ON 2-ARC-TRANSITIVE GRAPHS ADMITTING A VERTEX-TRANSITIVE SIMPLE GROUP

ZAI PING LU

Abstract. A graph Γ is said to be 2-arc-transitive if its automorphism group acts transitively on the set of 2-arcs of Γ. In this paper, we give a group-theoretic characterization of those connected 2-arc-transitive graphs which admit a vertex-transitive simple group.

Keywords. Simple group, quasisimple group, perfect group, arc-transitive, 2-arc-transitive.

1. Introduction

In this paper, all groups are assumed to be finite, and all graphs are assumed to be finite, simple and undirected.

Let $\Gamma = (V, E)$ be a regular graph with vertex set V and edge set E. Denote by $\mathsf{Aut}(\Gamma)$ the automorphism group of Γ, and let G be a subgroup of $\mathsf{Aut}(\Gamma)$. The graph Γ is called *G-vertex-transitive*, or G is called a *vertex-transitive group* of Γ, if G acts transitively on V, and called a Cayley graph of G if G acts regularly on V. Recall that an arc of Γ is an ordered pair of adjacent vertices, and a 2-arc is a triple (α, β, γ) of vertices with $\{\alpha, \beta\}$, $\{\beta, \gamma\} \in E$ and $\alpha \neq \gamma$. The graph Γ is called *G-arc-transitive* (or *$(G, 2)$-arc-transitive*) if it has no isolated vertex and G acts transitively on the set of arcs (or the set of 2-arcs). Note that 2-arc-transitivity leads to arc-transitivity, and arc-transitivity leads to vertex-transitivity.

In the literature, the solutions of quite a number of problems about arc-transitive graphs have been reduced or partially reduced into the class of graphs arising from (almost) simple groups. For example, the reduction for arc-transitive graphs of prime valency [25], the reduction for 2-arc-transitive graphs established in [27], the Weiss Conjecture [34, Conjecture 3.12] for non-bipartite locally primitive graphs [5], the normality of Cayley graphs of simple groups [10, 11], the existence and classification of edge-primitive graphs [13, 26], and so on. Certainly, the class of graphs admitting (almost) simple groups plays an important role in the theory of arc-transitive graphs.

In this paper, we focus on those arc-transitive graphs which admit a vertex-transitive simple group. One of our motivations comes from a problem in the study of the automorphism groups or the normality of arc-transitive Cayley graphs of finite nonabelian simple groups. Let $\Gamma = (V, E)$ be a connected G-arc-transitive graph of valency $d \geqslant 3$. Assume that either d is a prime or Γ is $(G, 2)$-arc-transitive, and G has a nonabelian simple subgroup T which acts regularly on V. Then the Weiss Conjecture is true for (Γ, G), that is, the orders of vertex-stabilizers have an upper bound depending only on the valency d, refer to [5]. This ensures that T is normal in

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G with a finite number of exceptions, see [10, Theorem 1.1]. An interesting problem, as proposed in [10], is to figure out the exceptions for T. This problem has been solved for $d \leqslant 5$ in several papers, refer to [8, 9, 10, 31]. In [32], the exceptions for T are determined under the assumption that d is a prime and a vertex-stabilizer is solvable. The other possible exceptions for T can be read out from a recent paper [21], which are alternating groups, simple groups with $|T| - 1 = d$ and, possibly, the simple orthogonal groups of minus type and characteristic 2. With these, we observe that if T is not normal in G then G is an almost simple group. This leads to another interesting problem. What will happen if we weaken the 'regularity' of T into 'transitivity'? Thus, in this paper, we consider those arc-transitive graphs satisfying the following assumptions:

Hypothesis 1.1. Γ is a connected G-arc-transitive graph of valency $d \geqslant 3$, G contains a vertex-transitive nonabelian simple subgroup T, and either d is a prime or Γ is $(G, 2)$-arc-transitive.

Recall that a group X is perfect if it equals to its derived subgroup. If a central extension of some simple group is perfect then it is called a quasisimple group or a covering group of the simple group. For a finite group X, denote by $\mathsf{rad}(X)$ and $\mathbf{O}_r(X)$, respectively, the maximal solvable normal subgroup and the maximal normal r-subgroup of X, where r is a prime divisor of $|X|$.

In Section 4, the following result is proved.

Theorem 1.2. *Assume that Γ, G and T are described as in Hypothesis* 1.1. *Then G has at most one transitive minimal normal subgroup, and one of the following holds:*

(1) $G \cong \mathrm{AGL}_3(2)$, *and Γ is the complete graph on* 8 *vertices;*

(2) *T is contained in a characteristic perfect subgroup N of G, and either*
   - (i) *N is quasisimple; or*
   - (ii) *$N/\mathbf{O}_r(N)$ is quasisimple, T and $N/\mathsf{rad}(N)$ are simple groups of Lie type over finite fields of characteristic r, and $|\mathsf{rad}(N)|$ is a divisor of $|T|$.*

In particular, if G has a transitive minimal normal subgroup M, then either $G \cong \mathrm{AGL}_3(2)$ or M is simple and $T \leqslant M$.

Theorem 1.2 is just the first step toward characterizing those simple groups which act transitively on the vertex set of a 2-arc-transitive graph or an arc-transitive graph of prime valency, and then classifying those graphs in Hypothesis 1.1 with T not normal in G. For (2)(i) and (ii) of Theorem 1.2 with $T \neq N$ (and so $N/\mathsf{rad}(N) \ncong T$), we observe that the simple group $N/\mathsf{rad}(N)$ has a factorization $N/\mathsf{rad}(N) = XY$ with $X \cong T$ and $Y \neq 1$. In a sequel, employing factorizations of finite (almost) simple groups, we shall work out a possible list for those simple groups T which are not normal in G.

2. Primes involved in some finite simple groups

In this section, we assume that n is a positive integer and r is a prime. Write

$$n = a_0 + a_1 r + \cdots + a_k r^k, \ s_r(n) = a_0 + a_1 + \cdots + a_k, \tag{2.1}$$

where a_i are integers with $0 \leqslant a_i < r$. For an integer x, denote by $\nu_r(x)$ the highest power of r that divides x. By Legendre's formula,

$$\nu_r(n!) = \frac{n - s_r(n)}{r - 1}. \tag{2.2}$$

In particular, $\nu_r(n!) \leqslant n-1$, where the equality holds if and only if $r = 2$ and n is a power of 2.

Recall that, for integers $l \geqslant 2$ and $q \geqslant 2$, a primitive prime divisor of $q^l - 1$ is a prime which divides $q^l - 1$ but does not divide $q^i - 1$ for any $0 < i < l$. If r is a primitive prime divisor of $q^l - 1$, then q has order l modulo r, and thus l is a divisor of $r-1$, in particular, $r \geqslant l+1$; if further $r \mid (q^m - 1)$ with $m \geqslant 1$ then $l \mid m$. Thus, by [12, Theorems 3.1 and 3.5], we have the following result, where $[x]$ denotes the integer part of a real number x.

Lemma 2.1. *Let $\Lambda_n(q) = \prod_{i=1}^{n}(q^i - 1)$, where n and q are integers no less than 2. Assume that r is a prime divisor of $\Lambda_n(q)$, and let l be the order of q modulo r. Then one of the following holds:*

(1) *r is odd or $q \equiv 1 \pmod 4$, and $\nu_r(\Lambda_n(q)) = [\frac{n}{l}]\nu_r(q^l - 1) + \nu_r([\frac{n}{l}]!)$;*
(2) *$r = 2$, $q \equiv 3 \pmod 4$, and $\nu_2(\Lambda_n(q)) = [\frac{n}{2}]\nu_2(q+1) + [\frac{n+a_0}{2}] + \nu_2(n!)$.*

Corollary 2.2. *Let n, q, r and $\Lambda_n(q)$ be as in Lemma 2.1. Then either*

(1) *$\nu_r(\Lambda_n(q)) < n\log_2(q) + \nu_r(n!) \leqslant q^{\frac{n}{2}} + n - 1$ for $(r,q) \neq (2,3)$; or*
(2) *$(r,q) = (2,3)$ and $\nu_2(\Lambda_n(q)) \leqslant \frac{5n-2}{2} \leqslant 3^{\frac{n}{2}} + n - 1$.*

In particular, $\nu_2(\Lambda_n(q)) = q^{\frac{n}{2}} + n - 1$ if and only if $(r,q,n) = (2,3,2)$.

Proof. Let l be the order of q modulo r.

Assume that (1) of Lemma 2.1 holds. Noting that $\nu_r(n!) \leqslant n-1$, we have

$$\begin{aligned}\nu_r(\Lambda_n(q)) &= [\frac{n}{l}]\nu_r(q^l - 1) + \nu_r([\frac{n}{l}]!) \leqslant [\frac{n}{l}]\log_r(q^l - 1) + \nu_r([\frac{n}{l}]!)\\ &< [\frac{n}{l}]\log_r(q^l) + \nu_r([\frac{n}{l}]!) \leqslant \log_r(q^n) + \nu_r(n!) \leqslant \log_2(q^n) + n - 1.\end{aligned}$$

It is easily shown that $x^{\frac{1}{2}} - \log_2(x)$ is nonnegative and monotonically increasing when $x \geqslant 16$. It follows that either $\log_2(q^n) \leqslant q^{\frac{n}{2}}$ or $q^n \leqslant 15$. The former case yields part (1) of this corollary. For $q^n \leqslant 15$, since either r is odd or $q \equiv 1 \pmod 4$, the only possibility is that $(q,n) = (2,2)$ or $(2,3)$; in this case, $r \in \{3,7\}$ and $\nu_r(\Lambda_n(q)) = 1$, which also meets (1) of the corollary.

Now let $r = 2$ and $q \equiv 3 \pmod 4$. If $q > 3$ then $n < \frac{n}{2}\log_2 q$, and so

$$\begin{aligned}\nu_2(\Lambda_n(q)) &\leqslant [\frac{n}{2}]\nu_2(q+1) + [\frac{n+a_0}{2}] + \nu_2(n)\\ &< [\frac{n}{2}]\log_2(2q) + [\frac{n+1}{2}] + \nu_2(n!)\\ &= [\frac{n}{2}]\log_2(q) + [\frac{n}{2}] + [\frac{n+1}{2}] + \nu_2(n!)\\ &= [\frac{n}{2}]\log_2(q) + n + \nu_2(n!) < n\log_2(q) + \nu_2(n!)\\ &\leqslant q^{\frac{n}{2}} + n - 1,\end{aligned}$$

desired as in (1) of this corollary. Assume that $q = 3$. Then

$$\nu_2(\Lambda_n(q)) = 2[\frac{n}{2}] + [\frac{n+a_0}{2}] + n - s_2(n).$$

Noting that $a_0 \in \{0,1\}$ and $s_2(n) \geqslant 1$, we have

$$\nu_2(\Lambda_n(q)) \leqslant 2[\frac{n}{2}] + [\frac{n+1}{2}] + n - 1 \leqslant \frac{5n-2}{2}.$$

It is easily shown that $3^x \geqslant 3x$ for $x \geqslant 1$. Thus $\frac{5n-2}{2} = 3 \cdot \frac{n}{2} + n - 1 \leqslant 3^{\frac{n}{2}} + n - 1$, and the corollary follows. □

For a group X, denote its derived subgroup by X'. For a finite simple group of Lie type in characteristic p, let $e(L)$ denote a lower bound, given as in [17, page 188, Table 5.3.A], on degrees of faithful projective s-modular representations of L with $s \neq p$.

Lemma 2.3. *Let L be a finite simple group of Lie type defined over a field of order $q = p^f$, where p is a prime. Assume that r is a prime divisor of $|L|$ with $r \neq p$. Then $\nu_r(|L|) < e(L)$ with the following exceptions:*

(1) $L = \mathrm{PSL}_2(9)$, $r = 2$, $\nu_r(|L|) = 3 = e(L)$;
(2) $L = \mathrm{Sp}_4(2)'$, $r = 3$, $\nu_r(|L|) = 2 = e(L)$;
(3) $L = \mathrm{PSU}_4(2)$, $r = 3$, $\nu_r(|L|) = 4 = e(L)$;
(4) $L = \mathrm{PSU}_4(3)$, $r = 2$, $\nu_r(|L|) = 7$ *and* $e(L) = 6$;
(5) $L = \mathrm{PSL}_2(5)$, $r = 2$, $\nu_r(|L|) = 2 = e(L)$;
(6) $L = \mathrm{PSL}_2(7)$, $r = 2$, $\nu_r(|L|) = 3 = e(L)$;
(7) $L = \mathrm{PSp}_4(3)$, $r = 2$, $\nu_r(|L|) = 6$ *and* $e(L) = 4$.

Proof. Suppose first that $(L, e(L))$ is a pair given as in the third column of [17, page 188, Table 5.3.A]. Then L, p, $e(L)$ and $|L|$ are listed in Table 2.1. Inspecting

L	p	$e(L)$	$\lvert L\rvert$
$\mathrm{PSL}_2(4)$	2	2	$p^2 \cdot 3 \cdot 5$
$\mathrm{PSL}_2(9)$	3	3	$p^2 \cdot 2^3 \cdot 5$
$\mathrm{PSL}_3(2)$	2	2	$p^3 \cdot 3 \cdot 7$
$\mathrm{PSL}_3(4)$	2	4	$p^6 \cdot 3^2 \cdot 5 \cdot 7$
$\mathrm{Sp}_4(2)'$	2	2	$p^3 \cdot 3^2 \cdot 5$
$\mathrm{PSp}_6(2)'$	2	7	$p^9 \cdot 3^4 \cdot 5 \cdot 7$
$\mathrm{PSU}_4(2)$	2	4	$p^6 \cdot 3^4 \cdot 5$
$\mathrm{PSU}_4(3)$	3	6	$p^6 \cdot 2^7 \cdot 5 \cdot 7$
$\mathrm{P\Omega}_8^+(2)$	2	8	$p^{12} \cdot 3^5 \cdot 5^2 \cdot 7$
$\Omega_7(3)$	3	27	$p^9 \cdot 2^9 \cdot 5 \cdot 7 \cdot 13$
$\mathrm{F}_4(2)$	2	$\geqslant 44$	$p^{24} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$
$\mathrm{G}_2(3)$	3	14	$p^6 \cdot 3^6 \cdot 7 \cdot 13$
$\mathrm{G}_2(4)$	2	12	$p^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$
$\mathrm{Sz}(8)$	2	8	$p^6 \cdot 5 \cdot 7 \cdot 13$

TABLE 2.1. Exceptions for $e(L)$

the groups in Table 2.1, we have $\nu_r(|L|) < e(L)$ unless $(L, r, \nu_r(|L|), e(L))$ is one of $(\mathrm{PSL}_2(9), 2, 3, 3)$, $(\mathrm{Sp}_4(2)', 3, 2, 2)$, $(\mathrm{PSU}_4(2), 3, 4, 4)$ and $(\mathrm{PSU}_4(3), 2, 7, 6)$.

We next deal with the case where $e(L)$ is listed in the second column of [17, page 188, Table 5.3.A]. We fix a Sylow r-subgroup R of L. Then $\nu_r(|L|) = \nu_r(|R|)$.

Case 1. Assume that $L = \mathrm{PSL}_2(q)$ and $e(L) = \frac{q-1}{(2,q-1)}$, where $4 < q \neq 9$. In this case, $|R|$ is a divisor of $\Lambda_2(q)$, and so $\nu_r(|L|) = \nu_r(|R|) \leqslant \nu_r(\Lambda_2(q))$. Since $q \neq 3$, by (1) of Corollary 2.2, $\nu_r(|L|) < 2\log_2(q) + 1$. If $q \leqslant 15$ then $q = 5$ or 7, which gives (5) or (6) of this lemma. Now let $q > 15$. Then $\log_2(q) \leqslant q^{\frac{1}{2}}$, and so $\nu_r(|L|) < 2\log_2(q) + 1 \leqslant 2q^{\frac{1}{2}} + 1$. Suppose that $\nu_r(|L|) \geq e(L)$. Then $2q^{\frac{1}{2}} + 1 > \frac{q-1}{2}$, and so $q^2 - 22q + 9 < 0$, yielding $q < 22$. Thus $q = 16$, 17 or 19, and then $e(L) \geqslant 8$; however, r^8 is not a divisor of $|\mathrm{PSL}_2(16)|$, $|\mathrm{PSL}_2(17)|$ or $|\mathrm{PSL}_2(19)|$, a contradiction. Then $\nu_r(|L|) < e(L)$, as desired.

Case 2. Assume that $L = \mathrm{PSL}_n(q)$ and $e(L) = q^{n-1} - 1$, where $n > 2$ and $(n, q) \neq (3, 2), (3, 4)$. Suppose that $q^{\frac{n-1}{4}} - 1 \leqslant 1$. Then $q^{n-1} \leqslant 16$, and so $(n, q) = (3, 3)$, $(4, 2)$ or $(5, 2)$. We have $e(L) \geqslant 7$, and $(|L|, p) = (2^4 \cdot 3^3 \cdot 13, 3)$, $(2^6 \cdot 3^2 \cdot 5 \cdot 7, 2)$ or $(2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31, 2)$. It follows that $\nu_r(|L|) < e(L)$.

Now let $q^{\frac{n-1}{4}} - 1 > 1$. Then $q^{n-1} - 1 = (q^{\frac{n-1}{2}} + 1)(q^{\frac{n-1}{4}} + 1)(q^{\frac{n-1}{4}} - 1) > (q^{\frac{n-1}{2}} + 1)(q^{\frac{n-1}{4}} + 1)$, and so

$$e(L) > (q^{\frac{n-1}{2}} + 1)(q^{\frac{n-1}{4}} + 1) = q^{\frac{3(n-1)}{4}} + q^{\frac{n-1}{2}} + q^{\frac{n-1}{4}} + 1 > q^{\frac{n}{2}} + 2^{\frac{n-1}{2}} + 2.$$

Noting that $|R|$ is a divisor of $\Lambda_n(q)$, we have $\nu_r(|L|) = \nu_r(|R|) \leqslant \nu_r(\Lambda_n(q))$. By Corollary 2.2, $\nu_r(|L|) < q^{\frac{n}{2}} + n - 1$. If $n = 4$ then $e(L) > q^{\frac{n}{2}} + 4 > \nu_r(|L|)$. If $n \neq 4$ then $2^{\frac{n-1}{2}} \geqslant n - 1$, and thus $e(L) > q^{\frac{n}{2}} + n - 1 + 2 > \nu_r(|L|)$.

Case 3. Assume that $L = \mathrm{PSp}_{2m}(q)$, where $m > 1$ and $(m, q) \neq (2, 2), (3, 2)$. Noting that $|R|$ is a divisor of $\Lambda_m(q^2)$, we have $\nu_r(|L|) = \nu_r(|R|) \leqslant \nu_r(\Lambda_m(q^2))$. By (1) of Corollary 2.2, since $q^2 \neq 3$, we have

$$\nu_r(|L|) < m \log_2(q^2) + \nu_r(m!) = 2 \log_2(q^m) + \nu_r(m!).$$

If $q^m \leqslant 15$ then $(m, q) = (2, 3)$; in this case, $r = 2$, $L = \mathrm{PSp}_4(3)$, $\nu_r(|L|) = 6$ and $e(L) = \frac{q^m-1}{2} = 4$, as in part (7). Thus we assume next that $q^m > 15$. Then $\log_2(q^m) \leqslant q^{\frac{m}{2}}$ and so $\nu_r(|L|) < 2q^{\frac{m}{2}} + m - 1$.

Suppose that q is odd. Then $e(L) = \frac{q^m-1}{2}$. If $m > 3$ then $m \leqslant 2^{\frac{m}{2}}$, and so

$$\nu_r(|L|) < 2q^{\frac{m}{2}} + m - 1 \leqslant 2q^{\frac{m}{2}} + 2^{\frac{m}{2}} - 1 < q^{\frac{m+2}{2}} - 1 \leqslant q^{m-1} - 1 < e(L).$$

Assume that $m \leqslant 3$. Then either $(m, q) = (3, 3)$ or $q \geqslant 5$. For $(m, q) = (3, 3)$, we have $\nu_r(|L|) \leqslant 9 < 13 = e(L)$. Now let $q \geqslant 5$. If $m = 2$ then $\nu_r(|L|) < 2q + 1$, yielding $\nu_r(|L|) \leqslant 2q \leqslant \frac{q-1}{2}q < \frac{q^2-1}{2} = e(L)$. If $m = 3$ then $\nu_r(|L|) < 2q^{\frac{3}{2}} + 2$, and thus $\nu_r(|L|) \leqslant 2q^{\frac{3}{2}} + 1 < q^2 + q + 1 \leqslant \frac{q^3-1}{2} = e(L)$.

Suppose that q is even. Then $e(L) = \frac{q^{m-1}(q^{m-1}-1)(q-1)}{2}$. If $m > 3$ then

$$\nu_r(|L|) < 2q^{\frac{m}{2}} + m - 1 \leqslant 2q^{\frac{m}{2}} + 2^{\frac{m}{2}} - 1 \leqslant 3q^{\frac{m}{2}} - 1 < q^{\frac{m}{2}+\frac{7}{4}} - 1 < q^m < e(L).$$

If $m = 2$ then $q \geqslant 4$ and $q^m > 15$, and so $\nu_r(|L|) < 2q + 1 < \frac{q(q-1)^2}{2} = e(L)$. If $m = 3$ then $q \geqslant 4$, and so $\nu_r(|L|) < 2q^{\frac{3}{2}} + 2 < q^2 + q + 2 < 2q^2 < \frac{q^2(q^2-1)(q-1)}{2} = e(L)$.

Case 4. Assume that $L = \mathrm{PSU}_n(q)$, where $n > 2$ and $(n, q) \neq (3, 2), (4, 2), (4, 3)$. Then $e(L) = \frac{q^n-1}{q+1}$ or $\frac{q^n-q}{q+1}$, where n is even or odd respectively. Since $|R|$ is a divisor of $\Lambda_n(q^2)$, we have $\nu_r(|L|) = \nu_r(|R|) \leqslant \nu_r(\Lambda_n(q^2))$. Since $q^2 \neq 3$, by (1) of Corollary 2.2, $\nu_r(|L|) < \log_2(q^{2n}) + n - 1$. If $n = 4$ then $q \geqslant 4$, and so $\nu_r(|L|) < 8q + 3 < (q^2 + 1)(q - 1) = e(L)$. If $n = 3$ then $\nu_r(|L|) < 6q + 2 < q(q - 1) = e(L)$ unless $q < 8$; for $q < 8$, we also have $\nu_r(|L|) < e(L)$ by calculation of the order of L. If $n = 5$ then $\nu_r(|L|) < 10q + 4 < (q^2 + 1)q(q - 1) = e(L)$ unless $q = 2$; for the exception $(n, q) = (5, 2)$, we have $r \in \{3, 5, 11\}$, and $\nu_r(|L|) \leqslant 5 < 10 = e(L)$. If $n = 6$ then $\nu_r(|L|) < 12q + 5 < (q^3 - 1)(q^2 - q + 1) = e(L)$ unless $q = 2$; for the exception $(n, q) = (6, 2)$, we have $r \in \{3, 5, 7, 11\}$, and $\nu_r(|L|) \leqslant 6 < 21 = e(L)$. Now let $n > 6$.

Then $\log_2(q^n) < q^{\frac{n}{2}}$ and $n < 2^{\frac{n}{2}}$, and so

$$\nu_r(|L|) < 2q^{\frac{n}{2}} + 2^{\frac{n}{2}} - 1 < 3q^{\frac{n}{2}} - \frac{2}{3} = \frac{2}{3}(\frac{9}{2}q^{\frac{n}{2}} - 1) < \frac{2}{3}(q^{\frac{2n+9}{4}} - 1) < \frac{q}{q+1}(q^{n-1} - 1) \leqslant e(L).$$

Case 5. Assume that $L = \mathrm{P}\Omega^{\epsilon}_{2m}(q)$, where $\epsilon = \pm$, $m > 3$ and $(m, q, \epsilon) \neq (4, 2, +)$. Then

$$e(L) = (q^{m-1} - 1)(q^{m-2} + 1),\ (q^{m-1} - 1)q^{m-2} \text{ or } (q^{m-1} + 1)(q^{m-2} - 1);$$

in particular, $e(L) > 3q^{m-2}$. Since $|R|$ is a divisor of $\Lambda_m(q^2)$, we have $\nu_r(|L|) = \nu_r(|R|) \leqslant \nu_r(\Lambda_m(q^2))$. Since $q^2 \neq 3$, by (1) of Corollary 2.2, $\nu_r(|L|) < m\log_2(q^2) + m - 1 = 2\log_2(q^m) + m - 1$. Noting that $q^m \geqslant 16$ and $m > 3$, we have $\log_2(q^m) \leqslant q^{\frac{m}{2}}$ and $m \leqslant 2^{\frac{m}{2}}$, and then

$$\nu_r(|L|) < 2\log_2(q^m) + m - 1 \leqslant 3q^{\frac{m}{2}} - 1 < 3q^{m-2} < e(L).$$

Case 6. Assume that $L = \Omega_{2m+1}(q)$, where q is odd, $m > 2$ and $(m, q) \neq (3, 3)$. Then $e(L) = q^{m-1}(q^{m-1} - 1)$ or $q^{2m-2} - 1$. Since $|R|$ is a divisor of $\Lambda_m(q^2)$, we have $\nu_r(|L|) = \nu_r(|R|) \leqslant \nu_r(\Lambda_m(q^2))$. By (1) of Corollary 2.2, $\nu_r(|L|) < m\log_2(q^2) + m - 1 = 2\log_2(q^m) + m - 1$. Since $m > 2$, we have $m < 3^{\frac{m}{2}}$. Noting that $q^m \geqslant 27$, we have $\log_2 q^m < q^{\frac{m}{2}}$, and thus

$$\nu_r(|L|) < 2\log_2 q^m + m - 1 < 2q^{\frac{m}{2}} + 3^{\frac{m}{2}} - 1 \leqslant 3q^{\frac{m}{2}} - 1 \leqslant q^{\frac{m+2}{2}} - 1 < e(L).$$

Case 7. Assume that L is an exceptional simple group of Lie type. Then $|R|$ is a divisor of $\Lambda_m(q^2)$ with m listed as follows:

L	$\mathrm{G}_2(q)$	$\mathrm{F}_4(q)$	$\mathrm{E}_6(q)$	$\mathrm{E}_7(q)$	$\mathrm{E}_8(q)$	${}^2\mathrm{B}_2(q)$	${}^2\mathrm{G}_2(q)$	${}^2\mathrm{F}_4(q)$	${}^3\mathrm{D}_4(q)$	${}^2\mathrm{E}_6(q)$
m	3	6	9	9	15	2	3	6	6	9

Noting that $q^2 \neq 3$, by (1) of Corollary 2.2, $\nu_r(|L|) < m\log_2(q^2) + 2 \leqslant 2mq + m - 1$. Comparing $2mq + m - 1$ and the values of $e(L)$ given in [17, page 188, Table 5.3.A], we have $\nu_r(|L|) < e(L)$, the details are omitted here. □

3. Simple subgroups in extensions of a simple group

Let X and Y be groups. Denote by $X.Y$ an extension of X by Y, while $X{:}Y$ stands for a split extension. By $X \leqslant Y$, $X \trianglelefteq Y$, $X\ \mathsf{char}\ Y$ and $X \lesssim Y$ we mean that X is a subgroup, a normal subgroup, a characteristic subgroup and isomorphic to a subgroup of Y, respectively. When $X \leqslant Y$ or $X \trianglelefteq Y$ but $X \neq Y$, we write $X < Y$ or $X \lhd Y$, respectively. We call X a section of Y if X is isomorphic a quotient group of some subgroup of Y. The automorphism group and inner automorphism group of X are denoted by $\mathsf{Aut}(X)$ and $\mathsf{Inn}(X)$, respectively, and let $\mathsf{Out}(X) = \mathsf{Aut}(X)/\mathsf{Inn}(X)$. As a consequence of the *Classification of Finite Simple Groups*, the *Schreier Conjecture* is true, see [7, Appendix A] for example. Thus, if X is a finite simple group then $\mathsf{Out}(X)$ is solvable. In addition, $\mathsf{Inn}(X) \cong X/\mathbf{Z}(X)$, where $\mathbf{Z}(X)$ is the center of X.

In the following, N is assumed to be a finite group. For $Y, X \leqslant N$, denote by $\mathbf{C}_X(Y)$ and $\mathbf{N}_X(Y)$ the centralizer and normalizer of Y in X, respectively. Clearly, $\mathbf{C}_X(Y) = \mathbf{C}_N(Y) \cap X$ and $\mathbf{N}_X(Y) = \mathbf{N}_N(Y) \cap X$. It is easily shown that both $\mathbf{C}_X(Y)$ and $\mathbf{N}_X(Y)$ are normal (or characteristic) subgroups of N provided that X and Y are normal (or characteristic) in N.

Lemma 3.1. *Assume that $K \trianglelefteq N$ and N/K is a nonabelian simple group. Suppose that $|K|^2$ divides of $|N|$. Then one of the following holds:*

1. *$N \cong K \times K$;*
2. *$K \,\mathsf{char}\, N$ and $N = KC$, where $C \,\mathsf{char}\, N$, $C = C'$ and $\mathsf{rad}(C) = K \cap C$.*

Proof. Assume first that $K^\sigma \neq K$ for some $\sigma \in \mathsf{Aut}(N)$. Clearly, $K^\sigma \trianglelefteq N^\sigma = N$, and so $K^\sigma K/K \trianglelefteq N/K$. Since N/K is simple, we have $N/K = (K^\sigma K)/K \cong K^\sigma/(K \cap K^\sigma)$. In particular, $|N| = |K||K^\sigma : (K \cap K^\sigma)|$. Noting that $|K|^2$ divides $|N|$, it follows that $K \cap K^\sigma = 1$ and $N = KK^\sigma = K \times K^\sigma$. Then part (1) of this lemma follows.

Now let $K \,\mathsf{char}\, N$. Choose a minimal member C among those characteristic subgroups of N with $N = KC$. Then $N/K = KC/K \cong C/(K \cap C)$, and $N/K = (N/K)' = (KC')/K$. In particular, $N = KC'$, and so $C = C'$ by the choice of C. We next show that $K \cap C$ is solvable. Note that $(K \cap C) \,\mathsf{char}\, N$.

Suppose that $K \cap C$ is insolvable. Choose I, $J \,\mathsf{char}\, (K \cap C)$ with $I < J$ and $J/I \cong T^l$, where $l \geqslant 1$ and T is a nonabelian simple group. Clearly, I, $J \,\mathsf{char}\, N$, and $\mathbf{C}_{C/I}(J/I) \cap (J/I) = 1$. Set $C_1/I = \mathbf{C}_{C/I}(J/I)$. Then $C_1 \,\mathsf{char}\, N$, $C_1 < C$, and $N \neq KC_1$ by the choice of C. Since N/K is simple, we have $(KC_1)/K = 1$, and so $C_1 \leqslant K \cap C$. Considering the action of C/I on J/I by conjugation, we have

$$C/(C_1J) \cong (C/I)/(C_1J/I) \lesssim \mathsf{Out}(T^l) = \mathsf{Out}(T)^l{:}\mathrm{S}_l,$$

where S_l is the symmetric group of degree l. Note that

$$N/K = KC/K \cong C/(K \cap C) \cong (C/(C_1J))/((K \cap C)/(C_1J)).$$

It follows that N/K is a section of $\mathsf{Out}(T)^l{:}\mathrm{S}_l$. Noting that $\mathsf{Out}(T)$ is solvable, it follows that N/K is a section of S_l, and so $|N/K|$ divides $l!$. Since $|K|^2$ divides $|N|$, we conclude that $|T|^l$ divides $|N/K|$, and thus $|T|^l$ divides $l!$. Then, for a prime divisor r of $|T|$, we have $l \leqslant \nu_r(|T|^l) \leqslant \nu_r(l!)$. By Legendre's formula, $\nu_r(l!) = \frac{l - s_r(l)}{r-1} \leqslant l - 1$, and so $l \leqslant l - 1$, a contradiction. Then $K \cap C$ is solvable, and part (2) of this lemma is true. □

For a finite group X, denote by $X^{(\infty)}$ the intersection of all subgroups appearing in the derived series of X.

Lemma 3.2. *Assume that N contains a normal subgroup $I \cong \mathbb{Z}_r^k$ and a nonabelian simple subgroup T such that r^k is a divisor of $|T|$, where r is a prime and $k \geqslant 1$. Suppose that N/I is a covering group of some simple group L. Then either $N = \mathbf{C}_N(I)$, or $\mathbf{C}_N(I) \leqslant \mathsf{rad}(N)$, $T \lesssim N/\mathbf{C}_N(I) \lesssim \mathrm{SL}_k(r)$ and one of the following holds:*

1. *$N = I{:}T = \mathbb{Z}_2^k{:}\mathrm{A}_{2^e}$, where $e \geqslant 3$, and either $k = 2^e - 2$ or $e = 3$ and $k \in \{4, 5\}$;*
2. *either $N = I{:}T \cong \mathrm{AGL}_3(2)$, or $N = I{:}T = \mathbb{Z}_2^6{:}\mathrm{PSp}_4(3) \lesssim \mathrm{AGL}_6(2)$;*
3. *L is a simple group of Lie type over a finite field of characteristic 2, $N \neq I{:}T = \mathbb{Z}_2^k{:}\mathrm{A}_{2^e}$, where $e \geqslant 3$, and either $k = 2^e - 2$ or $k \in \{4, 5\}$ and $e = 3$;*
4. *T and L are simple groups of Lie type over finite fields of characteristic r.*

Proof. Note that $\mathbf{C}_N(I)/I \trianglelefteq N/I$. Since N/I is quasisimple, either $\mathbf{C}_N(I)/I \leqslant \mathbf{Z}(N/I)$ or $\mathbf{C}_N(I)/I = N/I$, refer to [1, page 157, (31.2)]. For the latter, we have $N = \mathbf{C}_N(I)$. Thus we assume that $\mathbf{C}_N(I)/I \leqslant \mathbf{Z}(N/I)$. In particular, $\mathbf{C}_N(I) \leqslant \mathsf{rad}(N)$.

Now consider the action of N on I by conjugation, and let $\widehat{N}$ be the resulting subgroup of $\mathsf{Aut}(I)$. We have $\widehat{N} \cong N/C_N(I) \cong (N/I)/(C_N(I)/I)$. Then $\widehat{N}$ is a covering group of L, and N/I is a central extension of $\widehat{N}$. Let $\widehat{T}$ be the image of T in $\widehat{N}$. Since $T \cap \mathsf{rad}(N) = 1$, we have $\widehat{T} \cong T\mathbf{C}_N(I)/C_N(I) \cong T$, and so

$T \lesssim \widehat{N} \lesssim \mathrm{SL}_k(r)$. Since r^k is a divisor of $|T|$, noting that $T \cong T\mathsf{rad}(N)/\mathsf{rad}(N) \leqslant N/\mathsf{rad}(N) \cong L$, we have $k \leqslant \nu_r(|T|) \leqslant \nu_r(|L|)$. Further, if $T \cong L$ then $N = \mathsf{rad}(N){:}T$ and $N/I = (\mathsf{rad}(N)/I){:}(TI/I)$, since N/I is a covering group of $L \cong TI/I$, we have $N/I = (N/I)^{(\infty)} = TI/I$, yielding $\mathsf{rad}(N)/I = 1$, and so $I = \mathsf{rad}(N)$, and $L \cong T \cong TI/I = N/\mathbf{C}_N(I) \cong \widehat{N}$.

Case 1. Assume that $L \cong \mathrm{A}_n$ for some $n \geq 5$. Then

$$k \leqslant \nu_r(|L|) = \nu_r(\frac{n!}{2}) = \nu_r(n!) - (2 - (2, r-1)).$$

By Legendre's formula, we have $k \leqslant \frac{n-s_r(n)}{r-1} - (2-(2,r-1))$. On the other hand, since $\widehat{N} \lesssim \mathrm{SL}_k(r)$, a lower bound for k is given by [17, Propositions 5.3.2 and 5.3.7].

Suppose that $n \leqslant 8$. Check the subgroups of A_n with order divisible by r^k for all possible values of k. Using GAP [29], computation shows that $T \cong L = \mathrm{A}_8$, $r = 2$ and $k \in \{4, 5, 6\}$. Then $N = I{:}T$, desired as in (1) of this lemma.

Now let $n \geq 9$. Then $k \geqslant n-2$ by [17, page 186, Proposition 5.3.7], and thus $n-2 \leqslant k \leqslant \frac{n-s_r(n)}{r-1} - (2-(2,r-1))$. It follows that $k = n-2$, $r = 2$, n is a power of 2, and $\frac{|L|}{2^k}$ is odd. In particular, T is isomorphic to a simple subgroup of A_n with odd index. By [18, Theorem 1.2], we have $T \cong L = \mathrm{A}_n$, and thus $N = \mathbb{Z}_2^{n-2}{:}\mathrm{A}_n$ as in (1).

Case 2. Assume that L is one of the 26 sporadic simple groups. Then the lower bound for k is given as in [17, page 187, Proposition 5.3.8]. Checking the orders of sporadic simple groups, we conclude that $r = 2$ and one of the following holds: $L = \mathrm{M}_{12}$ with $k = 6$, $L = \mathrm{M}_{22}$ with $k \in \{6, 7\}$, $L = \mathrm{J}_2$ with $k \in \{6, 7\}$, $L = \mathrm{Suz}$ with $k \in \{12, 13\}$. Recall that $\widehat{N} \lesssim \mathrm{SL}_k(2)$ and $\widehat{N}$ is a covering group of L. Then $|L|$ is a divisor of $|\mathrm{SL}_k(2)|$, and so $|L : Q|$ is a divisor of $\Lambda_k(2)$, where Q is a Sylow 2-subgroup of L. If $k \in \{6, 7\}$ then $\Lambda_k(2)$ is not divisible by 5^2 or 11, and thus $L \neq \mathrm{M}_{12}$, M_{22} or J_2. This forces that $L = \mathrm{Suz}$ and $k \in \{12, 13\}$. By [23, Corollary 4.3], since $\widehat{N} \lesssim \mathrm{SL}_k(2)$, we have $|\mathrm{Suz}| \leqslant |\widehat{N}| < 2^{2k+4} \leqslant 2^{30}$, which is impossible.

Case 3. Assume that L is a simple group of Lie type over a finite field of characteristic p, and $L \ncong \mathrm{A}_n$ for any $n \geqslant 5$.

Subcase 3.1. Suppose first that $r \neq p$. Recalling that $\widehat{N} \lesssim \mathrm{SL}_k(r)$, by [17, Proposition 5.3.2 and Theoren 5.3.9], $k \geqslant e(L)$, where $e(L)$ is given as in [17, Table 5.3.A]. Then $e(L) \leqslant k \leqslant \nu_r(|T|) \leqslant \nu_r(|L|)$. Thus L appears in the exceptions listed in Lemma 2.3. Note that $|L|$ is a divisor of $|\mathrm{SL}_k(r)|$; in particular, $|L : Q|$ is a divisor of $\Lambda_k(r)$, where Q is a Sylow r-subgroup of L. In view this, the groups in (1), (2), (4) and (5) of Lemma 2.3 are easily excluded.

Assume that L is described as in (3), (6) or (7) of Lemma 2.3. Checking simple subgroups of L with order divisible by r^k, we conclude that $L \cong T \lesssim \mathrm{SL}_k(r)$, and thus $N = I{:}T$. For (3) of Lemma 2.3, we have $r = 3$ and $k = 4$; however, computation using GAP shows that $\mathrm{SL}_4(3)$ has no subgroup isomorphic to $\mathrm{PSU}_4(2)$. For (6) of Lemma 2.3, we have $r = 2$, $k = 3$ and $L = \mathrm{PSL}_2(7) \cong \mathrm{GL}_3(2)$. For (7) of Lemma 2.3, we have $r = 2$, $k = 6$ and $L = \mathrm{PSp}_4(3)$. Then part (2) of this lemma follows.

Subcase 3.2. Now let $r = p$. Assume that T is an alternating group or a sporadic simple group. Similarly as Cases 1 and 2, we have $r = 2$, $T \cong \mathrm{A}_{2^e}$ for some $e \geqslant 3$, and either $k = 2^e - 2$ or $k \in \{4, 5\}$ and $e = 3$. This gives part (3) of this lemma.

Assume that T is a simple group of Lie type over a finite field of characteristic p'. If $p' = r$ then part (4) of this lemma occurs. Now let $r \neq p'$. Then, by Lemma 2.3,

T and r are known. By a similar argument as in the case where $r \neq p$, we conclude that N is desired as in part (2) of this lemma. This completes the proof. □

Lemma 3.3. *Let N be a perfect group with $L := N/\mathsf{rad}(N)$ simple. Assume that N contains a nonabelian simple subgroup T such that $|\mathsf{rad}(N)|$ is a divisor of $|T|$. Then $N/\mathbf{O}_r(N)$ is a covering group of L for some prime divisor r of $|T|$, and either N is a covering group of L or one of the following holds:*

(1) *$N = \mathsf{rad}(N)T = [2^k]{:}\mathrm{A}_8$ or $\mathbb{Z}_2^{n-2}{:}\mathrm{A}_n$, where $k \in \{4,5,6\}$ and $n = 2^m$ for some integer $m \geqslant 4$;*
(2) *$N = IT = \mathbb{Z}_2^3{:}\mathrm{PSL}_3(2) \cong \mathrm{AGL}_3(2)$ or $N = IT = \mathbb{Z}_2^6{:}\mathrm{PSp}_4(3) \lesssim \mathrm{AGL}_6(2)$;*
(3) *L is a simple group of Lie type over a finite field of characteristic 2, $L \not\cong T$, and $\mathbf{O}_r(N)T = [2^k]{:}\mathrm{A}_8$ or $\mathbb{Z}_2^{n-2}{:}\mathrm{A}_n$, where k and n are as in part (1);*
(4) *T and L are simple groups of Lie type with characteristic r.*

Proof. Let $K = \mathsf{rad}(N)$, and choose $J \,\mathsf{char}\, K$ such that N/J is a covering group of L with maximal order as possible. If $J = 1$ then the lemma is true. Thus we assume that $J \neq 1$ in the following.

Let $J_0 \,\mathsf{char}\, J$ with $J/J_0 \cong \mathbb{Z}_r^k$ for some prime r and integer $k \geqslant 1$. Then Lemma 3.2 works for N/J_0, J/J_0 and TJ_0/J_0. Suppose that $N/J_0 = \mathbf{C}_{N/J_0}(J/J_0)$. Then N/J_0 is a perfect central extension of N/J. It follows that N/J_0 is a perfect central extension of L, refer to [1, page 167, (33.5)]. Thus N/J_0 is a covering group of L, which contradicts the choice of J. Therefore, $N/J_0 \neq \mathbf{C}_{N/J_0}(J/J_0)$. Let $\overline{N} = N/J_0$, $\overline{T} = TJ_0/J_0$ and $\overline{J} = J/J_0$. Then $T \cong \overline{T} \lesssim \overline{N}/\mathbf{C}_{\overline{N}}(\overline{J}) \lesssim \mathrm{SL}_k(r)$, $\overline{N}/\overline{J} \cong N/J$ and one of the following holds:

(i) $\overline{N} = \overline{J}\,\overline{T} = \mathbb{Z}_2^k{:}\mathrm{A}_n$, where $n = 2^m$ for some $m \geqslant 3$, and either $k = n-2$ or $k \in \{4,5\}$ with $n = 8$;
(ii) $\overline{N} = \overline{J}\,\overline{T} = \mathbb{Z}_2^3{:}\mathrm{PSL}_3(2)$ or $\mathbb{Z}_2^6{:}\mathrm{PSp}_4(3)$ with $k = 3$ or 6, respectively;
(iii) L is a simple group of Lie type over a finite field of characteristic 2, $\overline{J}\,\overline{T} = \mathbb{Z}_2^k{:}\mathrm{A}_n$, where $n = 2^m$ for some $m \geqslant 3$, and either $k = n-2$ or $k \in \{4,5\}$ with $n = 8$;
(iv) $\overline{T}$ and L are simple groups of Lie type over finite fields of characteristic r.

Case 1. Suppose that J is an r-group. Then $N/\mathbf{O}_r(N) \cong (N/J)/(\mathbf{O}_r(N)/J)$, and so $N/\mathbf{O}_r(N)$ is a covering group of L. For (iv), we get part (4) of this lemma. Assume that one of (i)-(iii) holds, in particular, $r = 2$. Then $\mathbb{Z}_2^k \cong \overline{J} = J/J_0 = \mathbf{O}_2(\overline{N}) = \mathbf{O}_2(N)/J_0$, and so $|\mathbf{O}_2(N)| = 2^k|J_0| = |J|$. Note that $\nu_2(|\mathrm{A}_n|) = n-2$, $\nu_2(|\mathrm{PSL}_3(2)|) = 3$ and $\nu_2(|\mathrm{PSp}_4(3)|) = 6$. It follows that either $\nu_2(|T|) = k$, or $T = \mathrm{A}_8$ and $k \in \{4,5\}$. Since $|\mathbf{O}_2(N)|$ is a divisor of $|T|$, we conclude that either $|\mathbf{O}_2(N)| = 2^k$, yielding $J_0 = 1$ and $\mathbf{O}_2(N) = J \cong \mathbb{Z}_2^k$, or $T \cong \mathrm{A}_8$ and $2^4 \leqslant |\mathbf{O}_2(N)| \leqslant 2^6$. Then one of (1)-(3) of this lemma holds.

Case 2. Suppose that J is not an r-group. Let $I = \mathbf{O}^r(J)$, the normal subgroup of J such that J/I is an r-group with maximal order. Then $1 \neq I \,\mathsf{char}\, N$. Choose $I_0 \,\mathsf{char}\, I$ such that $I/I_0 \cong \mathbb{Z}_p^l$ for some prime p and integer $l \geq 1$. By the choice of I, we have $r \neq p$. Assume that $TI_0/I_0 \leqslant \mathbf{C}_{N/I_0}(I/I_0)$. Since $(N/I_0)/(K/I_0)$ is simple and N/I_0 is perfect, we have $N/I_0 = (K/I_0)\mathbf{C}_{N/I_0}(I/I_0) = \mathbf{C}_{N/I_0}(I/I_0)$. In particular, I/I_0 lies in the center of J/I_0. Then $J/I_0 = \mathbf{O}_r(J/I_0) \times I/I_0$. Setting $\mathbf{O}_r(J/I_0) = J_1/I_0$, we have

$$N/J_1 \cong (N/I_0)/(J_1/I_0) = \mathbf{C}_{(N/I_0)/((J_1/I_0))}((I/I_0)(J_1/I_0)/(J_1/I_0)) \cong \mathbf{C}_{N/J_1}(J/J_1).$$

Thus N/J_1 is a perfect central extension of N/J. It follows that N/J_1 is a perfect central extension of L, which contradicts the choice of J. Therefore, $TI_0/I_0 \not\leqslant \mathbf{C}_{N/I_0}(I/I_0)$, and so $TI_0/I_0 \not\leqslant \mathbf{C}_{TI/I_0}(I/I_0)$. We have $T \cong TI_0/I_0 \lesssim \mathrm{SL}_l(p)$.

Now consider the group $TI/I_0 = (I/I_0){:}(TI_0/I_0)$. Applying Lemma 3.2 to the triple $(TI/I_0, TI_0/I_0, I/I_0)$, we conclude that one of the following holds:

(v) $p = 2$ and TI_0/I_0 is isomorphic to one of A_{2^e}, $\mathrm{PSL}_3(2)$ and $\mathrm{PSp}_4(3)$;
(vi) T is isomorphic to a simple group of Lie type with characteristic p.

Assume first that p is odd. Then T is isomorphic to a simple group of Lie type with characteristic p. Recall that either $r = 2$ and T is one of A_{2^m}, $\mathrm{PSL}_3(2)$ and $\mathrm{PSp}_4(3)$, or T is a simple group of Lie type with characteristic r, see (i)-(iv) above. It follows from [17, Proposition 2.9.1 and Theorem 5.1.1] that $r = 2$, and (T, p) is one of $(\mathrm{PSL}_2(4), 5)$, $(\mathrm{PSL}_3(2), 7)$, $(\mathrm{Sp}_4(2)', 3)$, $(\mathrm{PSU}_4(2), 3)$, $(\mathrm{PSL}_2(8), 3)$ and $(\mathrm{G}_2(2)', 3)$. Noting that $r^k p^l$ is a divisor of $|T|$, it follows that none of these groups satisfies both $T \lesssim \mathrm{SL}_k(r)$ and $T \lesssim \mathrm{SL}_l(p)$, a contradiction. Now let $p = 2$. Then r is odd as $r \neq p$, and so T is a simple group of Lie type over a finite field of characteristic r, which leads to a similar contradiction as above. This completes the proof. □

4. Proof of Theorem 1.2

In this section, we assume that $\Gamma = (V, E)$ *is a connected G-arc-transitive graph of valency $d \geqslant 3$, and either d is a prime or Γ is $(G, 2)$-arc-transitive.* For $\alpha \in V$, let $G_\alpha = \{g \in G \mid \alpha^g = \alpha\}$ and $\Gamma(\alpha) = \{\beta \in V \mid \{\alpha, \beta\} \in E\}$, called the *stabilizer* and *neighborhood* of α in G and in Γ, respectively. Then Γ is $(G, 2)$-arc-transitive if and only if G_α acts 2-transitively on $\Gamma(\alpha)$. Denote by $G_\alpha^{\Gamma(\alpha)}$ the permutation group induced by G_α on $\Gamma(\alpha)$. Then either $G_\alpha^{\Gamma(\alpha)}$ is 2-transitive on $\Gamma(\alpha)$, or d is a prime and $G_\alpha^{\Gamma(\alpha)} \leqslant \mathrm{AGL}_1(d)$, refer to [7, page 99, Corollary 3.5B]. In particular, by [7, page 107, Theorem 4.1B], the socle $\mathsf{soc}(G_\alpha^{\Gamma(\alpha)})$ is either simple or regular on $\Gamma(\alpha)$, and thus $\mathsf{soc}(G_\alpha^{\Gamma(\alpha)})$ is the unique minimal normal subgroup of $G_\alpha^{\Gamma(\alpha)}$. In addition, $\mathbf{C}_{G_\alpha^{\Gamma(\alpha)}}(\mathsf{soc}(G_\alpha^{\Gamma(\alpha)})) = 1$ or $\mathsf{soc}(G_\alpha^{\Gamma(\alpha)})$ by [7, page 114, Theorem 4.3B].

We shall proceed by analyzing the actions on V of normal subgroups of the group G. Let $N \trianglelefteq G$. By [28, Theorem 4.1], only one of the following holds:

(I) Γ is a bipartite graph, and the N-orbits are the two parts of the bipartition;
(II) N is semiregular and has at least three orbits on V, in particular, $|N|$ is a proper divisor of $|V|$;
(III) N is transitive on V; in this case, if K is an intransitive normal subgroup of N and N_α acts primitively on $\Gamma(\alpha)$ then (I) or (II) holds for Γ with G and N replaced by N and K, respectively.

In particular, if $N_\alpha \neq 1$ for some $\alpha \in V$ then N has at most two orbits on V.

Lemma 4.1. *Assume that $N \trianglelefteq G$ and $N_\alpha \neq 1$, where $\alpha \in V$. Then N has at most two orbits on V, N_α acts transitively on $\Gamma(\alpha)$, $\mathsf{soc}(N_\alpha^{\Gamma(\alpha)}) = \mathsf{soc}(G_\alpha^{\Gamma(\alpha)})$, and one of the following holds:*

(1) *N_α acts 2-transitively on $\Gamma(\alpha)$;*
(2) *N_α acts primitively on $\Gamma(\alpha)$, and either*
  (i) *$d = 28$, $N_\alpha^{\Gamma(\alpha)} = \mathrm{PSL}_2(8)$, $G_\alpha^{\Gamma(\alpha)} = \mathrm{P\Gamma L}_2(8)$; or*
  (ii) *$d = p^2$, $\mathbb{Z}_p^2{:}\mathrm{SL}_2(5) \trianglelefteq N_\alpha^{\Gamma(\alpha)} \trianglelefteq G_\alpha^{\Gamma(\alpha)} \trianglelefteq \mathbb{Z}_p^2{:}(\mathbb{Z}_{p-1}.\mathrm{PSL}_2(5))$, where $p \in \{19, 29, 59\}$;*

(3) $d = p^k$, $N_\alpha^{\Gamma(\alpha)} = \mathbb{Z}_p^k{:}H$, *where* H *is solvable and acts faithfully and semiregularly on* $\mathbb{Z}_p^k \setminus \{1\}$ *by conjugation, where* p *is a prime and* $k \geqslant 1$.

Proof. Since $N_\alpha \neq 1$, by [20, Lemma 2.5], N has at most two orbits on V, and N_α acts transitively on $\Gamma(\alpha)$. Note that $N_\alpha^{\Gamma(\alpha)}$ is a transitive normal subgroup of $G_\alpha^{\Gamma(\alpha)}$. Since $\mathsf{soc}(N_\alpha^{\Gamma(\alpha)})$ is a characteristic subgroup of $N_\alpha^{\Gamma(\alpha)}$, we have $\mathsf{soc}(N_\alpha^{\Gamma(\alpha)}) \trianglelefteq G_\alpha^{\Gamma(\alpha)}$, and so $\mathsf{soc}(N_\alpha^{\Gamma(\alpha)}) \cap \mathsf{soc}(G_\alpha^{\Gamma(\alpha)}) \trianglelefteq G_\alpha^{\Gamma(\alpha)}$. Recall that $\mathsf{soc}(G_\alpha^{\Gamma(\alpha)})$ is the unique minimal normal subgroup of $G_\alpha^{\Gamma(\alpha)}$. We have $\mathsf{soc}(N_\alpha^{\Gamma(\alpha)}) \geqslant \mathsf{soc}(G_\alpha^{\Gamma(\alpha)})$. Let K be an arbitrary minimal normal subgroup of $N_\alpha^{\Gamma(\alpha)}$. Since $\mathsf{soc}(G_\alpha^{\Gamma(\alpha)}) \cap K \trianglelefteq N_\alpha^{\Gamma(\alpha)}$, we have either $K \leqslant \mathsf{soc}(G_\alpha^{\Gamma(\alpha)})$ or $K \cap \mathsf{soc}(G_\alpha^{\Gamma(\alpha)}) = 1$. The latter case implies that $K \leqslant \mathbf{C}_{G_\alpha^{\Gamma(\alpha)}}(\mathsf{soc}(G_\alpha^{\Gamma(\alpha)})) = 1$ or $\mathsf{soc}(G_\alpha^{\Gamma(\alpha)})$, a contradiction. Thus $K \leqslant \mathsf{soc}(G_\alpha^{\Gamma(\alpha)})$. It follows that $\mathsf{soc}(N_\alpha^{\Gamma(\alpha)}) \leqslant \mathsf{soc}(G_\alpha^{\Gamma(\alpha)})$, and so $\mathsf{soc}(N_\alpha^{\Gamma(\alpha)}) = \mathsf{soc}(G_\alpha^{\Gamma(\alpha)})$.

Now we show that one of (1)-(3) holds. If $G_\alpha^{\Gamma(\alpha)}$ is not 2-transitive, then d is a prime, and part (3) occurs with $k = 1$, refer to [7, Corollary 3.5B]. Thus assume that $G_\alpha^{\Gamma(\alpha)}$ is 2-transitive. By [1, page 191, (35.25)] and [7, page 215, Theorem 7.2C], either $N_\alpha^{\Gamma(\alpha)}$ is a primitive subgroup of $G_\alpha^{\Gamma(\alpha)}$, or $N_\alpha^{\Gamma(\alpha)} = K{:}H$ with $K = \mathsf{soc}(G_\alpha^{\Gamma(\alpha)}) \cong \mathbb{Z}_p^k$ and H acting semiregularly on $K \setminus \{1\}$ by conjugation, where p is a prime and $k \geqslant 2$. Then the lemma follows from checking one by one the 2-transitive permutation groups listed in [3, pages 195-197, Tables 7.3 and 7.4], see also [22, Corollary 2.5]. □

Let $N \trianglelefteq G$. For $\alpha \in V$, let $N_\alpha^{[1]}$ be the kernel of N_α acting on $\Gamma(\alpha)$. Then $N_\alpha^{\Gamma(\alpha)} \cong N_\alpha/N_\alpha^{[1]}$. Let $\beta \in \Gamma(\alpha)$. We have $(N_\alpha^{\Gamma(\alpha)})_\beta = (N_{\alpha\beta})^{\Gamma(\alpha)} \cong N_{\alpha\beta}/N_\alpha^{[1]}$.

Lemma 4.2. *Let* $N \trianglelefteq G$ *and* $\{\alpha, \beta\} \in E$. *Then every insolvable composition factor of* N_α *is (isomorphic to) an insolvable composition factor of either* $N_\alpha^{\Gamma(\alpha)}$ *or* $(N_\alpha^{\Gamma(\alpha)})_\beta$. *In particular,* N_α *is solvable if and only if* $N_\alpha^{\Gamma(\alpha)}$ *is solvable.*

Proof. Pick $x \in G$ with $(\alpha, \beta)^x = (\beta, \alpha)$. Then

$$\Gamma(\alpha)^x = \Gamma(\beta),\ N_\beta = x^{-1}N_\alpha x,\ N_\beta^{[1]} = x^{-1}N_\alpha^{[1]}x \text{ and } N_{\alpha\beta} = x^{-1}N_{\alpha\beta}x.$$

It follows that

$$(N_\alpha^{\Gamma(\alpha)})_\beta \cong N_{\alpha\beta}/N_\alpha^{[1]} \cong N_{\alpha\beta}/N_\beta^{[1]} \cong (N_{\alpha\beta})^{\Gamma(\beta)} = (N_\beta^{\Gamma(\beta)})_\alpha.$$

Noting that $N_\alpha^{[1]} \trianglelefteq N_{\alpha\beta}$, we have $(N_\alpha^{[1]})^{\Gamma(\beta)} \trianglelefteq (N_{\alpha\beta})^{\Gamma(\beta)} = (N_\beta^{\Gamma(\beta)})_\alpha$. Put $N_{\alpha\beta}^{[1]} = N_\alpha^{[1]} \cap N_\beta^{[1]}$. Then $(N_\alpha^{[1]})^{\Gamma(\beta)} \cong N_\alpha^{[1]}N_\beta^{[1]}/N_\beta^{[1]} \cong N_\alpha^{[1]}/N_{\alpha\beta}^{[1]}$. Thus,

$$N_\alpha^{[1]}/N_{\alpha\beta}^{[1]} \cong (N_\alpha^{[1]})^{\Gamma(\beta)} \trianglelefteq (N_\beta^{\Gamma(\beta)})_\alpha \cong (N_\alpha^{\Gamma(\alpha)})_\beta. \tag{4.1}$$

By [14, Corollary 2.3], $G_{\alpha\beta}^{[1]}$ has a prime power order. Then $G_{\alpha\beta}^{[1]}$ is solvable, and so is $N_{\alpha\beta}^{[1]}$. Recalling that $N_\alpha^{\Gamma(\alpha)} \cong N_\alpha/N_\alpha^{[1]}$, the lemma follows from (4.1). □

Let $N \lhd G$, and suppose that N has at least three orbits on V. Set $V_N = \{\alpha^N \mid \alpha \in V\}$. Define the quotient graph $\Gamma_{G/N}$ with vertex set V_N and edge set $E_N := \{\{\alpha^N, \beta^N\} \mid \{\alpha, \beta\} \in E\}$. For $X \leqslant G$, let X^{V_N} be the subgroup of $\mathsf{Aut}(\Gamma_N)$ induced by X. By [28, Theorem 4.1], N is semiregular on V, and N is the kernel of G acting on V_N. Then $X^{V_N} \cong NX/N \cong X/(X \cap N)$. Further, we have the following lemma.

Lemma 4.3. *Let* $N \lhd G$ *and* $X \leqslant G$. *Assume that* N *has at least three orbits on* V. *Then the following statements hold:*

(1) *$X^{V_N} \cong NX/N$, N is semiregular on V, and $\Gamma_{G/N}$ has valency d; in particular, $|N|$ is a proper divisor of $|V|$; and*
(2) *$(NX)_\alpha \cong (X^{V_N})_{\alpha^N} \cong X_{\alpha^N}/(N\cap X)$, and if X is transitive on V then $|N|$ is a divisor of $|(X^{V_N})_{\alpha^N}||N\cap X|$; and*
(3) *$\Gamma_{G/N}$ is $(X^{V_N},2)$-arc-transitive if and only if Γ is $(NX,2)$-arc-transitive; and*
(4) *$\Gamma_{G/N}$ is $(G^{V_N},2)$-arc-transitive, or d is a prime and $\Gamma_{G/N}$ is G^{V_N}-arc-transitive.*

Proof. In view of [28, Theorem 4.1], we need only prove (2). Noting that $(NX)_{\alpha^N} = NX_{\alpha^N}$ and $N\cap X_{\alpha^N} = N\cap X$, we have $(X^{V_N})_{\alpha^N} \cong NX_{\alpha^N}/N \cong X_{\alpha^N}/(N\cap X)$. Since $(NX)_{\alpha^N} = N(NX)_\alpha$, we get

$$(NX)_\alpha \cong N(NX)_\alpha/N = (NX)_{\alpha^N}/N \cong (X^{V_N})_{\alpha^N} \cong X_{\alpha^N}/(N\cap X).$$

If X is transitive on V then $NX = X(NX)_\alpha$, and so

$$|N : N\cap X| = |NX : X| = |X(NX)_\alpha : X| = |(NX)_\alpha : X_\alpha|,$$

yielding $|N| = |(NX)_\alpha : X_\alpha||N\cap X| = \frac{|(X^{V_N})_{\alpha^N}||N\cap X|}{|X_\alpha|}$. Thus (2) holds. □

Lemma 4.4. *Let $K, N \trianglelefteq G$ and $I = K\cap N$. Assume that K has at least three orbits on V, and N is transitive on V. Then K/I is a homomorphic image of $(N^{V_K})_{\alpha^K}$.*

Proof. For $X \leqslant G$, let $\overline{X} = XI/I$, and identify $\overline{X}$ with a subgroup of $\mathsf{Aut}(\Gamma_{G/I})$. Then Lemma 4.3 (1) and (4) work for the triples (Γ, G, I) and $(\Gamma_{G/I}, \overline{G}, \overline{K})$. Let $\alpha\in V$ and $\overline{\alpha} = \alpha^I$. Then $\overline{K}$ is regular on $\overline{\alpha}^{\overline{K}}$, and $\overline{N}_{\overline{\alpha}^{\overline{K}}}$ acts transitively on $\overline{\alpha}^{\overline{K}}$. Noting that $(\overline{K}\,\overline{N})_{\overline{\alpha}^{\overline{K}}} = \overline{K}\,\overline{N}_{\overline{\alpha}^{\overline{K}}} = \overline{K}\times \overline{N}_{\overline{\alpha}^{\overline{K}}}$, it follows from [7, Theorem 4.2A] that $\overline{N}_{\overline{\alpha}^{\overline{K}}}$ induces a regular permutation group isomorphic to $\overline{K}$ on $\overline{\alpha}^{\overline{K}}$. Then $\overline{N}_{\overline{\alpha}^{\overline{K}}}$ has a quotient group isomorphic to $\overline{K}$. Clearly, α^K equals to the union of I-orbits involved in $\overline{\alpha}^{\overline{K}}$. It follows that $\overline{N}_{\overline{\alpha}^{\overline{K}}} = N_{\alpha^K}/I$. Then

$$\overline{N}_{\overline{\alpha}^{\overline{K}}} \cong \overline{K}\,\overline{N}_{\overline{\alpha}^{\overline{K}}}/\overline{K} = (K/I)(N_{\alpha^K}/I)/(K/I) \cong KN_{\alpha^K}/K \cong (N^{V_K})_{\alpha^K},$$

and the lemma follows. □

Recall that a permutation group is quasiprimitive if its minimal normal subgroups are all transitive.

Lemma 4.5. *The group G has at most one transitive minimal normal subgroup.*

Proof. Suppose that G has distinct transitive minimal normal subgroups M and N. Then $M\cap N = 1$, and so M and N centralize each other. Thus M and N are nonabelian and regular on V, and $\mathbf{C}_G(N) = M$, refer to [7, pp.108-109, Lemma 4.2A and Theorem 4.2A]. In particular, M and N are the only minimal normal subgroups of G. Then G is quasiprimitive on V. By [27, Theorem 2], Γ is not $(G,2)$-transitive; otherwise, G should have a unique minimal normal subgroup. Thus d is a prime and $G_\alpha^{\Gamma(\alpha)}$ is solvable, and hence G_α is solvable by Lemma 4.2, where $\alpha\in V$. Set $X = MN$. Then $X = MX_\alpha$, and we have $N \cong X/M = MX_\alpha/M \cong X_\alpha$. Thus X_α and hence G_α is insolvable, a contradiction. This completes the proof. □

By Lemma 4.5, we have the following corollary.

Corollary 4.6. *Assume that G contains a transitive simple subgroup T. If T is normal in a normal subgroup of G then T is normal in G.*

Proof. Let $T \trianglelefteq N \trianglelefteq G$. Then $T^g \trianglelefteq N$ for each $g\in G$. Since T is simple, both T and T^g are minimal normal subgroup of N. It follows that either $T = T^g$ or $T\cap T^g = 1$.

Suppose that $T \neq T^g$ for some $g \in G$. Then $T \cap T^g = 1$, and $TT^g = T \times T^g$. Since T is transitive on V, it follows from [7, pp.109, Theorem 4.2A] that both T and T^g are nonabelian and regular on V, and so $|T| = |V| > d$. Let $\alpha \in V$. Then $TT^g \trianglelefteq N = TN_\alpha$, and so $T^g \cong TT^g/T \trianglelefteq TN_\alpha/T \cong N_\alpha$. Thus N_α is insolvable, and so is $N_\alpha^{\Gamma(\alpha)}$ by Lemma 4.2. Of course, $G_\alpha^{\Gamma(\alpha)}$ is insolvable, and so $G_\alpha^{\Gamma(\alpha)}$ is 2-transitive on $\Gamma(\alpha)$. Then Γ is $(G, 2)$-arc-transitive, and (1) or (2) of Lemma 4.1 occurs for N.

Assume that (1) of Lemma 4.1 occurs, that is, N_α acts 2-transitively on $\Gamma(\alpha)$. Then, since N is transitive on V, we conclude that Γ is $(N, 2)$-transitive. By Lemma 4.5, N has at most one transitive minimal normal subgroup. Noting that T and T^g are minimal normal subgroups of N, we have $T = T^g$, a contradiction.

Assume that (2) of Lemma 4.1 occurs. Recalling that N_α has a normal simple subgroup isomorphic to T^g, by Lemma 4.2, T is isomorphic to a composition factor of either $N_\alpha^{\Gamma(\alpha)}$ or $(N_\alpha^{\Gamma(\alpha)})_\beta$. It follows that either $d = 28$ and $T \cong \mathrm{PSL}_2(8)$, or $d = p^2$ and $T \cong \mathrm{PSL}_2(5)$, where $p \in \{19, 29, 59\}$. The latter case forces that $|V| = |T| = 60 < d$, a contradiction. Therefore, we let $d = 28$ and $T = \mathrm{PSL}_2(8)$. Since T is regular on V, identifying V with T, the group N lies in the holomorph $T{:}\mathsf{Aut}(T)$ of T, where T acts on V by right multiplication. Letting α be the vertex corresponding to the identity of T, we have $N_\alpha \leqslant \mathsf{Aut}(T) \cong T.\mathbb{Z}_3$. Recall that N_α has a normal subgroup isomorphic to T. We conclude that $N_\alpha = \mathsf{Inn}(T)$ or $\mathsf{Aut}(T)$. Since $N_\alpha \neq 1$, by Lemma 4.1, $\Gamma(\alpha)$ is an N_α-orbit on V. Thus $\Gamma(\alpha)$, as a subset of T, is a conjugacy class of length 28 in T or under $\mathsf{Aut}(T)$, which is impossible by the Atlas [6].

The argument above shows that $T = T^g$ for all $g \in G$. Then $T \trianglelefteq G$, and the result follows. □

In the following, we always assume that G contains a transitive nonabelian simple subgroup T. Since Γ is connected and T is transitive on V, if Γ is a bipartite graph then T has a subgroup of index 2, which is impossible. Thus Γ is not bipartite. Then the next lemma follows at once from [28, Theorem 4.1], see also (I)-(III) above.

Lemma 4.7. *Assume that $N \trianglelefteq G$ and N contains a transitive nonabelian simple subgroup T. Let K be an intransitive normal subgroup of N, and $\alpha \in V$. If N_α acts primitively on $\Gamma(\alpha)$, then K is semiregular and has at least three orbits on V; in particular, $|K|$ is a proper divisor of $|V|$ and $|T|$.*

Lemma 4.8. *Assume that G is quasiprimitive on V, and G contains a transitive nonabelian simple subgroup T. Then either $\mathsf{soc}(G)$ is simple and $T \leqslant \mathsf{soc}(G)$, or Γ is the complete graph on 8 vertices, $T \cong \mathrm{PSL}_3(2)$ and $G \cong \mathrm{AGL}_3(2)$.*

Proof. Let $N = \mathsf{soc}(G)$. By Lemma 4.5, N is the unique minimal normal subgroup of G. Write $N = T_1 \times T_2 \times \cdots \times T_k$, where $k \geqslant 1$ and T_i are isomorphic simple groups.

Case 1. Assume first that N is abelian. Then G is primitive on V, $N \cong \mathbb{Z}_p^k$ and $G \lesssim \mathrm{AGL}_k(p)$ for some prime p. In this case, N is regular on V and $T \lesssim \mathrm{GL}_k(p)$, in particular, $k \geqslant 2$. If Γ is $(G, 2)$-arc-transitive then $p = 2$, refer to [16, Theorem 1]. If d is an odd prime then $|N| = |V|$ is even, and so $p = 2$.

Since T is transitive on V, we have $|T : T_\alpha| = 2^k$ for $\alpha \in V$. By [15], $k \geqslant 3$ and either $T = \mathrm{A}_{2^k}$, or $T = \mathrm{PSL}_n(q)$ with $\frac{q^n-1}{q-1} = 2^k$. Note that $\mathrm{A}_{2^k} \not\lesssim \mathrm{GL}_k(2)$, see [17, pp. 186, Proposition 5.3.7]. Then $T \cong \mathrm{PSL}_n(q)$, and $\frac{q^n-1}{q-1} = 2^k$. In particular, $q^n - 1$ has no primitive prime divisor. By Zsigmondy's Theorem, $n = 2$ and $q = 2^k - 1$. By [17, pp. 188, Theorem 5.3.9], we have $k \geqslant \frac{q-1}{(2,q-1)} = 2^{k-1} - 1$, yielding $k \leqslant 3$. Then

$k = 3$, $N \cong \mathbb{Z}_2^3$, $T \cong \mathrm{PSL}_3(2)$, and $G \cong \mathrm{AGL}_3(2)$. In particular, G is 3-transitive on V, and thus Γ is the complete graph on 8 vertices.

Case 2. Now assume that N is nonabelian. Suppose that $T \not\leqslant N$. Then $T \cap N = 1$, and $TN/N \cong T$. Since N is the unique minimal normal subgroup of G, we have $\mathbf{C}_G(N) = 1$, and thus T acts faithfully on $\{T_1, T_2, \ldots, T_k\}$ by conjugation. Then T is isomorphic to a subgroup of the symmetric group S_k. In particular, $|T|$ is a divisor of $k!$. Noting that $G = NG_\alpha$ for $\alpha \in V$, we have $T \cong TN/N \leqslant G/N \cong G_\alpha/(G_\alpha \cap N)$, and so G_α is insolvable. Then Γ is $(G, 2)$-arc-transitive, by [27, Theorem 2], G satisfies III(b)(i) or III(c) described as in [27, Section 2]. It follows that $|T_1|$ has a prime divisor p such that $|V|$ is divisible by p^k. Since T is transitive on V, it follows that p^k is a divisor of $|T|$. Thus $k!$ is divisible by p^k, and so $k \leqslant \nu_p(k!)$. By Legendre's formula, $\nu_p(k!) = \frac{k - s_p(k)}{p-1} \leqslant k - 1$, which lead to a contradiction. Therefore, $T \leqslant N$.

To complete the proof it remains to show that $k = 1$. Suppose on the contrary that $k > 1$, and consider the projections:

$$\phi_i : N \to T_i, x_1 \cdots x_k \mapsto x_i,\ x_j \in T_j,\ 1 \leqslant i, j \leqslant k.$$

Without loss of generality, we may let $\phi_1(T) \neq 1$. Then $T \cong \phi_1(T) \leqslant T_1$. Note that $T \neq N$, and so N is not regular on V. Let $\alpha \in V$. By Lemma 4.1, N_α acts transitively on $\Gamma(\alpha)$. Since N is transitive on V, we know that Γ is N-arc-transitive.

Recall that either Γ is $(G, 2)$-arc-transitive or the valency d of Γ is a prime. Suppose that d is a prime. Then Lemma 4.5 holds for the pair (N, Γ), and so N has at most one transitive minimal normal subgroup. Noting that $N = T_1 \times \cdots \times T_k$ with $k > 1$, it follows that every T_i is intransitive on V. Considering the quadruple (Γ, N, T, T_1), by Lemma 4.7, $|T_1|$ is a proper divisor of $|T|$, which contradicts that $T \cong \phi_1(T) \leqslant T_1$. Therefore, d is not a prime, and Γ is $(G, 2)$-arc-transitive.

Since N is not regular on V, by [27, Theorem 2], N satisfies III(b)(i) described as in [27, Section 2]. Then $N_\alpha \leqslant R_1 \times \cdots \times R_k$ for $\alpha \in V$, where $R_i = \phi_i(N_\alpha) < T_i$ for $1 \leqslant i \leqslant k$, and $R_1 \cong R_2 \cong \cdots \cong R_k$. In particular, $|N_\alpha|$ divides $|R_1|^k$. On the other hand, since $T \leqslant N$ and T is transitive on V, we have $N = TN_\alpha$, and so $N/T = TN_\alpha/T \cong N_\alpha/(N_\alpha \cap T)$. In particular, $|N/T|$ divides $|N_\alpha|$. Recalling that $T \lesssim T_1$ and $|N| = |T_1|^k$, it follows that $|T_1|^{k-1}$ divides $|N_\alpha|$, and hence $|T_1|^{k-1}$ divides $|R_1|^k$. Since $k > 1$, we have that $|T_1|$ divides $|R_1|^k$. Since $R_1 < T_1$, we conclude that a prime r is a divisor of $|T_1|$ if and only if r is a divisor of $|R_1|$. It follows from [24, Corollary 5 and Table 10.7] that R_1 is insolvable. Thus N_α is insolvable, and so $N_\alpha^{\Gamma(\alpha)}$ is insolvable by Lemma 4.2. Then N_α acts primitively on $\Gamma(\alpha)$ by Lemma 4.1.

Recalling that N is the unique minimal normal subgroup of G, we have $N \,\mathsf{char}\, G$. If T_1 is transitive on V then, applying Corollary 4.6 to the pair (G, T_1), we have $T_1 \trianglelefteq G$, contrary to the minimality of N. Thus T_1 is intransitive on V. Considering the quadruple (Γ, N, T, T_1), by Lemma 4.7, $|T_1|$ is a proper divisor of $|T|$, which contracts that $T \lesssim T_1$. Therefore, $k = 1$. This completes the proof. □

Corollary 4.9. *Assume that G contains a transitive minimal normal subgroup N and a transitive nonabelian simple subgroup T. Then either $d = 7$, $|V| = 8$ and $G \cong \mathrm{AGL}_3(2)$, or $T \leqslant N$ and N is simple.*

Proof. Choose a maximal intransitive normal subgroup K of G. Then $T \cap K = N \cap K = 1$; in particular, $KN = K \times N$. If $K = 1$ then G is quasiprimitive on V, and so the corollary is true by Lemma 4.8.

Assume that $K \neq 1$. Since $K \leq \mathbf{C}_G(N) \neq N$, by [7, Theorem 4.2A], N is nonabelian. Write $N = T_1 \times \cdots \times T_k$ for some integer $k \geqslant 1$ and isomorphic nonabelian simple groups T_i. Then G acts transitively on $\{T_1, \ldots, T_k\}$ by conjugation. It follows that G/K acts transitively on $\{T_1K/K, \ldots, T_kK/K\}$ by conjugation. Thus NK/K is a minimal normal subgroup of G/K. By Lemma 4.7, K has at least three orbits on V. Now consider the quotient graph $\Gamma_{G/K}$. Identifying G/K with a subgroup of $\mathsf{Aut}(\Gamma_{G/K})$, by Lemma 4.3 (1) and (4), we know that Lemma 4.8 works for $\Gamma_{G/K}$, G/K and TK/K. Noting that $N = T_1 \times \cdots \times T_k \cong NK/K \trianglelefteq G/K$, we have $G/K \ncong \mathrm{AGL}_3(2)$, and hence NK/K is simple and $TK/K \leqslant NK/K$. By Lemma 4.7, $|K|$ is a proper divisor of $|T|$. If $T \nleqslant N$ then $N \cap T = 1$ as T is simple, and so $T \cong TN/N \leqslant KN/N \cong K$, a contradiction. Thus $N \geqslant T$, and our result is true. ☐

Lemma 4.10. *Assume that G contains a transitive nonabelian simple subgroup T. Let K be a maximal intransitive normal subgroup of G. Then either*

(1) *$G \cong \mathrm{AGL}_3(2)$, $K = 1$, $|V| = 8$ and $d = 7$; or*

(2) *T is contained in a characteristic perfect subgroup N of G such that $N/\mathsf{rad}(N)$ is simple, $K \cap N = \mathsf{rad}(N)$ and $K/\mathsf{rad}(N) = \mathbf{C}_{G/\mathsf{rad}(N)}(N/\mathsf{rad}(N))$.*

Proof. By the choice of K, we know that G^{V_K} is a quasiprimitive permutation group on V_K. By Lemma 4.7, K is semiregular and has at least three orbits on V. It follows from (4) of Lemma 4.3 and Lemma 4.8 that either $d = 7$, $|V_K| = 8$ and $G^{V_K} \cong \mathrm{AGL}_3(2)$, or $\mathsf{soc}(G^{V_K})$ is a nonabelian simple group and $T^{V_K} \leqslant \mathsf{soc}(G^{V_K})$.

Case 1. Assume that $G^{V_K} \cong \mathrm{AGL}_3(2)$. Then $(G^{V_K})_{\alpha^K} \cong T \cong \mathrm{PSL}_3(2)$, where $\alpha \in V$. Let $I \lhd G$ with $K < I$ and $I/K \cong \mathbb{Z}_2^3$. Then $G = I{:}T$ and I is regular on V. In particular, $|V| = 8|K| = |I|$. Noting that $|V| = |T : T_\alpha|$, it follows that $|K|$ is a divisor of 21, and so K is solvable. Since $G/K \cong G^{V_K} \cong \mathrm{AGL}_3(2)$, we have $G^{(\infty)}/(G^{(\infty)} \cap K) \cong KG^{(\infty)}/K \cong (G/K)^{(\infty)} \cong \mathrm{AGL}_3(2) \cong G/K$. It follows that $G = KG^{(\infty)}$, and $G^{(\infty)}$ is a perfect extension of $(G^{(\infty)} \cap K){:}\mathbb{Z}_2^3$ by $\mathrm{PSL}_3(2)$. Noting that $(G^{(\infty)} \cap K){:}\mathbb{Z}_2^3$ is solvable, it follows from Lemma 3.3 that $G^{(\infty)} \cong \mathrm{AGL}_3(2)$, and $G^{(\infty)} \cap K = 1$. Since $G = KG^{(\infty)}$, we have $((G^{(\infty)})^{V_K})_{\alpha^K} = (G^{V_K})_{\alpha^K} \cong \mathrm{PSL}_3(2)$. By Lemma 4.4, K is isomorphic to a quotient group of $\mathrm{PSL}_3(2)$, and so $K = 1$ as $|K| < |T|$. Then $G = G^{(\infty)} \cong \mathrm{AGL}_3(2)$, and part (1) of this lemma follows.

Case 2. Assume that $T^{V_K} \leqslant \mathsf{soc}(G^{V_K})$ and $\mathsf{soc}(G^{V_K})$ is simple. In this case, we have $\mathsf{soc}(G^{V_K}) \cong \mathsf{soc}(G/K)$ and, letting $I = K \cap G^{(\infty)}$,

$$T \cong TK/K \leqslant \mathsf{soc}(G/K) = (G/K)^{(\infty)} = G^{(\infty)}K/K \cong G^{(\infty)}/I.$$

By Lemma 4.7, $|K|$ is a proper divisor of $|T|$. Then $|I|$ is a proper divisor of $|T|$. Since $T \cong G^{(\infty)}/I$, we know that $|I|^2$ is a proper divisor of $|G^{(\infty)}|$. In particular, $G^{(\infty)} \ncong I \times I$. Then, by Lemma 3.1, we may choose $N \,\mathsf{char}\, G^{(\infty)}$ such that $G^{(\infty)} = IN$ and $I \cap N = \mathsf{rad}(N)$. Clearly, $N \,\mathsf{char}\, G$, and $\mathsf{rad}(N) = I \cap N = K \cap N$. Let $\overline{G} = G/\mathsf{rad}(N)$, $\overline{N} = N/\mathsf{rad}(N)$ and $\overline{K} = K/\mathsf{rad}(N)$. We have $\overline{K}\,\overline{N} = \overline{K} \times \overline{N}$, that is, $\overline{K} \leqslant \mathbf{C}_{\overline{G}}(\overline{N})$.

Note that $\mathsf{rad}(N) \lhd G$ and $\mathsf{rad}(N)$ is intransitive on V. By (1) and (4) of Lemma 4.3, $\Gamma_{G/\mathsf{rad}(N)}$ has valency d and, identifying $\overline{G}$ with a subgroup of $\mathsf{Aut}(\Gamma_{G/\mathsf{rad}(N)})$, either d is a prime or $\Gamma_{G/\mathsf{rad}(N)}$ is $(\overline{G}, 2)$-arc-transitive. By the choice of N, we have

$$\overline{N} = N/\mathsf{rad}(N) \cong G^{(\infty)}/I \cong G^{(\infty)}K/K = \mathsf{soc}(G/K).$$

Then $\overline{N}$ is simple, and so $\overline{N}$ is a minimal normal subgroup of $\overline{G}$. Noting that $T \leqslant G^{(\infty)}$, we have $T \cong TK/K \leqslant G^{(\infty)}K/K \cong \overline{N}$. In particular, $|T|$ divides $|\overline{N}|$.

Let $\overline{T} = T\mathsf{rad}(N)/\mathsf{rad}(N)$. Then $\overline{T} \cong T$. Since T is transitive on V, it is easy to see that $\overline{T}$ acts transitively on $V_{\mathsf{rad}(N)}$; in particular, $|V_{\mathsf{rad}(N)}|$ is a divisor of $|\overline{T}|$. If $\overline{N}$ is intransitive on $V_{\mathsf{rad}(N)}$ then, by (1) of Lemma 4.3, $|\overline{N}|$ is a proper divisor of $|V_{\mathsf{rad}(N)}|$, and so $|\overline{N}| < |V_{\mathsf{rad}(N)}| \leqslant |\overline{T}| \leqslant |\overline{N}|$, a contradiction. Thus $\overline{N}$ is a transitive minimal normal subgroup of $\overline{G}$. By Corollary 4.9, we have $\overline{T} \leqslant \overline{N}$, yielding $T \leqslant N$.

Suppose that $\mathbf{C}_{\overline{G}}(\overline{N})$ is transitive on $V_{\mathsf{rad}(N)}$. Then both $\overline{N}$ and $\mathbf{C}_{\overline{G}}(\overline{N})$ are regular on $V_{\mathsf{rad}(N)}$, see [7, Theorem 4.2A]. This implies that $\overline{N} \cong \mathbf{C}_{\overline{G}}(\overline{N})$, refer to [7, Lemma 4.2A]. Thus $\mathbf{C}_{\overline{G}}(\overline{N})$ is simple, and hence $\mathbf{C}_{\overline{G}}(\overline{N})$ is a transitive minimal normal subgroup of $\overline{G}$. It follows from Lemma 4.5 that $\overline{N} = \mathbf{C}_{\overline{G}}(\overline{N})$, and so $\overline{N}$ is abelian, a contradiction.

Suppose that $\mathbf{C}_{\overline{G}}(\overline{N})$ is intransitive on $V_{\mathsf{rad}(N)}$. Set $\mathbf{C}_{\overline{G}}(\overline{N}) = C/\mathsf{rad}(N)$. Then C is intransitive on V. Recalling that $\overline{K} \leqslant \mathbf{C}_{\overline{G}}(\overline{N})$, we have $K \leqslant C$, and hence $K = C$ by the choice of K. Then part (2) of this lemma follows. □

Lemma 4.11. *Assume that G contains a transitive nonabelian simple subgroup T. Let N and K be as in* (2) *of Lemma* 4.10*. Then either N is quasisimple or* (4) *of Lemma* 3.3 *holds for N and T.*

Proof. By Lemma 4.7, $|K|$ is a divisor of $|T|$, and so $|\mathsf{rad}(N)|$ is a divisor of $|T|$ as $\mathsf{rad}(N) = K \cap N$. Then N, $\mathsf{rad}(N)$ and T are described as in Lemma 3.3. Thus it suffices to show N and T do not satisfy one of (1)-(3) given as in Lemma 3.3.

Again by Lemma 4.7, K has at least three orbits on V. Then Lemma 4.3 holds for (Γ, G, K, X), where $X \leqslant G$. For convenience, we put $\overline{X} = XK/K$ and identify $\overline{X}$ with a subgroup of $\mathsf{Aut}(\Gamma_{G/K})$. Then $\overline{T} \cong T$, $K \cap N = \mathsf{rad}(N)$ and $\overline{N} \cong N/\mathsf{rad}(N)$. Fix $\alpha \in V$, and let $B = \alpha^K$. Since $K \cap T = 1$, applying (2) of Lemma 4.3 to the pair (K, T), we conclude that $|\overline{T}_B|$ is divisible by $|K|$, and so $|\overline{N}_B|$ is divisible by $|K|$.

Case 1. Suppose that (1) or (2) of Lemma 3.3 holds for N and T. Then $N = \mathsf{rad}(N){:}T$, and so $\mathsf{soc}(\overline{G}) = \overline{N} = \overline{T} \cong T$. In this case, $|\overline{G} : \overline{N}| \leqslant 2$, we have $|\overline{G}_B : \overline{N}_B| \leqslant 2$. Thus $|\overline{N}_B|$ is divisible by every odd divisor of $|\overline{G}_B|$. In particular, $\overline{N}_B \neq 1$, and so Lemma 4.1 works for $(\Gamma_{G/K}, \overline{G}, \overline{N})$.

Subcase 1.1. Assume $N = [2^k]{:}\mathrm{A}_8$ with $k \in \{4, 5, 6\}$. Then $|K \cap N| = |\mathsf{rad}(N)| = 2^k$, $\mathsf{soc}(\overline{G}) = \overline{N} = \overline{T} \cong \mathrm{A}_8$, and $|\overline{N}_B|$ is divisible by 2^k.

Suppose that $\overline{N}_B$ is insolvable. Using GAP [29], we search the insoluble subgroups of A_8 with order divisible by 2^k. It follows that $\overline{N}_B \cong \mathrm{S}_6$ or $\mathbb{Z}_2^3{:}\mathrm{PSL}_3(2)$. Assume that $\overline{N}_B \cong \mathrm{S}_6$. Then the action of $\overline{N}$ on V_K is equivalent to the rank three action of A_8 on the 2-subsets of a 8-set. It follows that $d = 12$ or 15. In this case, $\Gamma_{G/K}$ is $(\overline{G}, 2)$-arc-transitive and of valency d, and then $d - 1$ is a divisor of $|\overline{G}_B|$. Recalling that $|\overline{N}_B|$ is divisible by every odd divisor of $|\overline{G}_B|$, it follows that $|\overline{N}_B|$ has a divisor 11 or 7, which is impossible as $\overline{N}_B \cong \mathrm{S}_6$. Thus, we have $\overline{N}_B \cong \mathbb{Z}_2^3{:}\mathrm{PSL}_3(2)$. Then the action of $\overline{N}$ on V_K is equivalent to the 2-transitive action of $\mathrm{PSL}_4(2)$ on the projective points or on hyperplanes. This implies that $\Gamma_{G/K}$ is the complete graph of order 15, and then $\overline{G}$ acts 3-transitively on V_K. Noting that $\overline{N}$ is not 3-transitive on V_K, we have $\overline{N} \neq \overline{G}$. Then $\overline{G} \cong \mathrm{S}_8$; however, S_8 has no transitive permutation representation of degree 15, a contradiction.

Next we suppose that $\overline{N}_B$ is solvable. By (3) of Lemma 4.1, d is a prime power. Since $\overline{N} = \overline{T} \cong \mathrm{A}_8$, considering the prime divisors of A_8, we conclude that $d \in \{2^l, 3, 5, 7, 9\}$, where $2 \leqslant l \leqslant 6$. Let $m = 2^k d$ if d is odd, or $m = 2^k(d-1)$ if d is even.

Then $|\overline{N}_B|$ is divisible by m. Searching by GAP the solvable subgroups of A_8 with order divisible by m, we conclude that $\overline{N}_B$ has the form of $[2^s]{:}S_3$ or $\mathbb{Z}_2^4{:}\mathbb{Z}_3^2{:}\mathbb{Z}_2^t$, where $s \geqslant 3$ and $0 \leqslant t \leqslant 2$. In particular, $d \in \{3, 4, 9\}$. Checking the vertex-stabilizers for connected arc-transitive graphs of valency 4, refer to [19, Lemma 2.6], we have $d \neq 4$. If $d = 3$ then $|\overline{N}_B| = 48$ by [33], and thus $|V_K| = 420$; however, by [4], there is no connected arc-transitive cubic graph of order 420.

Assume that $d = 9$. Then $|\mathbf{O}_2(\overline{N}_B)| \geqslant 2^4$. Noting that $\mathbf{O}_2(\overline{N}_B)\ \mathsf{char}\ \overline{N}_B$, it follows that $\mathbf{O}_2(\overline{N}_B) \trianglelefteq \overline{G}_B$, and then $\mathbf{O}_2(\overline{N}_B)$ lies in the kernel of $\overline{G}_B$ acting on $\Gamma_{G/K}(B)$. Since $\overline{G}_B$ acts 2-transitively on $\Gamma_{G/K}(B)$, we know that 72 is a divisor of $|\overline{G}_B^{\Gamma_{G/K}(B)}|$, and so $|\overline{G}_B|$ is divisible by $72|\mathbf{O}_2(\overline{N}_B)|$. Then $|\overline{G}_B|$ has a divisor $2^7 \cdot 3^2$. Noting that $\overline{G} \lesssim S_8$, it follows that $|\overline{G} : \overline{G}_B|$ is odd. Then $\Gamma_{G/K}$ has odd order and odd valency, which is impossible.

Subcase 1.2. Assume that $N = \mathbb{Z}_2^{n-2}{:}A_n$, where $n = 2^e$ for some $e \geqslant 4$. Then $\mathsf{soc}(\overline{G}) = \overline{N} = \overline{T} \cong A_n$, and $|K \cap N| = |\mathsf{rad}(N)| = 2^{n-2}$. By (2) of Lemma 4.3, $|\overline{T}_B|$ is divisible by 2^{n-2}, it follows that $\overline{T}_B$ has odd index in $\overline{T}$, and so $|V_K| = |\overline{T} : \overline{T}_B|$ is odd. Then $\Gamma_{G/K}$ is a $(\overline{G}, 2)$-arc-transitive graph of odd order. By [18, Theorem 1.1][1], n is odd, a contradiction.

Subcase 1.3. Assume that $N \cong \mathrm{AGL}_3(2)$. Then $\mathsf{soc}(\overline{G}) = \overline{N} = \overline{T} \cong \mathrm{PSL}_3(2)$, and $|K \cap N| = |\mathsf{rad}(N)| = 2^3$. By (2) of Lemma 4.3, $|\overline{N}_B|$ is divisible by 2^3. Checking the subgroups of $\mathrm{PSL}_3(2)$ with order divisible by 8, we have $\overline{N}_B \cong S_4$ or D_8. If $\overline{N}_B \cong D_8$ then, noting that $|\overline{G} : \overline{N}| \leqslant 2$, we have $|\overline{G}_B| \in \{8, 16\}$, which is impossible as Γ_K is $(\overline{G}, 2)$-arc-transitive. Thus $\overline{N}_B \cong S_4$ and, since $|\overline{N} : \overline{N}_B| = |V_K| = |\overline{G} : \overline{G}_B|$, we have $\overline{G} = \overline{N}$ by checking the subgroups of $\overline{G}$. Thus $\Gamma_{G/K}$ is the complete graph of order 7. From the 2-arc-transitivity of $\overline{G}$ on $\Gamma_{G/K}$, we conclude that $\mathrm{PSL}_3(2)$ has a 3-transitive permutation representation of degree 7, which is impossible.

Subcase 1.4. Assume that $N = \mathbb{Z}_2^6{:}\mathrm{PSp}_4(3) \lesssim \mathrm{AGL}_6(2)$. Then $\mathsf{soc}(\overline{G}) = \overline{N} = \overline{T} \cong \mathrm{PSp}_4(3)$, and $|K \cap N| = |\mathsf{rad}(N)| = 2^6$. By (2) of Lemma 4.3, $|\overline{N}_B|$ is divisible by 2^6. In particular, $|V_K| = |\overline{N} : \overline{N}_B|$ is odd, and so d is even. It follows that $\Gamma_{G/K}$ is $(\overline{G}, 2)$-arc-transitive, and Γ is $(G, 2)$-arc-transitive. If $d = 4$ or 6 then, by [19, Lemma 2.6] and [20, Theorem 3.4], $|\overline{G}_B|$ is indivisible by 2^6, a contradiction.

Now let $d \geqslant 8$. Checking the subgroups of $\mathrm{PSp}_4(3)$ with order divisible by 2^6, we conclude that $|\mathbf{O}_2(\overline{N}_B)| \geqslant 2^4$, and $\mathbb{Z}_2^4{:}\mathbb{Z}_2^2 \leqslant \overline{N}_B \leqslant \mathbb{Z}_2^4{:}A_5$. Recalling that $|\overline{N}_B|$ is divisible by every odd divisor of $|\overline{G}_B|$, it follows that $|\overline{N}_B|$ is divisible by $d-1$. Then the only possibility is that $d = 16$ and $\overline{N}_B = \mathbb{Z}_2^4{:}A_5$. By Lemma 4.2, $\overline{N}_B^{\Gamma_{G/K}(B)}$ is insolvable. It follows from Lemma 4.1 that $\overline{N}_B^{\Gamma_{G/K}(B)}$ is 2-transitive on $\Gamma_{G/K}(B)$, and so $\Gamma_{G/K}$ is $(\overline{N}, 2)$-arc-transitive as $\overline{N}$ is transitive on V_K. Then, by (3) of Lemma 4.3, Γ is $(KN, 2)$-arc-transitive.

By Lemma 4.4, $K/(K \cap N)$ is isomorphic to a quotient group of $\overline{N}_B$, it follows that $K/(K \cap N) = 1$, and so $K = K \cap N = \mathsf{rad}(N)$. Thus Γ is an $(N, 2)$-arc-transitive graph of valency 16. By (2) of Lemma 4.3, $N_\alpha \cong \overline{N}_B$, and so $N_\alpha \cong \mathbb{Z}_2^4{:}A_5$. Let $\beta \in \Gamma(\alpha)$, and $x \in N$ with $(\alpha, \beta)^x = (\alpha, \beta)$. Then $N_{\alpha\beta} \cong A_5$, $x \in \mathbf{N}_N(N_{\alpha\beta})$ and $x^2 \in N_{\alpha\beta}$. Since Γ is connected, $N = \langle x, N_\alpha \rangle$, refer to [2, page 118, 17B]. Recall that $N = \mathbf{O}_2(N){:}T = \mathbb{Z}_2^6{:}\mathrm{PSp}_4(3) \lesssim \mathrm{AGL}_6(2)$. By the Atlas [6], for $1 \leqslant l \leqslant 5$, we

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[1] In part (ii) of [18, Theorem 1.1], the value of n should be $2^{e+1} - 1$ but not $\binom{2^{e+1}-1}{2^e-1}$.

conclude that $\mathrm{SL}_l(2)$ has no subgroup isomorphic to $T = \mathrm{PSp}_4(3)$. It follows that T is an irreducible subgroup of $\mathrm{GL}_6(2)$, and thus we may consider N as an affine primitive permutation group of degree 2^6. Confirmed by GAP, N has a unique conjugacy class of subgroups isomorphic to N_α. This allows us the choose N_α as a subgroup of T. Then, by a further computation using GAP, we conclude that there is no desired x with $N = \langle x, N_\alpha\rangle$, a contradiction.

Case 2. Suppose that N and T satisfy (3) of Lemma 3.3. Then $\overline{N}$ is a simple group of Lie type with characteristic 2, and $\overline{N} \neq \overline{T} \cong T = \mathrm{A}_{2^e}$ for some $e \geqslant 3$. Noting that $\overline{N} = \overline{T}\,\overline{N}_B$, by [30, Theorem 1.1], $T = \mathrm{A}_8$ and one of the following holds:

(i) $\overline{N} \cong \mathrm{PSp}_6(2)$, and $\overline{N}_B \cong [3^3]{:}\mathbb{Z}_8{:}\mathbb{Z}_2$, $[3^3]{:}2\mathrm{S}_4$, $\mathrm{PSL}_2(8)$, $\mathrm{PSL}_2(8){:}3$, $\mathrm{PSU}_3(3){:}2$ or $\mathrm{PSU}_4(2){:}2$;
(ii) $\overline{N} \cong \mathrm{PSp}_8(2)$, and $\overline{N}_B \cong \mathrm{P\Omega}_8^-(2).2$;
(iii) $\overline{N} \cong \mathrm{P\Omega}_8^+(2)$, and $\overline{N}_B \cong \mathrm{Sp}_6(2)$, $\mathrm{PSU}_4(2)$, $\mathrm{PSU}_4(2){:}2$, $3 \times \mathrm{PSU}_4(2)$, $(3 \times \mathrm{PSU}_4(2)){:}2$ or A_9.

By Lemma 3.3, $\overline{N} \lesssim \mathrm{PSL}_l(2)$ for some l with $2^l \leqslant |\mathbf{O}_2(N)| \in \{2^4, 2^5, 2^6\}$. It follows from [17, page 200, Proposition 5.4.13] that $l = 6$ and $\overline{N} \cong \mathrm{PSp}_6(2)$. Then $\overline{G} = \overline{N}$. Recalling that $|\mathsf{rad}(N)|$ is a divisor of $|\overline{T}_B|$, it follows that 2^6 is a divisor of $|\overline{G}_B|$. This forces that $\overline{G}_B \cong \mathrm{PSU}_3(3){:}2$ or $\mathrm{PSU}_4(2){:}2$. By the 2-arc-transitivity of $\overline{G}$ on $\Gamma_{G/K}$, either $\mathrm{PSU}_3(3){:}2$ or $\mathrm{PSU}_4(2){:}2$ has a 2-transitive permutation representation of degree d, which is impossible by [3, Table 7.4]. This completes the proof. □

Proof of Theorem 1.2. Let $\Gamma = (V, E)$ be a connected G-arc-transitive graph of valency $d \geqslant 3$. Assume that G contains a vertex-transitive nonabelian simple subgroup T, and that either d is a prime or Γ is $(G, 2)$-arc-transitive. By Lemma 4.5, G has at most one transitive minimal normal subgroup. If G has a transitive minimal normal subgroup M then, by Corollary 4.9, either (1) of Theorem 1.2 holds or M is simple and $T \leqslant M$. In the general case, taking a maximal intransitive normal subgroup K of G, by Lemma 4.10, either (Γ, G) is described as in (1) of Theorem 1.2, or G has a characteristic perfect subgroup N such that $T \leqslant N$, $N/\mathsf{rad}(N)$ is simple, $K \cap N = \mathsf{rad}(N)$ and $K/\mathsf{rad}(N) = \mathbf{C}_{G/\mathsf{rad}(N)}(N/\mathsf{rad}(N))$. For the latter case, $|\mathsf{rad}(N)|$ is a divisor of $|T|$ by Lemma 4.7, and we obtain (2)(i) or (ii) of Theorem 1.2 from Lemma 4.11. This completes the proof. □

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Z.P. Lu, Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, P. R. China

Email address: lu@nankai.edu.cn