathrm{mod} \, 10). \end{cases}$$

**Theorem 6.5.** *Assume that $L \cong \mathrm{S}_3$. Then $\Gamma$ is isomorphic to one of $\nu_1$ non-isomorphic connected edge-transitive cubic bipartite graphs described as follows:*

(1) *$\frac{|\varepsilon+\eta|}{2}$ semisymmetric graphs with automorphism group isomorphic to $\mathbb{Z}_2 \times T$;*
(2) *a unique symmetric graph with automorphism graph isomorphic to $\mathbb{Z}_2 \times \mathrm{PGL}_2(p)$;*
(3) *$\nu_1 - 1 - \frac{|\varepsilon+\eta|}{2}$ non-isomorphic symmetric graphs, $\frac{p+\eta-4}{8}$ of these graphs have automorphism group isomorphic to $\mathrm{PGL}_2(p)$, and the others have automorphism group isomorphic to $\mathbb{Z}_2 \times T$.*

*Proof.* Let $R_0, R_1, \ldots, R_{\frac{p+\eta}{2}-1}$ be defined as in Lemma 6.3. Put $I = \{0, 1, 2 \ldots, \frac{p+\eta}{2} - 1\}$, and choose an $i_0 \in I$ with $\langle L, R_{i_0} \rangle \cong \mathrm{A}_5$. For each $i \in I$, by Lemma 6.4, $\langle L, R_i \rangle = T$ if and only if $i \in I_0 := I \setminus \{i_0, \frac{p+\eta}{2} - i_0\}$. Then $|I_0| = 2\nu_1 - 1 - \frac{|\varepsilon+\eta|}{2}$, and we get $|I_0|$ distinct connected $T$-semisymmetric cubic graphs $\Gamma_i := \mathsf{BC}(T, L, R_i)$, where $i$ runs over $I_0$. Moreover, $\Gamma \cong \Gamma_i$ for some $i \in I_0$.

By Theorem 2.10, since $\Gamma_i$ is $T$-semisymmetric, $T$ is the unique insolvable minimal normal subgroup of $\mathrm{Aut}\Gamma_i$. In particular, by Lemma 3.8, $\mathrm{Aut}\Gamma_i = T\{\mathsf{conj}(g)_{\{L,R\}} \mid g \in$

$\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R_i\})\}$. Let $c_i = a^i c$. It follows from Lemmas 3.9 and 6.3 that

$$\mathrm{Aut}\Gamma_i = \begin{cases} T \times \langle \hat{c}\tilde{c} \rangle \cong T \times \mathbb{Z}_2 & \text{if } \varepsilon = \eta, \ i = 0; \\ T\langle \mathsf{conj}(c)_{\{L,R_i\}} \rangle \times \langle \hat{c}_i \tilde{c}_i \rangle \cong \mathrm{PGL}_2(p) \times \mathbb{Z}_2 & \text{if } \varepsilon \neq \eta, \ i = \frac{p+\eta}{4}; \\ T\langle \mathsf{conj}(c_i)_{\{L,R_i\}} \rangle \times \langle \hat{c}\tilde{c} \rangle \cong \mathrm{PGL}_2(p) \times \mathbb{Z}_2 & \text{if } \varepsilon = \eta, \ i = \frac{p+\eta}{4}; \\ T \times \langle \hat{c}_i \tilde{c}_i \rangle \cong T \times \mathbb{Z}_2 & \text{if } i \neq \frac{p+\eta}{4}, i + \frac{\varepsilon+\eta}{2} \text{ is odd}; \\ T\langle \mathsf{conj}(c_i)_{\{L,R_i\}} \rangle \cong \mathrm{PGL}_2(p) & \text{if } i \neq \frac{p+\eta}{4}, i + \frac{\varepsilon+\eta}{2} \text{ is even}. \end{cases}$$

Clearly, $\Gamma_0 \not\cong \Gamma_{\frac{p+\eta}{4}}$, and if $i \in I_1 := I_0 \setminus \{0, \frac{p+\eta}{4}\}$ then $\Gamma_i \not\cong \Gamma_0$ or $\Gamma_{\frac{p+\eta}{4}}$. Thus, it remains to consider the isomorphisms among $2\nu_1 - 2 - |\varepsilon + \eta|$ graphs $\Gamma_i$, where $i \in I_1$.

Let $I_2 = \{i \in I_1 \mid \mathrm{Aut}\Gamma_i \cong \mathrm{PGL}_2(p)\}$ and $I_3 = I_1 \setminus I_2$. Then $\Gamma_i \not\cong \Gamma_j$ for all $i \in I_2$ and $j \in I_3$. It is easily shown that $|I_2| = \frac{p+\eta}{4} - 1$. Let $i, j \in I_2$ or $I_3$ with $i \neq j$. Recall that $\mathbf{N}_{\mathrm{PGL}_2(p)}(L, P) = \langle c, a^{\frac{p+\eta}{2}} \rangle$. It follows from Lemma 3.6 that $\Gamma_i \cong \Gamma_j$ if and only if $cR_i c = R_j$, i.e., $ca^{-i}La^i c = a^{-j}La^j$. Noting that $ca^{-i}La^i c = a^i La^{-i}$, it is easily shown that $ca^{-i}La^i c = a^{-j}La^j$ if and only if $j \equiv p + \eta - i \pmod{\frac{p+\eta}{2}}$, see the proof of Lemma 6.2. Since $1 \leqslant i, j < \frac{p+\eta}{2}$, if $j \equiv p + \eta - i \pmod{\frac{p+\eta}{2}}$ then $i + j = \frac{p+\eta}{2}$. Thus $\Gamma_i \cong \Gamma_j$ if and only if $i + j = \frac{p+\eta}{2}$. On the other hand, it is easy to check that $I_2 = \{\frac{p+\eta}{2} - i \mid i \in I_2\}$ and $I_3 = \{\frac{p+\eta}{2} - i \mid i \in I_3\}$. Then we have $\frac{|I_2|}{2}$ or $\frac{|I_3|}{2}$ non-isomorphic graphs $\Gamma_i$ when $i$ runs over $I_2$ or $I_3$, respectively. This completes the proof. $\qquad\square$

**6.2. $|L| = 12$.** Assume that $L \cong \mathrm{D}_{12}$. Then $p \equiv \pm 7 \pmod{16}$ and $\varepsilon = \eta$, see (6.1). In addition, $R \cong \mathrm{D}_{12}$ or $\mathrm{A}_4$, and $P \cong \mathbb{Z}_2^2$. It is easily shown that $\mathbf{N}_{\mathrm{PGL}_2(p)}(P) = \mathbf{N}_T(P) \cong \mathrm{S}_4$, $\mathbf{N}_{\mathrm{PGL}_2(p)}(L) = \mathbf{N}_T(L) \cong \mathrm{D}_{24}$, and $\mathbf{N}_{\mathrm{PGL}_2(p)}(L, R) \leqslant \mathbf{N}_T(L, P) \cong \mathrm{D}_8$. Write $\mathbf{N}_T(P) = P{:}\langle a, b \rangle$, where $a$ has order 3 and $b$ is an involution such that $\mathbf{N}_T(L, P) = P : \langle b \rangle$.

**Lemma 6.6.** *Assume that $L \cong \mathrm{D}_{12}$. Then $\mathcal{R} = \{P{:}\langle a \rangle, a^{-1}La, aLa^{-1}\}$.*

*Proof.* Let $R \in \mathcal{R}$. If $R \cong \mathrm{A}_4$ then $R \leqslant \mathbf{N}_{\mathrm{PGL}_2(p)}(P) = P{:}\langle a, b \rangle$, yielding $R = P{:}\langle a \rangle$. Suppose that $R \cong \mathrm{D}_{12}$. Then $R = x^{-1}Lx$ for some $x \in \mathrm{PGL}_2(p)$. We have $P, xPx^{-1} \leqslant L$, and so $xPx^{-1} = y^{-1}Py$ for some $y \in L$. Then $yx \in \mathbf{N}_{\mathrm{PGL}_2(p)}(P) = P{:}\langle a, b \rangle$. It follows that $R = x^{-1}Lx = z^{-1}Lz$ for some $z \in \langle a, b \rangle$. Noting that $bLb = L$, we have $R = P{:}\langle a \rangle$, $a^{-1}La$ or $aLa^{-1}$. Clearly, $P{:}\langle a \rangle \neq a^{-1}La$ or $aLa^{-1}$. If $a^{-1}La = aLa^{-1}$ then $a \in \mathbf{N}_T(L)$, yielding $\mathrm{A}_4 \cong P{:}\langle a \rangle \leqslant \mathbf{N}_T(L) \cong \mathrm{D}_{24}$, a contradiction. Then the lemma follows. $\qquad\square$

**Theorem 6.7.** *Assume that $L \cong \mathrm{D}_{12}$. Then $\Gamma$ is isomorphic to one of two edge-transitive cubic graphs with automorphism group isomorphic to $T \times \mathbb{Z}_2$, one of them is semisymmetric and the other one is symmetric.*

*Proof.* Inspecting the subgroups of $T$, we deduce that $\langle L, R \rangle = T$ for all $R \in \mathcal{R}$. Up to isomorphism of graphs, write $\Gamma = \mathsf{BC}(T, L, R)$ for some $R \in \mathcal{R}$. By Theorem 2.10 and Lemma 3.8, we have $\mathrm{Aut}\Gamma = T\{\mathsf{conj}(g)_{\{L,R\}} \mid g \in \mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R\})\}$.

Assume that $R = P{:}\langle a \rangle$. Then $L \not\cong R$, and so $\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R\}) = \mathbf{N}_{\mathrm{PGL}_2(p)}(L, R)$. We have $P{:}\langle b \rangle \leqslant \mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R\}) = \mathbf{N}_{\mathrm{PGL}_2(p)}(L, R) \leqslant \mathbf{N}_T(L, P) = P{:}\langle b \rangle$, yielding $\mathbf{N}_{\mathrm{PGL}_2(p)}(L, R) = P{:}\langle b \rangle < T$. Then $\mathrm{Aut}\Gamma = T \times \langle \hat{b}\tilde{b} \rangle$, and $\Gamma$ is semisymmetric.

Assume that $R \neq P{:}\langle a \rangle$. Noting that $ba^{-1}Lab = aLa^{-1}$, we have $\mathsf{BC}(T, L, a^{-1}La) \cong \mathsf{BC}(T, L, aLa^{-1})$. Thus, we may choose $R = a^{-1}La$. Note that $P \leqslant \mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R\}) \leqslant \mathbf{N}_{\mathrm{PGL}_2(p)}(P) = \mathbf{N}_T(P) = P{:}\langle a, b \rangle$. Calculation shows that $\mathbf{N}_{\mathrm{PGL}_2(p)}(L, R) = P$ and $\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R\}) = P \times \langle ba \rangle$. We get $\mathrm{Aut}\Gamma = T\langle \mathsf{conj}(ba)_{\{L,R\}} \rangle = T \times \langle \widehat{ba}\,\widetilde{\mathsf{conj}(ba)}_{\{L,R\}} \rangle$.

Noting that $\mathsf{conj}(ba)_{\{L,R\}}$ interchanges two parts of $\Gamma$, it follows that $\Gamma$ is symmetric. Then the result follows. □

6.3. $|L| = 24$. Assume that $L \cong \mathrm{S}_4$. Then $p \equiv \pm 15 \,(\mathrm{mod}\ 32)$ by (6.1). In addition, $\mathbf{N}_{\mathrm{PGL}_2(p)}(L) = L$, $P \cong \mathrm{D}_8$, and $\mathbf{N}_{\mathrm{PGL}_2(p)}(P) = \mathbf{N}_T(P) \cong \mathrm{D}_{16}$. For each $R \in \mathcal{R}$ we have $R \cong \mathrm{S}_4$ or $\mathrm{D}_{24}$, and it is easily shown that $T = \langle L, R \rangle$. Note, if $R \cong \mathrm{D}_{24}$ then $\varepsilon = \eta$. Write $\mathbf{N}_T(P) = P{:}\langle b \rangle$, where $b$ is an involution in $T$.

Let $R \in \mathcal{R}$. Since $L$ is self-normalized in $\mathrm{PGL}_2(p)$, we have $R_1 := bLb \neq L$. If $R \cong \mathrm{S}_4$ then $R = R_1$ by Lemma 6.1. Assume that $R \cong \mathrm{D}_{24}$. Then $\varepsilon = \eta$, and $\mathbf{N}_{\mathrm{PGL}_2(p)}(R) = \mathbf{N}_T(R) \cong \mathrm{D}_{48}$. We deduce from Lemma 5.2 that $T$ has two classes of subgroups $\mathrm{D}_8$ and two classes subgroups $\mathrm{D}_{24}$. Note that all subgroups $\mathrm{D}_8$ in $\mathrm{D}_{24}$ are conjugate. It follows that, for the given pair $(L, P)$, there exists a unique subgroup $R_0 < T$ with $R_0 \cong \mathrm{D}_{24}$ and $R_0 \cap L = P$. Thus $\mathcal{R} = \{R_0, R_1\}$.

Note that $\mathbf{N}_{\mathrm{PGL}_2(p)}(L) = L$ and $|\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R_i\}) : \mathbf{N}_{\mathrm{PGL}_2(p)}(L, R_i)| \leqslant 2$. We have $\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R_0\}) = \mathbf{N}_{\mathrm{PGL}_2(p)}(L, R_0) = P$, and $\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R_1\}) = P{:}\langle b \rangle$. Then, by Theorem 2.10 and Lemma 3.8, we have the following result.

**Theorem 6.8.** *Assume that $L \cong \mathrm{S}_4$. Then $\Gamma$ is isomorphic to one of two edge-transitive cubic graphs, one of them is semisymmetric with automorphism group $\mathrm{PSL}_2(p)$, and the other one is symmetric with automorphism group $\mathrm{PSL}_2(p) \times \mathbb{Z}_2$.*

## 7. Proof of Theorem 1.1

Let $\Gamma = (V, E)$ be a connected edge-transitive cubic graph of order $2n$ with $n$ even and square-free, and let $A = \mathrm{Aut}\Gamma$. If $A$ is solvable then $\Gamma \cong \mathsf{K}_4$ by Theorem 2.5. Assume that $A$ is insolvable, and let $T = A^{(\infty)}$. By Theorem 2.10, either $T$ is one of $\mathrm{J}_1$ and $\mathrm{PSL}_2(p)$, or $\Gamma$ is described as in Lines 1, 2 of Table 1 and Line 1 of Table 2. If $T = \mathrm{J}_1$ then Line 3 of Table 1 and Line 2 of Table 2 follow from Theorem 4.2. If $T = \mathrm{PSL}_2(p)$ and $\Gamma$ is $T$-edge-transitive then we get Lines 4-10 of Table 1 by Theorems 5.7, 5.12-5.14, and Lines 3-10 of Table 2 by Theorems 6.5, 6.7 and 6.8.

In the following, we assume that $T = \mathrm{PSL}_2(p)$, and $\Gamma$ is not $T$-edge-transitive. Fix an edge $\{u, w\} \in E$, and let $A^* = \langle A_u, A_w \rangle$. By Lemma 2.9, $|\mathsf{rad}(A^*)| \in \{3, 6\}$, $\Gamma$ is $\mathsf{rad}(A^*)T$-edge-transitive, and one of the following holds:

(i) $T$ is transitive on one part say $W$ of $\Gamma$ and has three orbits on the other part $U$;
(ii) $T$ is regular on $V$, and $p \equiv \pm 3 \,(\mathrm{mod}\ 8)$.

Let $M = \langle z \rangle$ be the unique Sylow 3-subgroup of $\mathsf{rad}(A^*)$, and put $G = MT$. For each $g \in \mathrm{PGL}_2(p)$, extend $\mathsf{conj}(g)$ to an automorphism of $G$ by setting $y^{\mathsf{conj}(g)} = y$ for $y \in M$. Let $\mathrm{Aut}(M) = \langle \tau \rangle$, and extend $\tau$ to an automorphism of $G$ by setting $x^\tau = x$ for $x \in T$. Then

$$\mathrm{Aut}(G) = \langle \tau \rangle \times \{\mathsf{conj}(g) \mid g \in \mathrm{PGL}_2(p)\}.$$

Clearly, $G$ acts transitively on each $A^*$-orbit. This implies that $\Gamma$ is $G$-edge-transitive. Let $\bar{T}$ be the subgroup of $\mathrm{Aut}\Gamma_M$ induced by $T$. For $v \in V$, let $\bar{v}$ be the $M$-orbit containing $v$. Then $T_{\bar{v}} \cong G_v \cong \bar{T}_{\bar{v}}$, see (2.1). We next discuss in two cases.

**Case 1.** Assume that (i) occurs, $u \in U$ and $w \in W$. Then $n = 3|T : T_u| = |T : T_w|$, and so $|T_u| = 3|T_w|$. Recall that $\bar{T}_{\bar{w}} \cong T_{\bar{w}}$, $T_w \trianglelefteq T_{\bar{w}}$ and $M \cong T_{\bar{w}}/T_w$, see (2.2). Since

$M \cong \mathbb{Z}_3$, it follows from Lemma 2.1 that either $G_w \cong \bar{T}_{\bar{w}} \cong \mathbb{Z}_6$ and $G_u \cong \bar{T}_{\bar{u}} \cong \mathrm{S}_3$, or $G_w \cong \bar{T}_{\bar{w}} \cong \mathrm{A}_4$ and $G_u \cong \bar{T}_{\bar{u}} \cong \mathrm{D}_{12}$, and so $T_w \cong \mathbb{Z}_2$ or $\mathbb{Z}_2^2$, respectively. In particular, $G_w \cap T = G_u \cap G_w = T_w$. Since $|T_u| = 3|T_w|$, we have $|T_u| = |T_{\bar{u}}|$. Then $T_u = T_{\bar{u}} \cong G_u$, yielding $G_u = T_{\bar{u}} < T$. It is easy to see that those subgroups of $T$ isomorphic to $\bar{T}_{\bar{u}}$ are all conjugate under $\mathrm{Aut}(G)$. Up to isomorphism of graphs, we fix a subgroup $L < T$ and Sylow 2-subgroup $P$ of $L$, and write $\Gamma \cong \mathsf{BC}(G, L, R)$, where $L \cong \bar{T}_{\bar{u}}$, $R \cong \bar{T}_{\bar{w}}$, $R \cap T = P$, and $\langle L, R \rangle = G$.

Noting that $P$ is the unique Sylow 2-subgroup of $R$, we write $R = P{:}\langle yx \rangle$, where $y \in M$ and $x \in T$ with $\langle yx \rangle \cong \mathbb{Z}_3$. Since $\langle L, R \rangle = G$, we deduce that $M = \langle y \rangle$, and so $R = P{:}\langle zx \rangle$ or $P{:}\langle z^{-1}x \rangle$. Clearly, $\tau \in \mathrm{Aut}(G, L, P)$, and $(P{:}\langle zx \rangle)^\tau = P{:}\langle z^{-1}x \rangle$. Thus, up to isomorphism of graphs, we further choose $R = P{:}\langle zx \rangle$, and then $\Gamma$ is determined completely by $R_0 := P{:}\langle x \rangle$.

Again by $\langle L, R \rangle = G$, we have that $\langle L, x \rangle = T$ and $x$ has order 3. Then $\Gamma_0 := \mathsf{BC}(T, L, R_0)$ is a connected $T$-semisymmetric cubic graph, and $R_0 \cong R \cong G_w$. Conversely, if $\Gamma_0$ is connected then it is easily shown that $\mathsf{BC}(G, L, R)$ is also connected.

Let $A = \mathrm{Aut}\mathsf{BC}(G, L, R)$. Then $T, G \trianglelefteq A$ by Theorem 2.10. Noting that the normal subgroup $T$ is transitive on one part of $\mathsf{BC}(G, L, R)$ but not transitive on the other one, it follows that $\mathsf{BC}(G, L, R)$ is semisymmetric. Further, by Lemma 3.8, we deduce that $A = G\{\sigma_{\{L,R\}} \mid \sigma \in \mathrm{Aut}(G, L, R)\}$. Clearly, $\mathrm{Aut}(G, L, R) \leqslant \langle \tau \rangle \times \mathrm{Aut}(T, L, R_0)$.

Suppose that $L \cong \mathrm{S}_3$ and $R \cong \mathbb{Z}_6$. By Lemma 6.2, $\varepsilon = \eta$, and $R_0$ is uniquely determined by $L$. By Lemma 6.3, we have $\mathrm{Aut}(G, L, R_0) = \{\mathsf{conj}(g) \mid g \in P \times \langle c \rangle\}$, where $c$ generates the center of $\mathbf{N}_T(L)$ and $\langle R_0, c \rangle \cong \mathrm{D}_{12}$. Calculation shows that $\mathrm{Aut}(G, L, R) = \{\mathsf{conj}(g), \tau\mathsf{conj}(cg) \mid g \in P\}$. Noting that $\tau\mathsf{conj}(c)$ inverses $z$ and centralizes $T$, we have $A = G\{\sigma_{\{L,R\}} \mid \sigma \in \mathrm{Aut}(G, L, R)\} \cong \mathrm{S}_3 \times T$, and then $\Gamma$ is described as in Line 11 of Table 2.

Suppose that $L \cong \mathrm{D}_{12}$ and $R \cong \mathrm{A}_4$. Using Lemma 6.6 and Theorem 6.7, by a similar argument as above, we deduce that $R_0$ is uniquely determined by $L$, and $A \cong \mathrm{S}_3 \times T$. Then $\Gamma$ is described as in Line 12 of Table 2.

**Case 2**. Assume that (ii) occurs. Then $G_v \cong \mathbb{Z}_3$, and $\Gamma \cong \mathsf{Cos}(G, H, 1, o)$, where $o$ is an involution, $H \cong \mathbb{Z}_3$ and $\langle H, o \rangle = G$. Clearly, $o \in T$. Write $H = \langle yx \rangle$, where $y \in M$ and $x \in T$. Since $\langle yx, o \rangle = \langle H, o \rangle = G$, we deduce that $M = \langle y \rangle$, and $\langle x, o \rangle = T$. In particular, $\mathsf{Cos}(T, \langle x \rangle, 1, o)$ is a connect $T$-symmetric cubic graph. Conversely, for a connect $T$-symmetric cubic graph $\mathsf{Cos}(T, \langle x \rangle, 1, o')$, since $G = M \times T = \langle y \rangle \times T$, it is easily shown that $\langle yx, o' \rangle$ has a homomorphic image $\langle x, o' \rangle = T$. Then $|G : \langle yx, o' \rangle|$ is a divisor of $|G : T| = |M| = 3$, and hence either $G = \langle yx, o' \rangle$ or $|G : \langle yx, o' \rangle| = 3$. The latter case implies that $\langle yx, o' \rangle \cong T$ is simple, since $\langle yx, o' \rangle \nleqslant T$ and $T$ is normal in $G$, we have $\langle yx, o' \rangle \cap T = 1$, and hence $3|T| = |G| \geqslant |T\langle yx, o' \rangle| = |T|^2$, yielding $|T| \leqslant 3$, a contradiction. Thus $G = \langle yx, o' \rangle$, and so $\mathsf{Cos}(G, H, 1, o')$ is connected.

Recalling that $\langle y \rangle = M = \langle z \rangle$, we have $y = z$ or $z^{-1}$. By the definition of $\tau$, we have $y^\tau = y^{-1}$, $(yx)^\tau = y^{-1}x$, and $o^\tau = o$. Then $\mathsf{Cos}(G, H, 1, o) \cong \mathsf{Cos}(G, H^\tau, 1, o)$, see (III) in Subsection 3.2. Thus, up to isomorphism of graphs, we may choose $H = \langle zx \rangle$. Moreover, all elements of $T$ with order 3 are all conjugate, this allows we fix an element $x \in T$ of order 3. Noting that $\mathsf{Cos}(T, \langle x \rangle, 1, o)$ is a connect $T$-symmetric cubic graph, the argument in Subsection 5.1 is available for $\mathsf{Cos}(T, \langle x \rangle, 1, o)$. In particular, we assume

that $\mathsf{Cos}(T, \langle x \rangle, 1, o)$ is one of $\omega_0$ non-isomorphic symmetric cubic graphs, $\frac{p-\eta-6}{4}$ of them have automorphism group $T\langle\mathsf{conj}(b)_{\langle x \rangle}\rangle \cong \mathrm{PGL}_2(p)$, and the others have automorphism group $\langle \hat{ab} \rangle \times T$, where $\omega_0$, $o \in O_0$, $a$ and $b$ are defined as in Subsection 5.1.

Let $A = \mathrm{Aut}\mathsf{Cos}(G, H, 1, o)$. By Theorem 2.10, we have $T, G \trianglelefteq A$. It follows from Lemma 3.4 that $A = G\{\sigma_H \mid \sigma \in \mathrm{Aut}(G, H, HoH)\}$. Recall that $\mathrm{Aut}(G) = \langle \tau \rangle \times \{\mathsf{conj}(g) \mid g \in \mathrm{PGL}_2(p)\}$. It is easily shown that $\mathrm{Aut}(G, H, HoH) \leqslant \langle\tau\rangle\times\mathrm{Aut}(T, \langle x \rangle, \langle x \rangle o\langle x \rangle) = \langle\tau\rangle\times\{\mathsf{conj}(g) \mid g \in \mathbf{N}_{\mathrm{PGL}_2(p)}(\langle x \rangle, \langle x \rangle o\langle x \rangle)\}$. By calculation, see the proof of Theorem 5.7, we have $\mathbf{N}_{\mathrm{PGL}_2(p)}(\langle x \rangle, \langle x \rangle o\langle x \rangle) = \langle x \rangle\langle b \rangle$ or $\langle x \rangle\langle ab \rangle$ when $\mathrm{Aut}\mathsf{Cos}(T, \langle x \rangle, 1, o) \cong \mathrm{PGL}_2(p)$ or $\mathbb{Z}_2 \times \mathrm{PSL}_2(p)$, respectively. It follows that $\mathrm{Aut}(G, H, HoH) = \{\tau\mathsf{conj}(g) \mid g \in \langle x \rangle\langle b \rangle\}$ or $\{\tau\mathsf{conj}(g) \mid g \in \langle x \rangle\langle ab \rangle\}$, respectively. Since $ab \in T$ and $g\hat{g} = \mathsf{conj}(g)_H$ for $g \in \mathbf{N}_G(H)$, we have $A = G\{\sigma_H \mid \sigma \in \mathrm{Aut}(G, H, HoH)\} = G\langle\tau\mathsf{conj}(b)_H\rangle$ or $G\langle\tau\hat{ab}\rangle$, which is isomorphic to $(\mathrm{PSL}_2(p) \times \mathbb{Z}_3){:}\mathbb{Z}_2$ or $\mathrm{PSL}_2(p) \times \mathrm{S}_3$, respectively.

Finally, suppose that $\mathsf{Cos}(G, H, 1, o_1) \cong \mathsf{Cos}(G, H, 1, o_2)$ for $o_1, o_2 \in O_0$. Then, by Lemma 3.3, there is $\sigma \in \mathrm{Aut}(G, H)$ such that $Ho_1^\sigma H = Ho_2 H$. This implies that $\langle x \rangle o_1^{\mathsf{conj}(g)}\langle x \rangle = \langle x \rangle o_2\langle x \rangle$ for some $g \in \mathrm{PGL}_2(p)$. Then $\mathsf{Cos}(T, \langle x \rangle, 1, o_1) \cong \mathsf{Cos}(T, \langle x \rangle, 1, o_2)$. By Theorem 5.7, we have $o_1 = o_2$. Thus distinct involutions $o$ in $O_0$ produce non-isomorphic symmetric graphs $\mathsf{Cos}(G, H, 1, o)$. Therefore, $\Gamma$ is described as in Lines 11 or 12 of Table 1. This completes the proof of Theorem 1.1.

## References

[1] M. Aschbacher, Finite Group Theory, Cambridge University Press, Cambridge, 1993.

[2] N. Biggs, *Algebraic Graph Theory*, 2nd ed., Cambridge University Press, New York, 1992.

[3] P.J. Cameron, H.R. MaiMani, G.R. Omidi and B. Tayfeh-Rezaie, 3-*design from* $\mathrm{PSL}(2, q)$, Discrete Math. **306** (2006), 3062–3073.

[4] P.J. Cameron, G.R. Omidi and B. Tayfeh-Rezaie, 3-*design from* $\mathrm{PGL}(2, q)$, Electron. J. Combin. **13** (2006), #R50.

[5] M. Conder, *Arc-transitive trivalent Cayley graphs*, J. Algebra. **610** (2022), 896–910.

[6] M. Conder and P. Dobcsányi, *Trivalent symmetric graphs on up to* 768 *vertices*, J. Combin. Math. Combin. Comput. **40** (2002), 41–63.

[7] M. Conder and P. Lorimer, *Automorphism groups of symmetric graphs of valency* 3, J. Combin. Theory Ser. B. **47** (1989), 60–72.

[8] M. Conder, A.Malnič, D. Marušič and P. Potočnik, *A census of semisymmetric cubic graphs on up to* 768, J. Algebraic Combin. **23** (2006), 255–294.

[9] M. Conder and R. Nedela, *Symmetric cubic graphs of small girth*, J. Combin. Theory Ser. B. **97** (2007), 757–768.

[10] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *Atlas of Finite Groups*, Oxford University Press, Oxford, 1985.

[11] J.D. Dixon and B. Mortimer, *Permutation Groups*, Springer-Verlag, New York, 1996.

[12] D.Ž. Djoković and G.L. Miller, *Regular groups of automorphisms of cubic graphs*, J. Combin. Theory Ser. B. **29** (1980), 195–230.

[13] S.F. Du and M.Y. Xu, *A classification of semisymmetric graphs of order* 2pq, Comm. Algebra. **28** (2002), 2685–2714.

[14] The GAP Group, *GAP-Groups, Algorithms, and Programming*, Version 4.11.1, 2021. http://www.gap-system.org

[15] M. Giudici, C. H. Li and C. E. Praeger, *Analysing finite locally s-arc transitive graphs*, Trans. Amer. Math. Soc. **365** (2004), 291–317.

[16] D.M. Goldschmidt, *Automorphisms of trivalent graphs*, Ann. Math. **111** (1980), 377–406.

[17] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin, 1967.

[18] M. E. Iofinova and A. A. Ivanov, *Biprimitive cubic graphs*, Investigation in Algrbraic Theory of Combinatorial Objects, Proceedings of the seminar, Institute for System Studies, Moscow, 1985, Kluwer Academic Publishers, London, 1994, pp 459–472.

[19] P. Kleidman and M. Liebeck, *The Subgroup Structure of The Finite Classical Groups*, Cambridge University Press, Cambridge, 1990.

[20] C.H. Li, Z. P. Lu and D. Marušič, *On primitive permutation groups with small suborbits and their orbital graphs*, J. Algebra **279** (2004), 749–770.

[21] C.H. Li, Z.P. Lu and G.X. Wang, *Vertex-transitive cubic graphs of square-free order*, J. Graph Theory **75** (2014), 1–19.

[22] C. H. Li, Z. P. Lu and G. X. Wang, *Arc-transitive graphs of square-free order and small valency*, Discrete Math. **339** (2016), 2907–2918.

[23] G.X. Liu and Z.P. Lu, *On edge-transitive cubic graphs of square-free order*, European J. Combin. **45** (2015), 41–46.

[24] J.J. Li and Z.P. Lu, *Cubic s-arc transitive Cayley graphs*, Discrete Math. **309** (2009), 6014–6025.

[25] S. Lipschutz and M.Y. Xu, *Note on infinite families of travalent semisymmetric graphs*, European J. Combin. **23** (2002), 707–717.

[26] A. Malnič, D. Marušič and P. Potočnik, *On cubic graphs admitting an edge-transitive solvable Group*, J. Algebraic Combin. **20** (2004), 99–113.

[27] R.C. Miller, *The trivalent symmetric graphs of girth at most six*, J. Combin. Theory Ser. B. **10** (1971), 163–182.

[28] C. W. Parker, *Semisymmetric cubic graphs of twice odd order*, European J. Combin. **28** (2007), 572–591.

[29] W.T. Tutte, *A family of cubical graphs*, Proc. Cambridge Philos. Soc. **43** (1947), 459–474.

[30] W.T. Tutte, *On the symmetry of cubic graphs*, Canad. J. Math. **11** (1959), 621–634.

[31] W.T. Tutte, *Connectivity in Graphs*, University of Toronto Press, Toronto, 1966.

[32] R. Weiss, *s-Transitive graphs*, Algebraic methods in graph theory, vols. I, II, Szeged, 1978, Colloq. Soc. Janos Bolyai, vol. 25, North-Holland, Amsterdam, New York, 1981, pp 827–847.

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